

Analisi di Reti (modulo II)

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Algorithmic Game Theory

Algorithmic Issues in Non-cooperative (i.e.,
strategic) Distributed Systems

Two Research Traditions

- Theory of Algorithms: computational issues
 - What can be feasibly computed?
 - How much does it take to compute a solution?
 - Which is the quality of a computed solution?
- Game Theory: interaction between self-interested individuals
 - What is the outcome of the interaction?
 - Which social goals are compatible with selfishness?

one of the foremost mathematicians
of the 20th century



*John von Neumann
(1903-1957)*

*Games and Economic Behavior
(1944, with O. Morgenstern)*

Different Assumptions

- Theory of Algorithms (in distributed systems):
 - Processors are *obedient*, *faulty*, or *adversarial*
 - *Large* systems, *limited* computational resources
- Game Theory:
 - Players are *strategic* (selfish)
 - *Small* systems, *unlimited* computational resources

The World of Modern Networks

- Agents often autonomous (users)
 - Users have their own individual goals
 - Network components owned by providers
 - Often involve **large** networks
 - Massive systems
 - Limited **communication/computational** resources
- ⇒ *Both **strategic** and **computational** issues!*

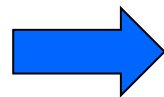
$$\text{Algorithmic Game Theory} = \text{Theory of Algorithms} + \text{Game Theory}$$

- **Game Theory** provides a bunch of tools useful for addressing **computational problems** in non-cooperative scenarios
 - networks used by self-interested users
- **Theory of algorithms** sheds light on results of **Game Theory**
 - for several results on the existence of equilibria/mechanisms we have that such an equilibrium/mechanism cannot be found/implemented efficiently

Basics of Game Theory: games & equilibria

A game

- A game consists of:
 - A set of **players**
 - A set of rules of encounter: **Who** should act **when**, and **what** are the possible actions (**strategies**)
 - A specification of **payoffs** for each combination of strategies
- **Game Theory** attempts to predict the **outcome** of the game (**solution**) by taking into account the individual behavior of the players



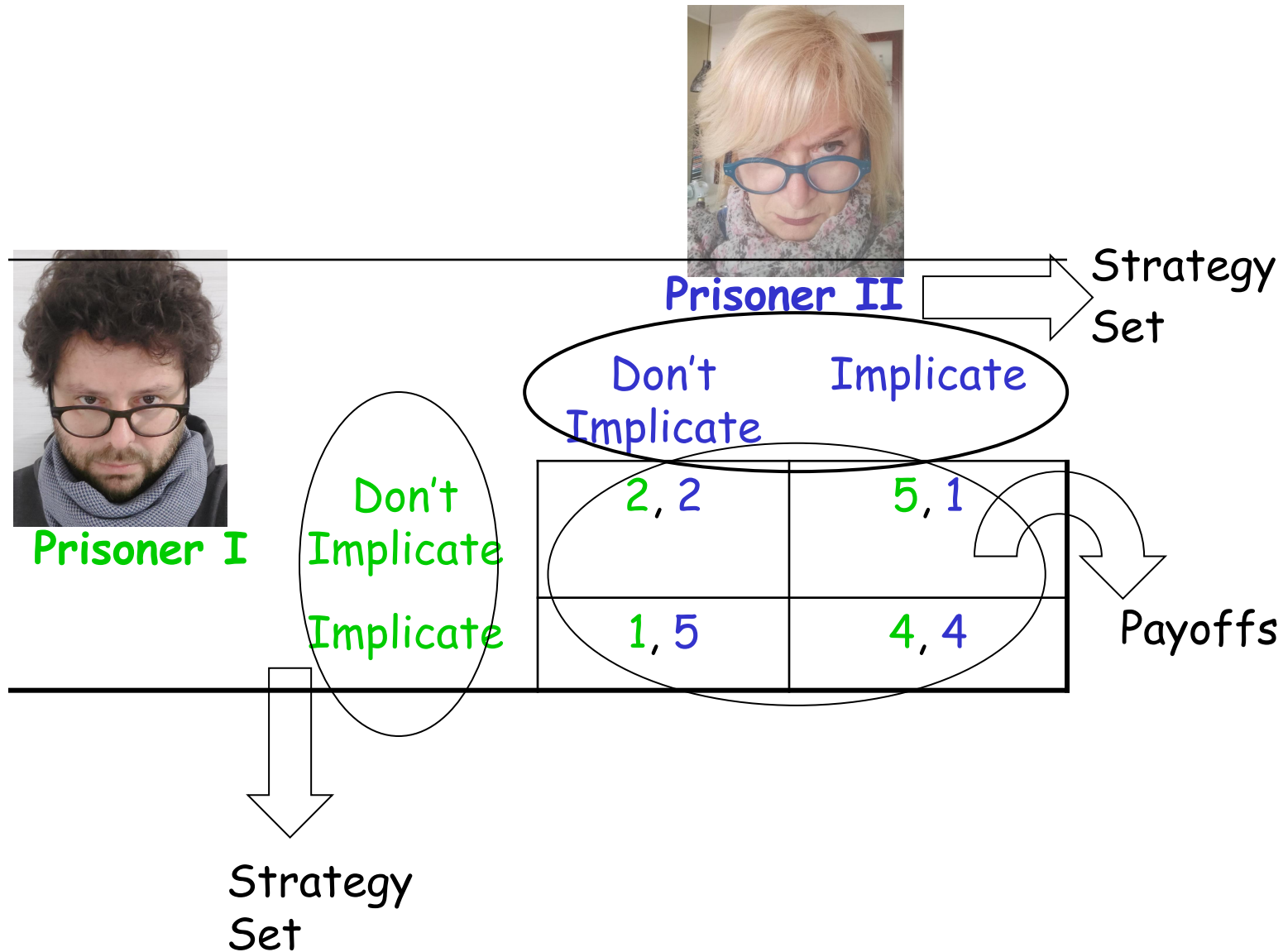
an equilibrium!

