

Network Formation Games

Network Formation Games

- NFGs model distinct ways in which *selfish* agents might create and evaluate networks
- We'll see two models:
 - Global Connection Game
 - Local Connection Game
- Both models aim to capture two competing issues: players want
 - to minimize the cost they incur in building the network
 - to ensure that the network provides them with a high quality of service

Motivations

- NFGs can be used to model:
 - social network formation (edge represent social relations)
 - how subnetworks connect in computer networks
 - formation of networks connecting users to each other for downloading files (P2P networks)

Setting

- What is a stable network?
 - we use a NE as the solution concept
 - we refer to networks corresponding to Nash Equilibria as being stable
- How to evaluate the overall quality of a network?
 - we consider the *social cost*: the sum of players' costs
- **Our goal**: to bound the efficiency loss resulting from stability

Global Connection Game

E. Anshelevich, A. Dasgupta, J. Kleinberg, E. Tardos, T. Wexler, T. Roughgarden,
[The Price of Stability for Network Design with Fair Cost Allocation](#), FOCS'04

The model

- $G=(V,E)$: directed graph
- c_e : non-negative cost of the edge $e \in E$
- k players
- player i has a source node s_i and a sink node t_i
- player i 's **goal**: to build a network in which t_i is reachable from s_i while paying as little as possible
- Strategy for player i : a path P_i from s_i to t_i

The model

- Given a strategy vector S , the constructed network will be $N(S) = \cup_i P_i$
- The cost of the constructed network will be shared among all players as follows:

$$\text{cost}_i(S) = \sum_{e \in P_i} c_e / k_e(S)$$

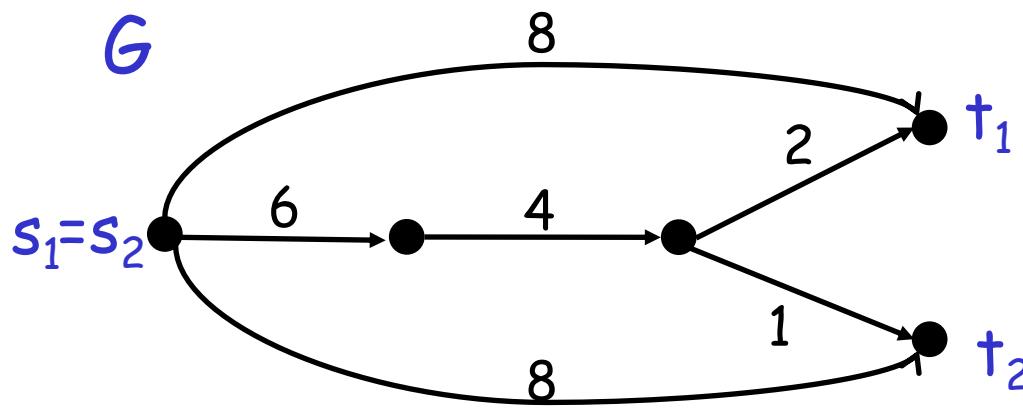
$k_e(S)$: number of players whose path contains e

sometimes we write k_e instead of $k_e(S)$
when S is clear from the context

this cost-sharing scheme is called
fair or *Shapley cost-sharing mechanism*

Remind

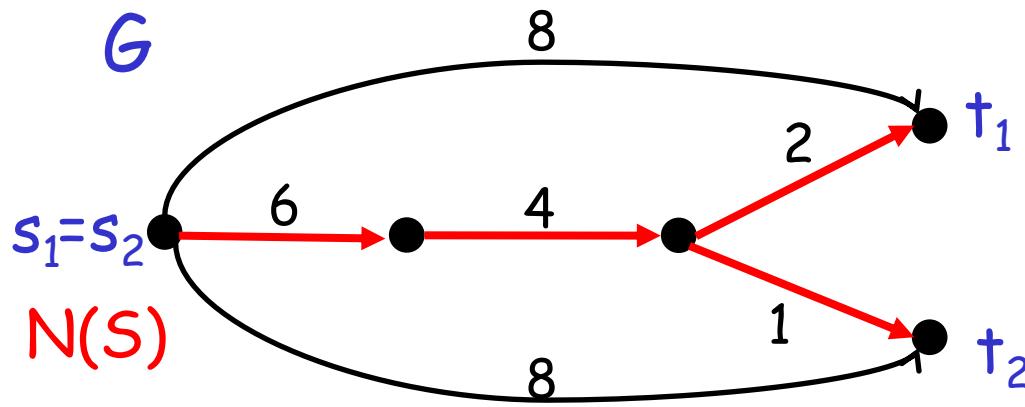
- We use Nash equilibrium (NE) as the solution concept
- A strategy vector S is a NE if no player has convenience to change its strategy
- Given a strategy vector S , $N(S)$ is *stable* if S is a NE
- To evaluate the overall quality of a network, we consider the *social cost*, i.e. the sum of all players' costs
$$\text{cost}(S) = \sum_i \text{cost}_i(S)$$
- a network is *optimal* or *socially optimal* if it minimizes the social cost



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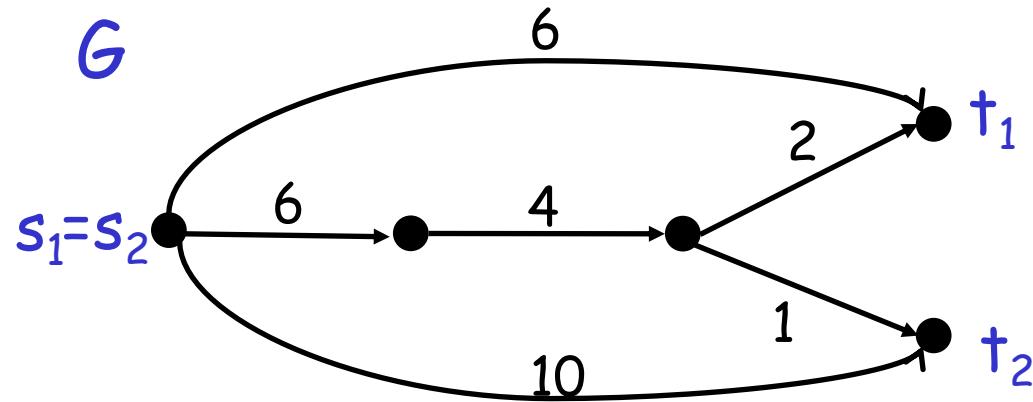
Notice: $\text{cost}(S) = \sum_{e \in N(S)} c_e$



the optimal network is a cheapest subgraph of G containing a path from s_i to t_i , for each i

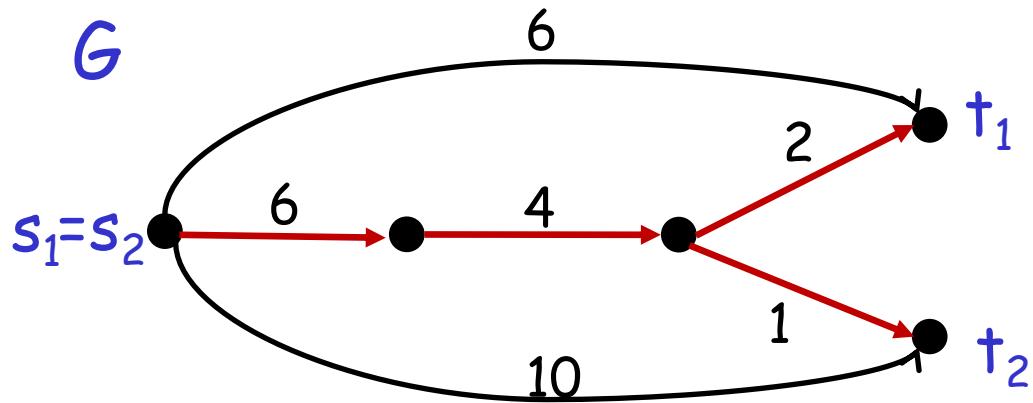
$$\begin{aligned}\text{cost}_1 &= 7 \\ \text{cost}_2 &= 6\end{aligned}$$

an example



what is the socially
optimal network?

an example



what is the socially optimal network?

cost of the social optimum: 13

is it stable?

