Range Trees

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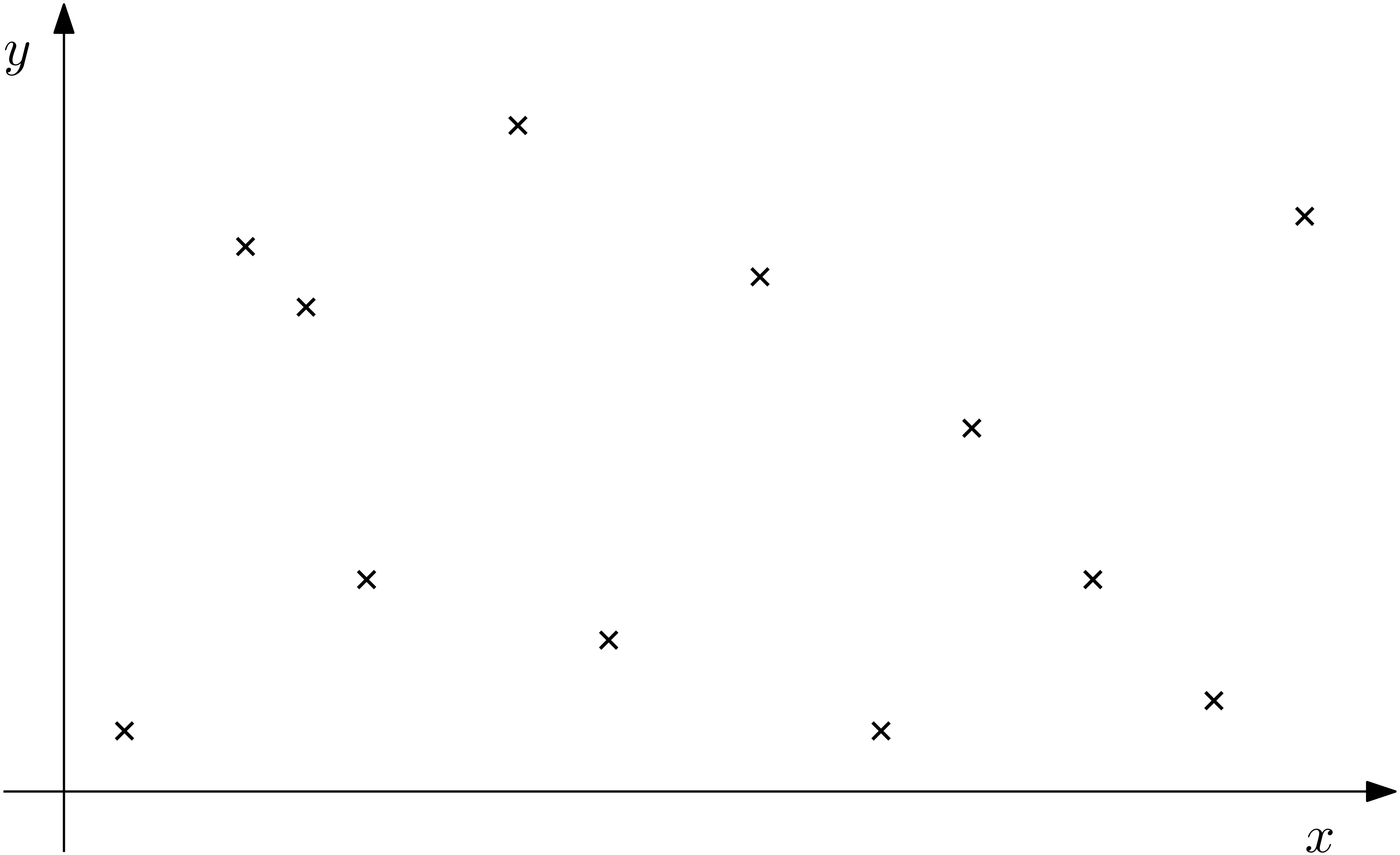

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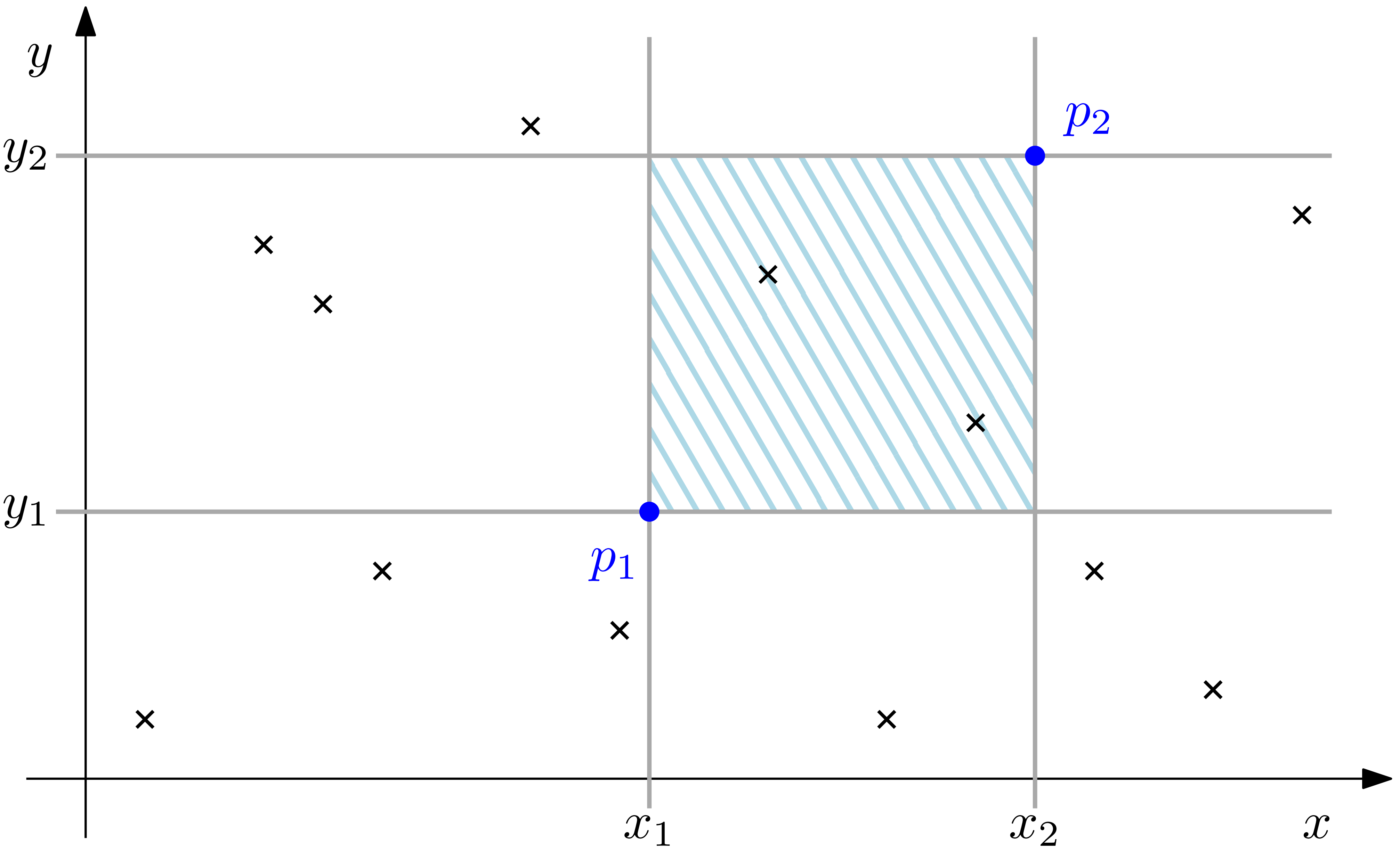

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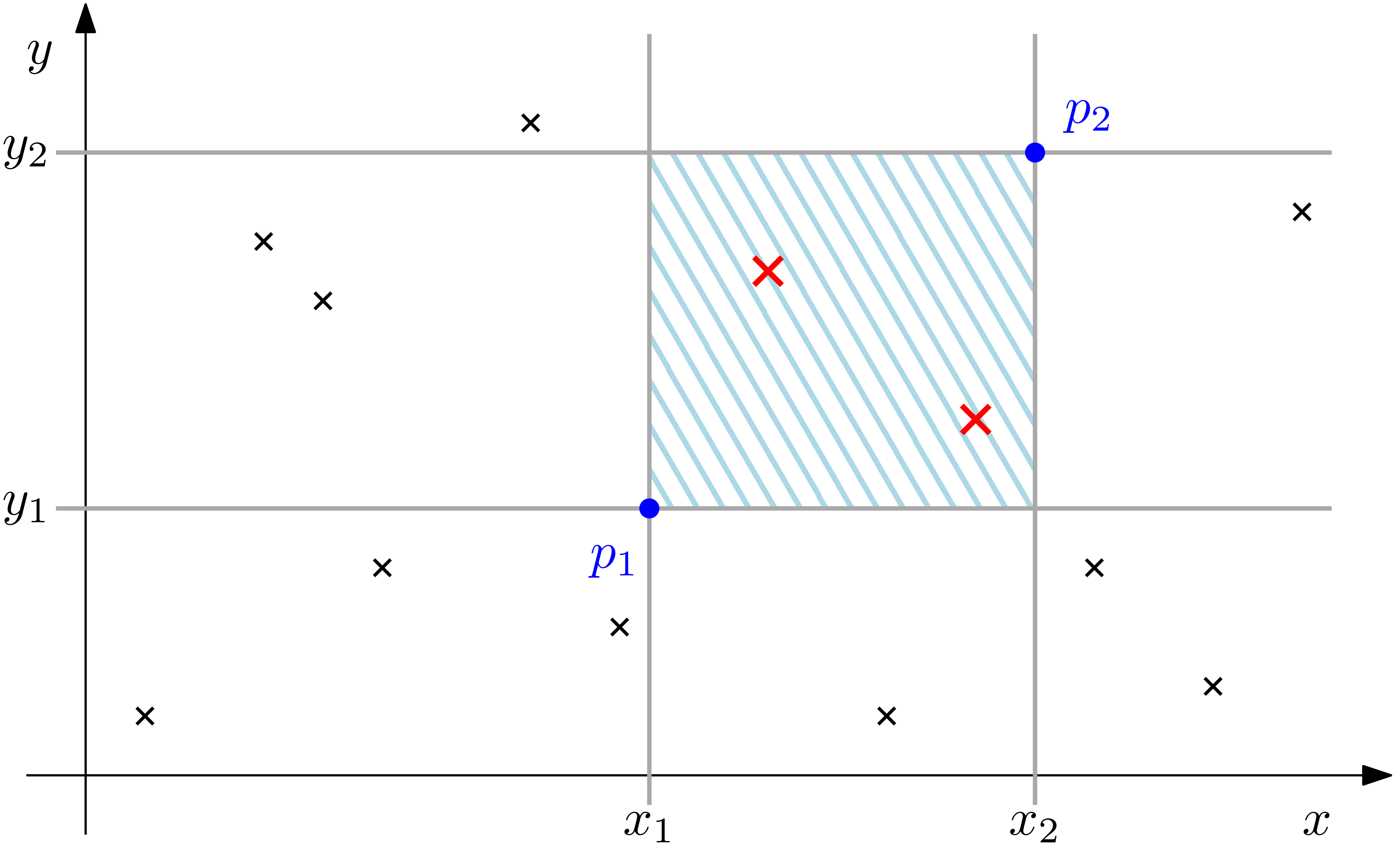


Range Trees

Input:

A set S of n D-dimensional points.

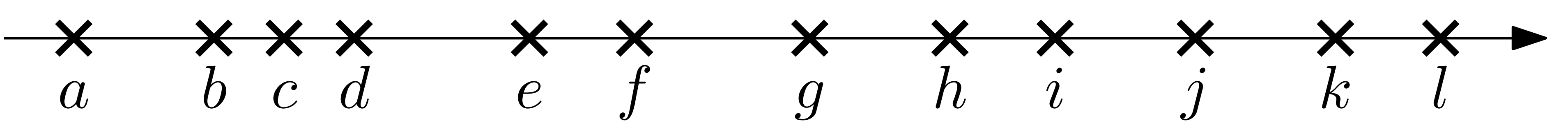
Goal:

Design a data stucture that, given $p_1 \in \mathbb{Z}^D, p_2 \in \mathbb{Z}^D$ can:

- Report *the number* of points $q \in S$ such that $p_1 \leq q \leq p_2$.
- Report *the set* of points $q \in S$ such that $p_1 \leq q \leq p_2$.
- Report the point $q \in S$, $p_1 \leq q \leq p_2$, with *smallest* D-th coordinate.
- ...

An easy case: $D = 1$

- Points are integers
- Store points in a sorted array (in time $O(n \log n)$).
- Perform queries by binary searching for p_1 and p_2



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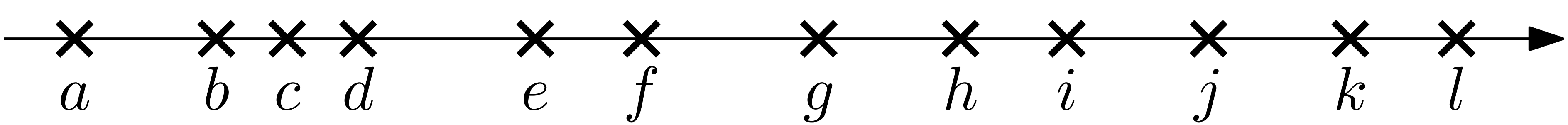
Query time: $O(\log n + k)$ $k =$ "size" of the output.

- $k = \#$ reported points.
- $k = \Theta(1)$ if we only care about the *number* of points.



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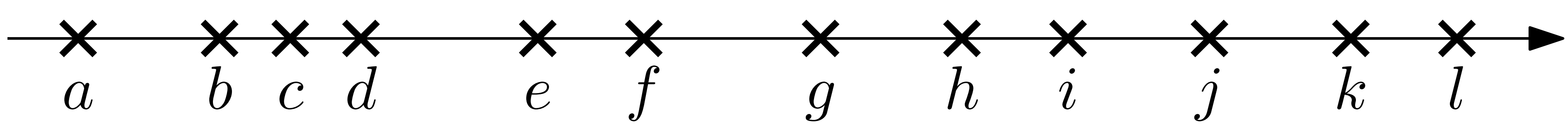
Space complexity: $O(n)$



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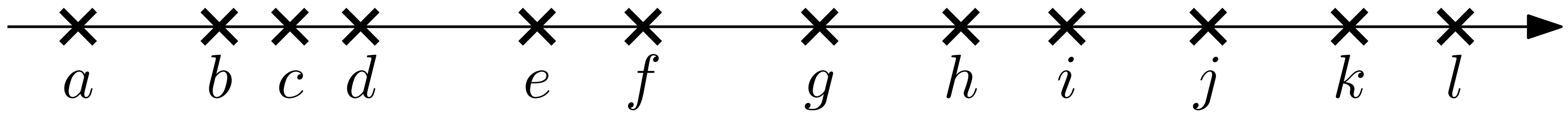

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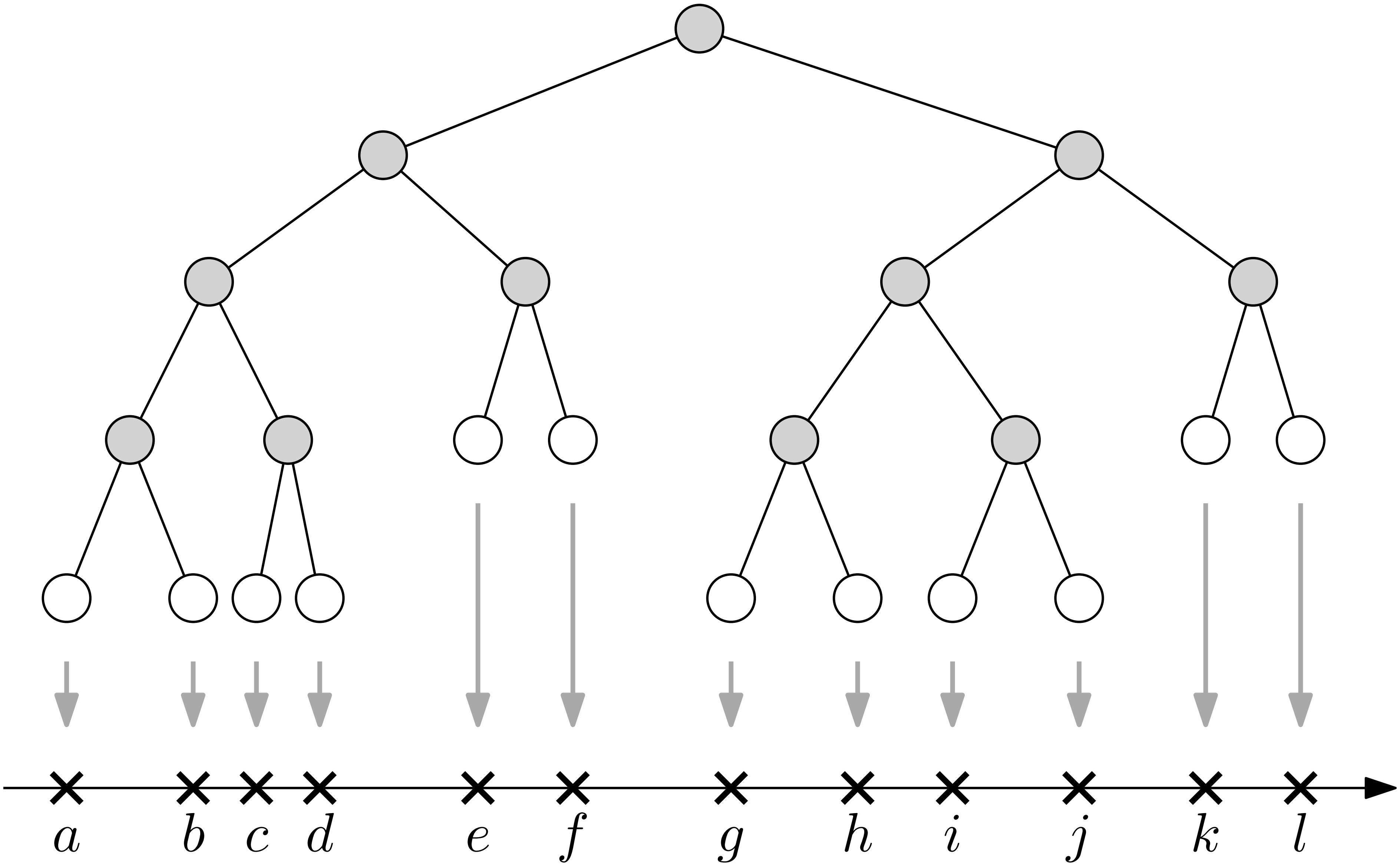

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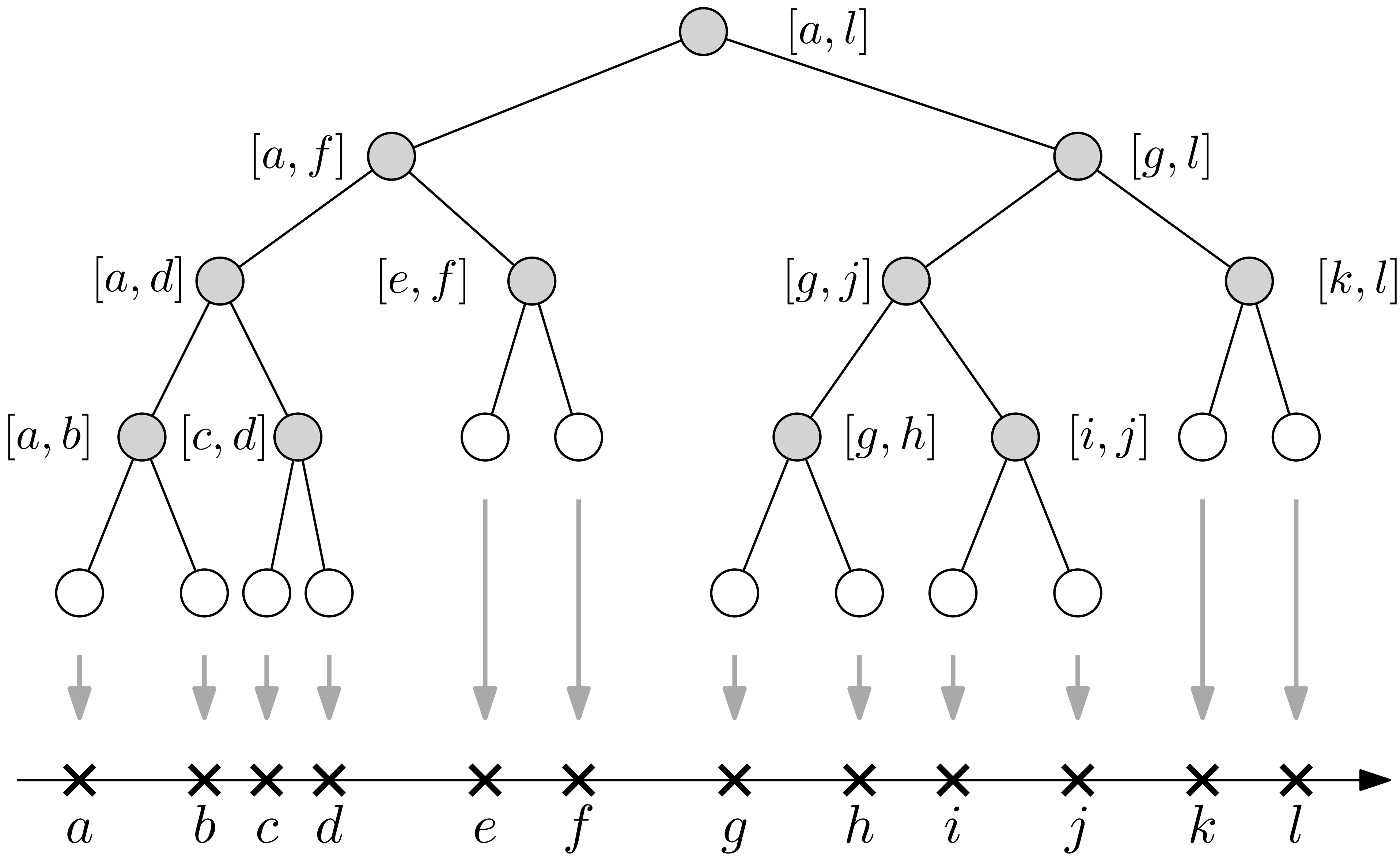