A famous one-shot game: the Prisoner's Dilemma

...the story of two strange and dangerous fellows...



A famous one-shot game: the Prisoner's Dilemma



Prisoner I's decision



Prisoner I Don't Implicate
 Implicate

Prisoner II

Don't Implicate

Implicate

2, 2	5, 1
1, 5	4, 4



■ Prisoner I's decision:

- If II chooses Don't Implicate then it is best to Implicate
- If II chooses Implicate then it is best to Implicate
- It is best to Implicate for I, regardless of what II does:
Dominant Strategy

Prisoner II's decision



Prisoner I

Don't Implicate

Implicate



Prisoner II

Don't Implicate

Implicate

2, 2

5, 1

1, 5

4, 4

■ Prisoner II's decision:

- If I chooses Don't Implicate then it is best to Implicate
- If I chooses Implicate then it is best to Implicate
- It is best to Implicate for II, regardless of what I does:
Dominant Strategy

Hence...



Prisoner I

Don't
Implicate
Implicate

Prisoner II

Don't
Implicate

Implicate



2, 2	5, 1
1, 5	<u>4, 4</u>

- It is best for both to implicate **regardless** of what the other one does
- **Implicate** is a Dominant Strategy for both
- (**Implicate**, **Implicate**) becomes the **Dominant Strategy Equilibrium**
- Note: If they might collude, then it's beneficial for both to **Not Implicate**, but it's not an equilibrium as both have incentive to deviate

A network game

C, S : peering points

two Internet Service Providers (ISP):

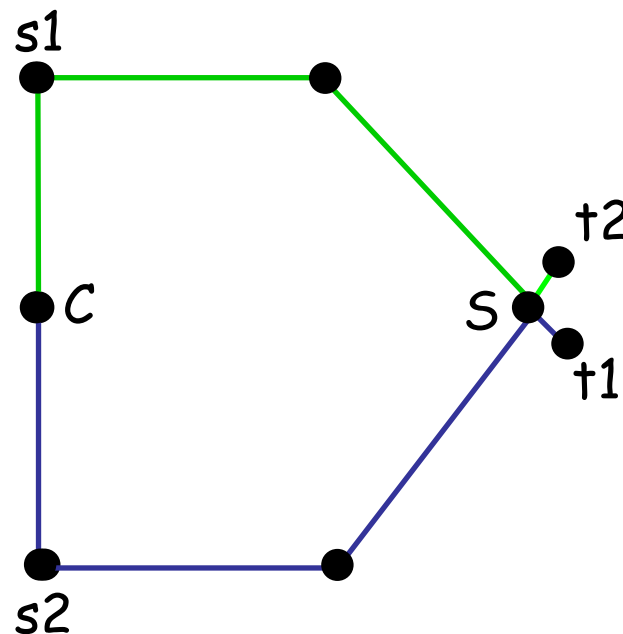
$ISP1$ e $ISP2$

$ISP1$ wants to send traffic from $s1$ to $t1$

$ISP2$ wants to send traffic from $s2$ to $t2$

(long) links have cost 1
(for ISP owning the link)

Each ISP_i can use two paths: the one passing through C o
the one passing through S