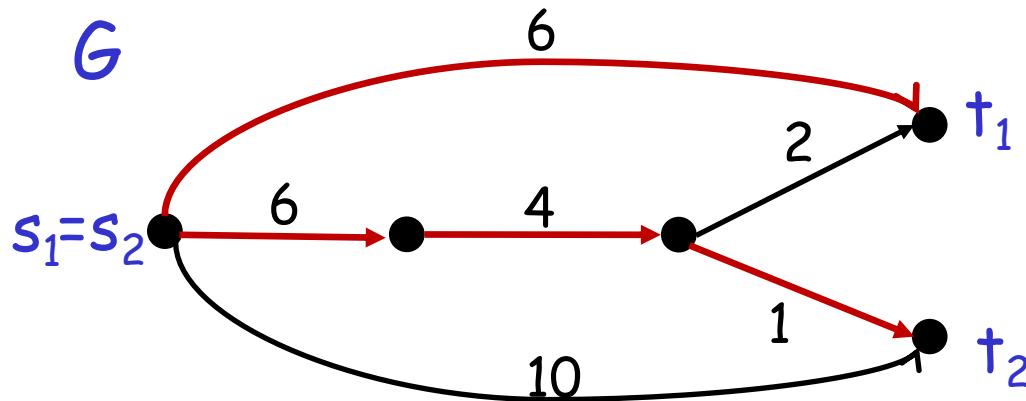
...no!

social cost
of the network
13

13

$\text{cost}_1 = 7$
 $\text{cost}_2 = 6$

an example



what is the socially
optimal network?

cost of the social
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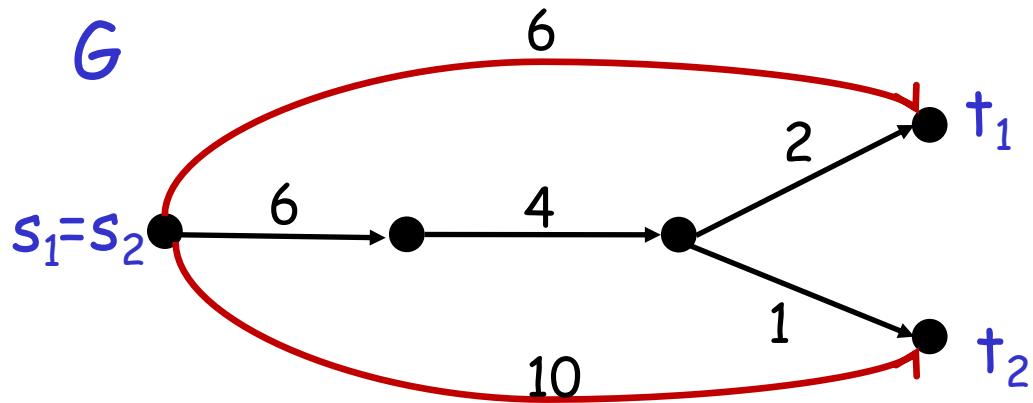
...no!

social cost
of the network
17

17

$\text{cost}_1 = 6$
 $\text{cost}_2 = 11$

an example



what is the socially optimal network?

cost of the social optimum: 13

is it stable?

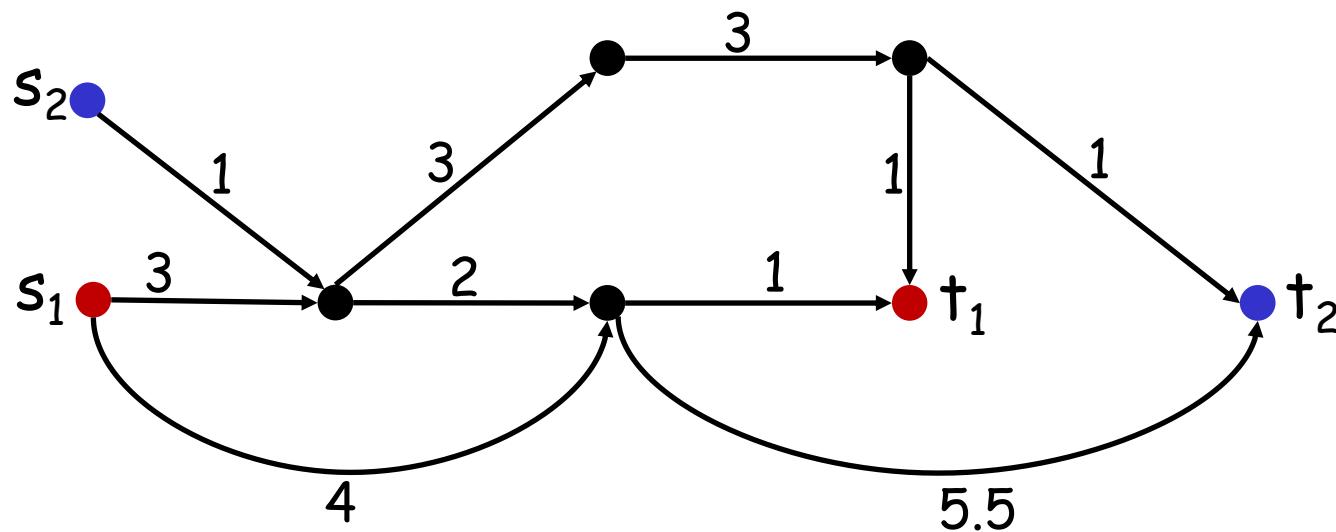
...yes!

social cost
of the network
16

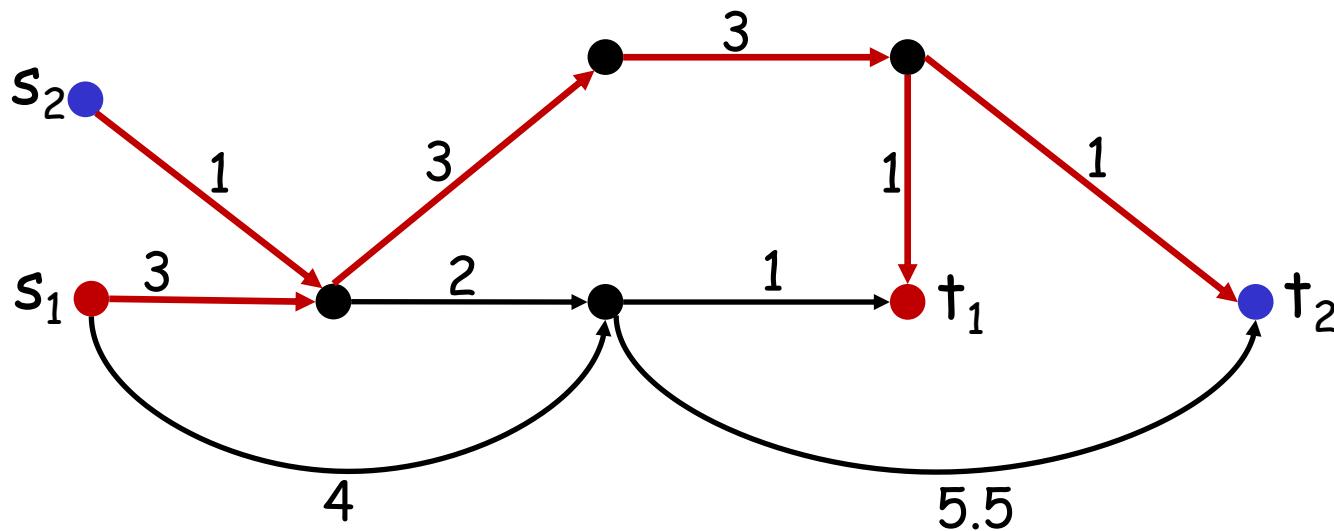
16

$\text{cost}_1 = 6$
 $\text{cost}_2 = 10$

one more example



one more example



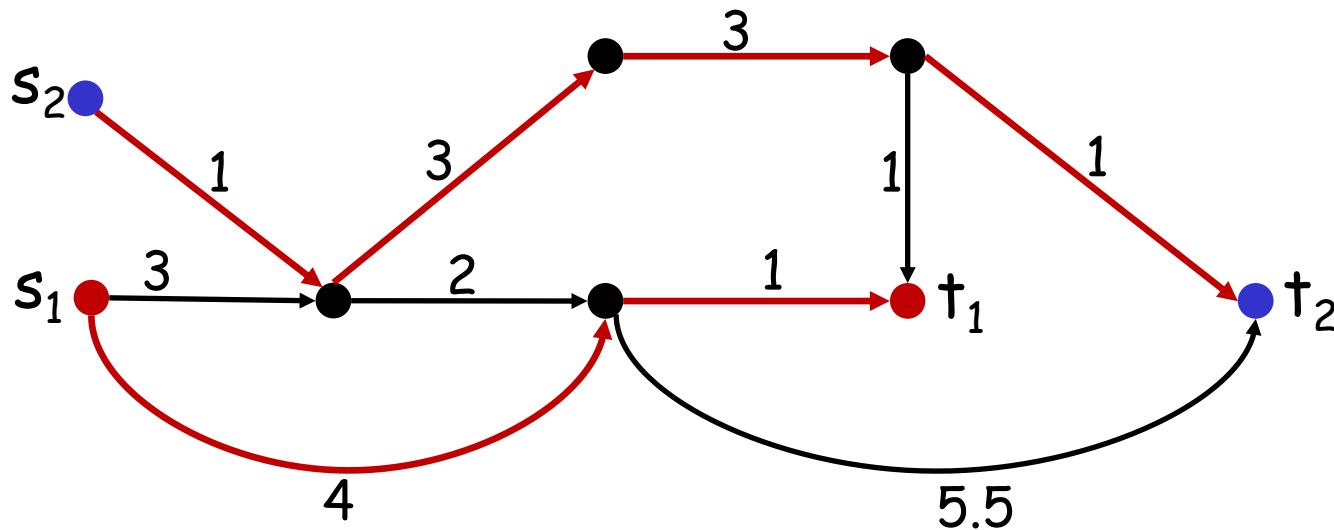
optimal network has cost 12

$$\text{cost}_1 = 7$$

$$\text{cost}_2 = 5$$

is it stable?