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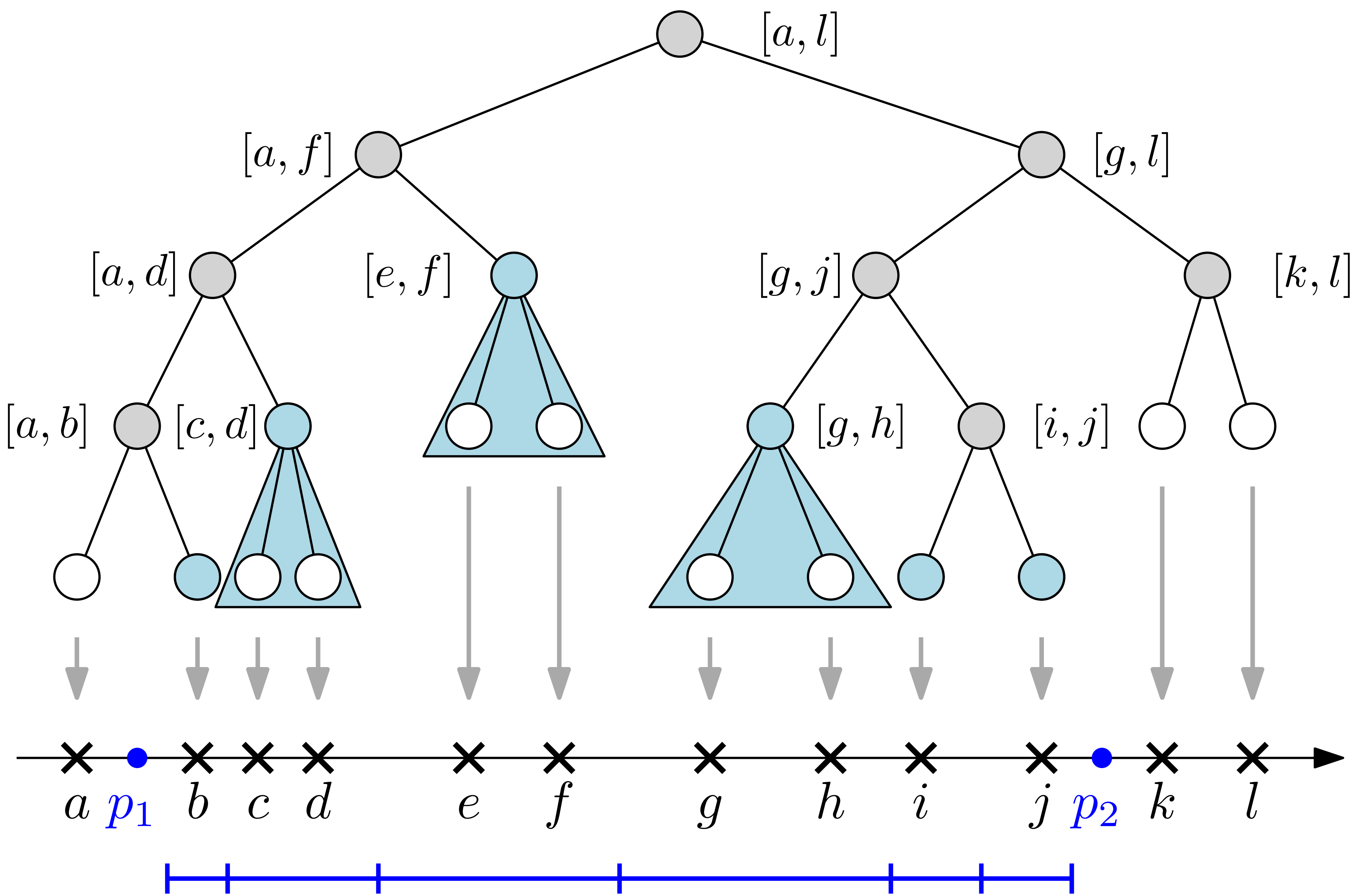

Range Trees: $D = 1$

Construction:

- **Preliminarily** sort S (only once!)
- Split S into S_1 and S_2 of $\approx \frac{n}{2}$ elements each. $O(1)$
- Recursively build T_1 and T_2 from S_1 and S_2, respectively.
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$$O(n \log n)$$

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What if S is already sorted?

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$$O(n \log n)$$

What if S is already sorted? $O(n)$ (we will need this later)

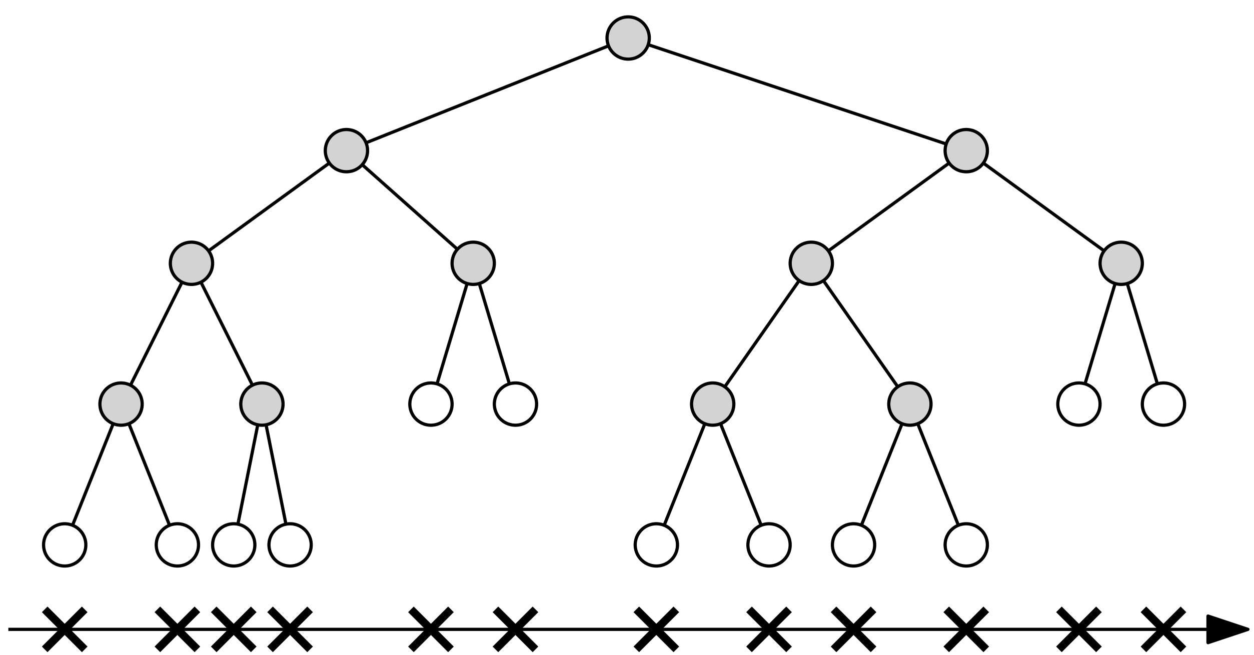
Range Trees: $D = 1$

Preprocessing time: $O(n \log n)$

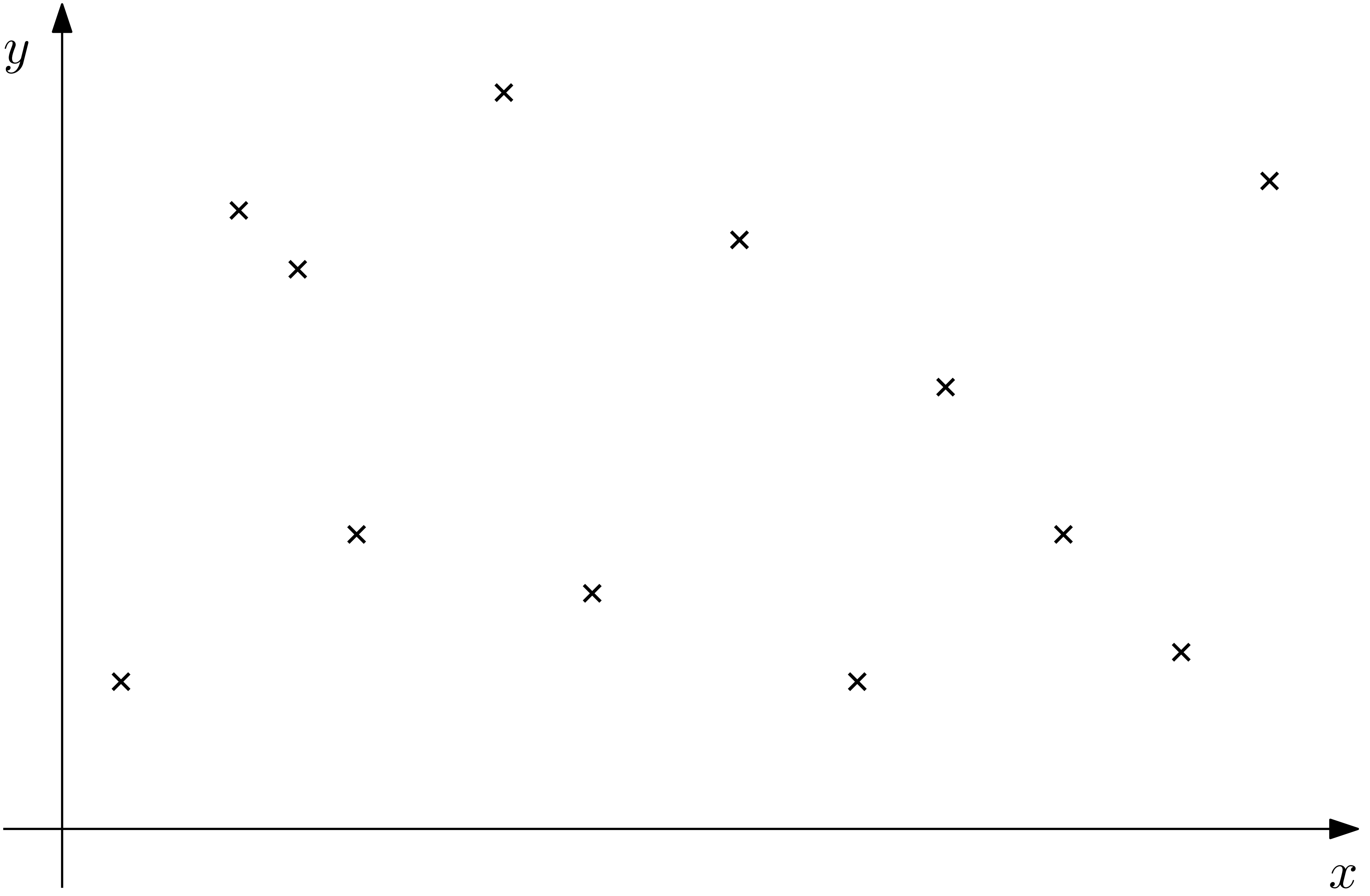
Query time: $O(\log n + k)$

- $k = \#$ reported points.
- $k = \Theta(1)$ if we only care about the *number* of points.

Space complexity: $O(n)$



Range Trees: $D = 2$



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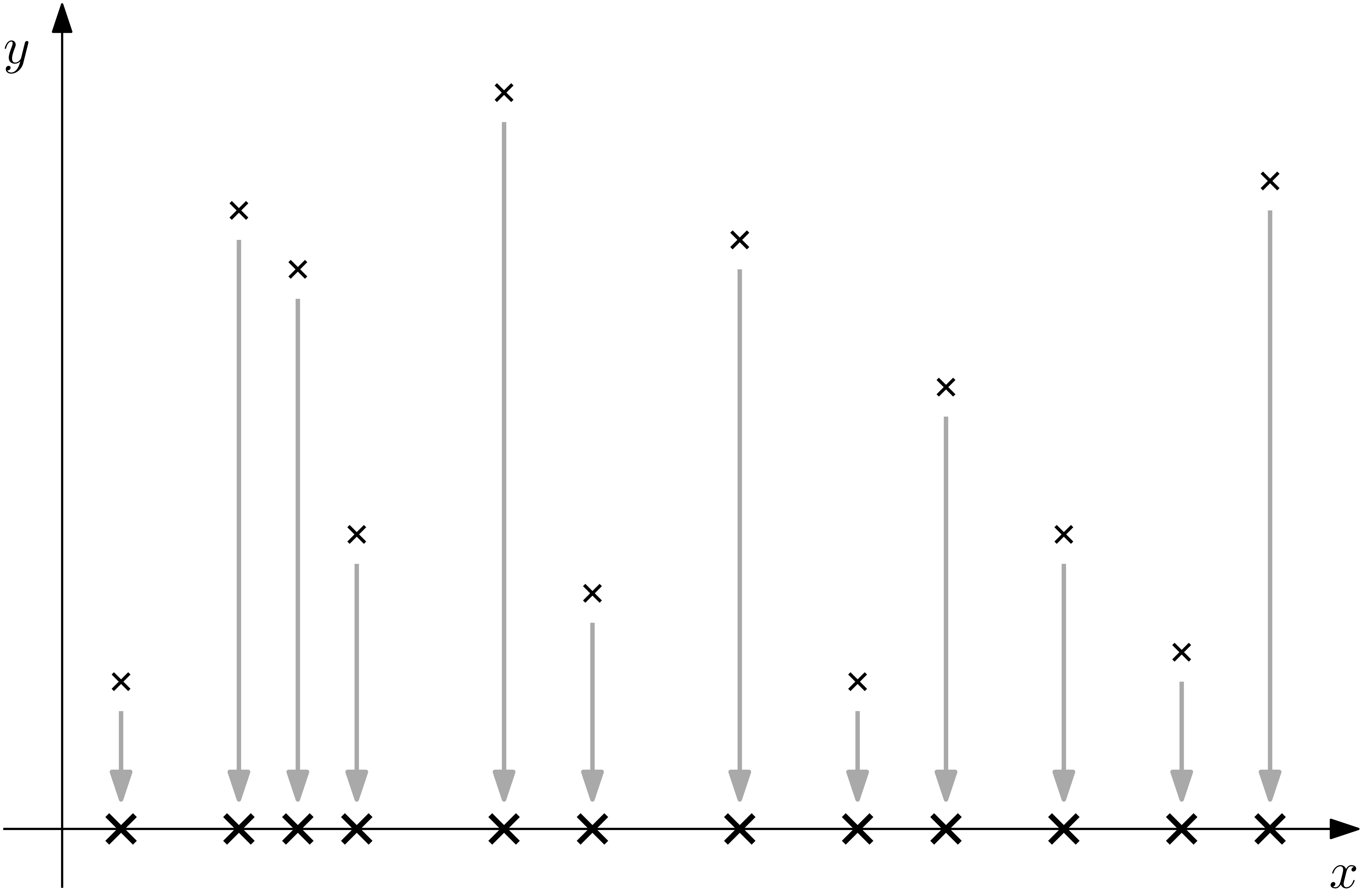

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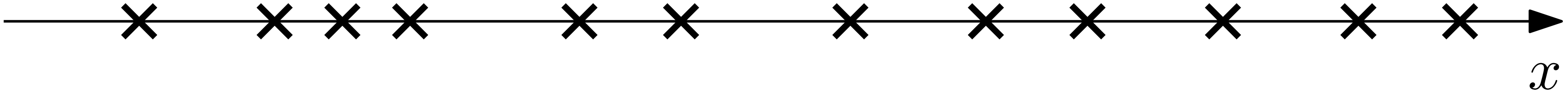
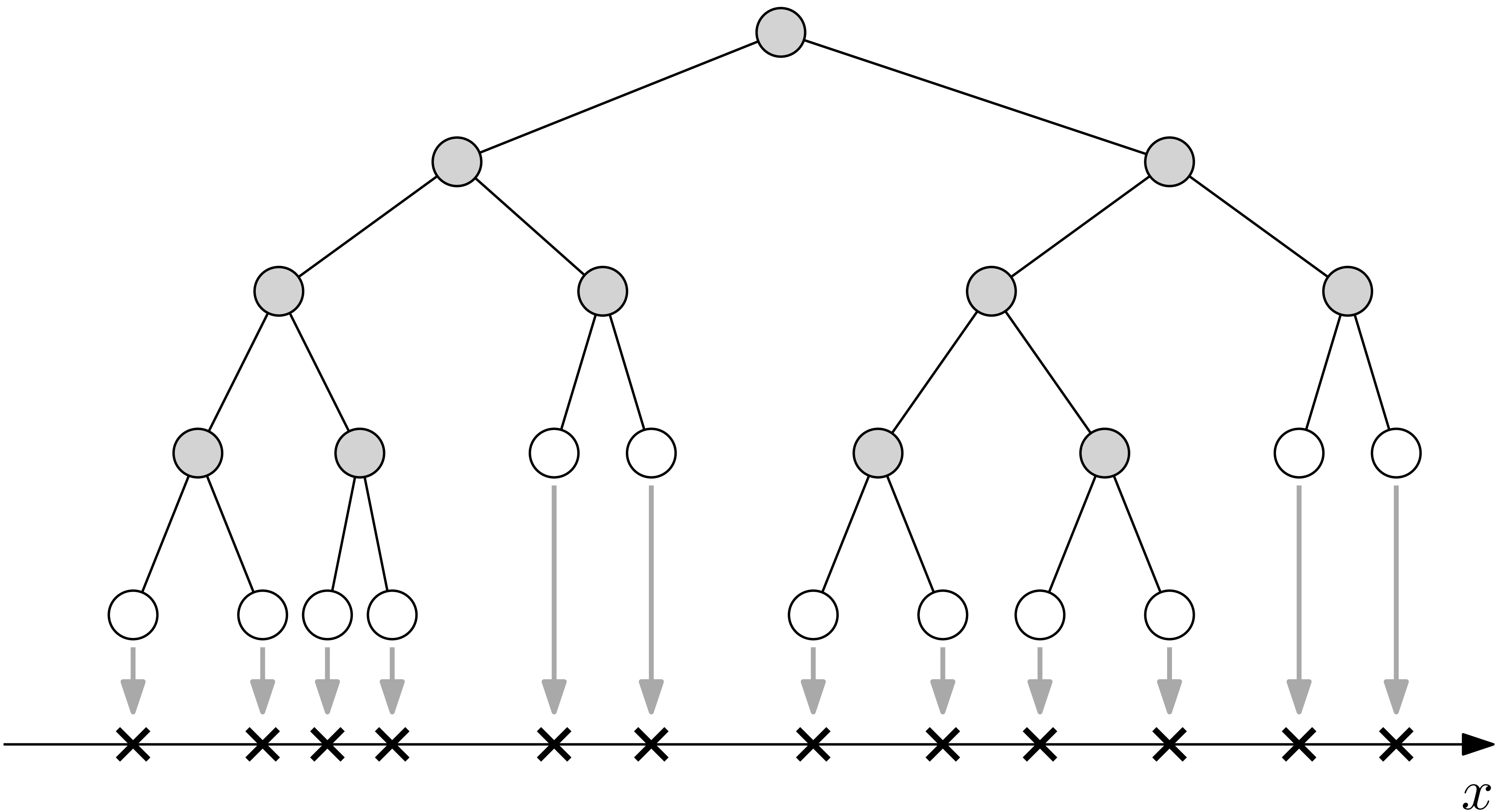


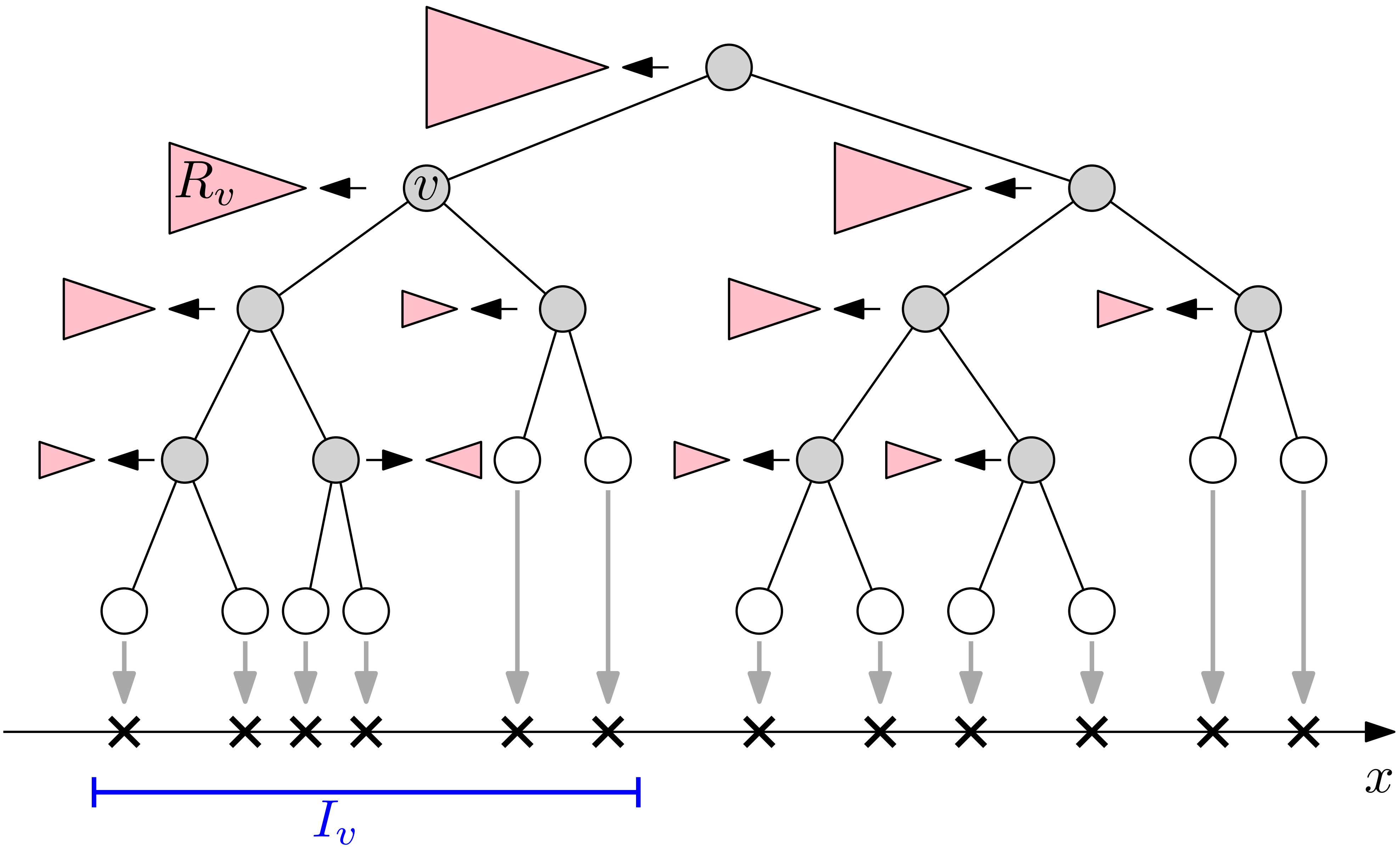
Range Trees: $D = 2$

Build a range tree on the set of x-coordinates of the points in S



Range Trees: $D = 2$

For each node v representing an interval $I_v = [x_1, x_2]$, build a range tree R_v on the y coodinates of the points in S with x-coordinate in I_v



