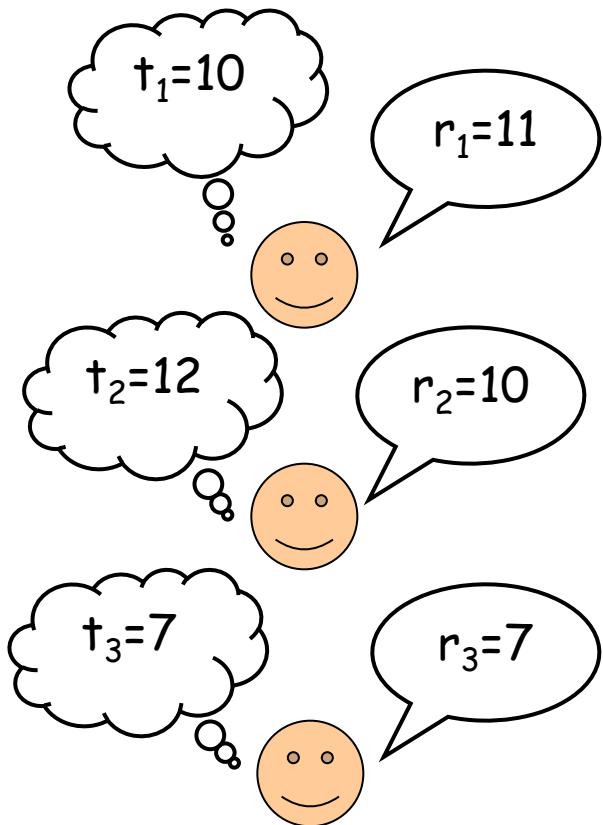


Combinatorial Auction

A single item auction



t_i : is the **maximum** amount of money player i is willing to pay for the painting

If player i wins and has to pay p
its utility is $u_i = t_i - p$

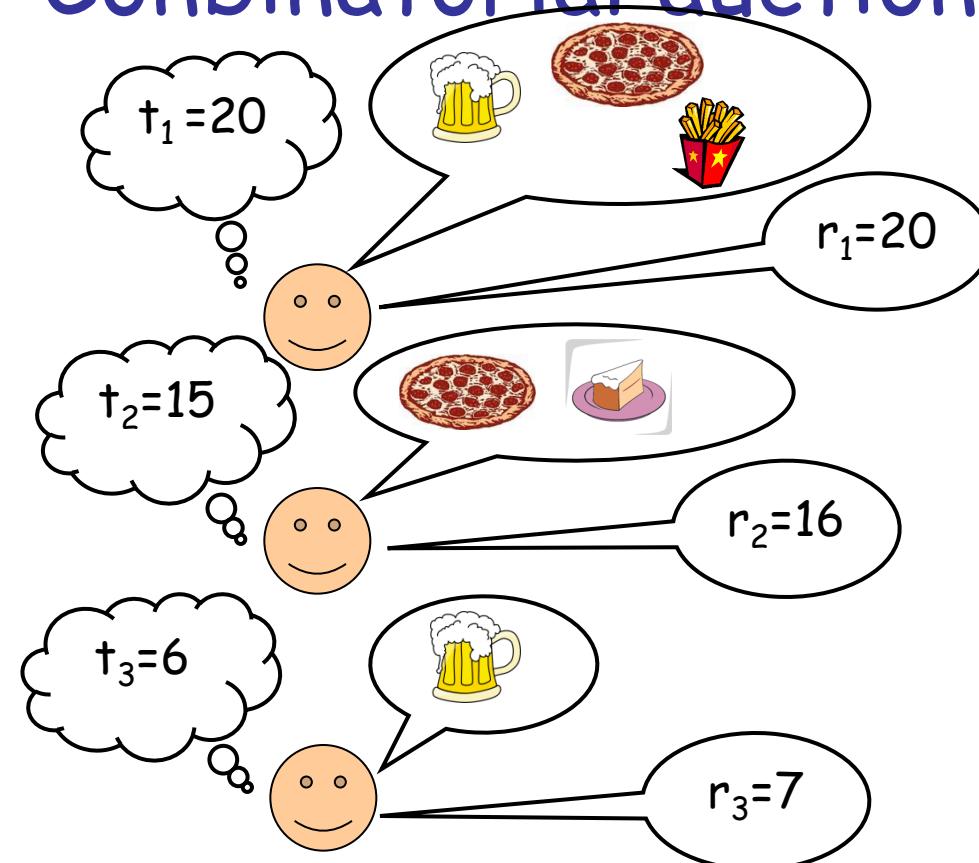
Social-choice function:
the winner should be the guy having in mind the highest value for the painting