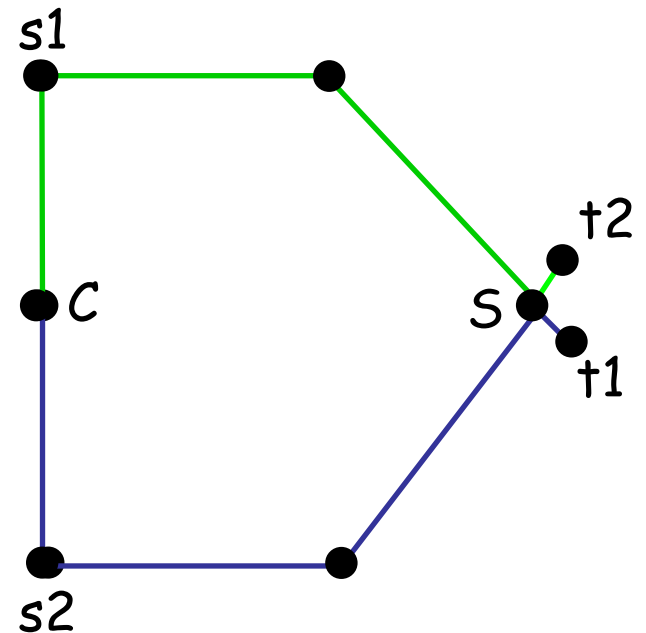


A network game

C, S : peering points

Cost Matrix

		ISP2	
		through S	through C
ISP1	through S	2, 2	5, 1
	through C	1, 5	4, 4



Formal representation of a game: Normal Form

- N rational players
- S_i = Strategy set of player i
- The strategy combination (s_1, s_2, \dots, s_N) gives payoff (p_1, p_2, \dots, p_N) to the N players
 $\Rightarrow S_1 \times S_2 \times \dots \times S_N$ payoff matrix

Dominant Strategy Equilibrium

- **Dominant Strategy Equilibrium**: is a strategy combination $s^* = (s_1^*, s_2^*, \dots, s_N^*)$, such that s_i^* is a **dominant strategy** for each i , namely, for **any possible alternative strategy profile** $s = (s_1, s_2, \dots, s_i, \dots, s_N)$:
 - if p_i is a **utility**, then $p_i(s_1, s_2, \dots, s_i^*, \dots, s_N) \geq p_i(s_1, s_2, \dots, s_i, \dots, s_N)$
 - if p_i is a **cost**, then $p_i(s_1, s_2, \dots, s_i^*, \dots, s_N) \leq p_i(s_1, s_2, \dots, s_i, \dots, s_N)$
- Dominant Strategy is the **best response** to any strategy of other players
- If a game has a DSE, then players will immediately converge to it
- Of course, not all games (only very few in the practice!) have a dominant strategy equilibrium

A more relaxed solution concept: Nash Equilibrium [1951]

- **Nash Equilibrium**: is a strategy combination

$s^* = (s_1^*, s_2^*, \dots, s_N^*)$ such that for each i , s_i^* is a **best response** to $(s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_N^*)$, namely,

for **any possible alternative strategy** s_i of player i

- if p_i is a **utility**, then $p_i(s_1^*, s_2^*, \dots, s_i^*, \dots, s_N^*) \geq p_i(s_1^*, s_2^*, \dots, s_i, \dots, s_N^*)$
- if p_i is a **cost**, then $p_i(s_1^*, s_2^*, \dots, s_i^*, \dots, s_N^*) \leq p_i(s_1^*, s_2^*, \dots, s_i, \dots, s_N^*)$

Nash Equilibrium: The Battle of the Sexes (coordination game)

		Woman	
		Stadium	Cinema
Man	Stadium	6, 5	2, 2
	Cinema	1, 1	5, 6

- (Stadium, Stadium) is a NE: Best responses to each other
- (Cinema, Cinema) is a NE: Best responses to each other

☹ but they are not Dominant Strategy Equilibria ... are we really sure they will eventually go out together????

A similar game: routing congestion game

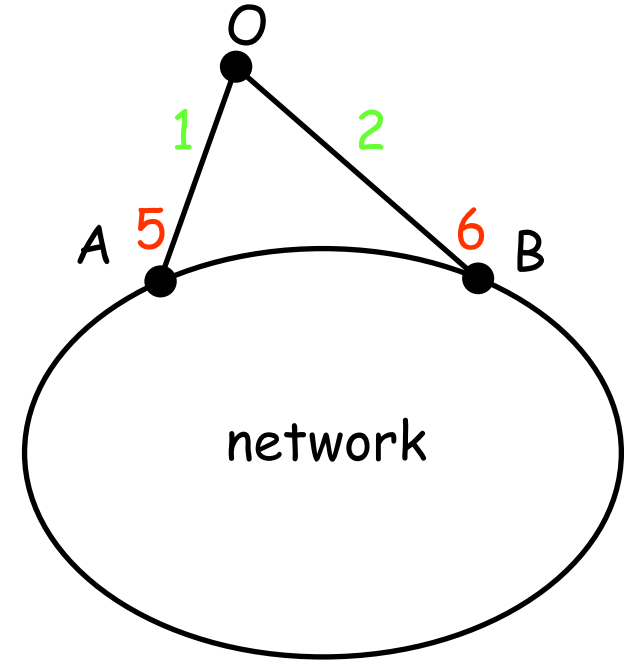
two traffic streams originated at node O need to be routed to the rest of the network