one more example



...no!, player 1 can decrease its cost

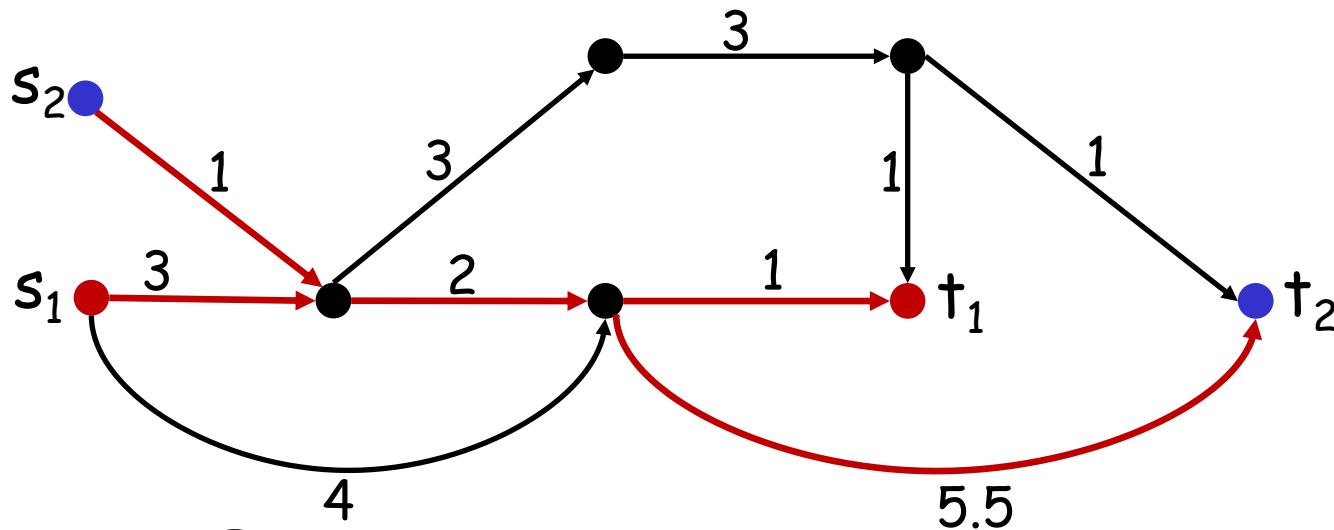
$$\text{cost}_1=5$$

$$\text{cost}_2=8$$

is it stable? ...yes!

the social cost is 13

one more example



...a better NE...

$$\text{cost}_1 = 5$$

$$\text{cost}_2 = 7.5$$

the social cost is 12.5

Addressed issues

- Does a stable network always exist?
- Can we bound the price of anarchy (PoA)?
- Can we bound the price of stability (PoS)?
- Does the repeated version of the game always converge to a stable network?

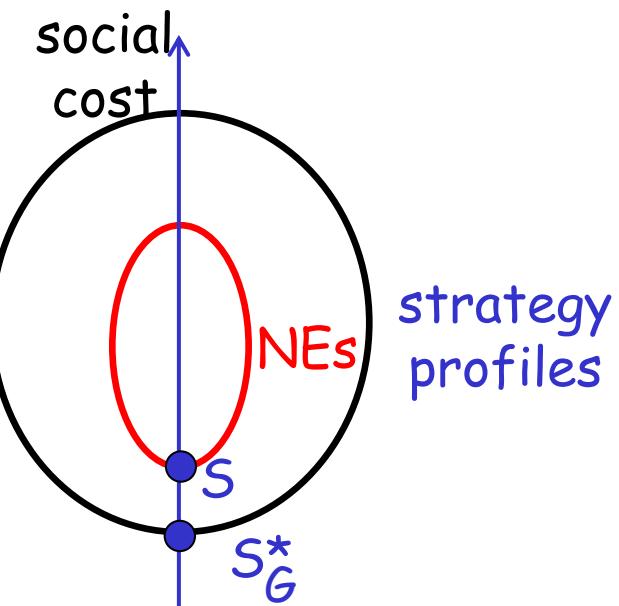
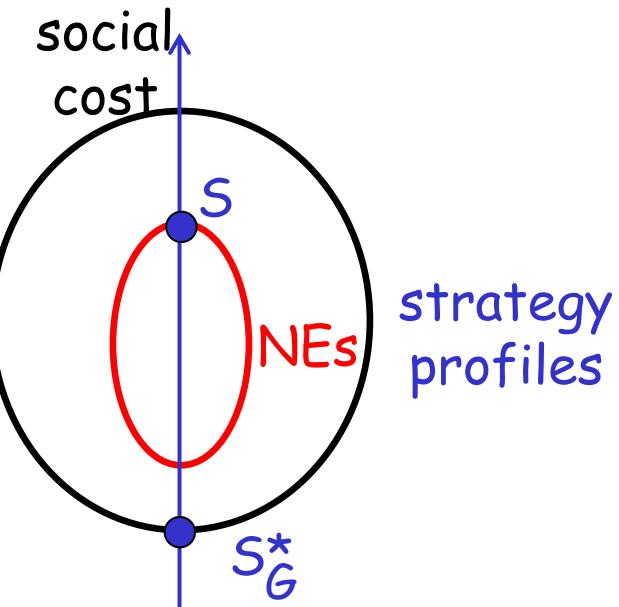
PoA and PoS

for a given network G , we define:

$$\text{PoA of the game in } G = \max_{\substack{S \text{ s.t.} \\ S \text{ is a NE}}} \frac{\text{cost}(S)}{\text{cost}(S_G^*)}$$

$$\text{PoS of the game in } G = \min_{\substack{S \text{ s.t.} \\ S \text{ is a NE}}} \frac{\text{cost}(S)}{\text{cost}(S_G^*)}$$

S_G^* : socially optimum for G



PoA and PoS

we want to bound PoA and PoS in the **worst case**:

PoA of the game = \max_G PoA in G

PoS of the game = \max_G PoS in G

some notations

we use:

$$x = (x_1, x_2, \dots, x_k); \quad x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k); \quad x_i = (x_{-i}, x_i)$$

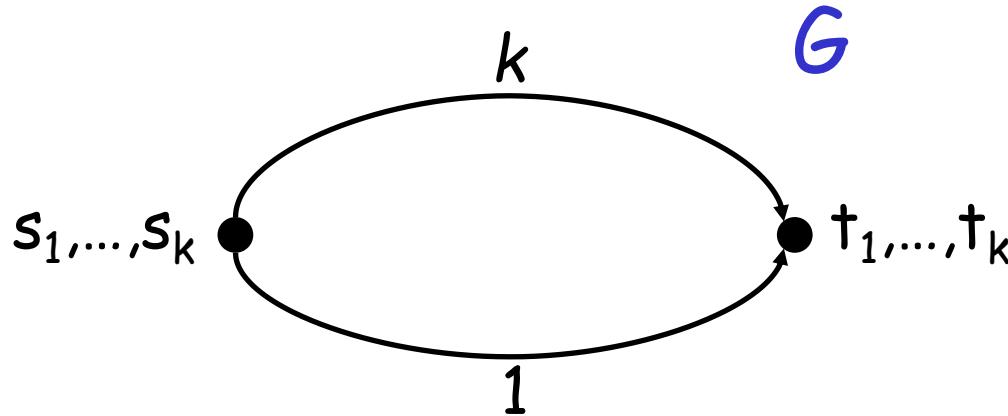
G : a weighted directed network

cost or length of a path π in G : $\sum_{e \in \pi} c_e$

$d_G(u, v)$: distance in G from a node u to a node v : length of any shortest path in G from u to v

Price of Anarchy

Price of Anarchy: a lower bound



optimal network has cost 1

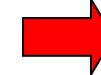
best NE: all players use the lower edge



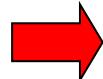
PoS in G is 1



worst NE: all players use the upper edge



PoA in G is k



PoA of the
game is $\geq k$

Theorem

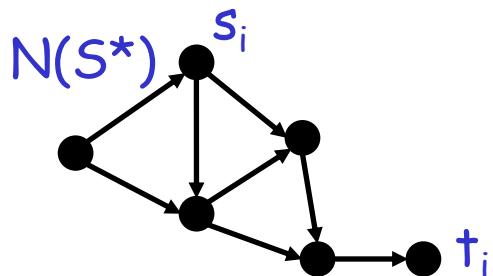
The price of anarchy in the global connection game with k players is **at most k**

proof

S : a NE S^* : a strategy profile minimizing the social cost
for each player i ,
 let π_i be a shortest path in G from s_i to t_i

we have

$$\text{cost}_i(S) \leq \text{cost}_i(S_{-i}, \pi_i) \leq d_G(s_i, t_i) \leq \text{cost}(S^*)$$