r_i : is the amount of money player i bids (in a sealed envelope) for the painting

The mechanism tells to players:
(1) How the item will be allocated (i.e., who will be the **winner**), depending on the received bids
(2) The payment the winner has to return, as a function of the received bids

Combinatorial auction



Each player wants a bundle of objects

t_i : value player i is willing to pay for its bundle

if player i gets the bundle at price p
his utility is $u_i = t_i - p$

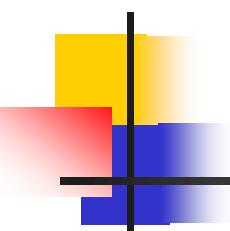


the mechanism decides the set of winners and the corresponding payments

$F = \{ W \subseteq \{1, \dots, N\} : \text{winners in } W \text{ are compatible}\}$

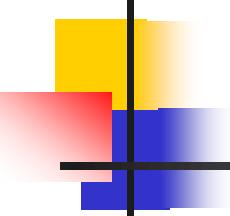
Combinatorial Auction (CA) problem - single-minded case

- Input:
 - n buyers, m indivisible objects
 - each buyer i :
 - Wants a subset S_i of the objects
 - has a value t_i for S_i
- Solution:
 - $W \subseteq \{1, \dots, n\}$, such that for every $i, j \in W$, with $i \neq j$, $S_i \cap S_j = \emptyset$
- Measure (to maximize):
 - Total value of W : $\sum_{i \in W} t_i$



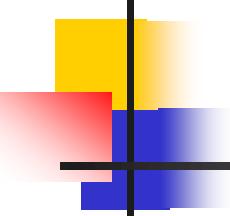
CA game

- each buyer i is selfish
- Only buyer i knows t_i (while S_i is public)
- We want to compute a "good" solution w.r.t. the true values
- We do it by designing a mechanism
- Our mechanism:
 - Asks each buyer to report its value r_i
 - Computes a solution using an output algorithm $g(\cdot)$
 - takes payments p_i from buyer i using some payment function p



More formally

- Type of agent buyer i :
 - t_i : value of S_i
 - Intuition: t_i is the maximum value buyer i is willing to pay for S_i
- Buyer i 's valuation of $W \in F$:
 - $v_i(t_i, W) = t_i$ if $i \in W$, 0 otherwise
- SCF: a good allocation of the objects w.r.t. the true values



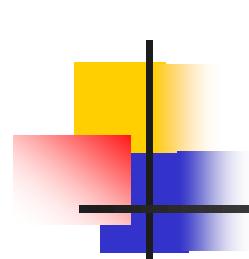
How to design a truthful mechanism for the problem?

Notice that:
the (true) total value of a feasible W is:

$$\sum_{i \in W} t_i = \sum_i v_i(t_i, W)$$

the problem is **utilitarian**!

...VCG mechanisms apply



VCG mechanism

- $M = \langle g(r), p(x) \rangle$:

- $g(r)$: $x^* = \arg \max_{x \in F} \sum_j v_j(r_j, x)$

- $p_i(r)$: for each i :

$$p_i(r) = \sum_{j \neq i} v_j(r_j, g(r_{-i})) - \sum_{j \neq i} v_j(r_j, x^*)$$

$g(r)$ has to compute an optimal solution...

...can we do that?