Costs without congestion:

$$c(O,A)=1 \quad c(O,B)=2$$

Costs with congestion:

$$c(O,A)=5 \quad c(O,B)=6$$

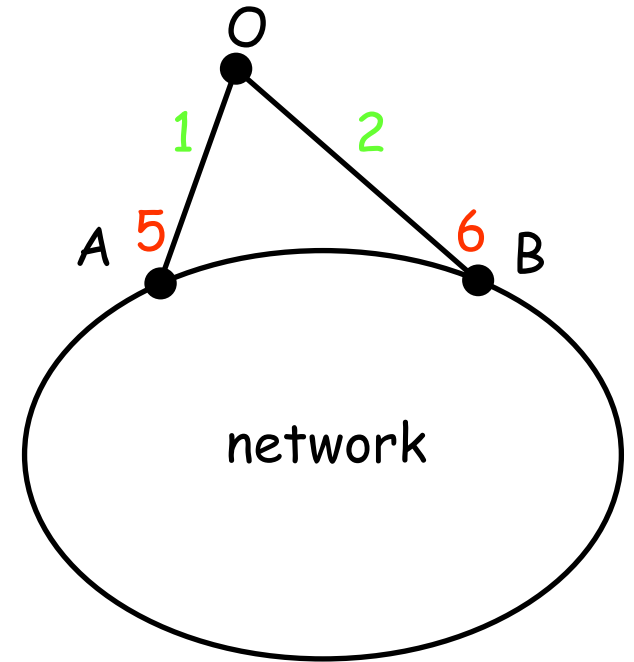


Each stream can use two paths: the one passing through A or the one passing through B

A similar game: routing congestion game

Cost Matrix

		stream 2	
		through A	through B
stream 1	through A	5, 5	1, 2
	through B	2, 1	6, 6



Nash Equilibrium

- In a NE no agent can unilaterally deviate from its strategy given others' strategies as fixed
- Agent has to deliberate about the strategies of the other agents
- Dominant Strategy Equilibrium \Rightarrow Nash Equilibrium (but the converse is not true)
- If the game is played **repeatedly** and players converge to a solution, then it has to be a NE

A big game theoretic issue: the existence of a NE

- Unfortunately, for pure strategies games (as those seen so far), it is easy to see that we cannot have a general result of existence
- In other words, there may be no, one, or many NE, depending on the game

A conflictual game: Matching pennies

		Player II	
		Head	Tail
Player I	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- ⇒ In any configuration, one of the players prefers to change his strategy
- ⇒ no NE!

On the existence of a NE (2)

- However, when a player can select his strategy **randomly** by using a **probability distribution** over his set of possible strategies (**mixed strategy**), then the following general result holds:
- **Theorem (Nash, 1951)**: Any game with a finite set of players and a finite set of strategies has a NE of **mixed strategies** (i.e., the **expected payoff** cannot be improved by changing unilaterally the selected probability distribution).
- **Matching pennies**: if each player sets $p(\text{Head})=p(\text{Tail})=1/2$, then the expected payoff of each player is 0, and this is a NE, since no player can improve on this by choosing a different probability distribution!

Analysis of a game: some typical issues

1. Establish whether the game always has a NE
2. Find a NE, once it does exist
3. In a repeated game, establish whether and in how many steps the system will eventually converge to a NE
4. Establish the quality of a NE

On the quality of a NE

- How inefficient is a NE in comparison to an ideal situation in which the players would strive to collaborate with the common goal of choosing the best outcome?
Best outcome w.r.t. what?
- we need a social-choice function C mapping strategy profiles into real numbers
 - C measures the overall quality of an outcome s
 - e.g. $C(s)$: sum of all players' costs/utilities

A worst-case perspective: the Price of Anarchy (PoA)

- **Definition** (Koutsopias & Papadimitriou, 1999): Given a game G and a **social-choice function** C , let S be the set of all NE. If the payoff represents a **cost** (resp., a **utility**) for a player, let OPT be the outcome of G **minimizing** (resp., **maximizing**) C . Then, the **Price of Anarchy (PoA)** of G w.r.t. C is

$$PoA_G(C) = \sup_{s \in S} \frac{C(s)}{C(OPT)} \left(\text{resp.}, \inf_{s \in S} \frac{C(s)}{C(OPT)} \right)$$

The price of stability (PoS)

- **Definition** (Schulz & Moses, 2003)

Given a game G and a **social-choice function** C , let S be the set of all NE. If the payoff represents a **cost** (resp., a **utility**) for a player, let OPT be the outcome of G **minimizing** (resp., **maximizing**) C . Then, the **Price of Stability (PoS)** of G w.r.t. C is:

$$PoS_G(C) = \inf_{s \in S} \frac{C(s)}{C(OPT)} \left(\text{resp.}, \sup_{s \in S} \frac{C(s)}{C(OPT)} \right)$$