Theorem

The price of anarchy in the global connection game with k players is **at most k**

proof

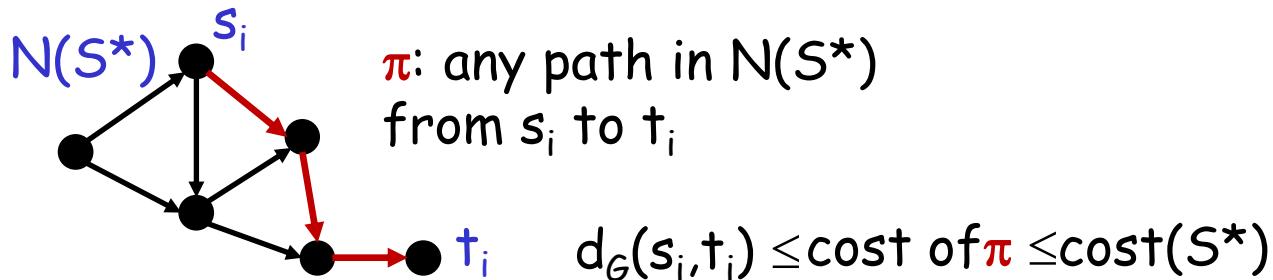
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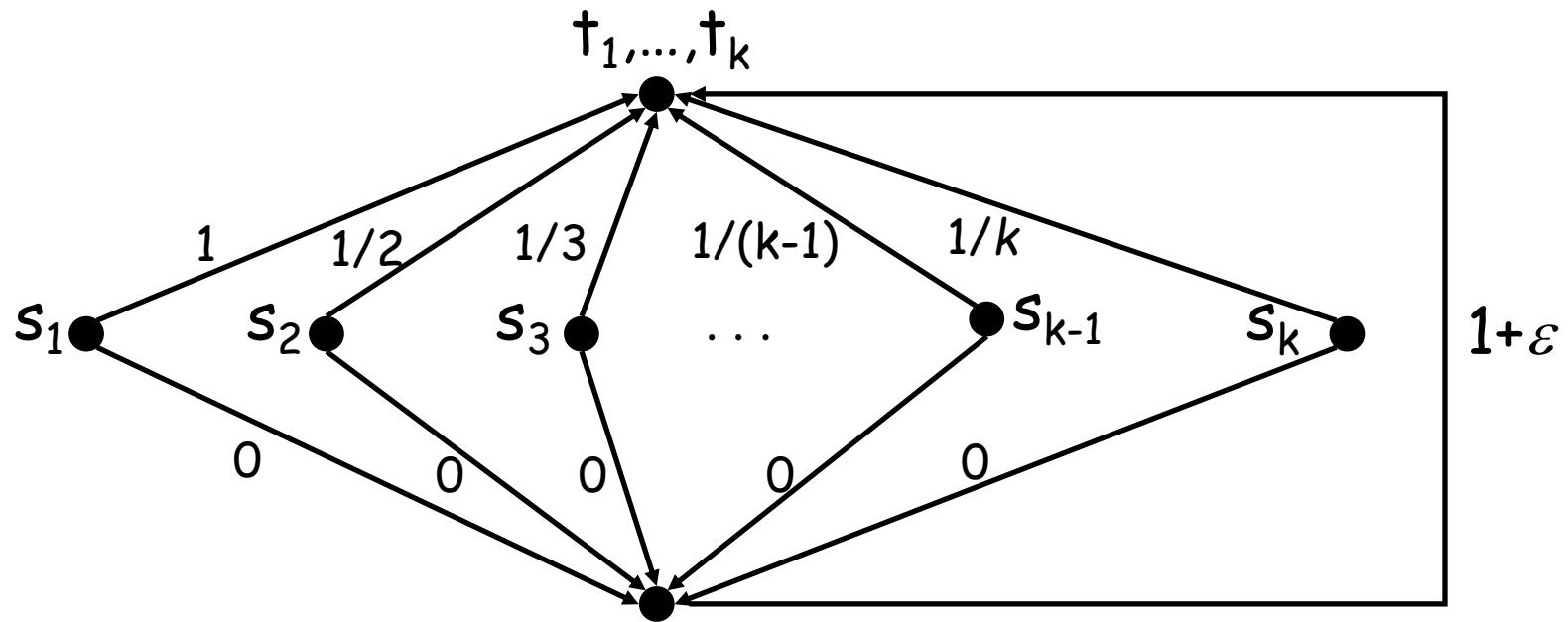


$$\text{cost}(S) = \sum_i \text{cost}_i(S) \leq k \text{cost}(S^*)$$

Price of Stability & potential function method

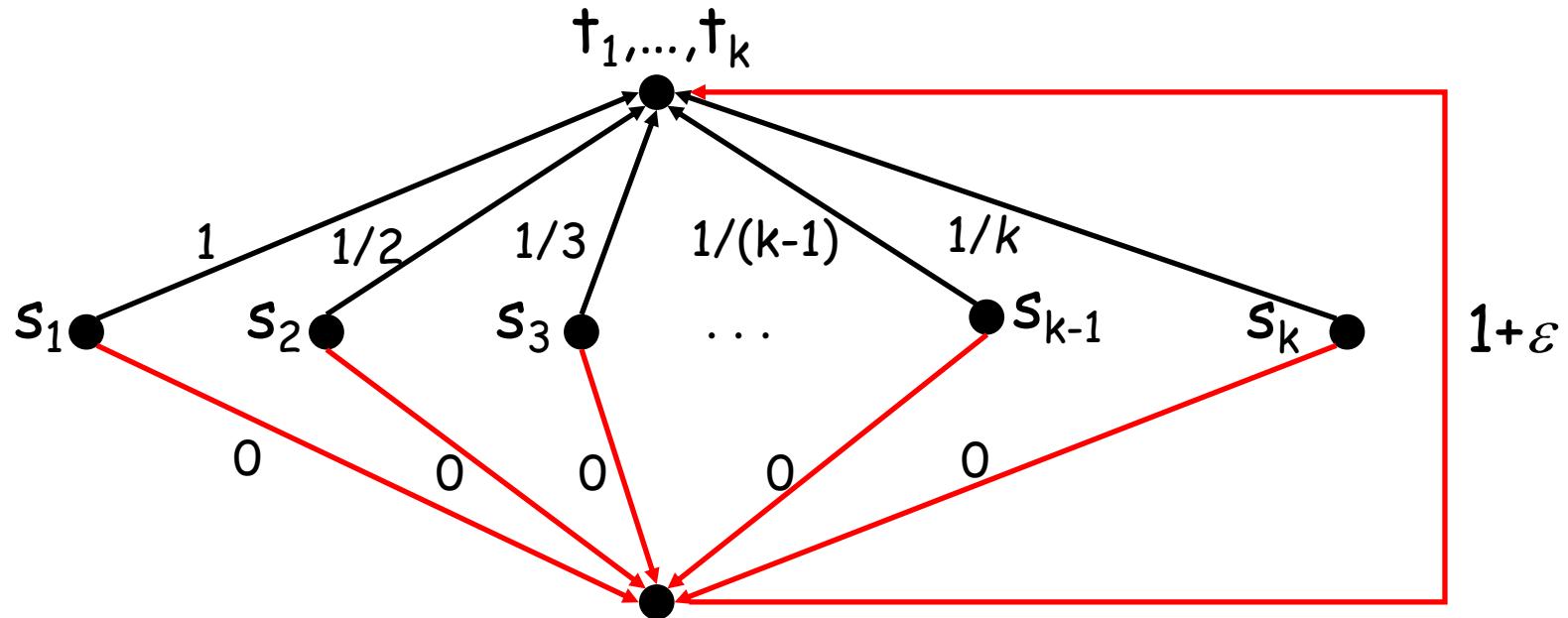
Price of Stability: a lower bound

$\varepsilon > 0$: small value



Price of Stability: a lower bound

$\varepsilon > 0$: small value

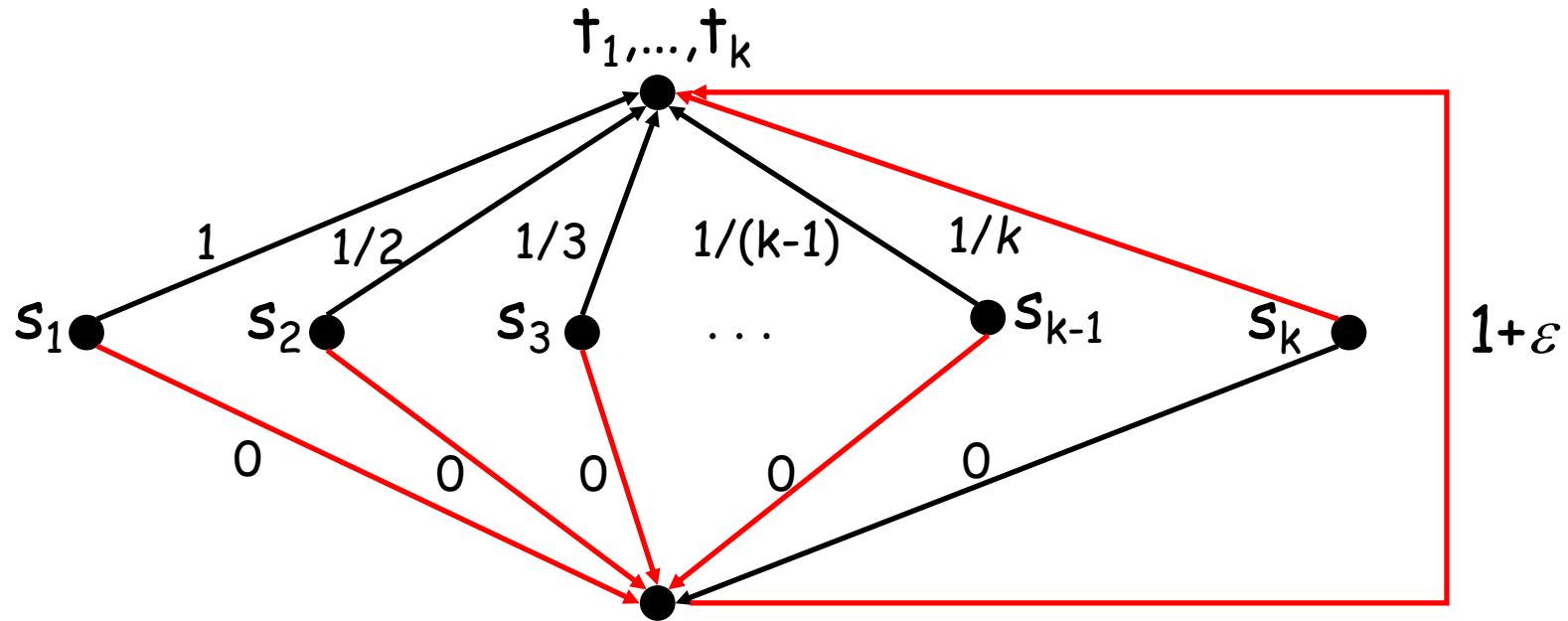


The optimal solution has a cost of $1+\varepsilon$

is it stable?