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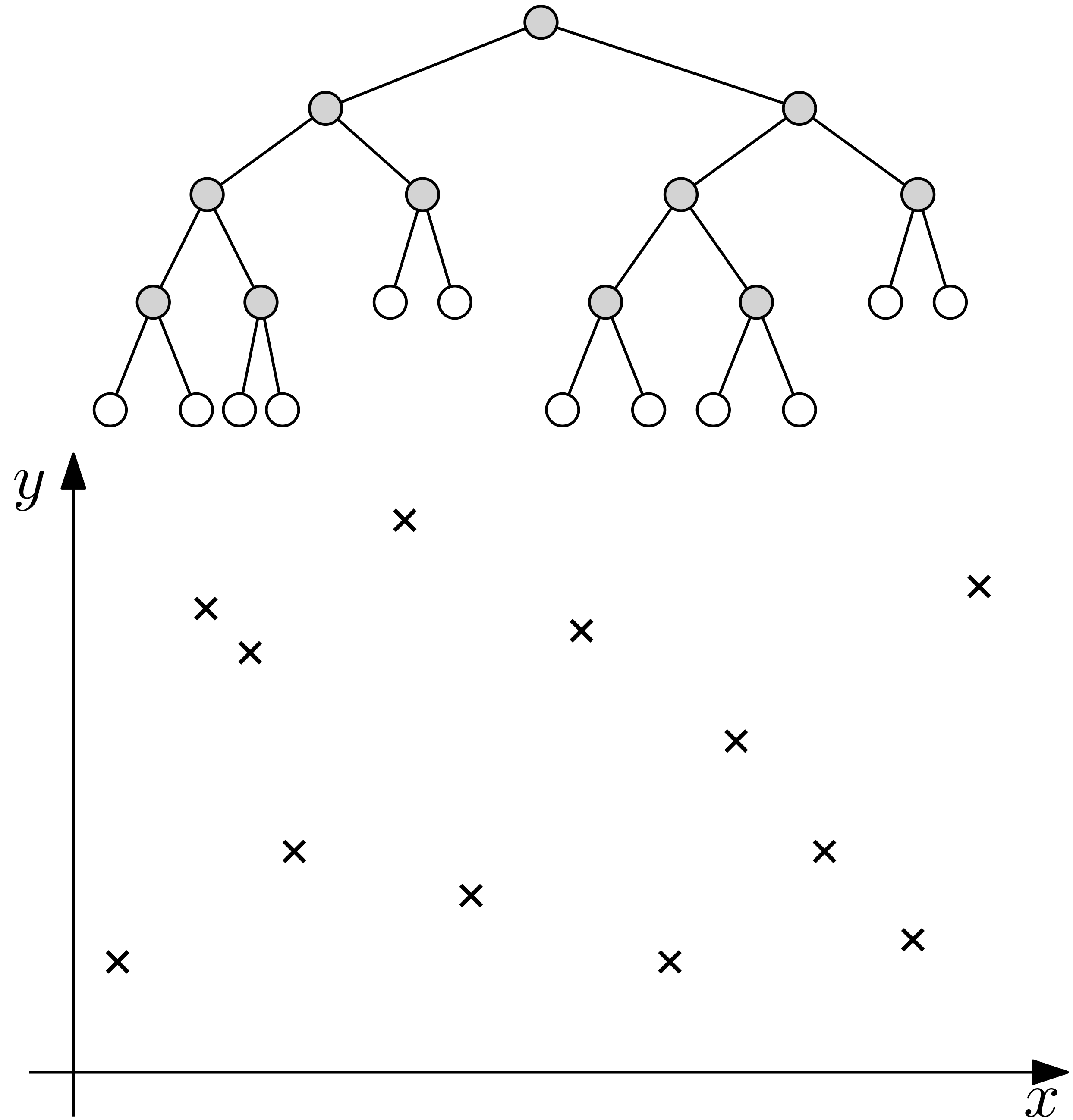

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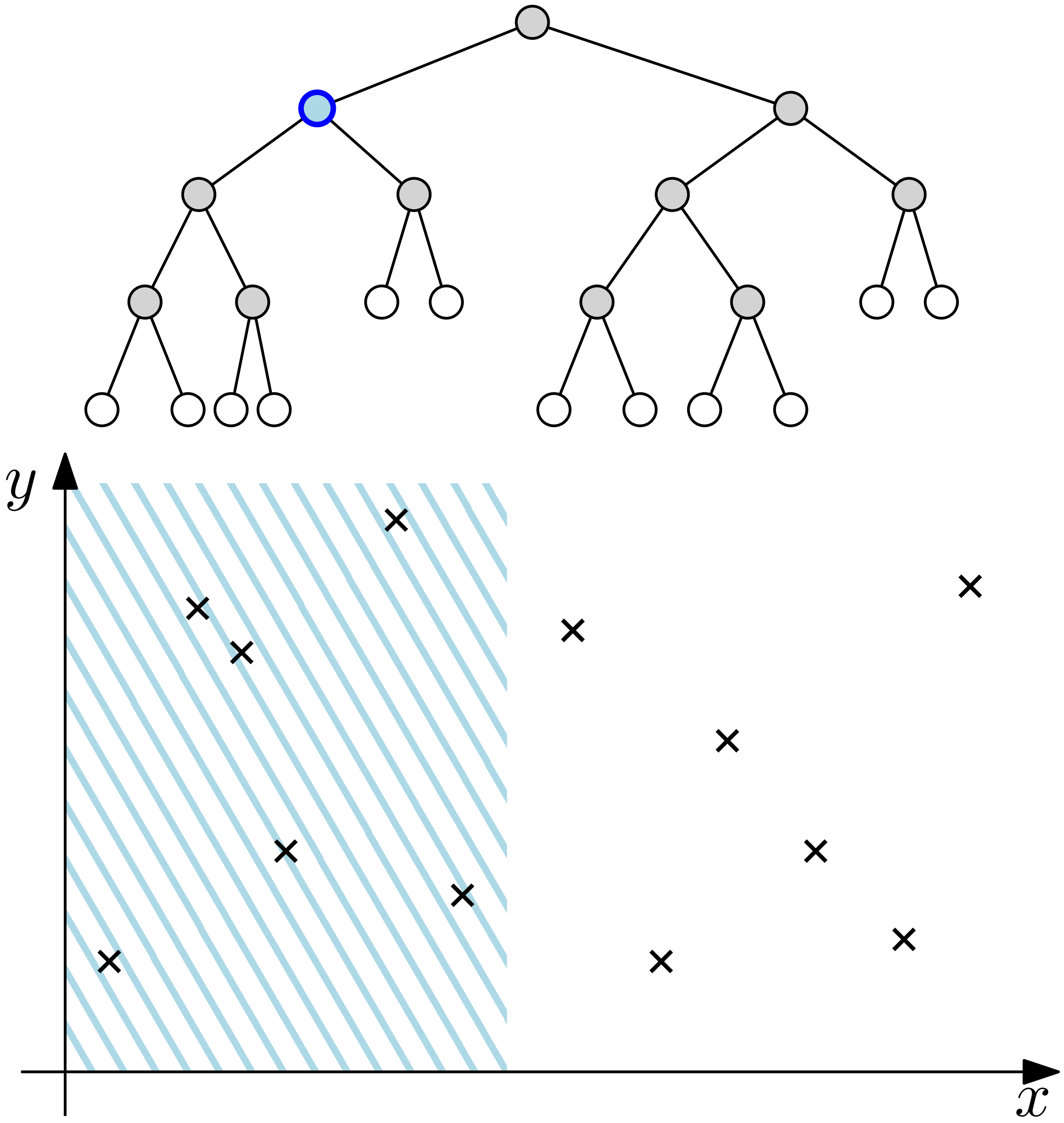

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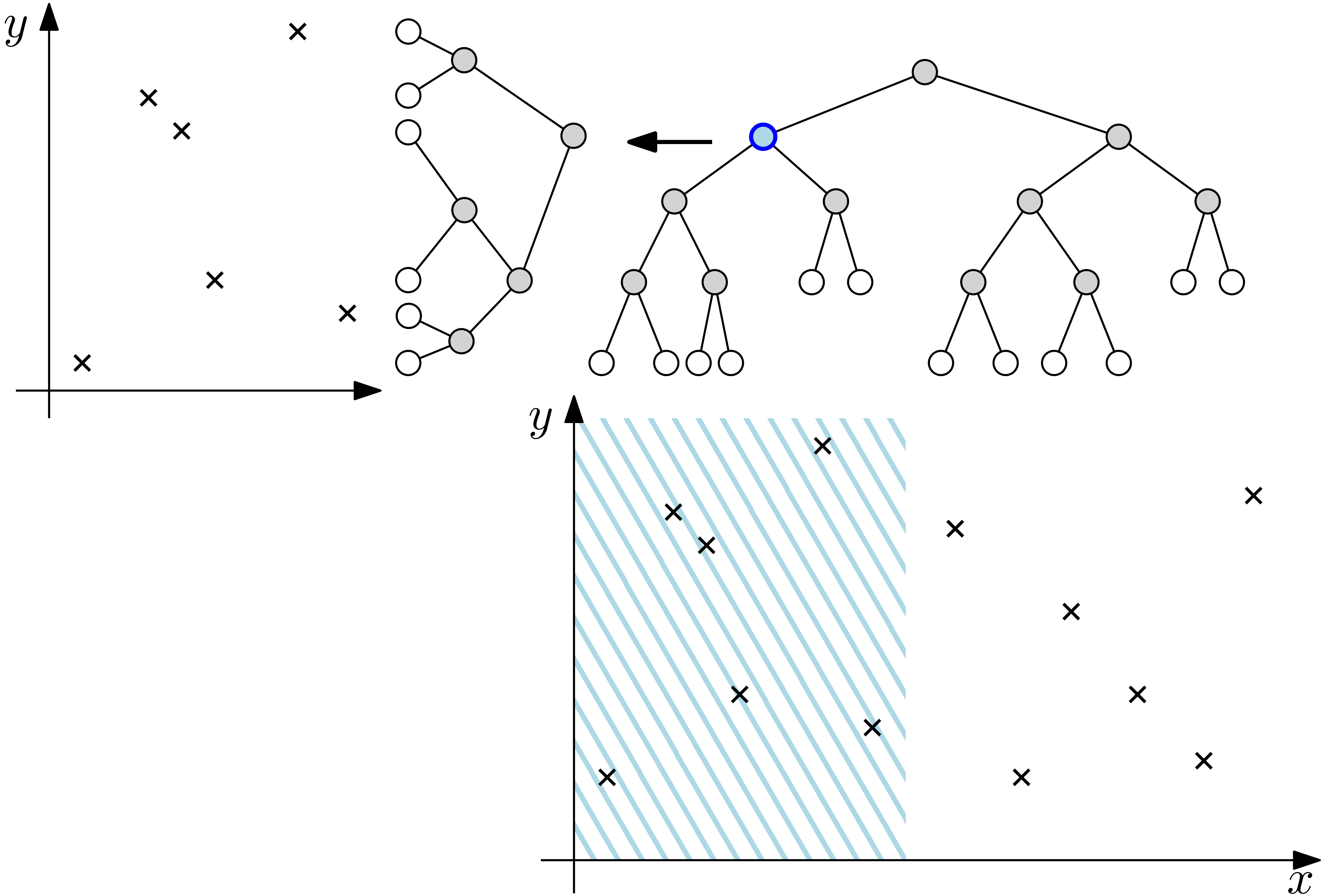

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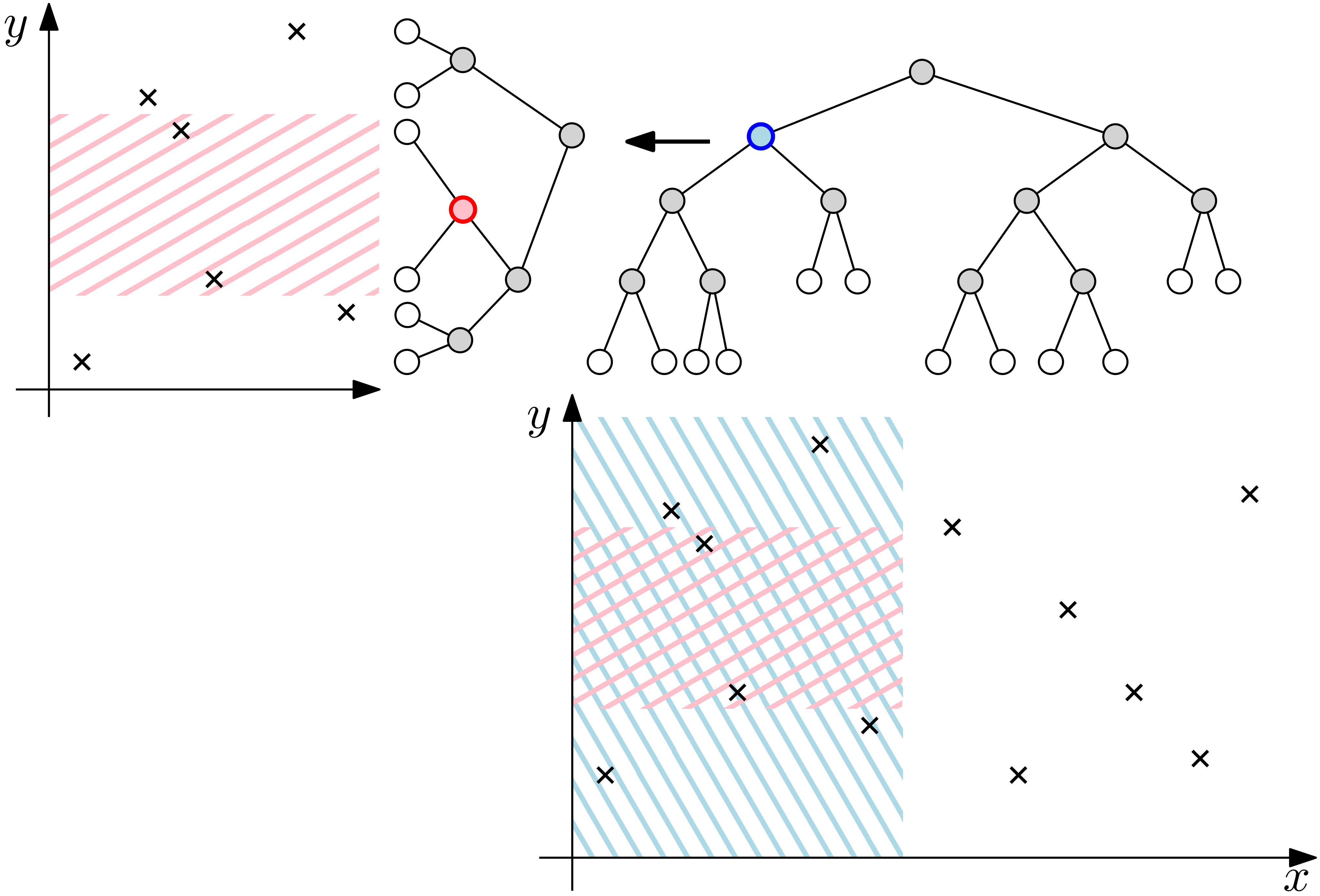

Range Trees: $D = 2$

Construction:

- **Preliminarily** sort S on the x-coordinate.
- Split S into S_1 and S_2 of $\approx \frac{n}{2}$ elements each.
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- Store, in v, a pointer to a new 1D Range Tree on S
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Time: $O(n \log n) + T(n)$, where $T(n) = 2 \cdot T(\frac{n}{2}) + O(n \log n)$

$$O(n \log^2 n)$$

Range Trees: $D = 2$

S^y is the set S sorted on the y-coordinate

Construction:

- **Preliminarily** sort S on the x-coordinate.
- Split S into S_1 and S_2 of $\approx \frac{n}{2}$ elements each.
- Recursively build (T_1, S_1^y) and (T_2, S_2^y) from S_1 and S_2, respectively.
- The root v of T has T_1 and T_2 as its left and right subtrees.
- Merge S_1^y and S_2^y into S^y.
- Store, in v, a pointer to a new 1D Range Tree on S^y
- Return (T, S^y)

Range Trees: $D = 2$

S^y is the set S sorted on the y-coordinate

Construction:

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Time: $O(n \log n) + T(n)$, where $T(n) = 2 \cdot T(\frac{n}{2}) + O(n)$

$$O(n \log n)$$

Range Trees: $D = 2$

To report the points $p_1 = (x_1, y_1) \leq q \leq p_2 = (x_2, y_2)$:

- Use T to find the $h = O(\log n)$ subtrees $R_1, \ldots, R_h$ that store the points $q = (x, y)$ with $x_1 \leq x \leq x_2$.

- For each tree $R_j \in \{R_1, \ldots, R_h\}$ representing the x-interval I_j:
  - Query R_j to report the number of/set of points $q = (x, y)$ with $x \in I_j$ and $y_1 \leq y \leq y_2$.

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Time complexity:

$$O(\log n) \cdot O(\log n) + O(k) = O(\log^2 n + k)$$

Number of R_is — Time to query R_i — "size" of the output

Range Trees: $D = 2$

Preprocessing time: $O(n \log n)$

Query time: $O(\log^2 n + k)$

- $k = \#$ reported points.
- $k = \Theta(1)$ if we only care about the *number* of points.

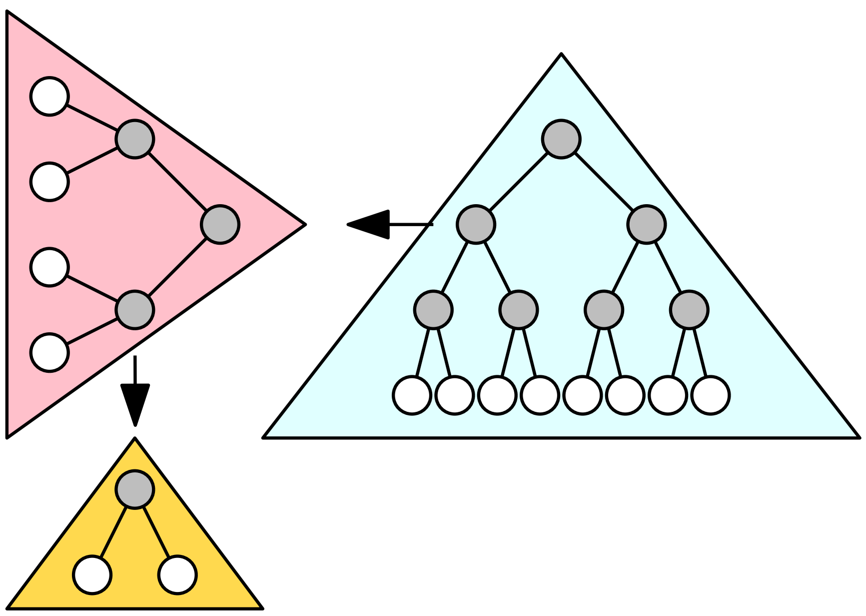
Space complexity:

- Bounded by the overall size of 1D Range Trees
- Each point belongs to $O(\log n)$ 1D Range Tees
- Total space: $O(n \log n)$

Higher dimensions: construction

To store points $p = (x, y, z, w, ...)$ in $D > 2$ dimensions:
Recursive construction:

- Build a Range Tree T on the first coordinate x of the points:
- For each subtree T_v of T associated with the interval $I_v = [x_1, x_2]$:
  - Construct a range tree R_v on the last $D-1$ coordinates $(y, z \ldots)$ of the set of points $p = (x, y, \ldots)$ with $x \in I_v$.
  - Store, in v, a pointer to R_v.



Time: $O(n \log^{D-1} n)$.

Space: $O(n \log^{D-1} n)$.

Higher dimensions: query

Let $p_1 = (x_1, y_1, z_1, \dots)$, $p_2 = (x_2, y_2, z_2, \dots)$.

To report the points $p_1 \leq q \leq p_2$:

- Use T to find the $h = O(\log n)$ subtrees $R_1, \dots, R_h$ that store the points $q = (x, y, z, \dots)$ with $x_1 \leq x \leq x_2$.
- For each tree $R_j \in \{R_1, \dots, R_h\}$ representing the x-interval I_j:
  - Recursively query R_i to report the number/set of points q s.t. $x \in I_j$ and $(y_1, z_1, \dots) \leq q \leq (y_2, z_2, \dots)$.

Query time: $O(\log^D n + k)$.

Recap

D	Size	Preprocessing Time	Query Time	Notes
1	$O(n)$	$O(n \log n)$	$O(\log n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n + k)$	
> 2	$O(n \log^{D-1} n)$	$O(n \log^{D-1} n)$	$O(\log^D n + k)$	

Fractional Cascading

Fractional Cascading: The problem

Input:

k sorted arrays $A_1, \ldots, A_k$ of n elements each:

A_1	4	9	15	22	23	38	41	50	53	58
A_2	3	7	10	11	15	17	20	36	62	64
A_3	21	23	29	35	37	40	52	57	61	66
A_4	2	5	6	15	24	27	39	50	54	76

Query:

Given x report, for $i = 1, \ldots, k$, x if $x \in A_i$ or its *predecessor* if $x \notin A_i$.