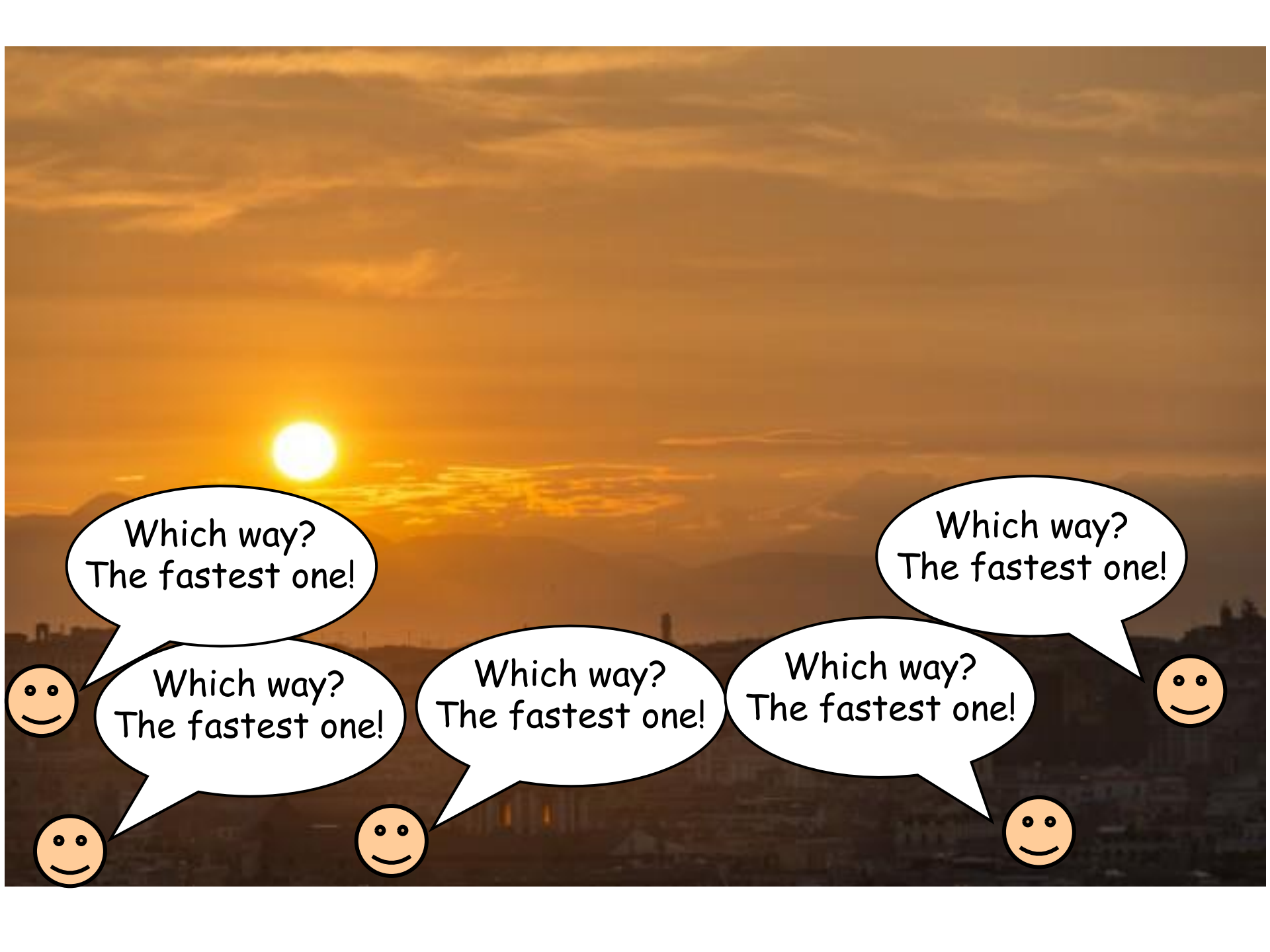
Some remarks

- PoA and PoS are
 - ≥ 1 for minimization problems
 - ≤ 1 for maximization problems
- PoA and PoS are small when they are close to 1
- PoS is at least as close to 1 than PoA
- In a game with a unique equilibrium $\text{PoA} = \text{PoS}$
- PoA is similar to the concept of **approximation ratio** of a heuristic
- a bound on the PoS provides a significantly weaker guarantee than a bound on the PoA
- Why to study the PoS?
 - sometimes a nontrivial bound is possible only for PoS
 - PoS quantifies the necessary degradation in quality under the game-theoretic constraint of stability

An example: Selfish Routing



Which way?
The fastest one!