Price of Stability: a lower bound

$\varepsilon > 0$: small value

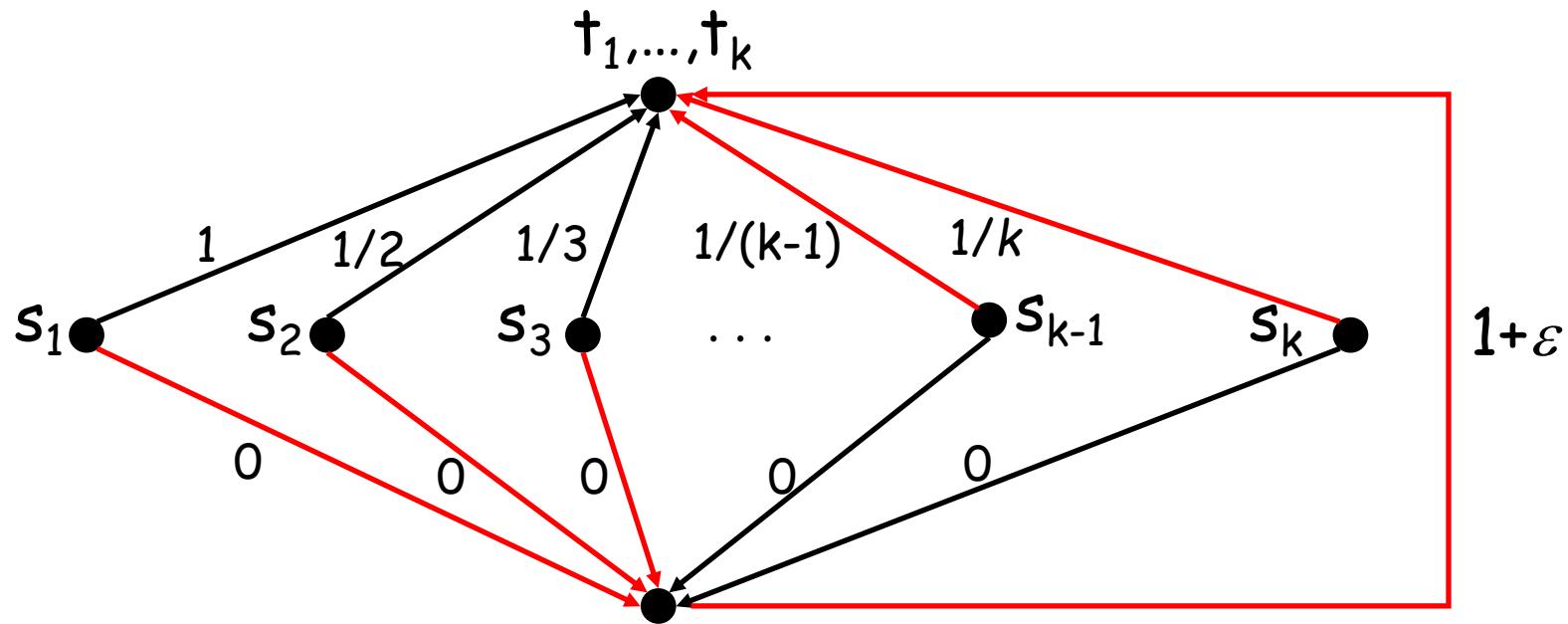


...no! player k can decrease its cost...

is it stable?

Price of Stability: a lower bound

$\varepsilon > 0$: small value

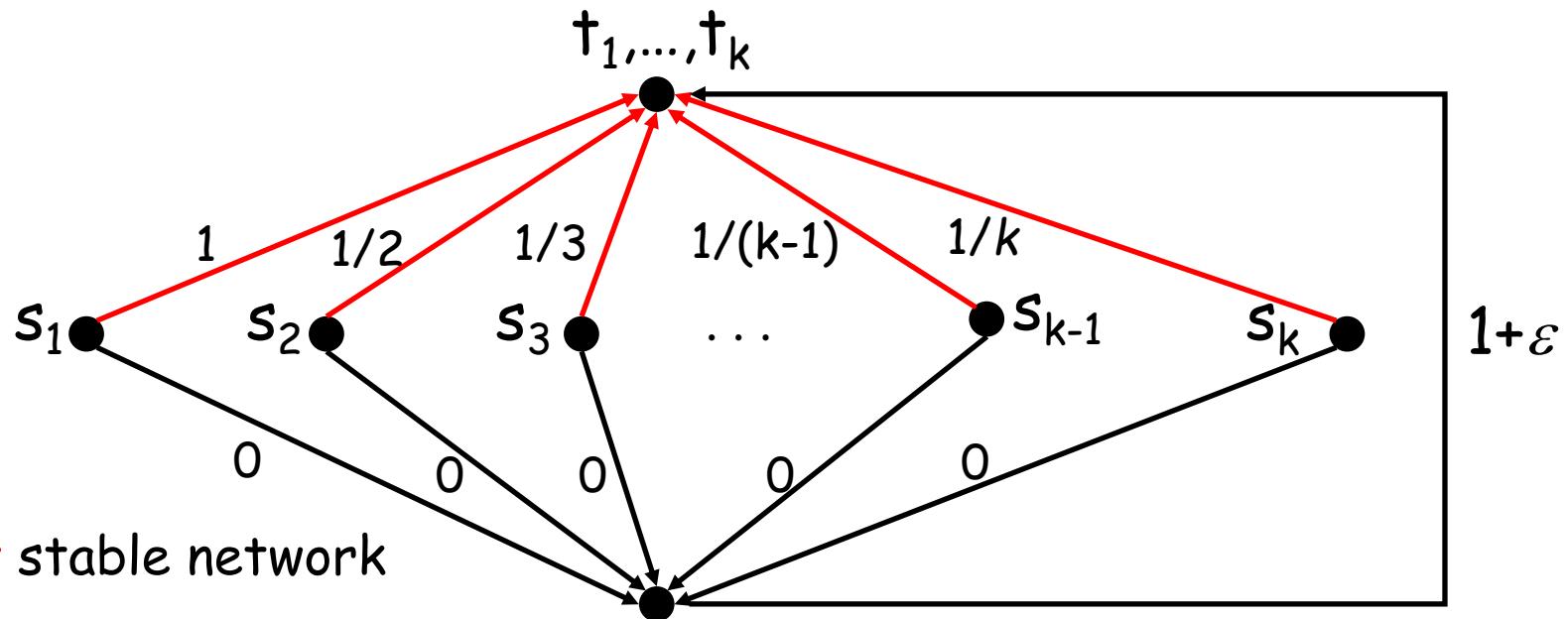


...no! player $k-1$ can decrease its cost...

is it stable?

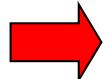
Price of Stability: a lower bound

$\varepsilon > 0$: small value



social cost: $\sum_{j=1}^k 1/j = H_k \leq \ln k + 1$ k-th *harmonic number*

the **optimal** solution
has a cost of $1+\varepsilon$



PoS of the
game is $\geq H_k$

Theorem

Any instance of the global connection game has a pure Nash equilibrium, and better response dynamic always converges

Theorem

The price of stability in the global connection game with k players is at most H_k , the k -th harmonic number

To prove them we use the
Potential function method

Notation:

$$x = (x_1, x_2, \dots, x_k); \quad x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k); \quad x_i = (x_{-i}, x_i)$$

Definition

For any finite game, an *exact potential function* Φ is a function that maps every strategy vector S to some real value and satisfies the following condition:

$\forall S = (S_1, \dots, S_k)$, $S'_i \neq S_i$, let $S' = (S_{-i}, S'_i)$, then

$$\Phi(S) - \Phi(S') = \text{cost}_i(S) - \text{cost}_i(S')$$

A game that posses an exact potential function
is called *potential game*

Theorem

Every potential game has at least one pure Nash equilibrium, namely the strategy vector S that minimizes $\Phi(S)$

proof

consider any move by a player i that results in a new strategy vector S'

we have:

$$\underbrace{\Phi(S) - \Phi(S')}_{\leq 0} = \text{cost}_i(S) - \text{cost}_i(S')$$

$$\text{cost}_i(S) \leq \text{cost}_i(S')$$

player i cannot decrease its cost, thus S is a NE



Theorem

In any finite potential game, better response dynamics always converge to a Nash equilibrium

proof

better response dynamics simulate local search on Φ :

1. each move strictly decreases Φ
2. finite number of solutions



Note: in our game, a best response can be computed in polynomial time

Theorem

Suppose that we have a potential game with potential function Φ , and assume that for any outcome S we have

$$\text{cost}(S)/A \leq \Phi(S) \leq B \text{ cost}(S)$$

for some $A, B > 0$. Then the price of stability is at most AB

proof

Let S' be the strategy vector minimizing Φ

Let S^* be the strategy vector minimizing the social cost

we have:

$$\text{cost}(S')/A \leq \Phi(S') \leq \Phi(S^*) \leq B \text{ cost}(S^*)$$



...turning our attention to
the global connection game...