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k sorted arrays $A_1, \ldots, A_k$ of n elements each:

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$x = 31$

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$x = 58$

Query:

Given x report, for $i = 1, \ldots, k$, x if $x \in A_i$ or its *predecessor* if $x \notin A_i$.

Fractional Cascading: A Trivial solution

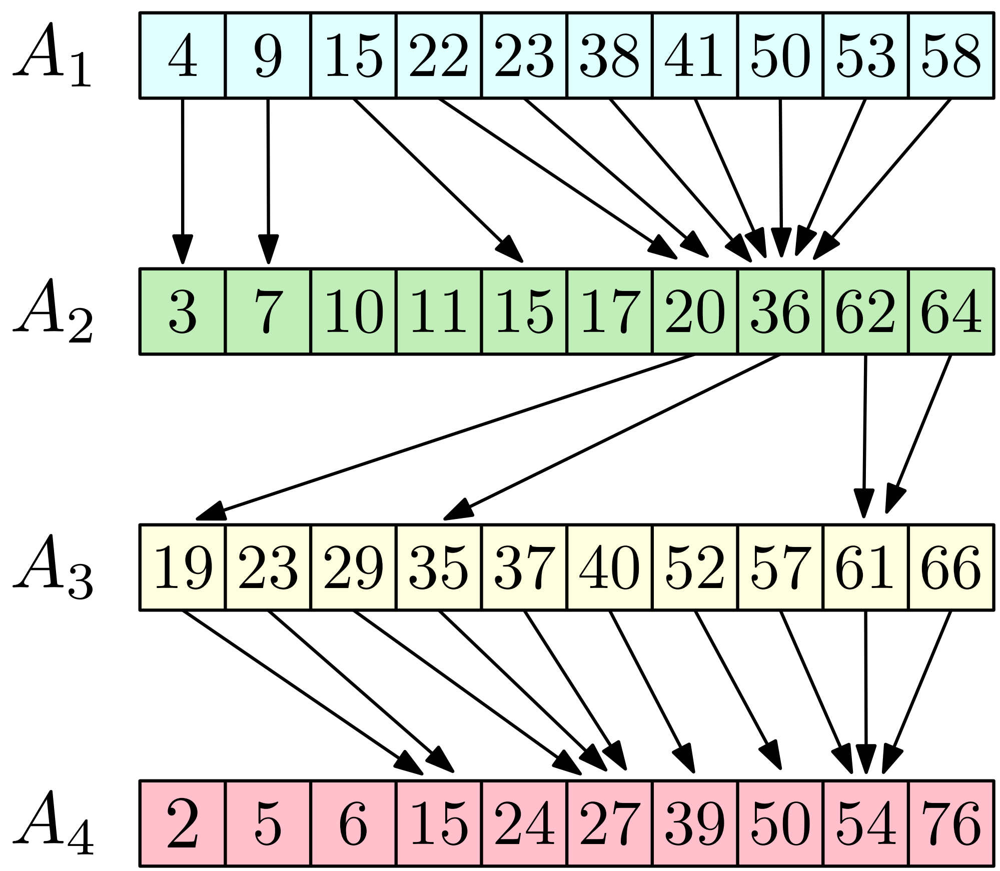
- For $i = 1, \ldots, k$:
  - Binary search for x in A_i

Time: $O(k \log n)$

Fractional Cascading

First idea: cross linking

Keep pointers from $A_i[j]$ to the predecessor of $A_i[j]$ in A_{i+1}.



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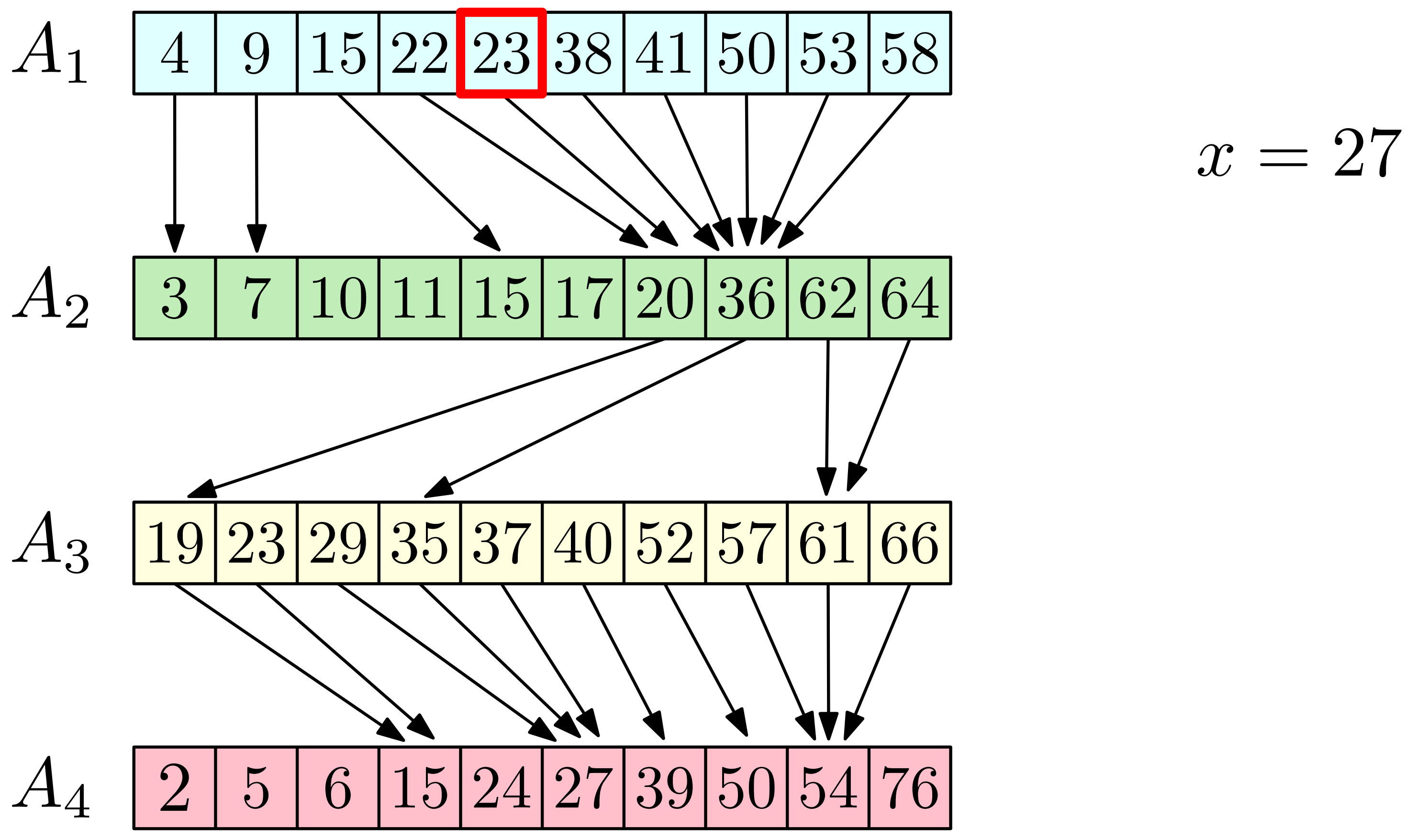

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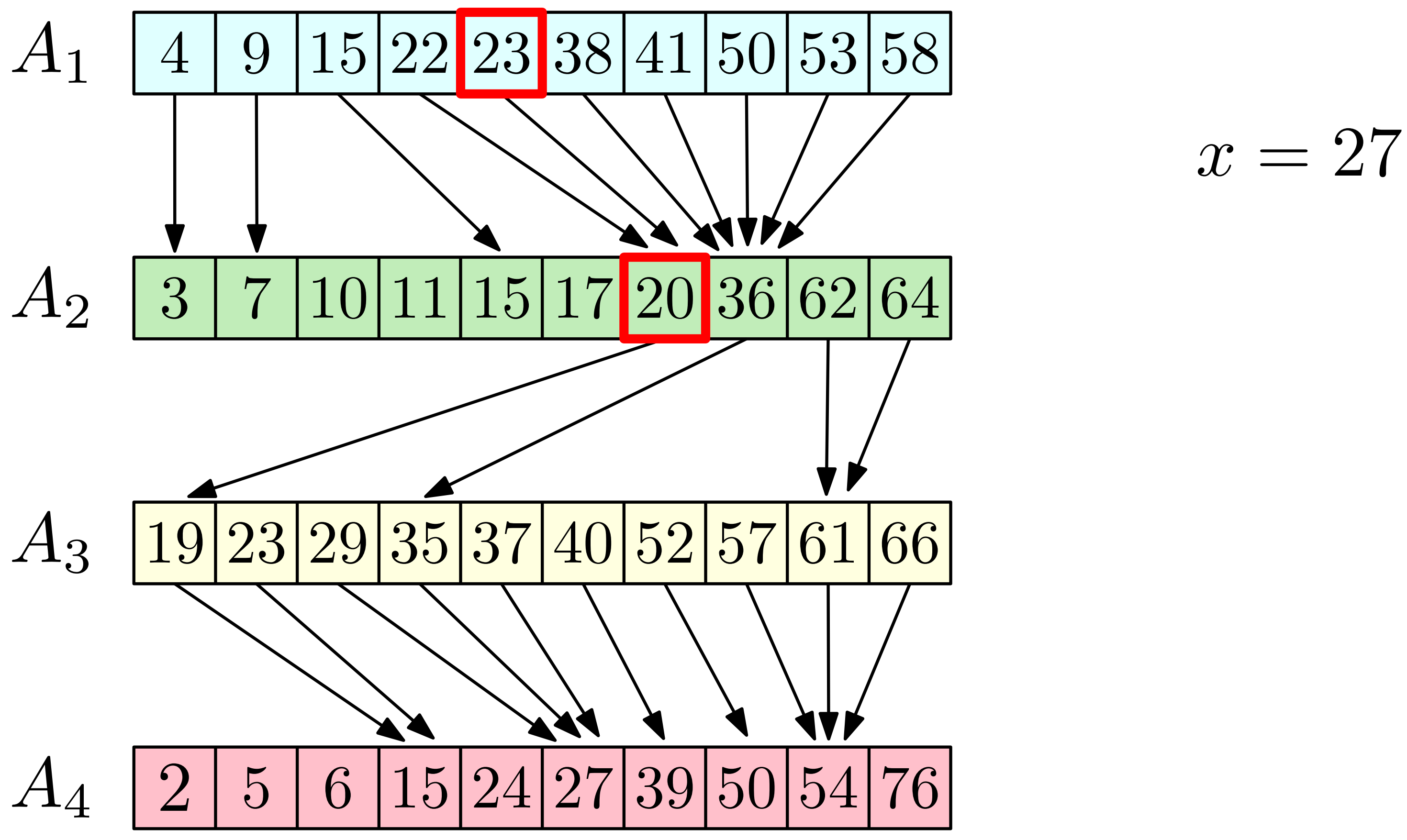

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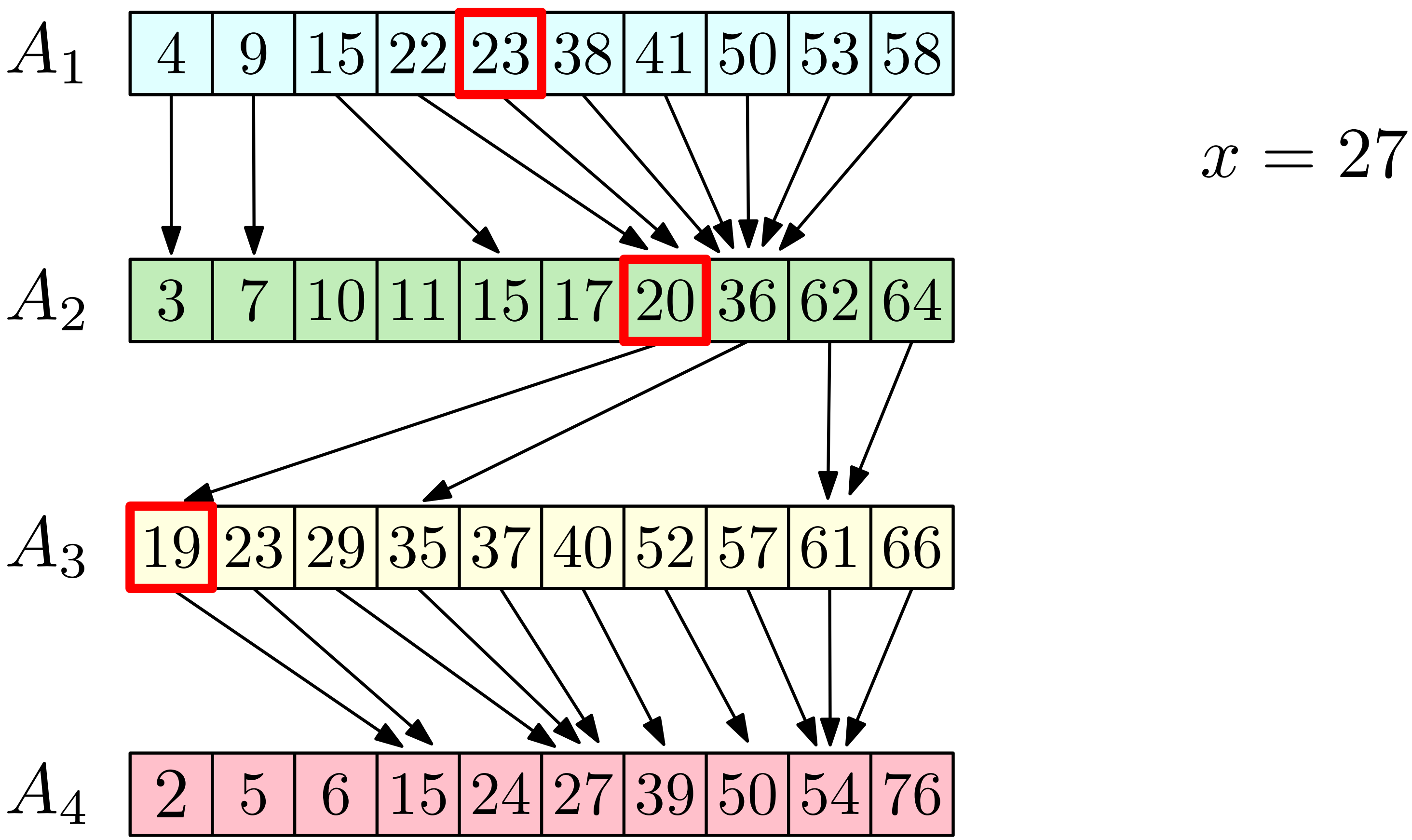

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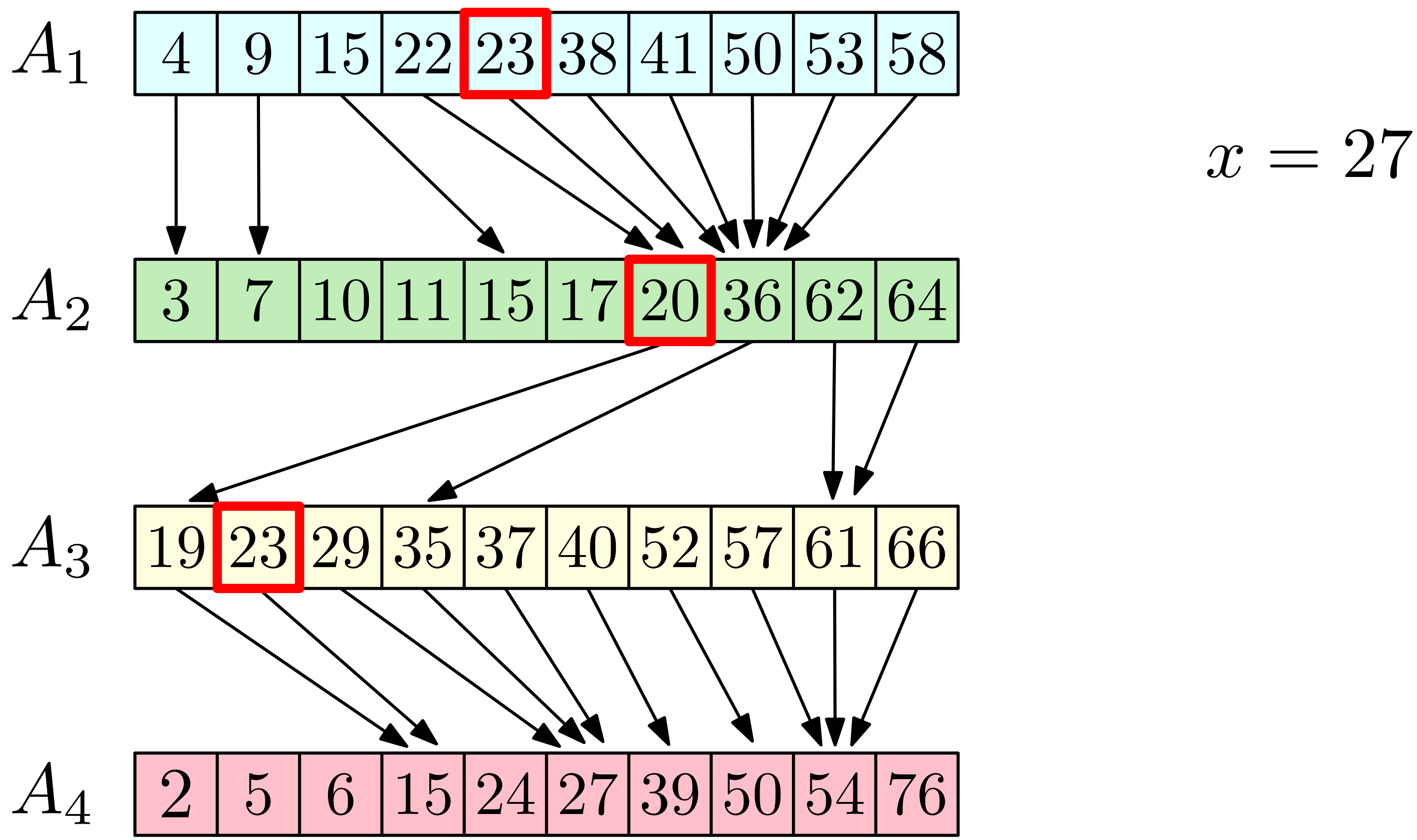

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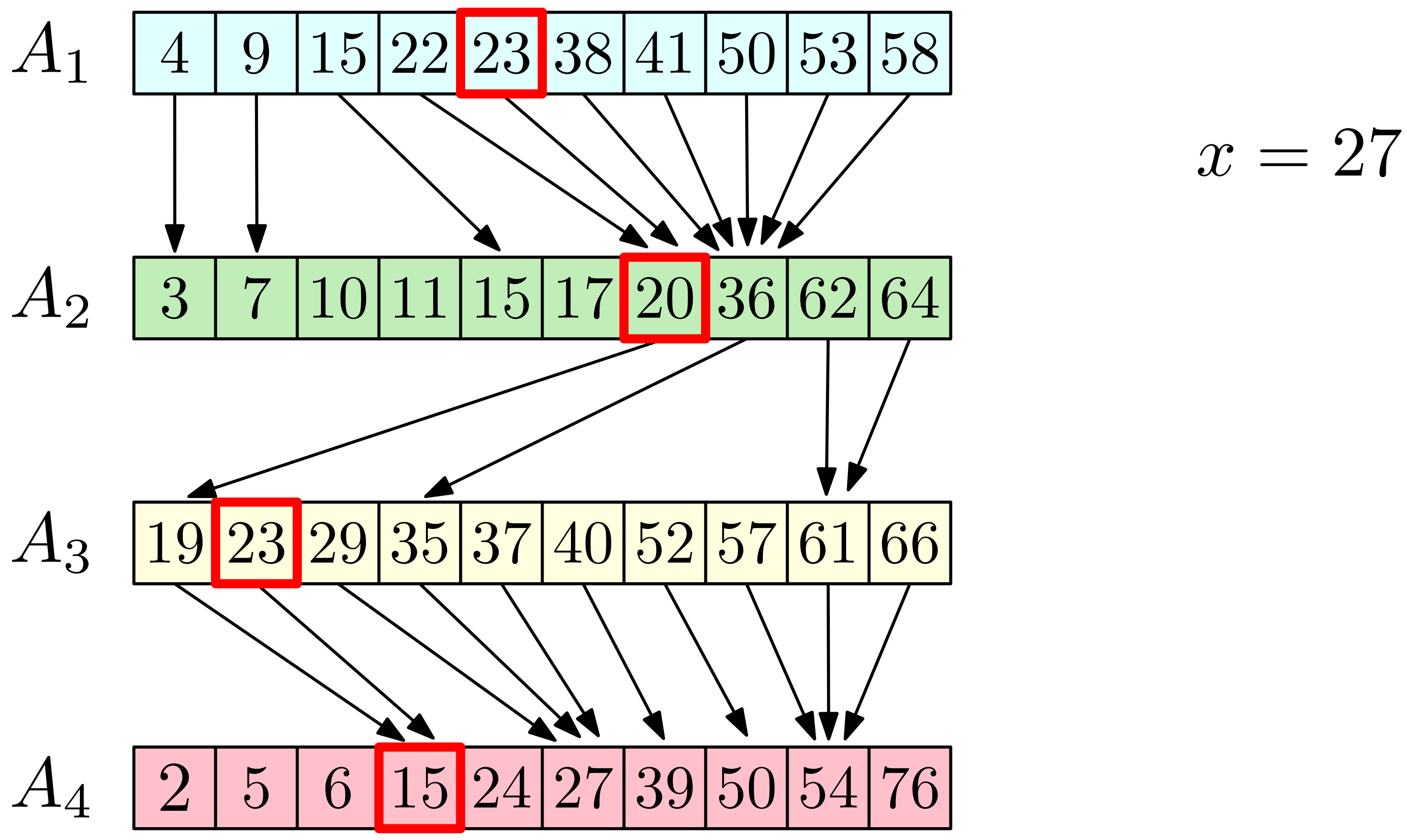