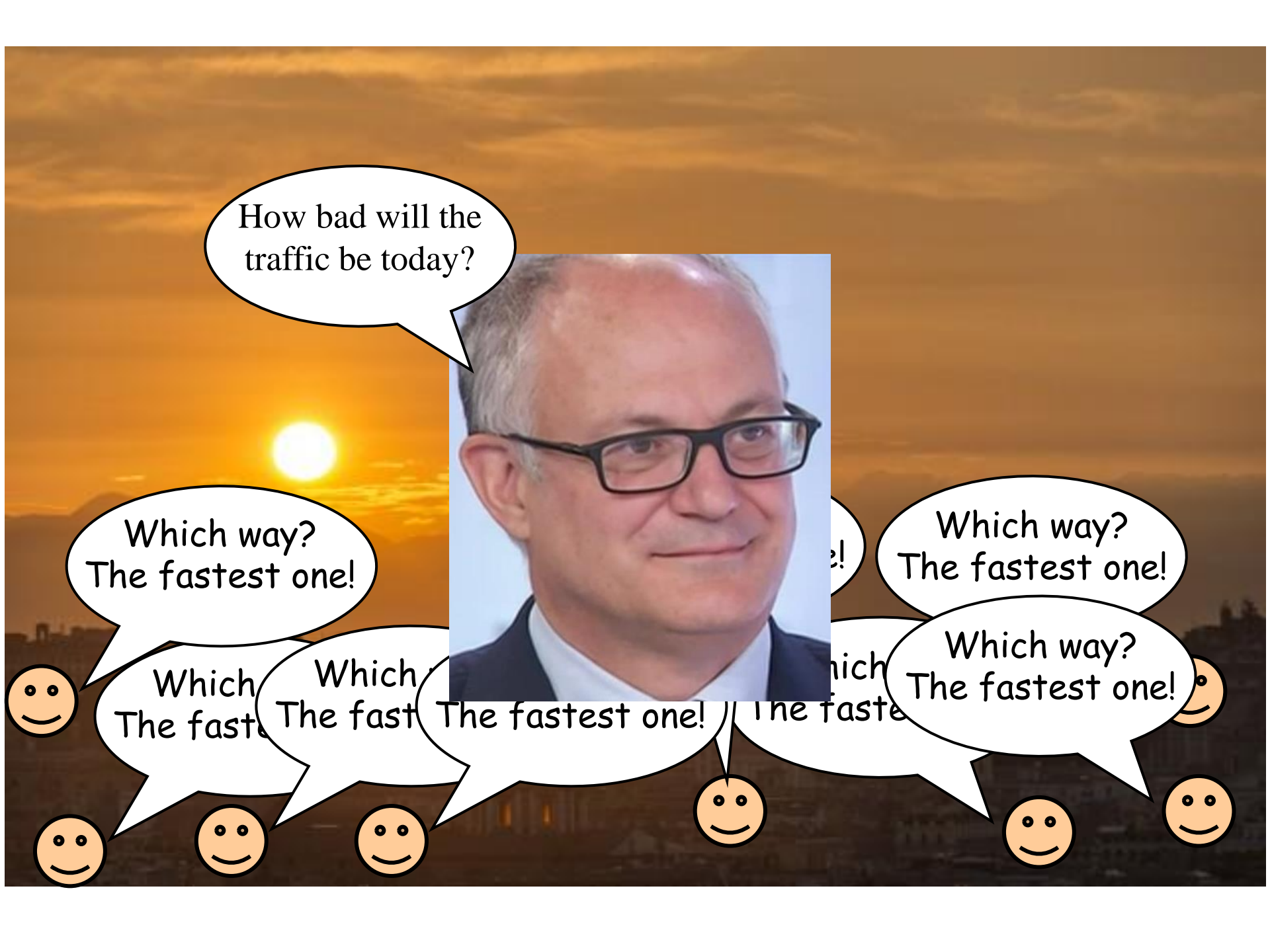
Which way?
The fastest one!

Which way?
The fastest one!

Which way?
The fastest one!

Which way?
The fastest one!

Which way?
The fastest one!



How bad will the traffic be today?

Which way?
The fastest one!

Which way?
The fastest one!

Which way?
The fastest one!

Which way?
The fastest one!

Which way?
The fastest one!

Which way?
The fastest one!

selfish routing

A large network can be modelled by using game theory

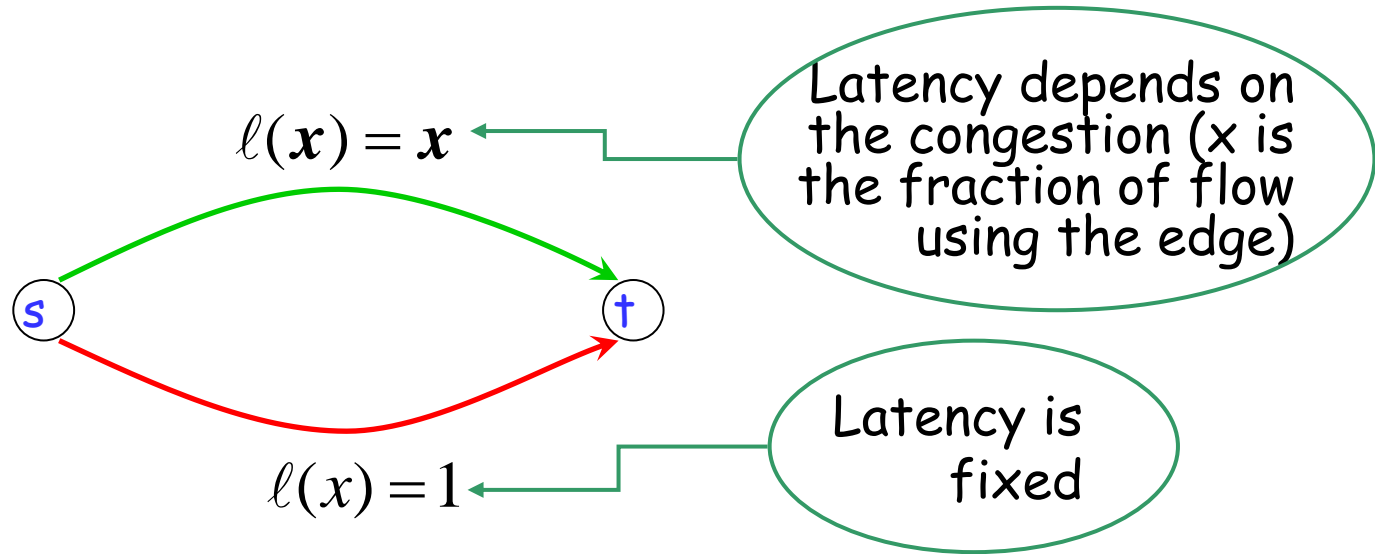
players	→	users
strategies	→	paths over which users can route their traffic