Let Φ be the following function mapping any strategy vector S to a real value:

$$\Phi(S) = \sum_{e \in E} \Phi_e(S)$$

where

$$\Phi_e(S) = c_e H_{k_e(S)}$$

$$H_k = \sum_{j=1}^k 1/j \quad \text{k-th harmonic number}$$

[we define $H_0 = 0$]

Lemma 1

Let $S = (P_1, \dots, P_k)$, let P'_i be an alternative path for some player i , and define a new strategy vector $S' = (S_{-i}, P'_i)$. Then:

$$\Phi(S) - \Phi(S') = \text{cost}_i(S) - \text{cost}_i(S')$$

Lemma 2

For any strategy vector S , we have:

$$\text{cost}(S) \leq \Phi(S) \leq H_k \text{cost}(S)$$

...from which we have:

PoS of the game is $\leq H_k$

Lemma 2

For any strategy vector S , we have:

$$\text{cost}(S) \leq \Phi(S) \leq H_k \text{cost}(S)$$

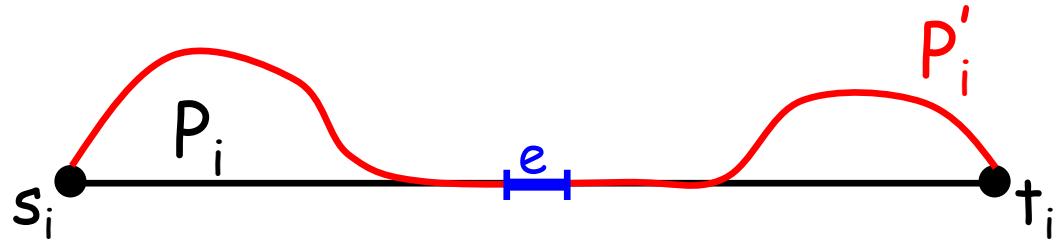
proof

$$\begin{aligned} \text{cost}(S) \leq \Phi(S) &= \sum_{e \in E} c_e H_{k_e(S)} \\ &= \sum_{e \in N(S)} c_e H_{k_e(S)} \leq \sum_{e \in N(S)} c_e H_k = H_k \text{cost}(S) \end{aligned}$$

$$1 \leq k_e(S) \leq k \quad \text{for } e \in N(S)$$



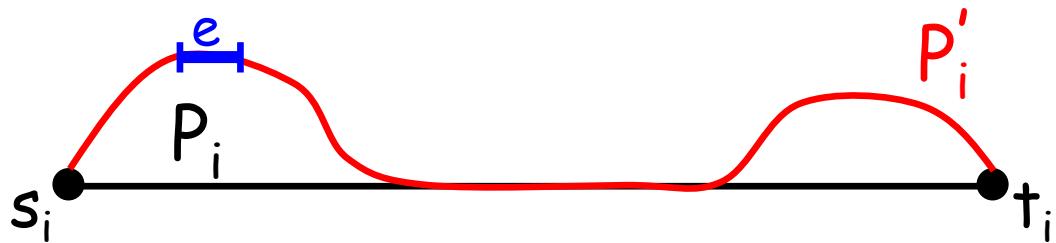
(proof of Lemma 1)



for each $e \in P_i \cap P'_i$

term e of $\text{cost}_i()$ & potential Φ_e remain the same

(proof of Lemma 1)



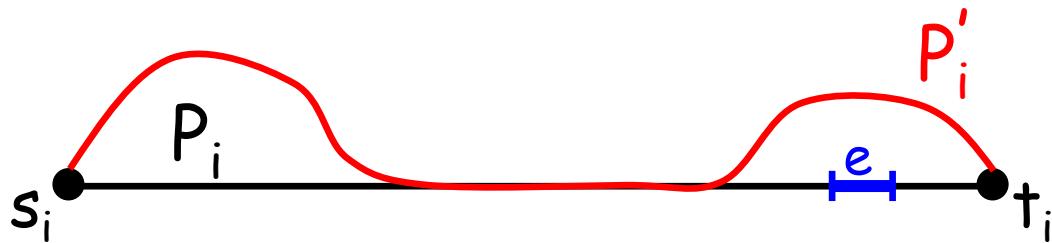
for each $e \in P'_i \setminus P_i$

term e of $\text{cost}_i()$ increases by $c_e / (k_e(S) + 1)$

potential Φ_e increases from $c_e \left(1 + \frac{1}{2} + \dots + \frac{1}{k_e(S)} \right)$
to $c_e \left(1 + \frac{1}{2} + \dots + \frac{1}{k_e(S)} + \frac{1}{k_e(S) + 1} \right)$

$$\rightarrow \Delta \Phi_e = c_e / (k_e(S) + 1)$$

(proof of Lemma 1)



for each $e \in P_i \setminus P'_i$

term e of $\text{cost}_i()$ decreases by $c_e / k_e(S)$

potential Φ_e decreases from $c_e \left(1 + \frac{1}{2} + \dots + \frac{1}{k_e(S)-1} + \frac{1}{k_e(S)}\right)$

to $c_e \left(1 + \frac{1}{2} + \dots + \frac{1}{k_e(S)-1}\right)$

$$\rightarrow \Delta \Phi_e = -c_e / k_e(S)$$



Theorem

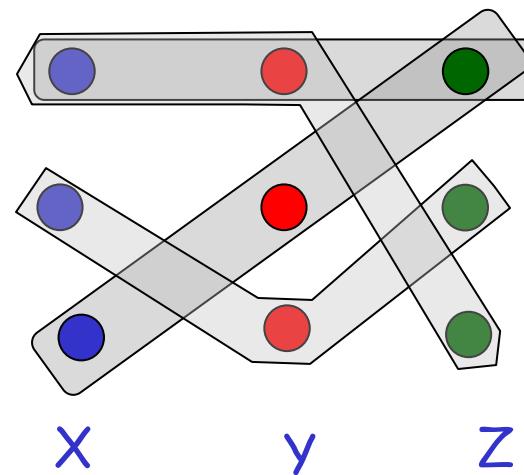
Given an instance of a GC Game and a value C , it is NP-complete to determine if a game has a Nash equilibrium of cost at most C .

proof

Reduction from 3-dimensional matching problem

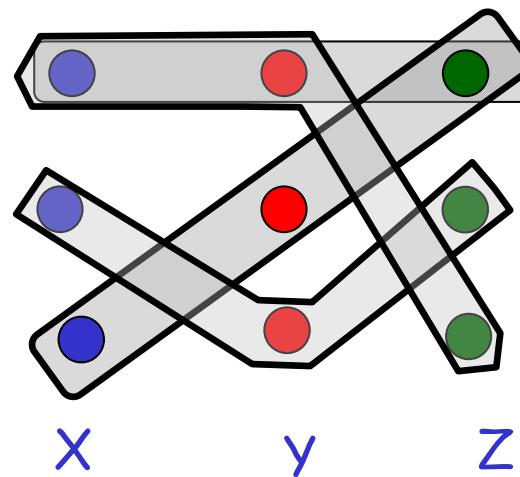
3-dimensional matching problem

- Input:
 - disjoint sets X, Y, Z , each of size n
 - a set $T \subseteq X \times Y \times Z$ of ordered triples
- Question:
 - does there exist a set of n triples in T so that each element of $X \cup Y \cup Z$ is contained in exactly one of these triples?