Theorem

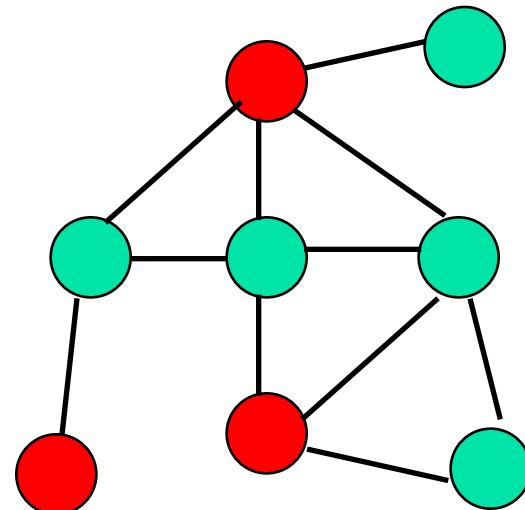
Approximating CA problem within a factor better than $m^{1/2-\varepsilon}$ is NP-hard, for any fixed $\varepsilon > 0$.

proof

Reduction from maximum independent set problem

Maximum Independent Set (IS) problem

- **Input:**
 - a graph $G=(V,E)$
- **Solution:**
 - $U \subseteq V$, such that no two vertices in U are jointed by an edge
- **Measure:**
 - Cardinality of U

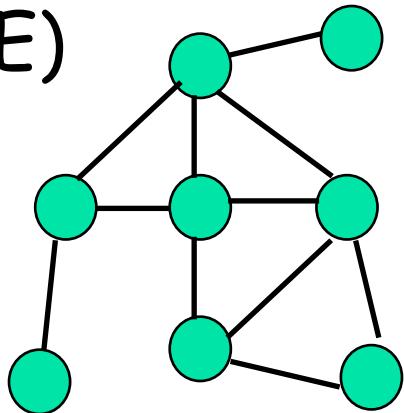


Theorem (J. Håstad, 2002)

Approximating IS problem within a factor better than $n^{1-\varepsilon}$ is NP-hard, for any fixed $\varepsilon > 0$.

the reduction

$G=(V,E)$



each edge is an object

each node i is a buyer with:

S_i : set of edges incident to i

$t_i=1$

CA instance has a solution of total value $\geq k$ if and only if there is an IS of size $\geq k$

A solution of value k for the instance of CA with $\text{Opt}_{CA}/k \leq m^{\frac{1}{2}-\varepsilon}$ for some $\varepsilon > 0$

would imply

A solution of value k for the instance of IS and hence:

$$\text{Opt}_{IS}/k = \text{Opt}_{CA}/k \leq m^{\frac{1}{2}-\varepsilon} \leq n^{1-2\varepsilon}$$

since $m \leq n^2$



How to design a truthful mechanism for the problem?

Notice that:
the (true) total value of a feasible W is:

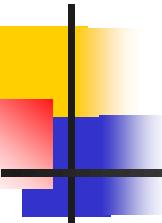
$$\sum_i v_i(t_i, W)$$

the problem is **utilitarian**!

...but a VCG mechanism is not computable
in polynomial time!

what can we do?

...fortunately, our problem is **one parameter**!



A problem is **binary demand (BD)** if

1. a_i 's type is a **single parameter** $t_i \in \mathbb{R}$
2. a_i 's valuation is of the form:

$$v_i(t_i, o) = t_i w_i(o),$$

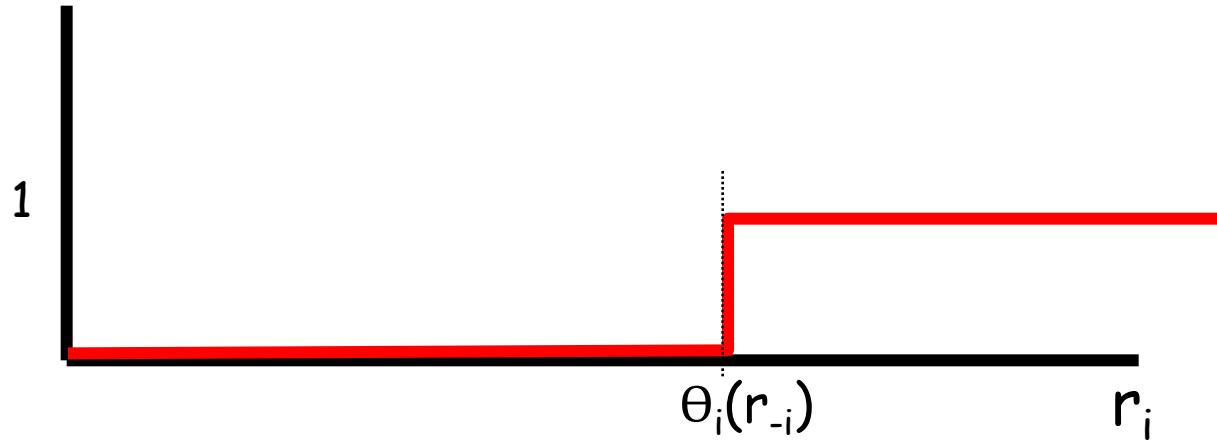
$w_i(o) \in \{0, 1\}$ work load for a_i in o

when $w_i(o) = 1$ we'll say that a_i is
selected in o

Definition

An algorithm $g()$ for a maximization BD problem is **monotone** if

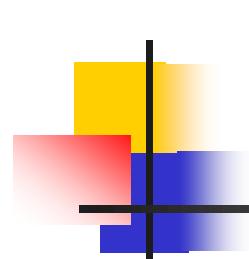
\forall agent a_i , and for every $r_{-i} = (r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_N)$, $w_i(g(r_{-i}, r_i))$ is of the form:



$\theta_i(r_{-i}) \in \mathbb{R} \cup \{+\infty\}$: **threshold**

payment from a_i is:

$$p_i(r) = \theta_i(r_{-i})$$



- Our goal: to design a mechanism satisfying:
 1. $g(\cdot)$ is monotone
 2. Solution returned by $g(\cdot)$ is a "good" solution, i.e. an approximated solution
 3. $g(\cdot)$ and $p(\cdot)$ computable in polynomial time

A greedy \sqrt{m} -approximation algorithm

1. reorder (and rename) the bids such that

$$r_1/\sqrt{|S_1|} \geq r_2/\sqrt{|S_2|} \geq \dots \geq r_n/\sqrt{|S_n|}$$

2. $W \leftarrow \emptyset; X \leftarrow \emptyset$
3. for $i=1$ to n do
 1. if $S_i \cap X = \emptyset$ then $W \leftarrow W \cup \{i\}; X \leftarrow X \cup S_i$
4. return W

Lemma

The algorithm $g()$ is monotone

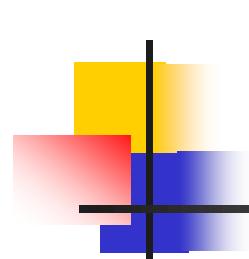
proof

It suffices to prove that, for any selected agent i , we have that i is still selected when it raises its bid

$$r_1/\sqrt{|S_1|} \geq \dots \geq r_i/\sqrt{|S_i|} \geq \dots \geq r_n/\sqrt{|S_n|}$$

Increasing r_i can only move bidder i up in the greedy order, making it easier to win





Computing the payments

...we have to compute for each selected bidder i its threshold value

How much can bidder i decrease its bid before being non-selected?