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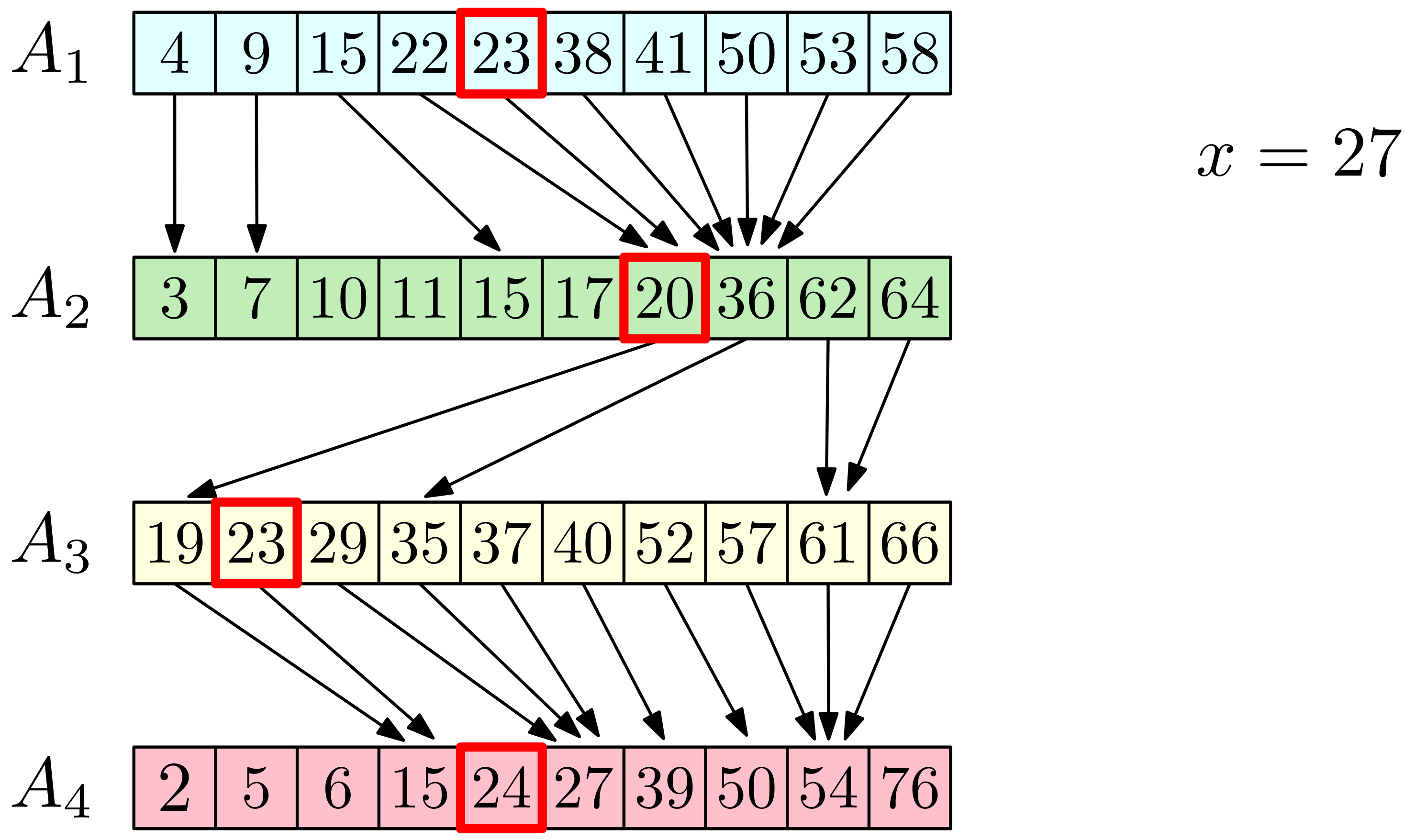

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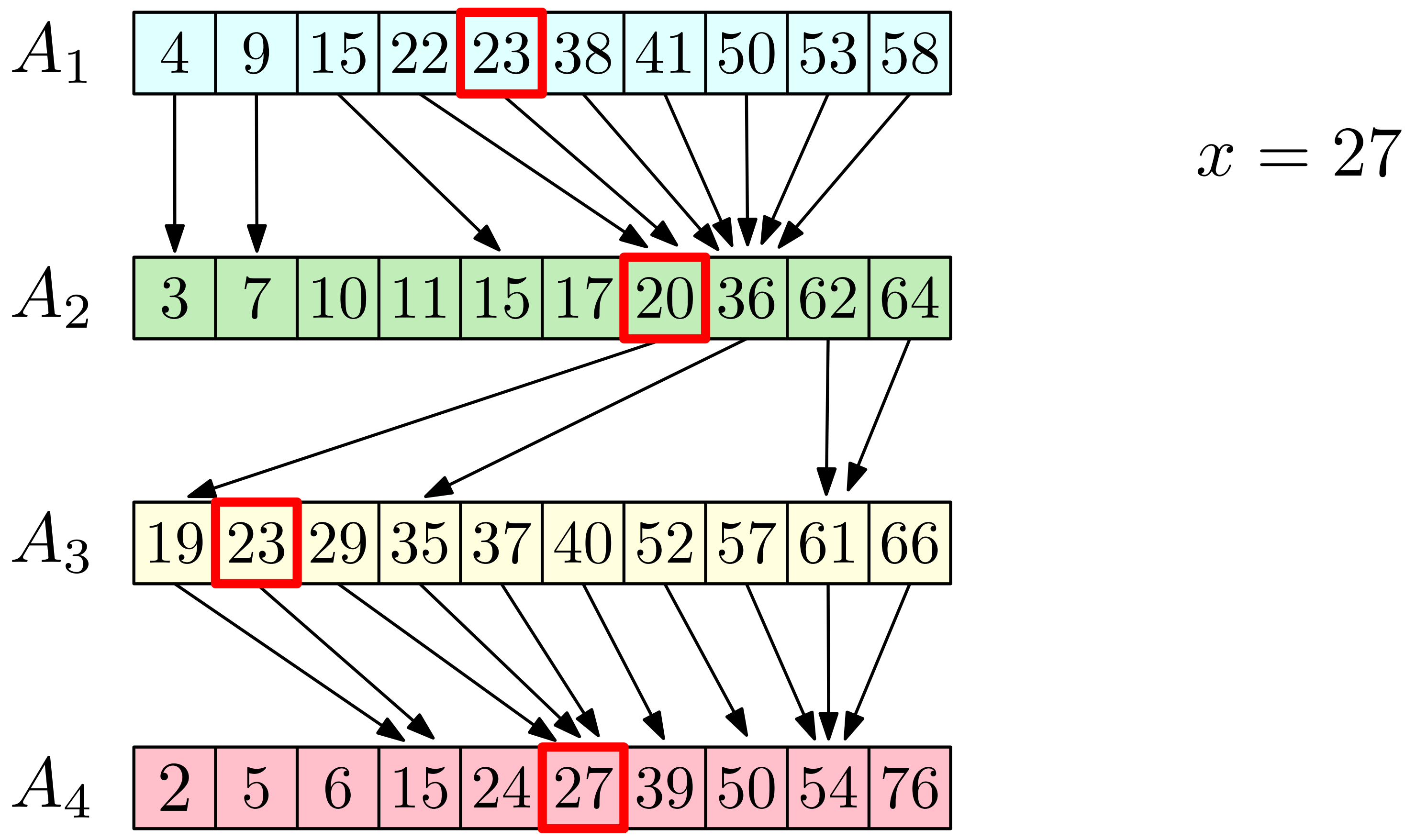
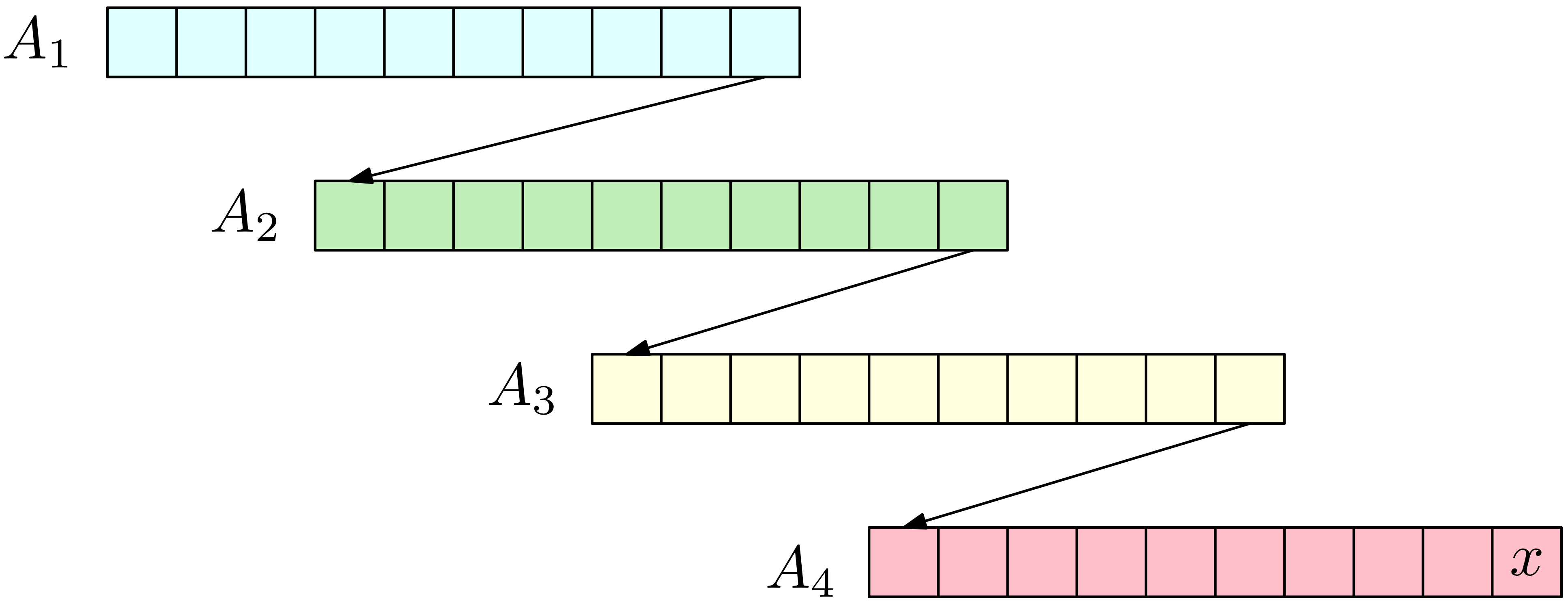


Fractional Cascading

How much time does it take?



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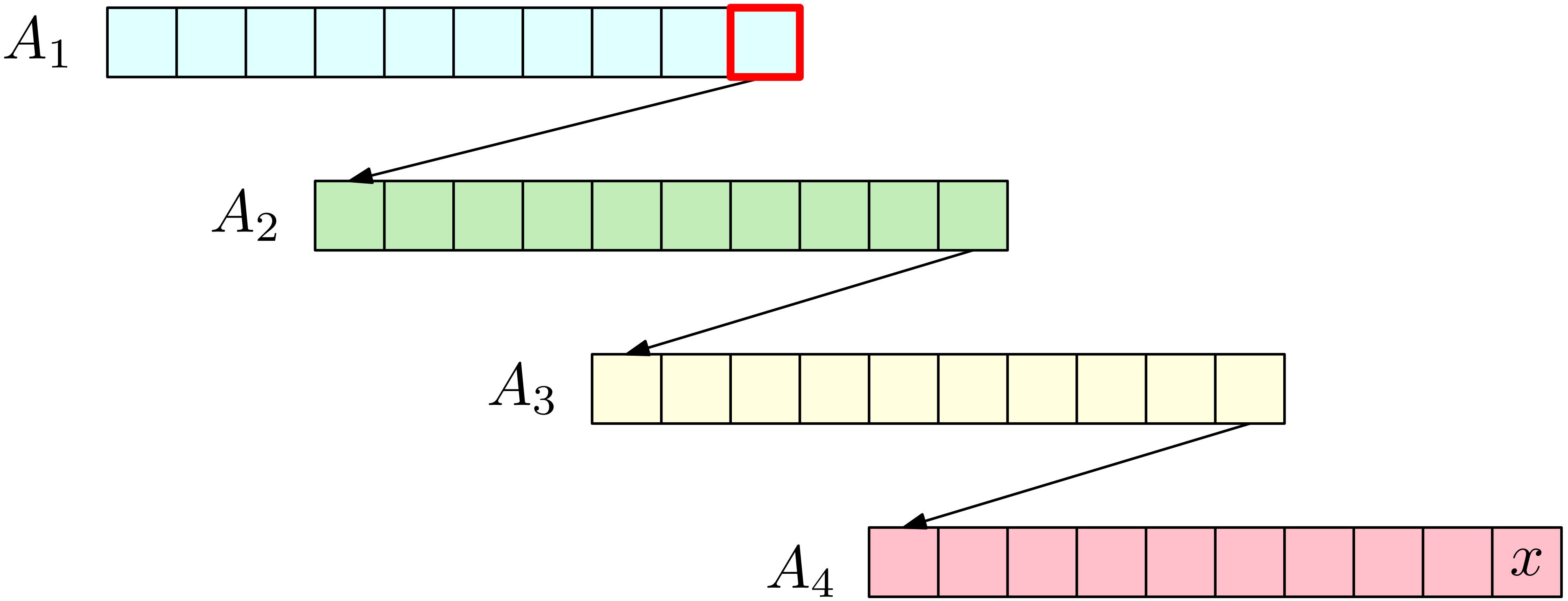

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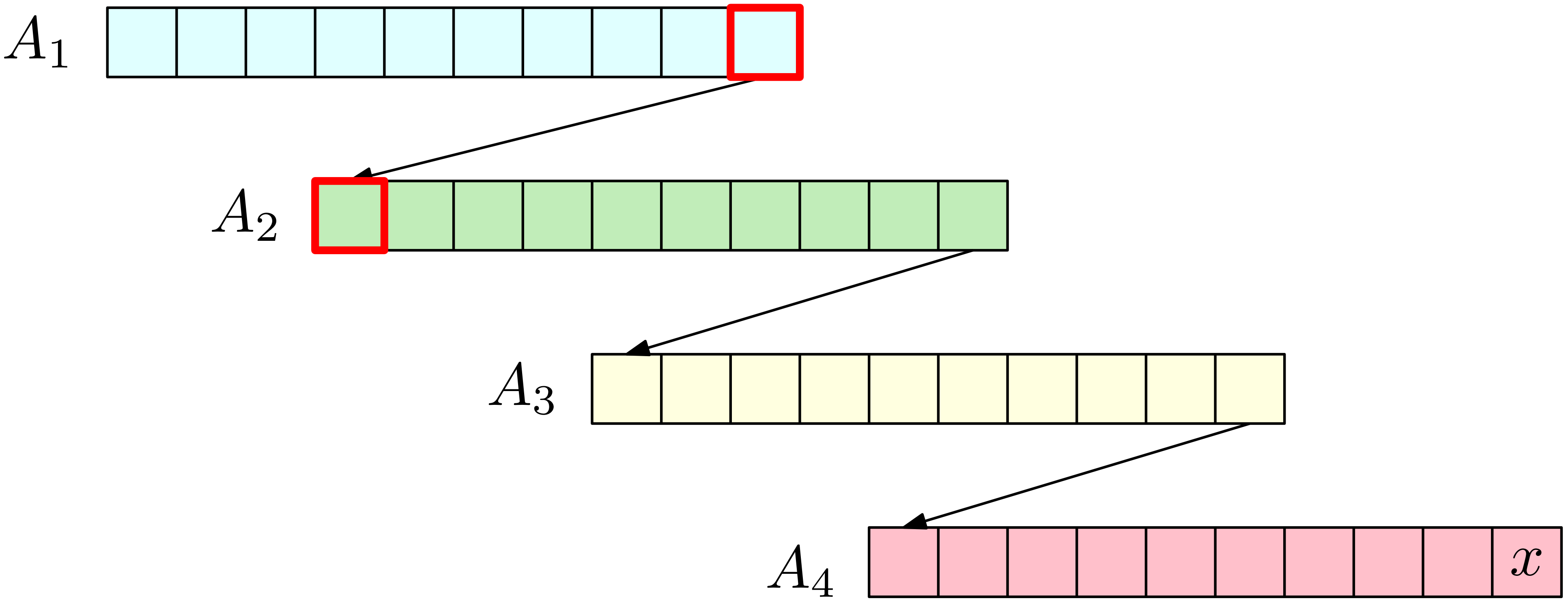

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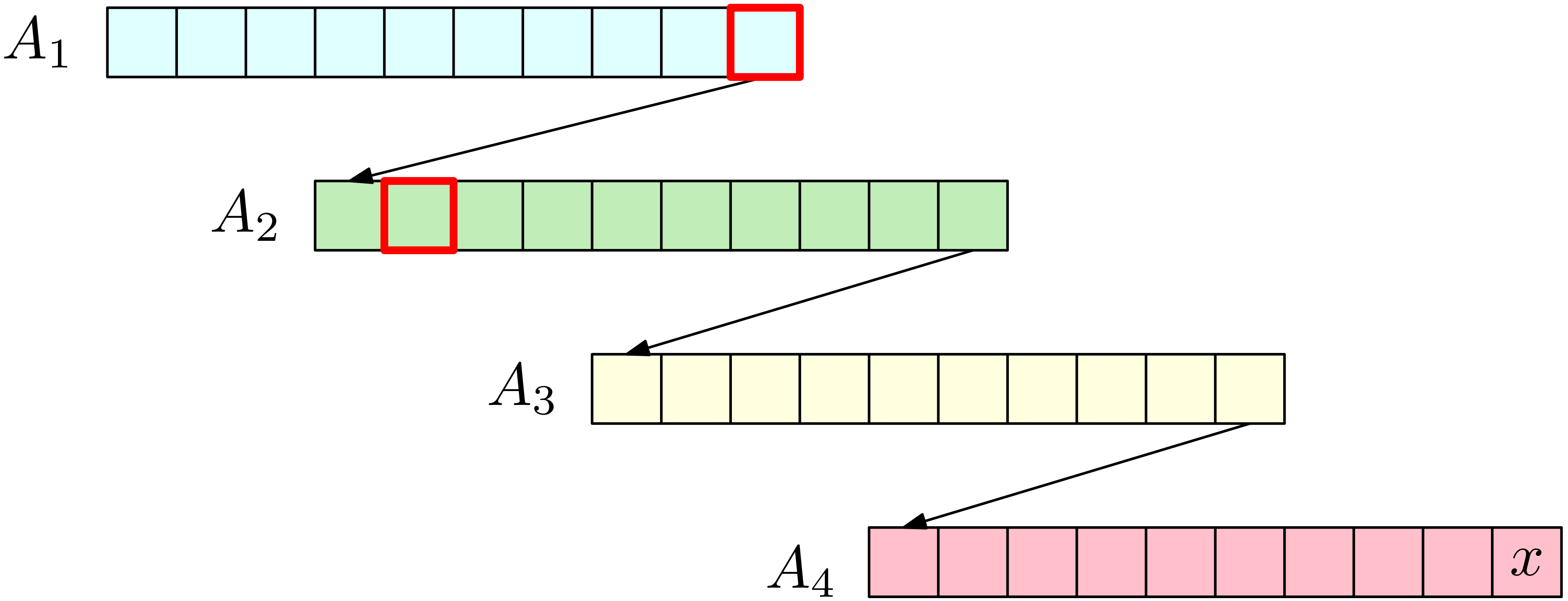

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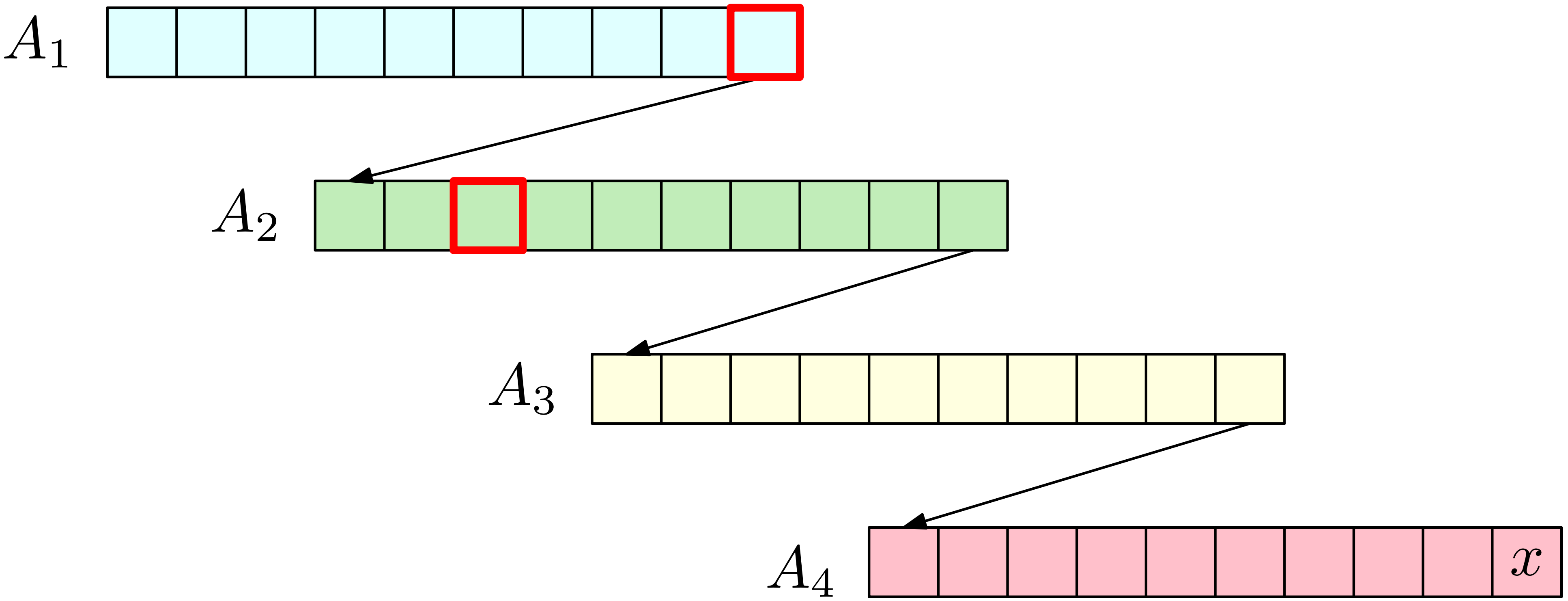