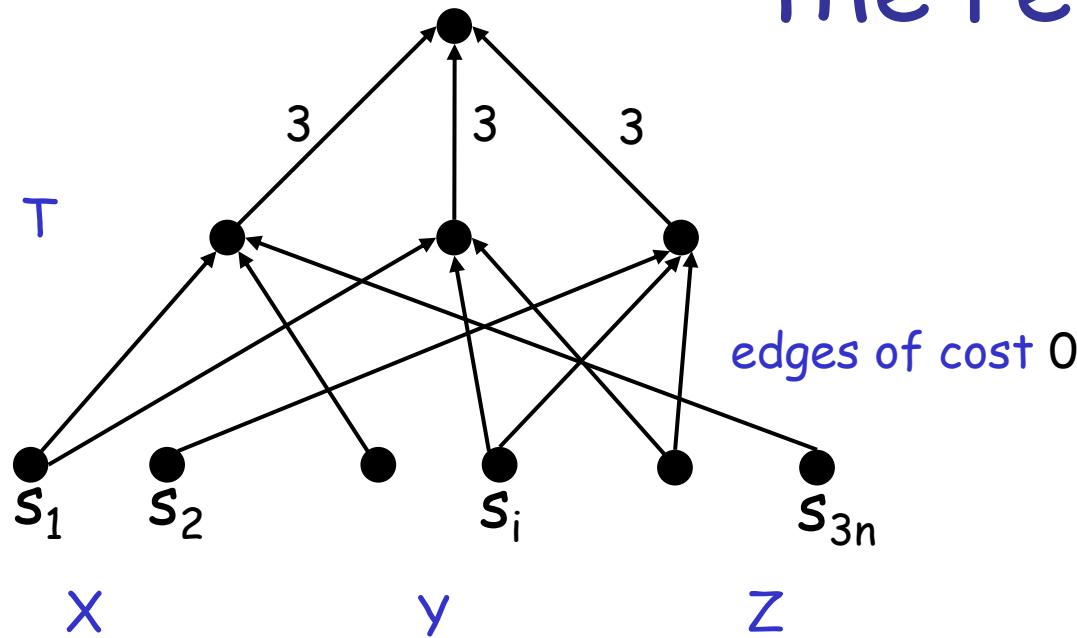


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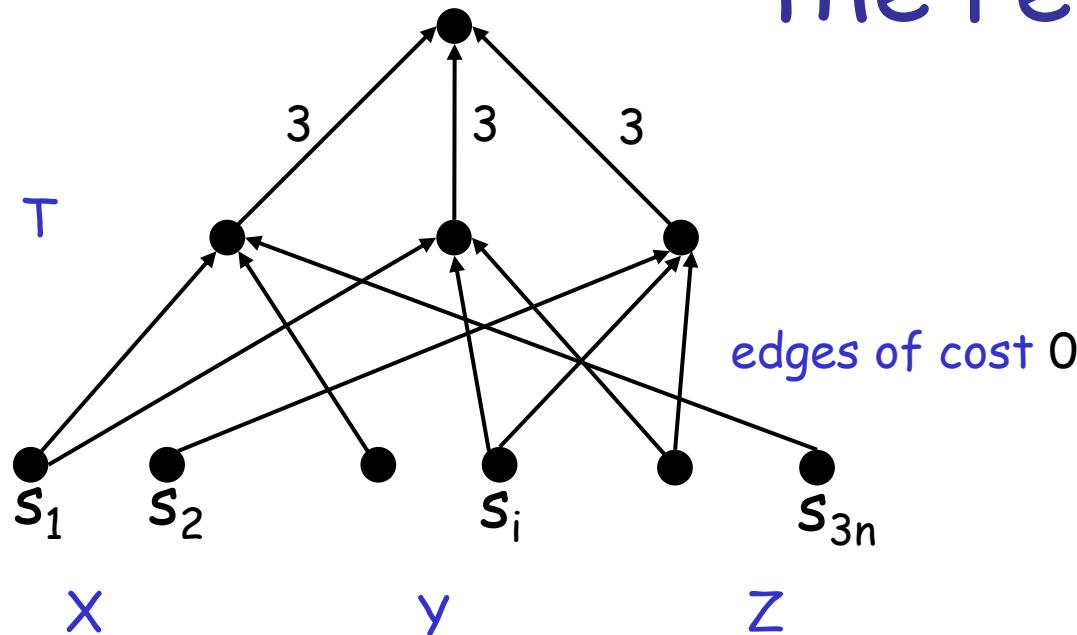


the reduction



There is a 3D matching if and only if there is a NE of cost at most $C=3n$

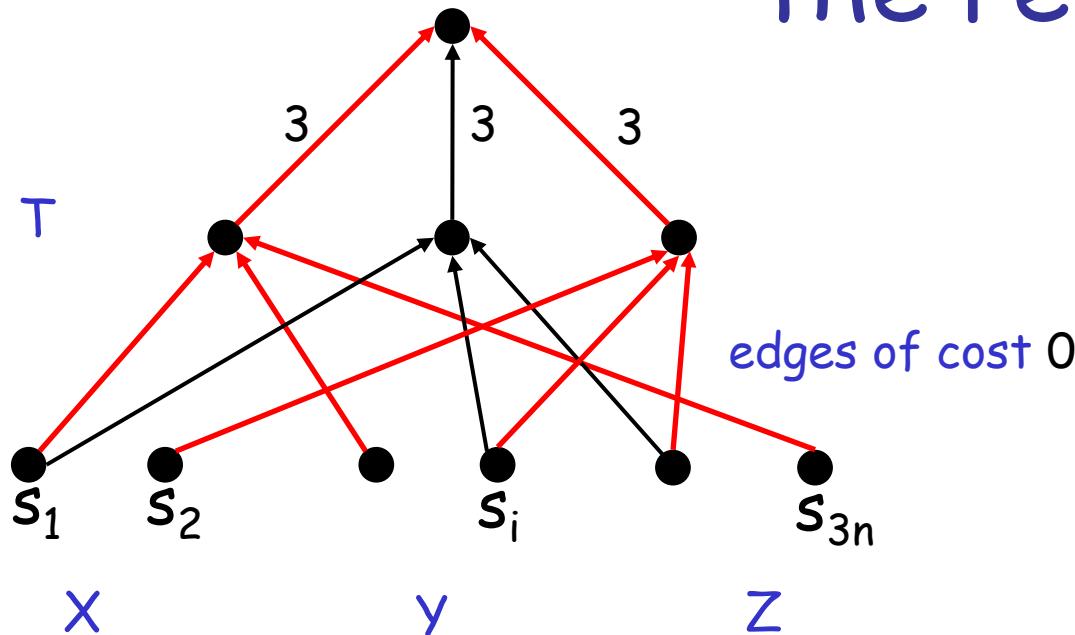
the reduction



Assume there is a 3D matching.

S: strategy profile in which each player choose a path passing through the triple of the matching it belongs to

the reduction



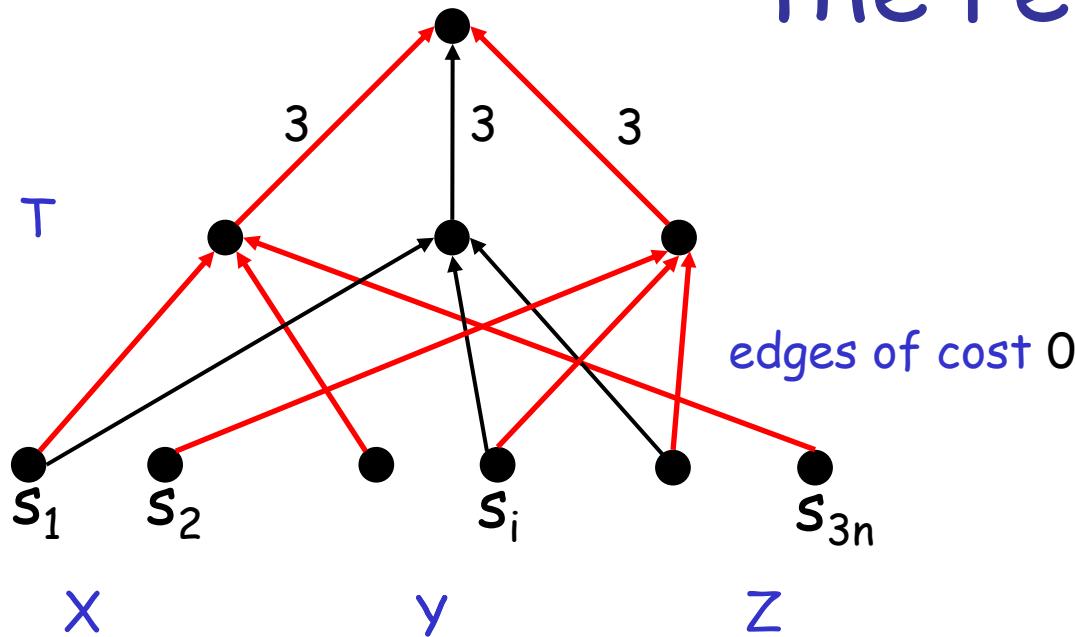
Assume there is a 3D matching.

S : strategy profile in which each player choose a path passing through the triple of the matching it belongs to

$\text{cost}(S) = 3n$

S is a NE

the reduction



Assume there is a NE of cost $\leq 3n$

$N(S)$ uses at most n edges of cost 3

each edge of cost 3 can "serve" at most 3 players

then, the edge of cost 3 are exactly n

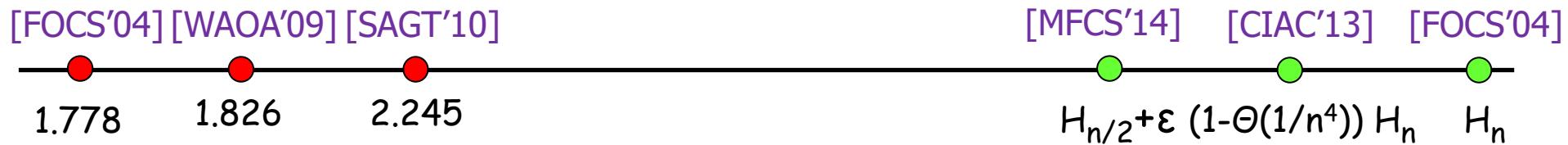
...and they define a set of triples that must be a 3D-matching



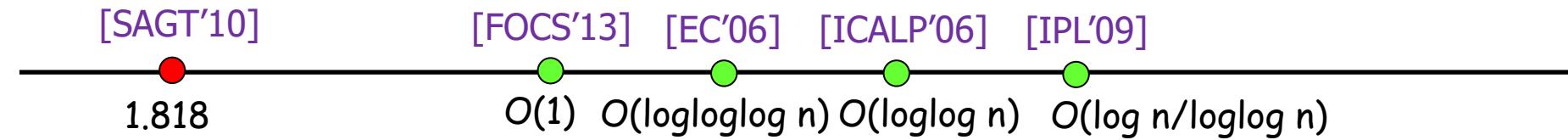
What is the PoS of the
game for undirected
networks?