Computing payment p_i

Consider the greedy order without i

$$r_1/\sqrt{|S_1|} \geq \dots \geq r_i/\sqrt{|S_i|} \geq \dots \geq r_n/\sqrt{|S_n|}$$

↑
index j

Use the greedy algorithm to find the smallest index j (if any) such that:

1. j is selected
2. $S_j \cap S_i \neq \emptyset$

$$p_i = r_j \sqrt{|S_i|} / \sqrt{|S_j|}$$

$p_i = 0$ if j doesn't exist

Lemma

Let OPT be an optimal solution for CA problem, and let W be the solution computed by the algorithm, then

$$\sum_{i \in OPT} r_i \leq \sqrt{m} \sum_{i \in W} r_i$$

proof

$$\forall i \in W \quad OPT_i = \{j \in OPT : j \geq i \text{ and } S_j \cap S_i \neq \emptyset\}$$

since

$$\bigcup_{i \in W} OPT_i = OPT \quad \text{it suffices to prove: } \sum_{j \in OPT_i} r_j \leq \sqrt{m} r_i \quad \forall i \in W$$

$$\sum_{j \in OPT} r_j \leq \sum_{i \in W} \sum_{j \in OPT_i} r_j \leq \sum_{i \in W} \sqrt{m} r_i \leq \sqrt{m} \sum_{i \in W} r_i$$

Lemma

Let OPT be an optimal solution for CA problem, and let W be the solution computed by the algorithm, then

$$\sum_{i \in OPT} r_i \leq \sqrt{m} \sum_{i \in W} r_i$$

proof

$$\forall i \in W \quad OPT_i = \{j \in OPT : j \geq i \text{ and } S_j \cap S_i \neq \emptyset\}$$

since

$$\bigcup_{i \in W} OPT_i = OPT \quad \text{it suffices to prove: } \sum_{j \in OPT_i} r_j \leq \sqrt{m} r_i \quad \forall i \in W$$

crucial observation
for greedy order we have

$$r_j \leq \frac{r_i \sqrt{|S_j|}}{\sqrt{|S_i|}} \quad \forall j \in OPT_i$$

proof

$\forall i \in W$

$$\sum_{j \in OPT_i} r_j \leq \frac{r_i}{\sqrt{|S_i|}} \sum_{j \in OPT_i} \sqrt{|S_j|} \leq \sqrt{m} r_i$$

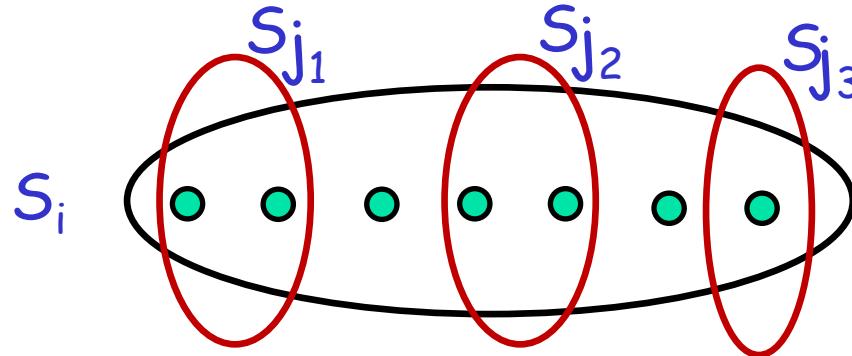
we can bound

Cauchy-Schwarz
inequality

$$\sum_{j \in OPT_i} \sqrt{|S_j|} \leq \sqrt{|OPT_i|} \sqrt{\sum_{j \in OPT_i} |S_j|} \leq \sqrt{|S_i|} \sqrt{m}$$

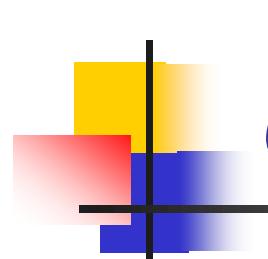
$\leq |S_i|$

$\leq m$



$$OPT_i = \{j_1, j_2, j_3\}$$





Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^n x_i y_i \right) \leq \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \left(\sum_{i=1}^n y_i^2 \right)^{1/2}.$$

...in our case...

$$n = |\text{OPT}_i|$$

$$\begin{aligned} x_j &= 1 \\ y_j &= \sqrt{|S_j|} \quad \text{for } j=1, \dots, |\text{OPT}_i| \end{aligned}$$