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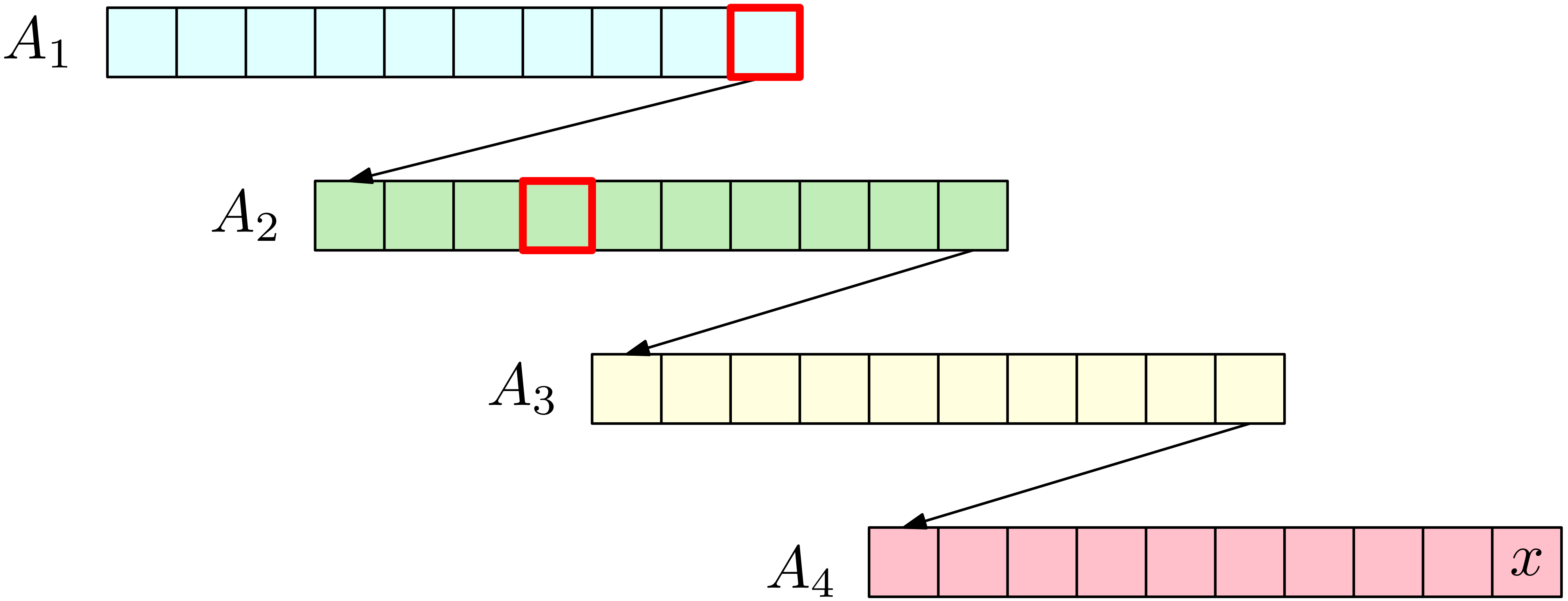

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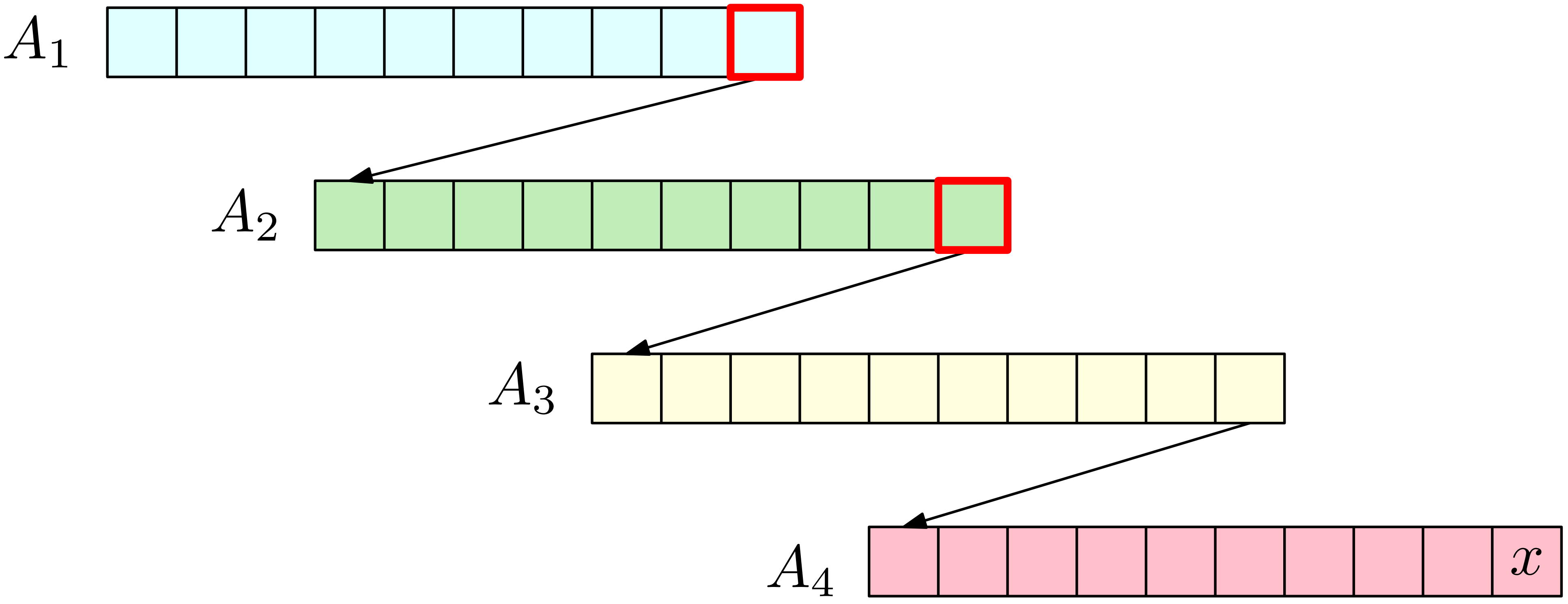
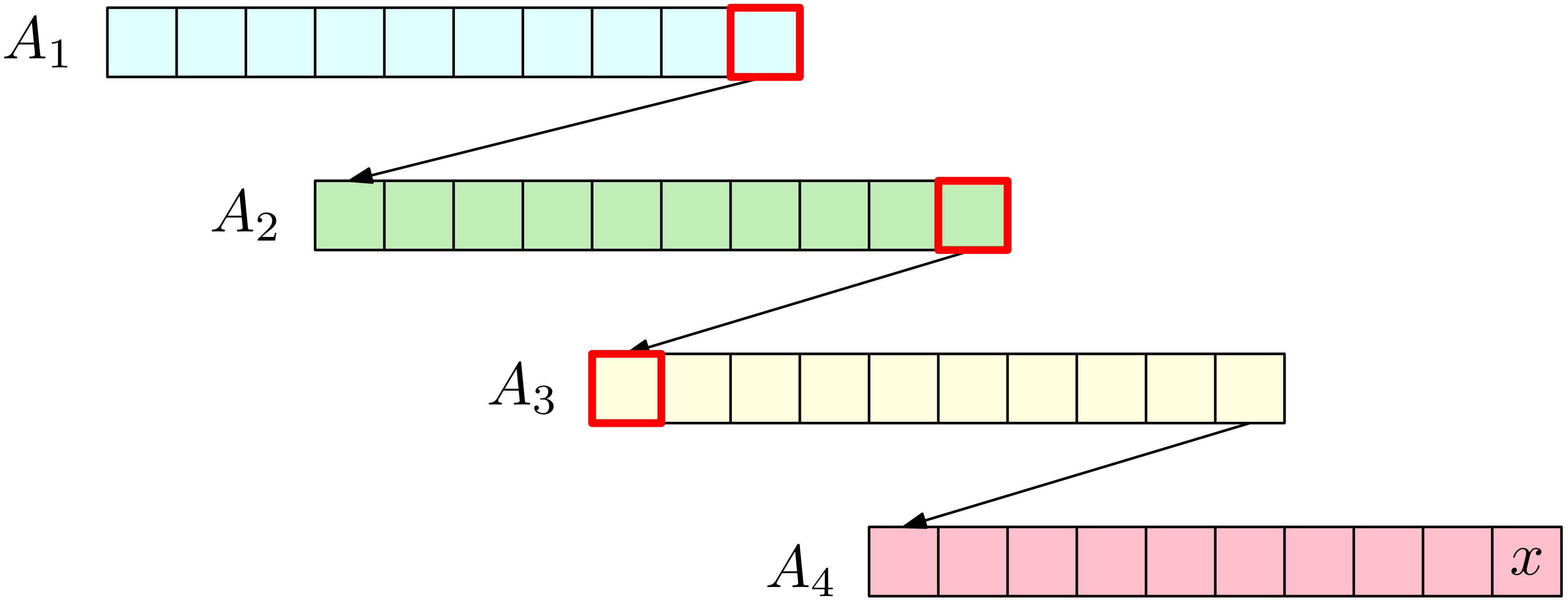


Fractional Cascading

How much time does it take?



Fractional Cascading

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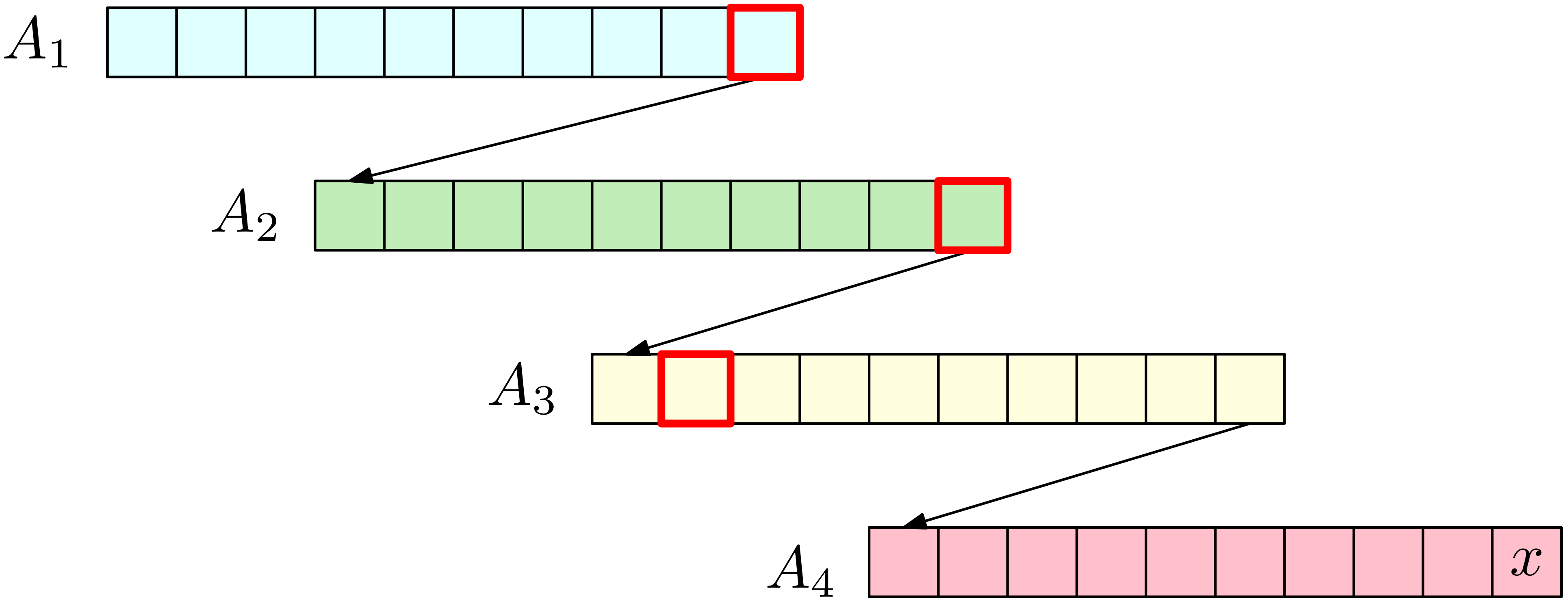

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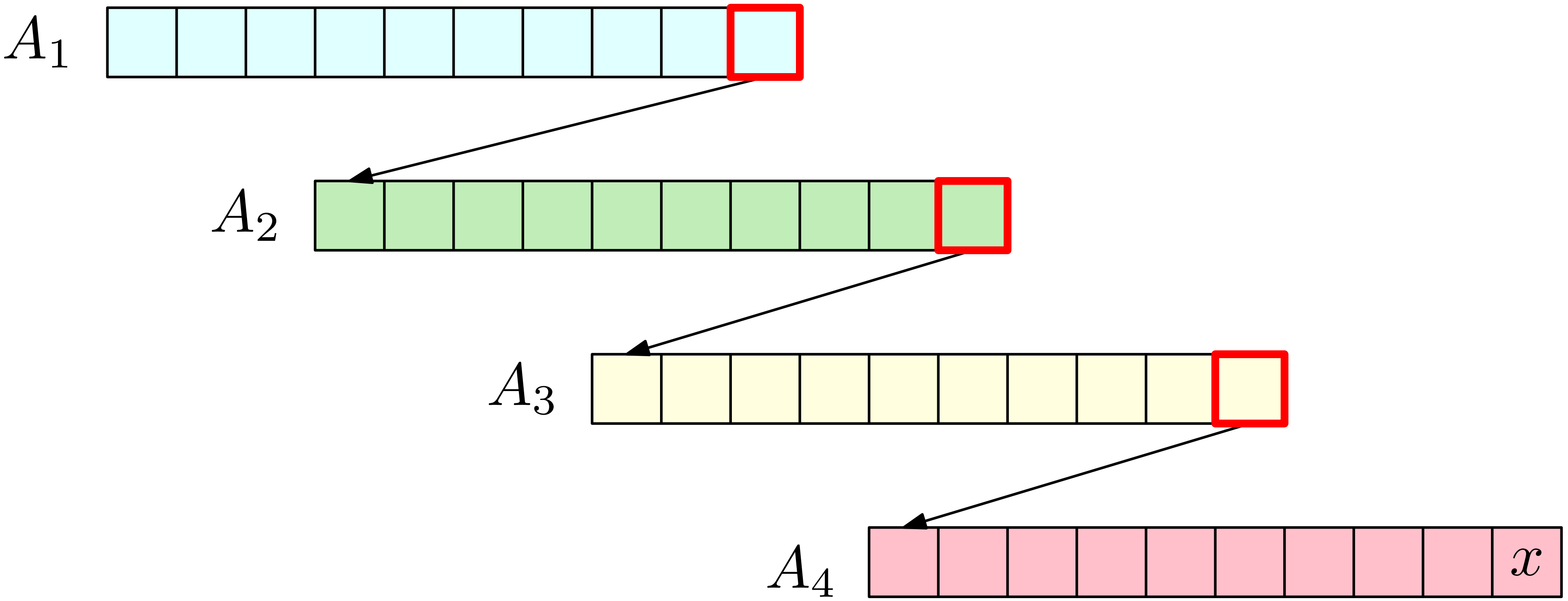

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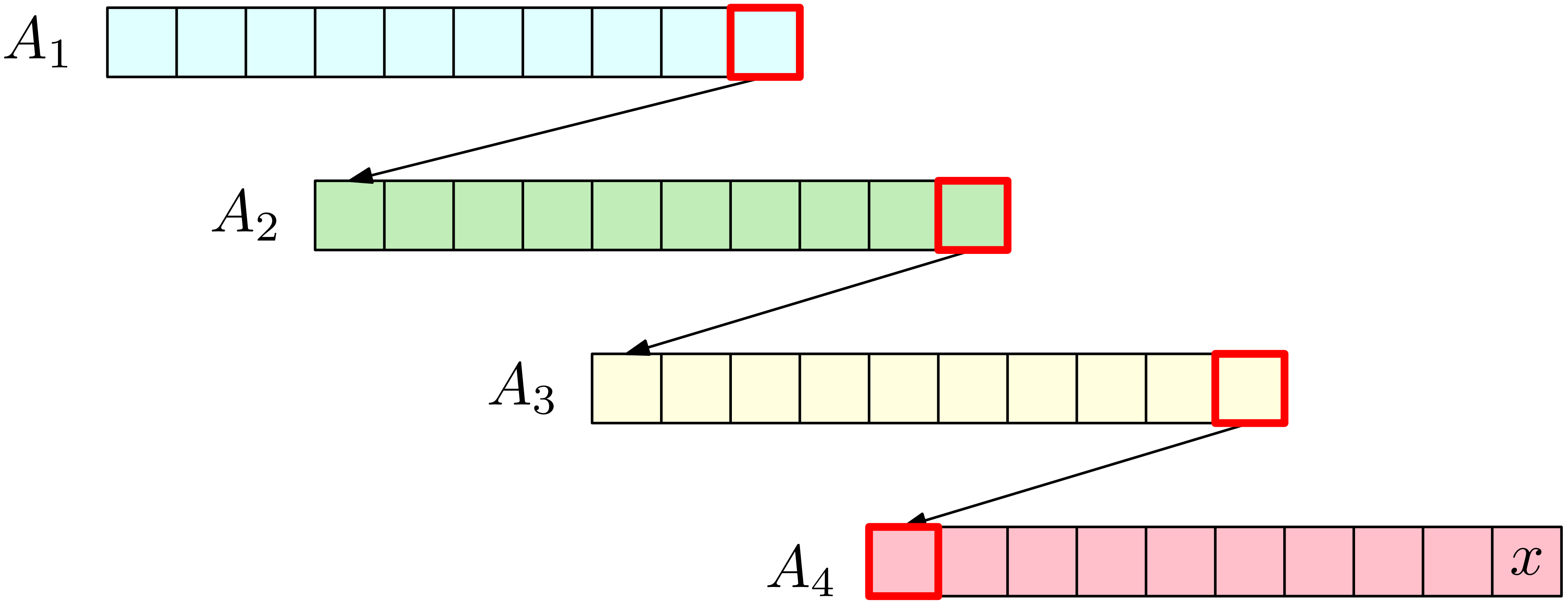

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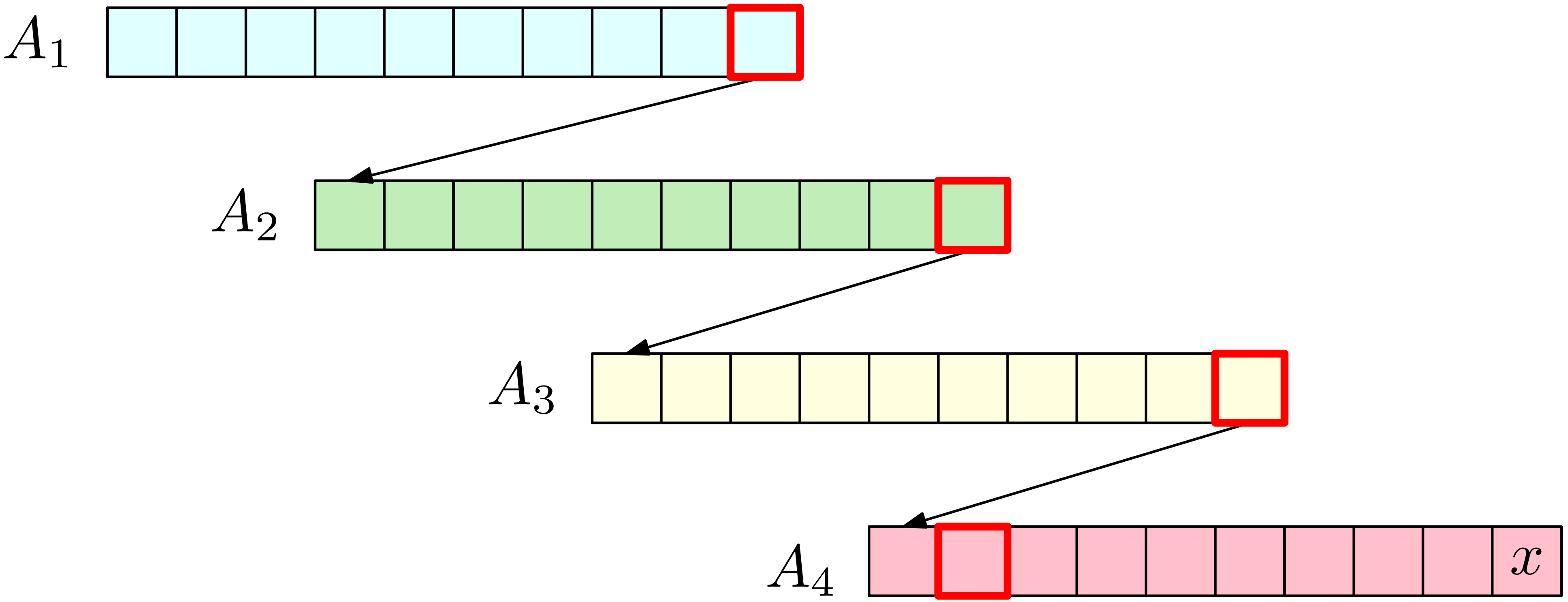