Non-atomic Selfish Routing:

- there is a large number of (selfish) users
- every user controls a tiny fraction of the traffic
- each edge has a cost function measuring the travel time as function of amount of traffic on the edge
- every user tries to minimize his travel time
- social-choice function (to minimize): average travel time incurred by players

Example: Pigou's game [1920]

One unit
of traffic



- What is the NE of this game?
- Trivial: all the fraction of flow tends to travel on the upper edge \Rightarrow the cost of the flow is $1 \cdot 1 + 0 \cdot 1 = 1$
- How bad is this NE?
- The optimal solution is the minimum of $C(x) = x \cdot x + (1-x) \cdot 1 \Rightarrow C'(x) = 2x - 1 \Rightarrow \text{OPT} = 1/2 \Rightarrow C(\text{OPT}) = 1/2 \cdot 1/2 + (1 - 1/2) \cdot 1 = 0.75$

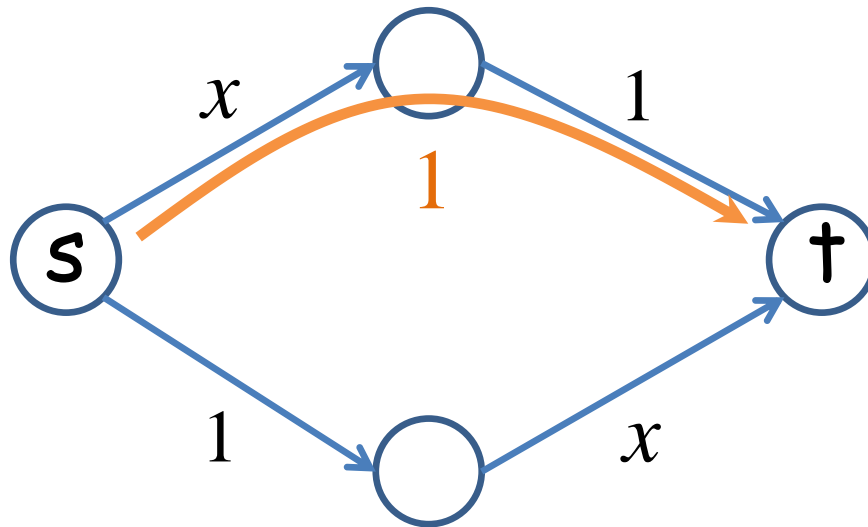
ratio between
the two costs
(NE vs Opt) $= 1/0.75 = 4/3$

Do we have to take into account selfish behaviour of the users when we design a network?

The Braess's paradox

One unit
of traffic

average
travel time = 2

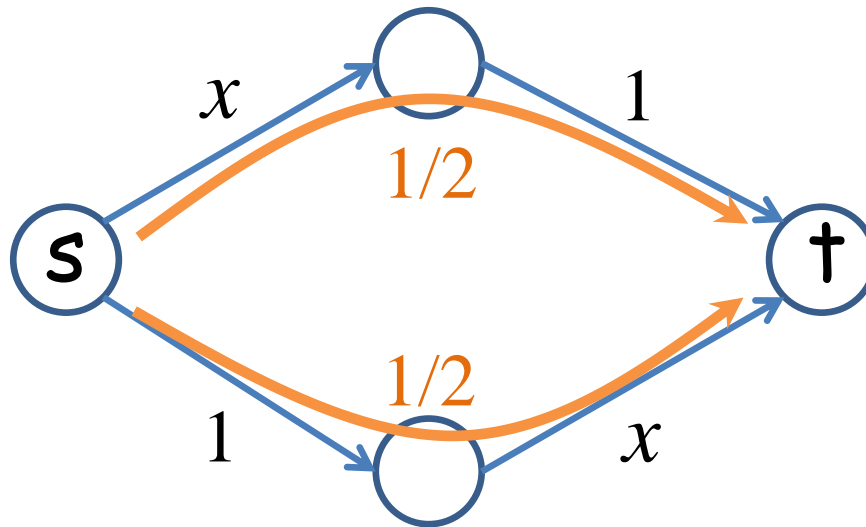


is it a NE?
...no!