PoS for undirected graphs: State of the art

● UB ● LB

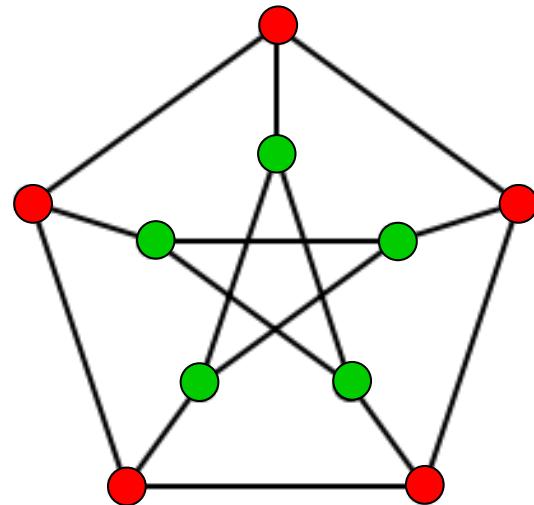


one single terminal (multicast)
+ all sources (broadcast)



Max-cut game

- $G=(V,E)$: undirected graph
- Nodes are (selfish) players
- Strategy S_u of u is a color {red, green}
- player u 's payoff in S (to maximize):
 - $p_u(S) = |\{(u,v) \in E : S_u \neq S_v\}|$

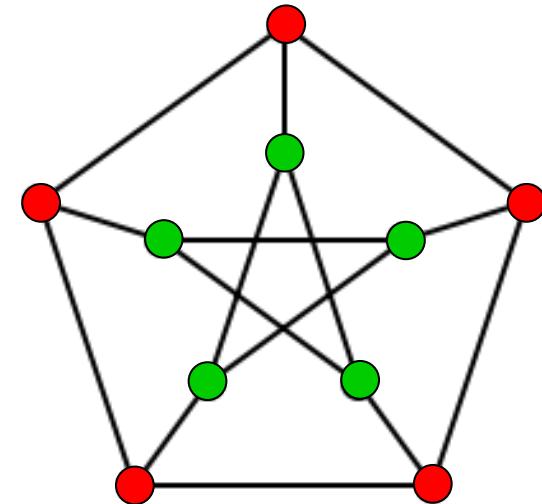


social welfare of
strategy vector S
 $\sum_u p_u(S) =$
2 #edges crossing
the red-green cut

Max-cut game

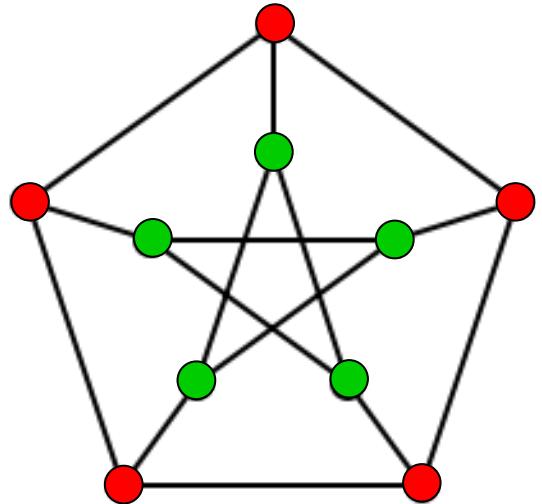
does a Nash Equilibrium
always exist?

how bad a Nash
Equilibrium
Can be?



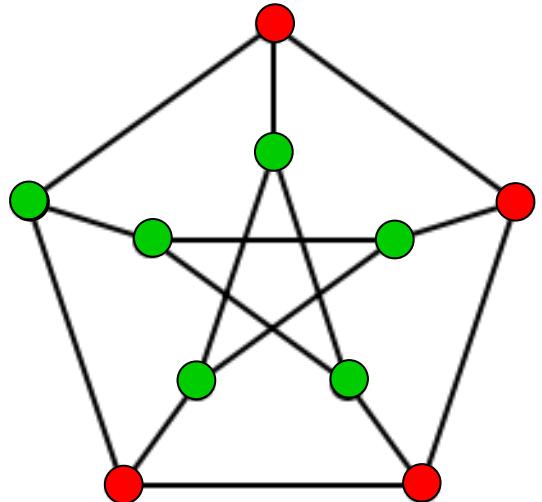
does the repeated
game always
converge to a
Nash Equilibrium?

...let's play Max-cut game
on Petersen Graph



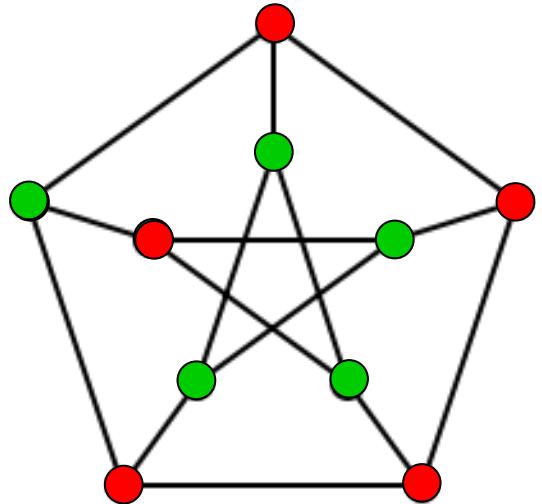
...is it a NE?

...let's play Max-cut game
on Petersen Graph



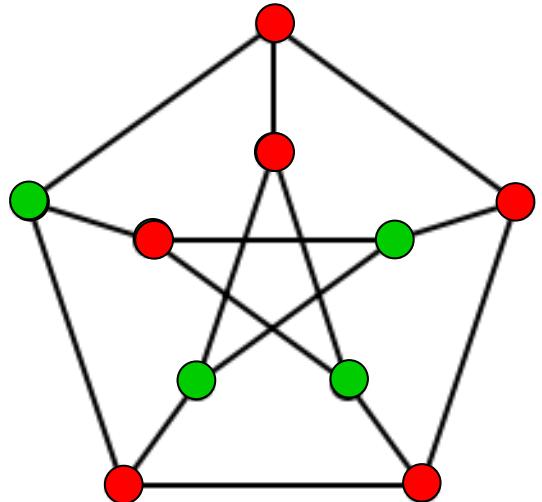
...is it a NE?

...let's play Max-cut game
on Petersen Graph



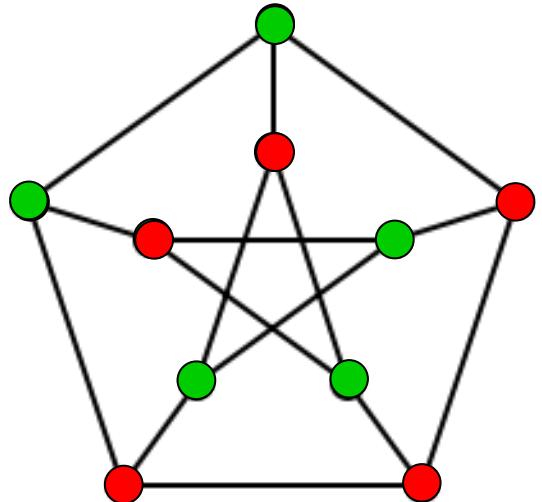
...is it a NE?

...let's play Max-cut game
on Petersen Graph



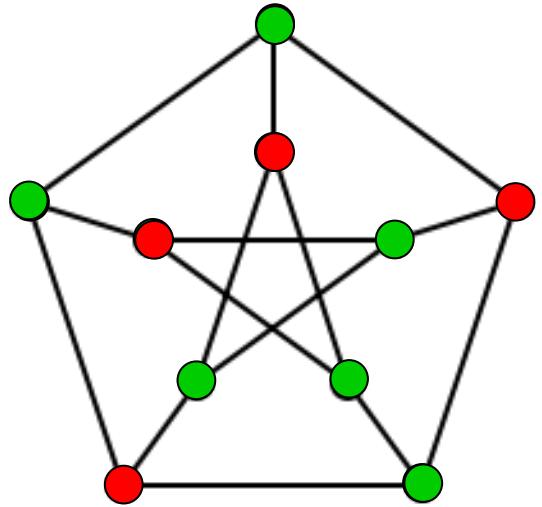
...is it a NE?

...let's play Max-cut game
on Petersen Graph



...is it a NE?

...let's play **Max-cut game**
on Petersen Graph



...is it a NE?

...yes!

of edges crossing
the cut is 12

Exercise

Show that:

- (i) Max-cut game is a potential game
- (ii) PoS is 1
- (iii) PoA $\geq \frac{1}{2}$
- (iv) there is an instance of the game having a NE with social welfare of $\frac{1}{2}$ the social optimum