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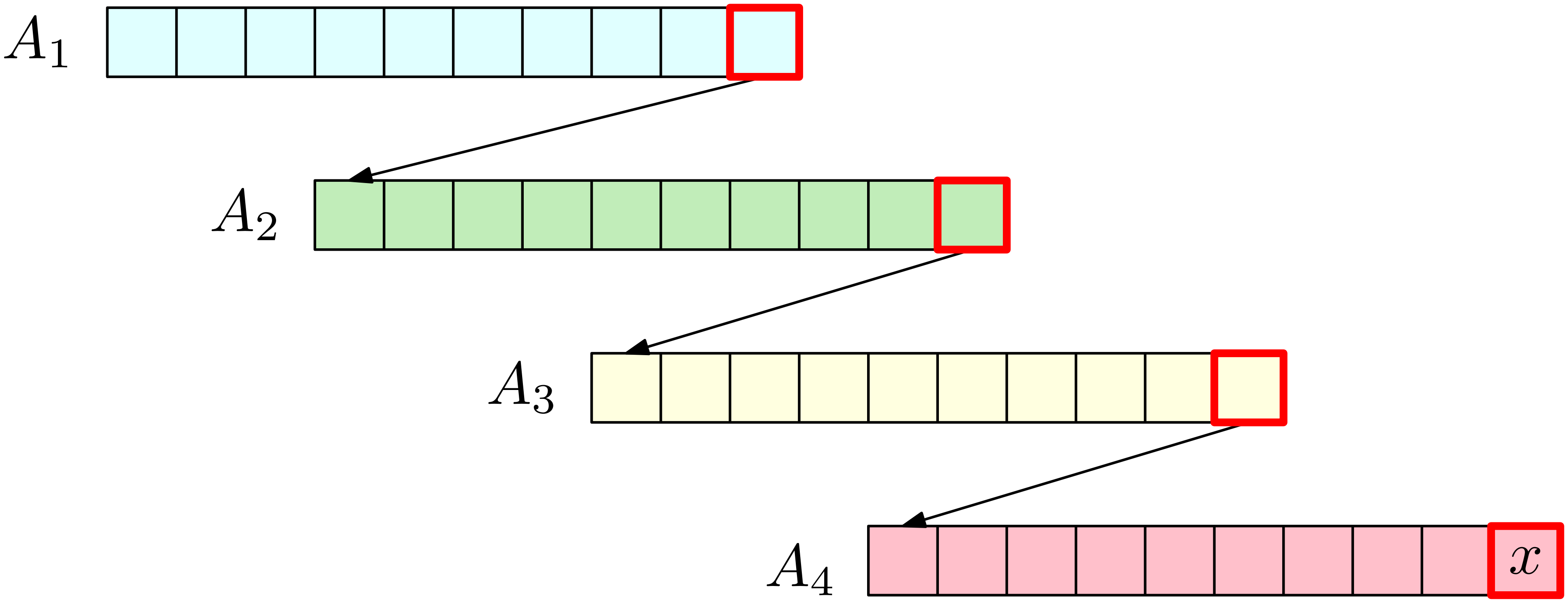

Fractional Cascading

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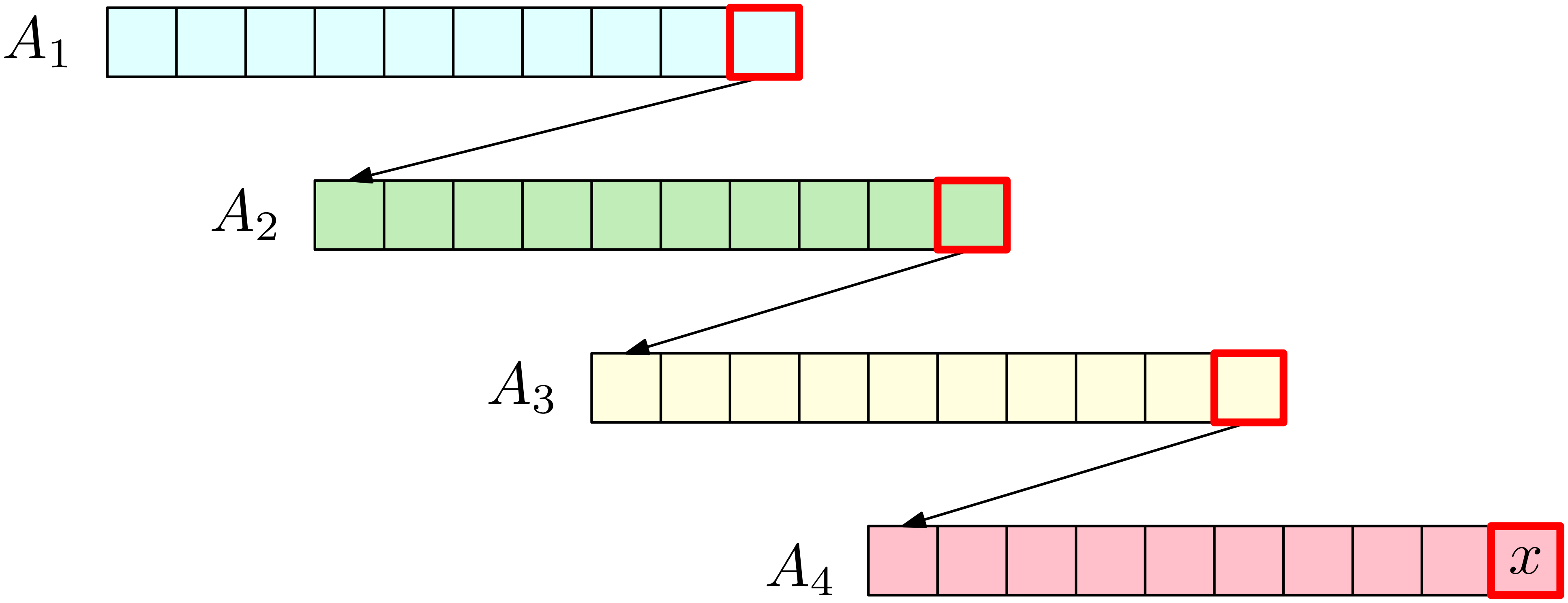
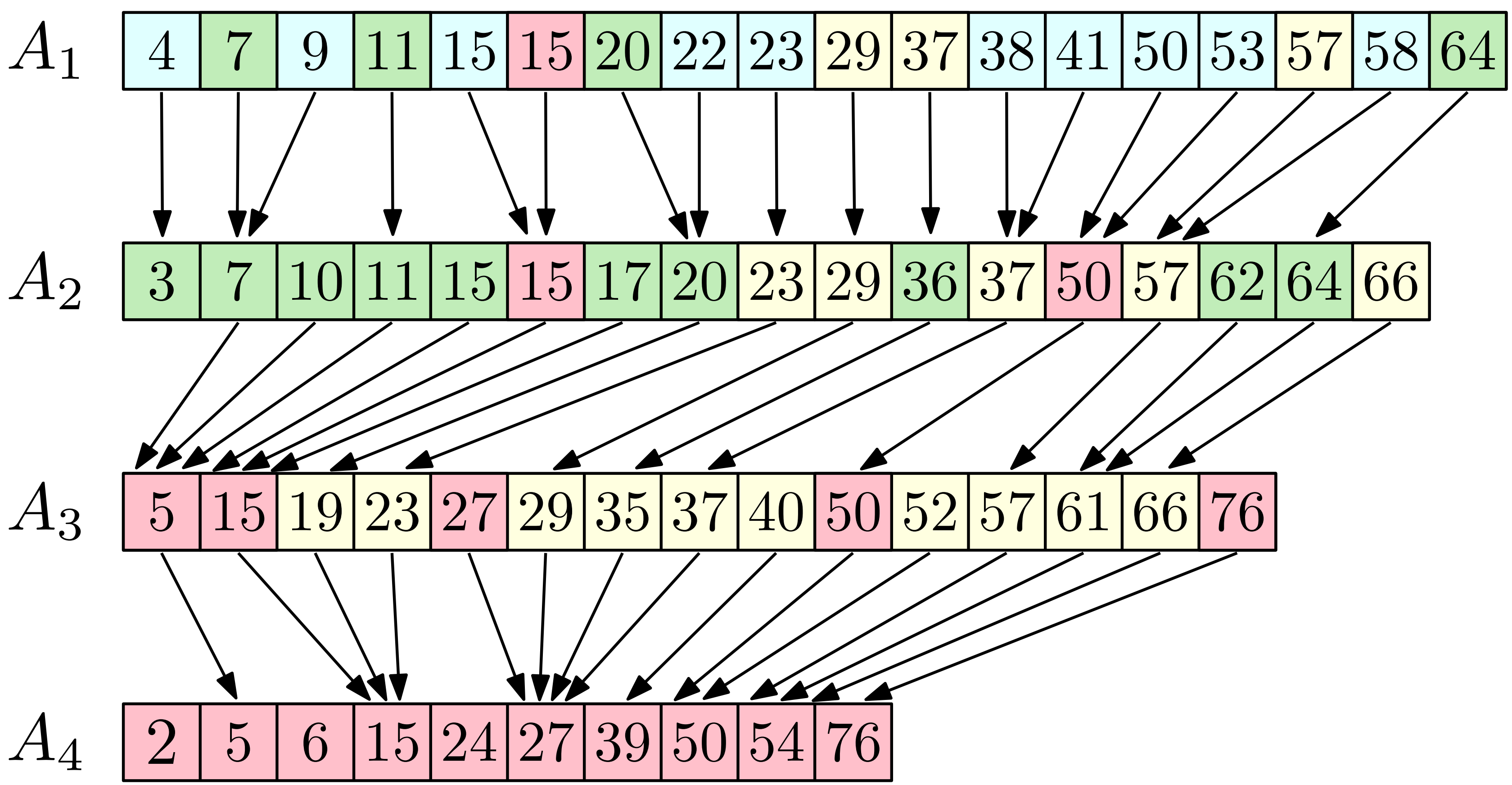


Worst-case time: $O(kn)$

Fractional Cascading

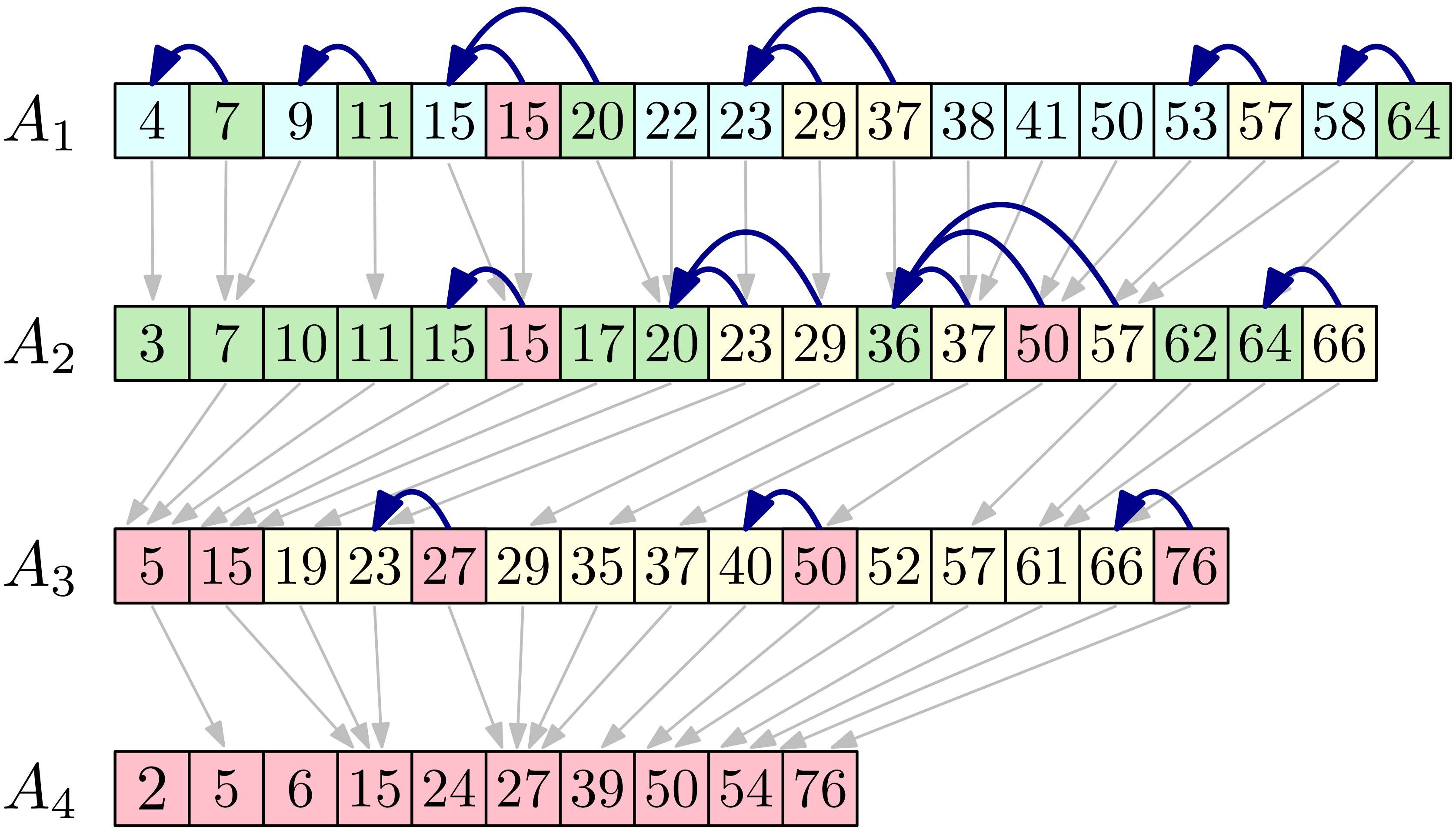
Second idea: fractional cascading

For $i = k, k-1, \ldots, 2$: Add every other element of A_i to A_{i-1}.



Fractional Cascading

Keep pointers from newly added elements to A_i to their predecessor among the original elements of A_i



Fractional Cascading

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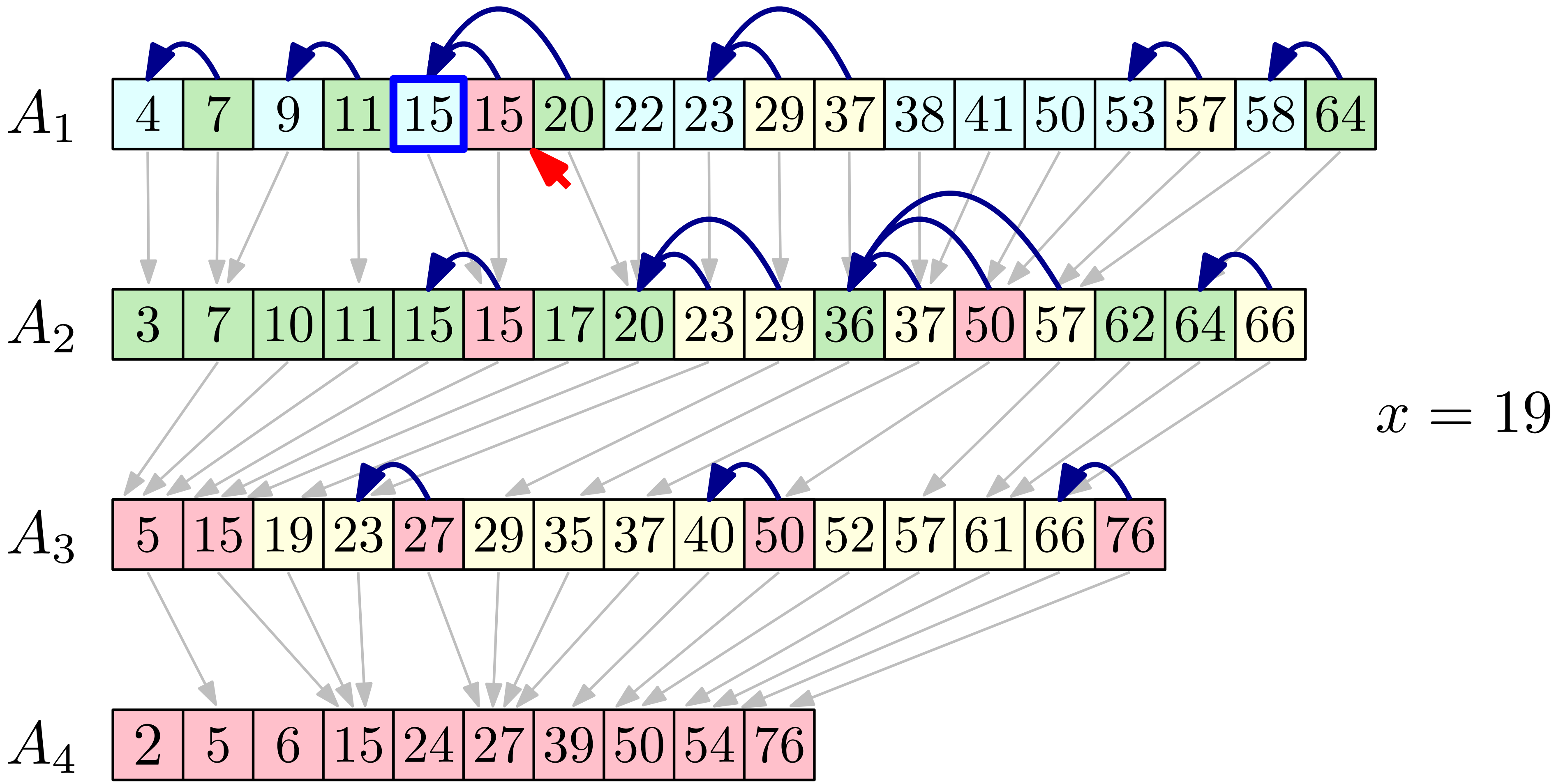

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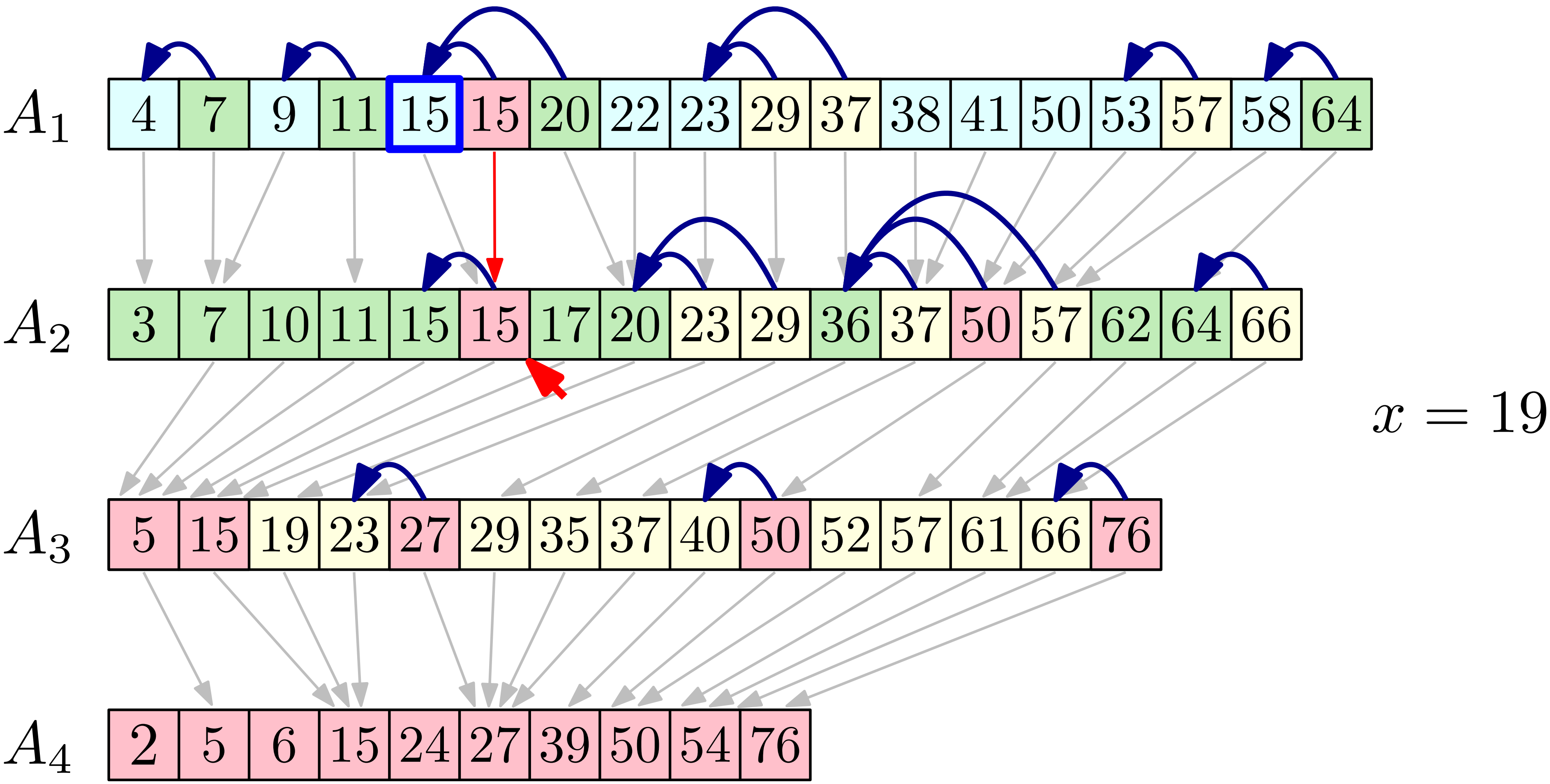