The Braess's paradox

One unit
of traffic

average
travel time = 1.5



the only NE

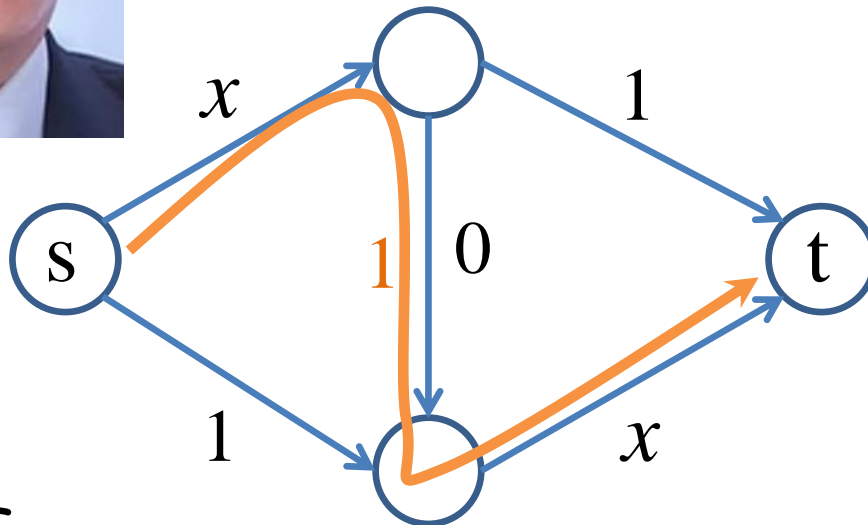
Notice: this
is also the
optimal
outcome.

The Braess's paradox

average
travel time = 2

To reduce the
traffic, I will
build a new
road.

One unit
of traffic

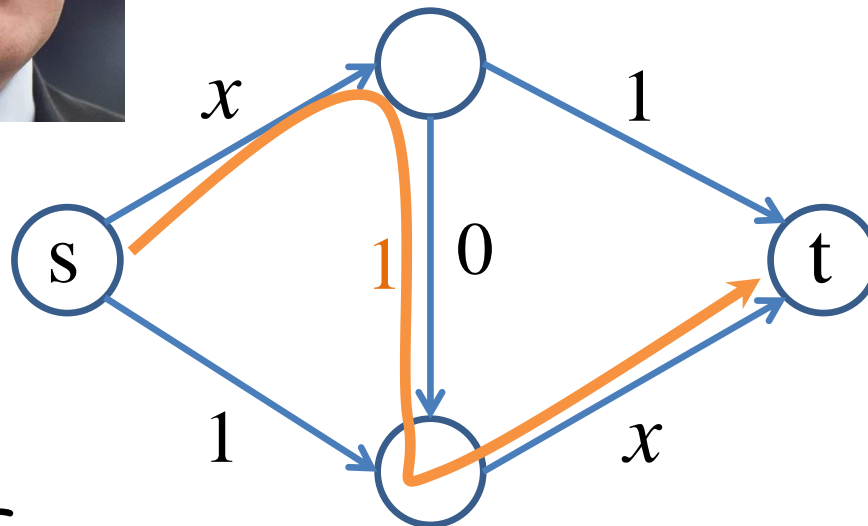


the only NE

The Braess's paradox



One unit
of traffic



the only NE

average
travel time = 2

Notice:

- the optimal outcome as before:
- $\frac{1}{2}$ up & $\frac{1}{2}$ down
- average travel time of 1.5

$\frac{4}{3}$ as in the
Pigou's example

ratio between
the two costs
(NE vs Opt) = $2/1.5 = \frac{4}{3}$

Theorem (Roughgarden&Tardos 2000)

The Price of Anarchy of the Selfish Routing Game with linear latency function is at most $4/3$

Pollution game

There are n countries. Each country faces the choice of either passing legislation to control pollution or not. Assume that pollution control has a cost of 3 for the country, but each country that pollutes adds 1 of all countries (in term of added health costs). The cost of controlling pollution is 3.

...notice that the cost of controlling pollution is considerably larger than the cost a country pays for being socially irresponsible...

can we bound the PoA?
And the PoS?

Tragedy of commons

There are n players. Each player wants to send information along a shared channel of known maximum capacity 1. Player i 's strategy is to send x_i units of flow along the channel, for some $x_i \in [0,1]$.

Each player would like to have a large fraction of the bandwidth but the quality of the channel deteriorates as the total assigned bandwidth increases. More precisely, the value of a player i is $x_i(1 - \sum_j x_j)$.

can we bound the PoA?
And the PoS?