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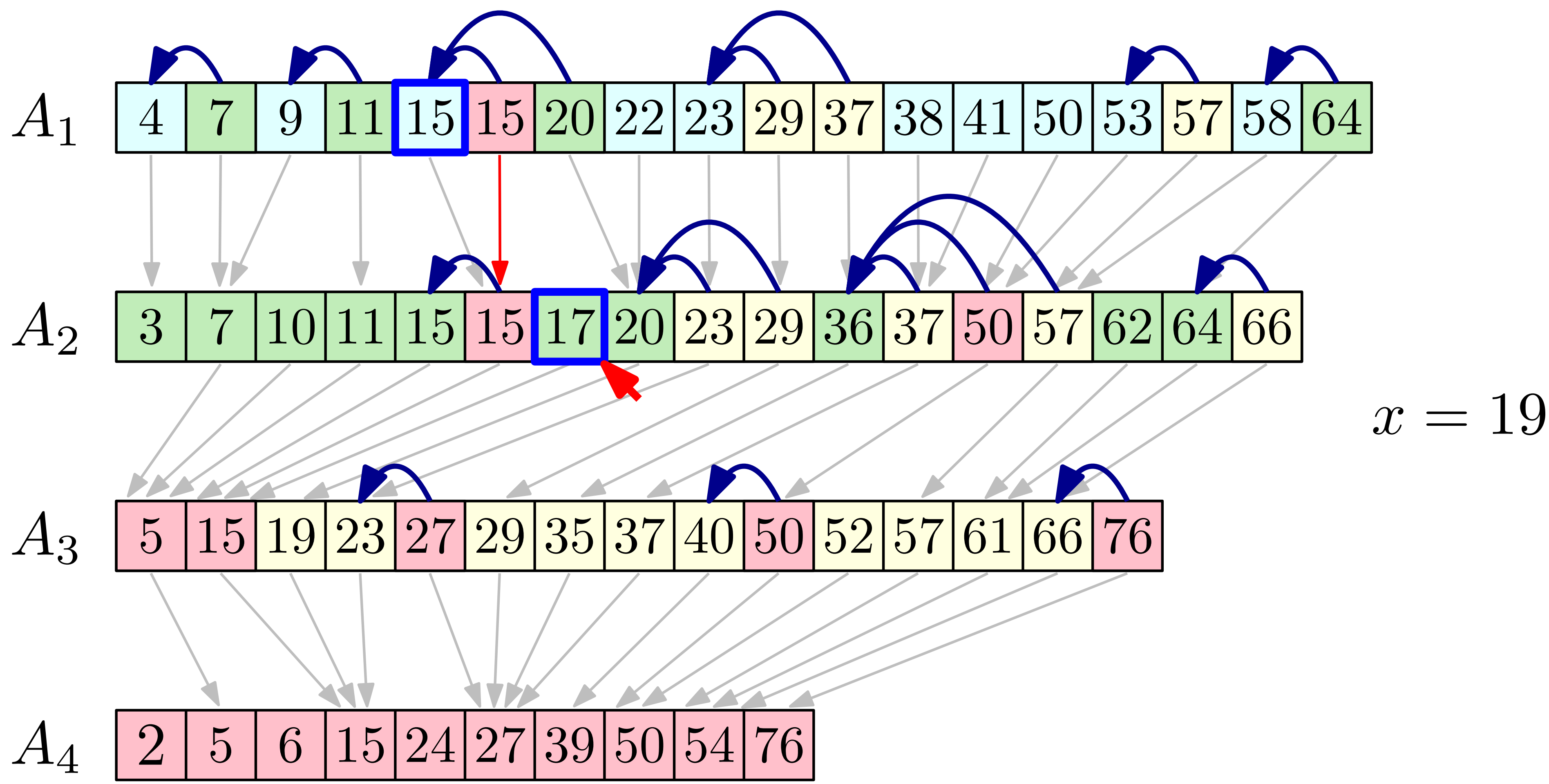

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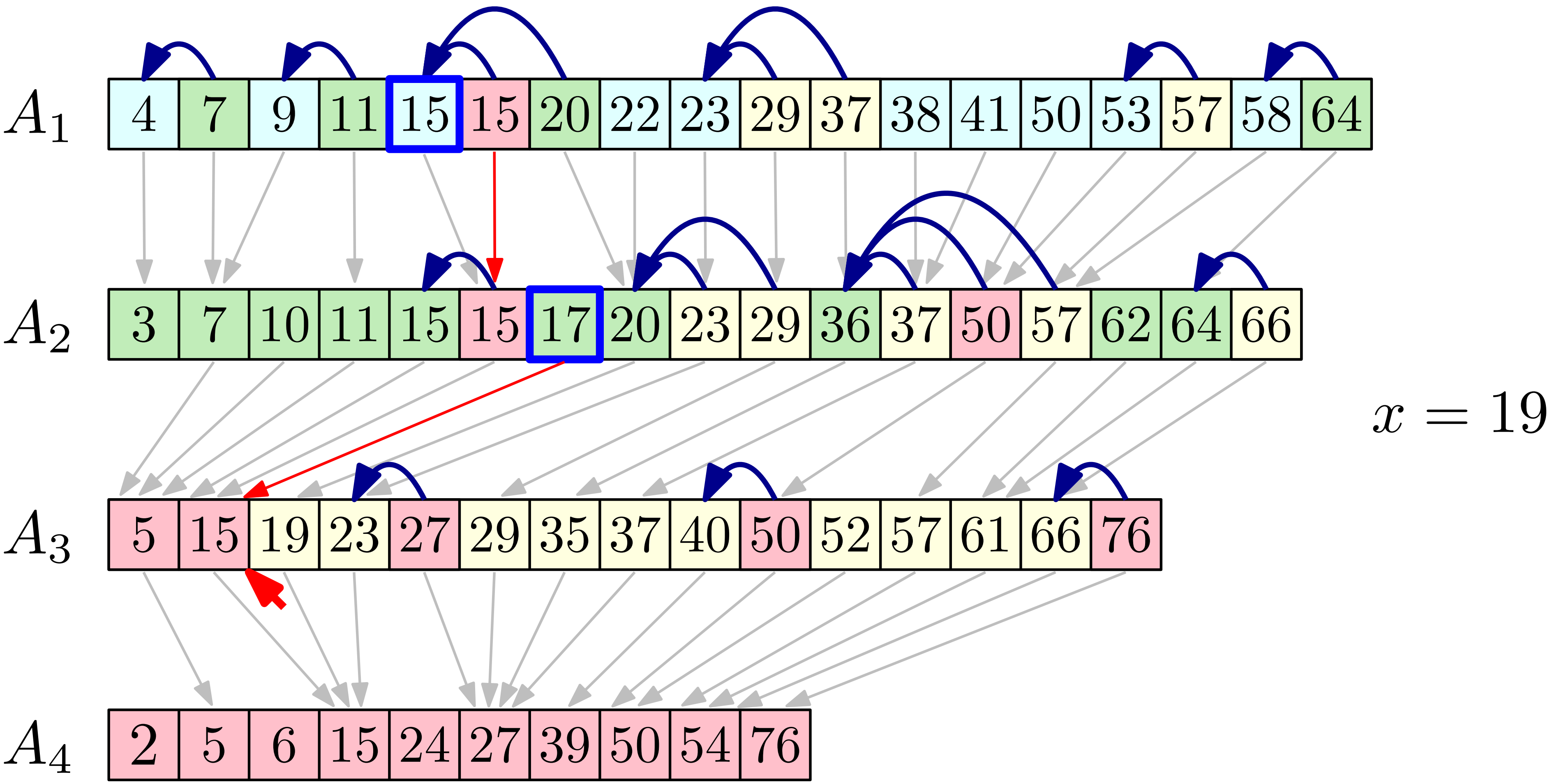

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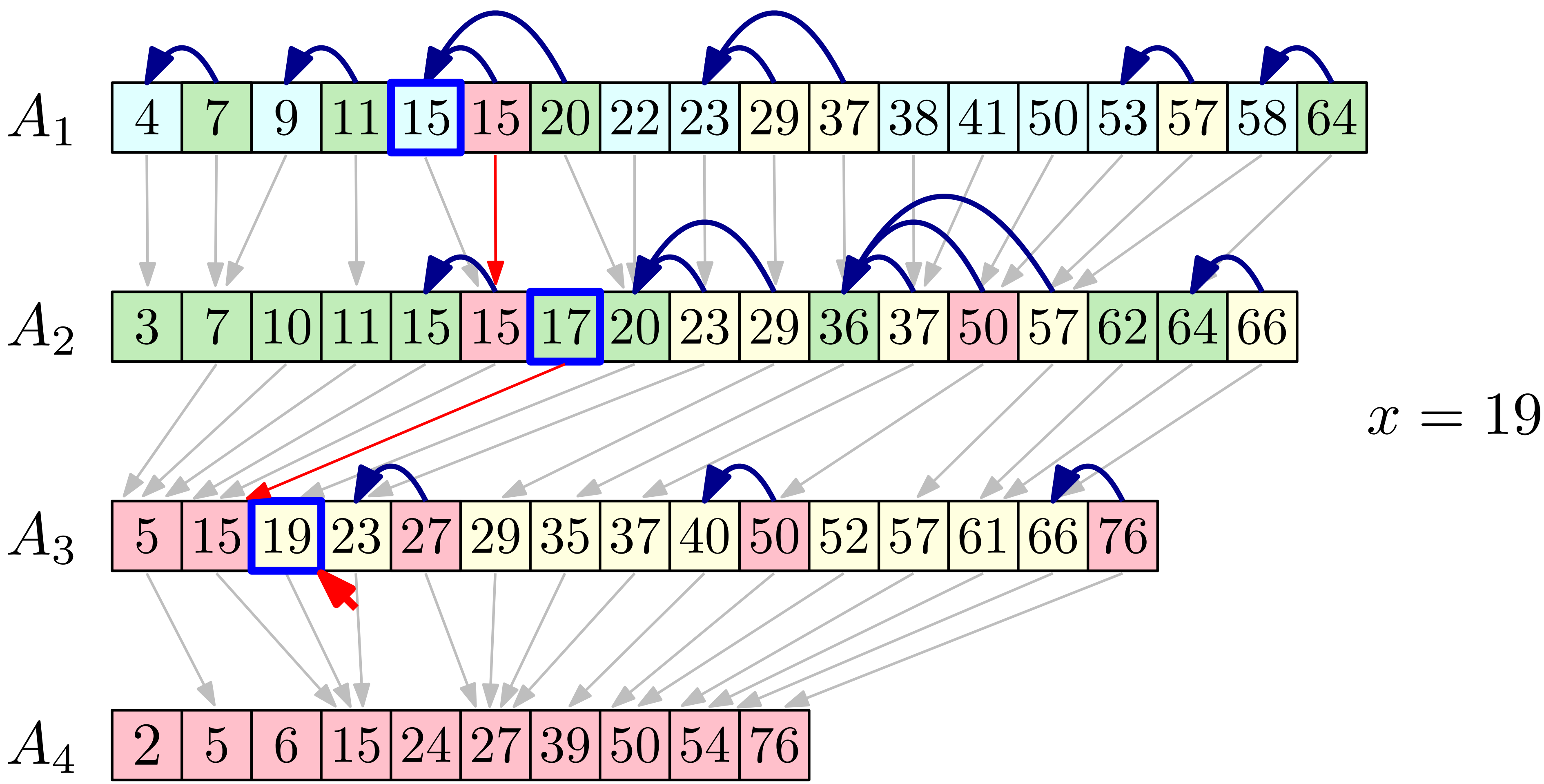

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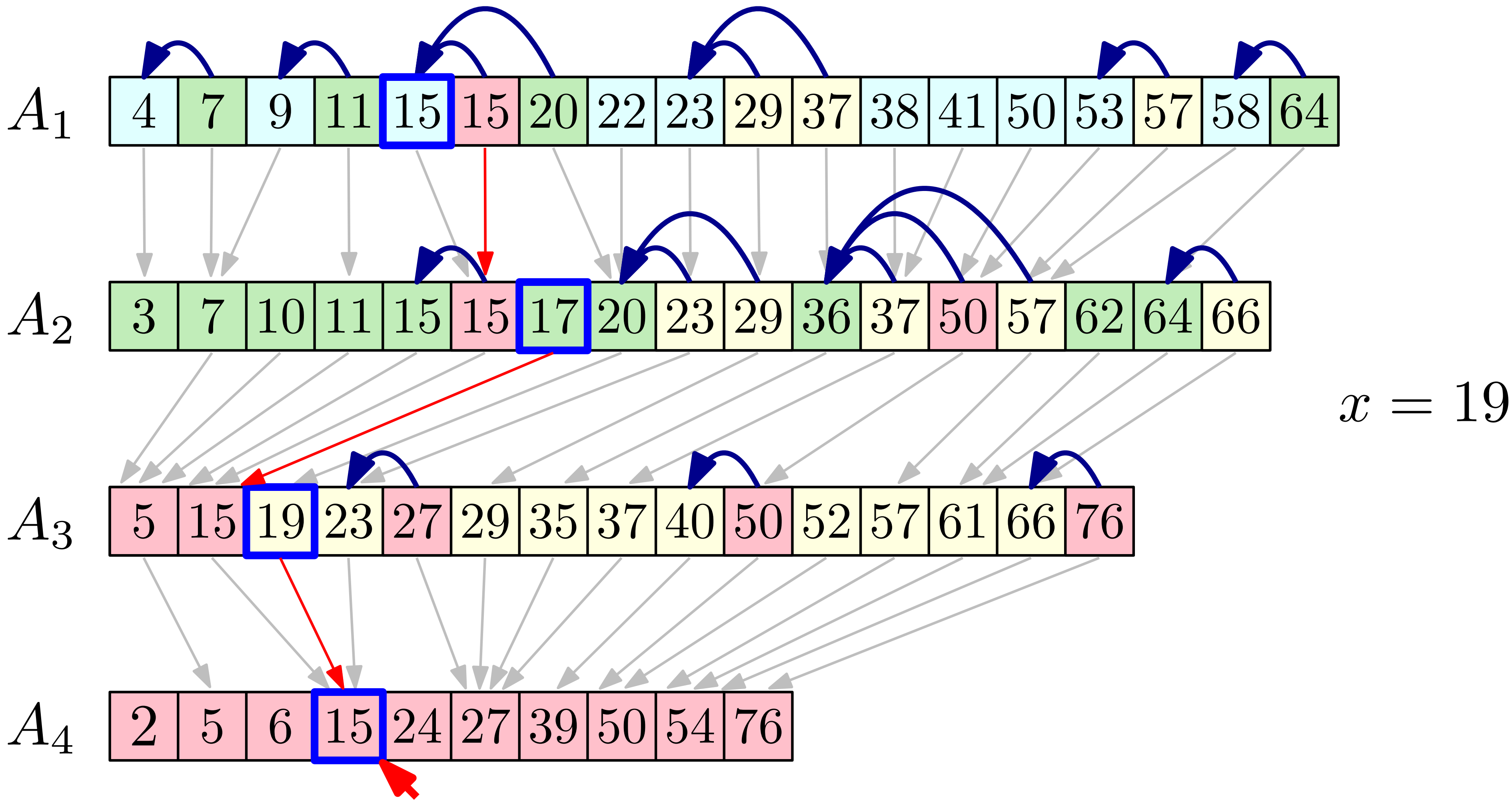

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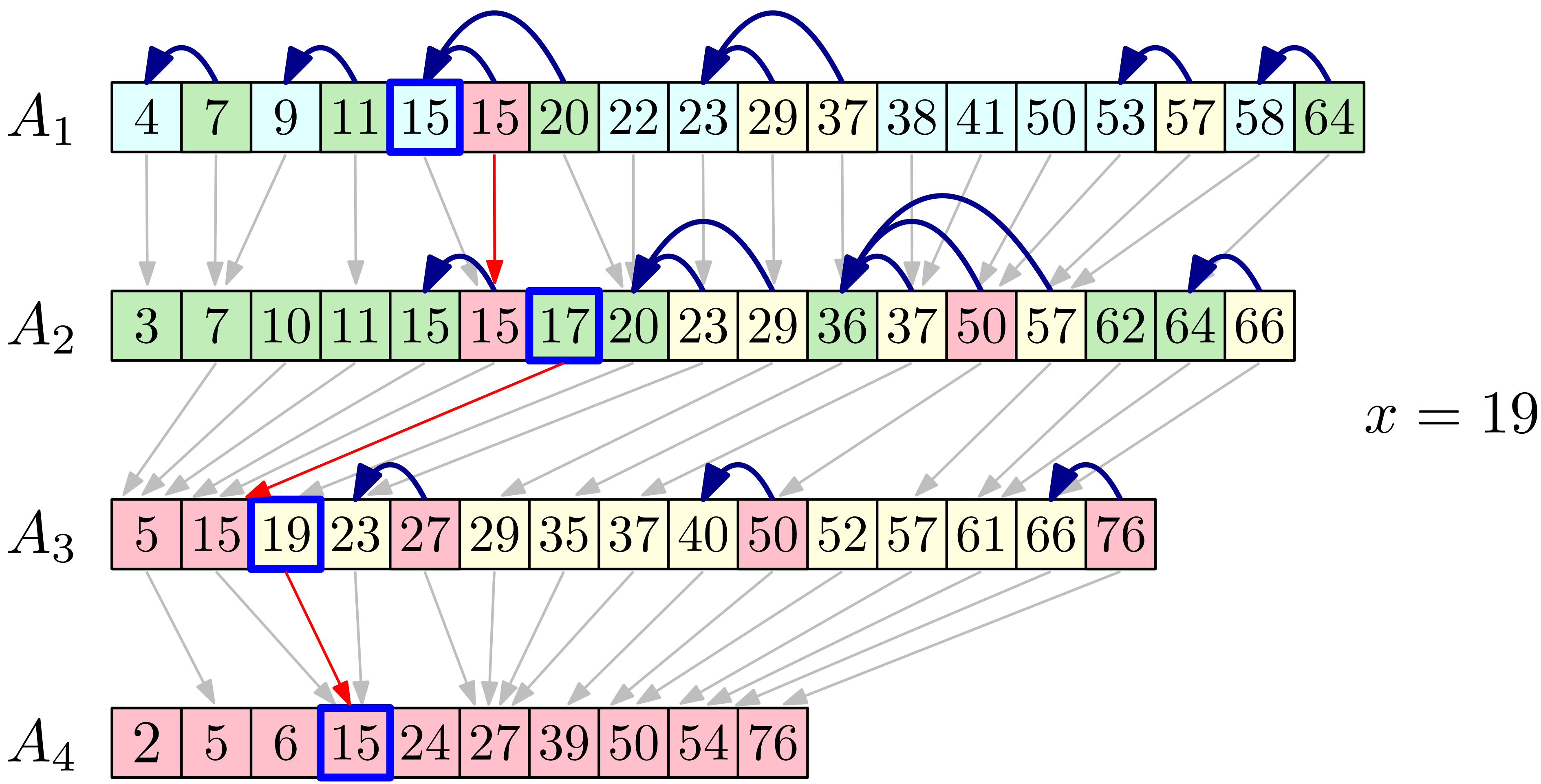
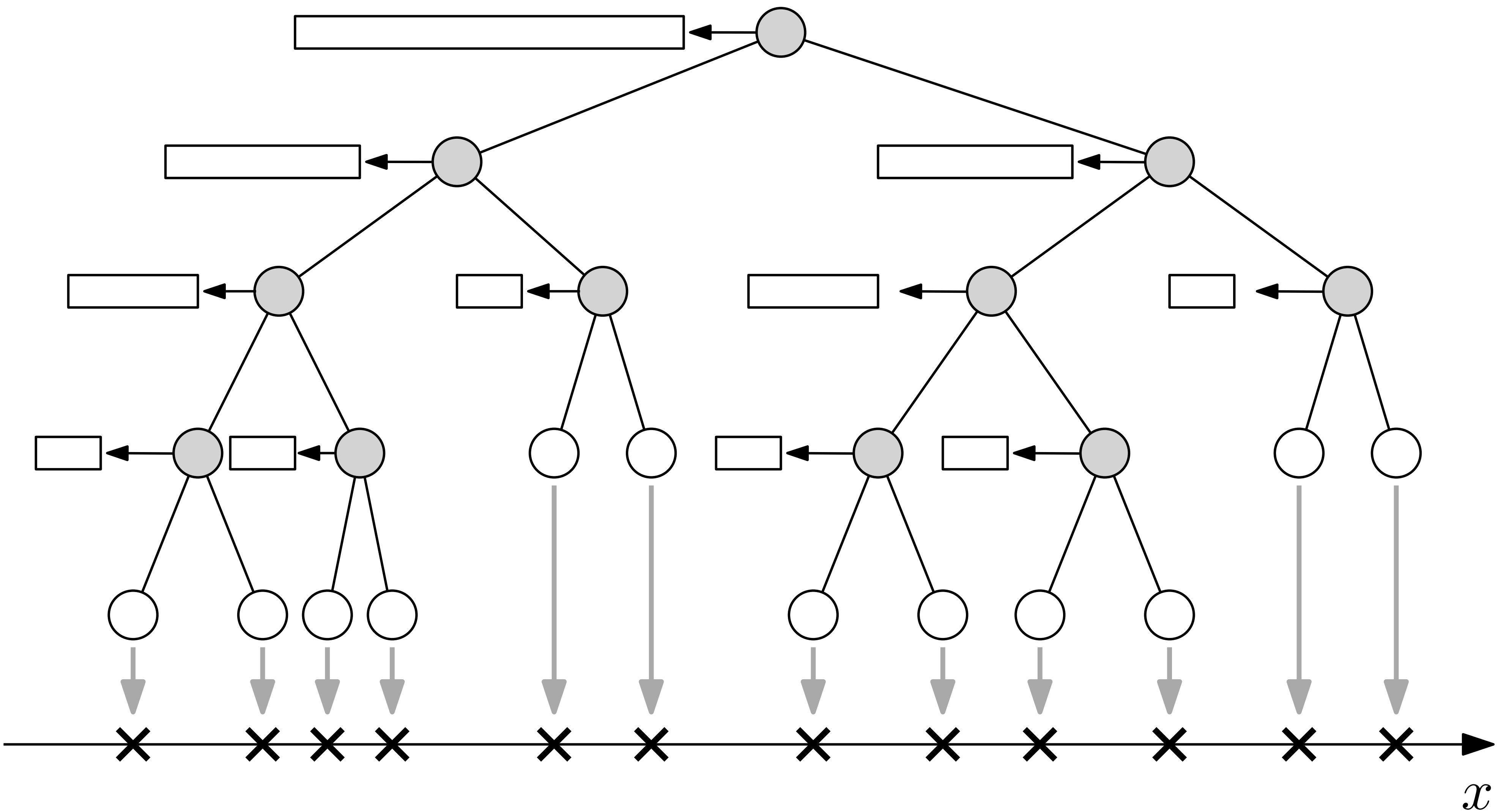


Size $O(kn)$ Preprocessing $O(kn)$ Query: $O(k + \log n)$

Layered Range Trees

Layered Range Trees, $D = 2$

Reuse the cross-linking idea from fractional cascading



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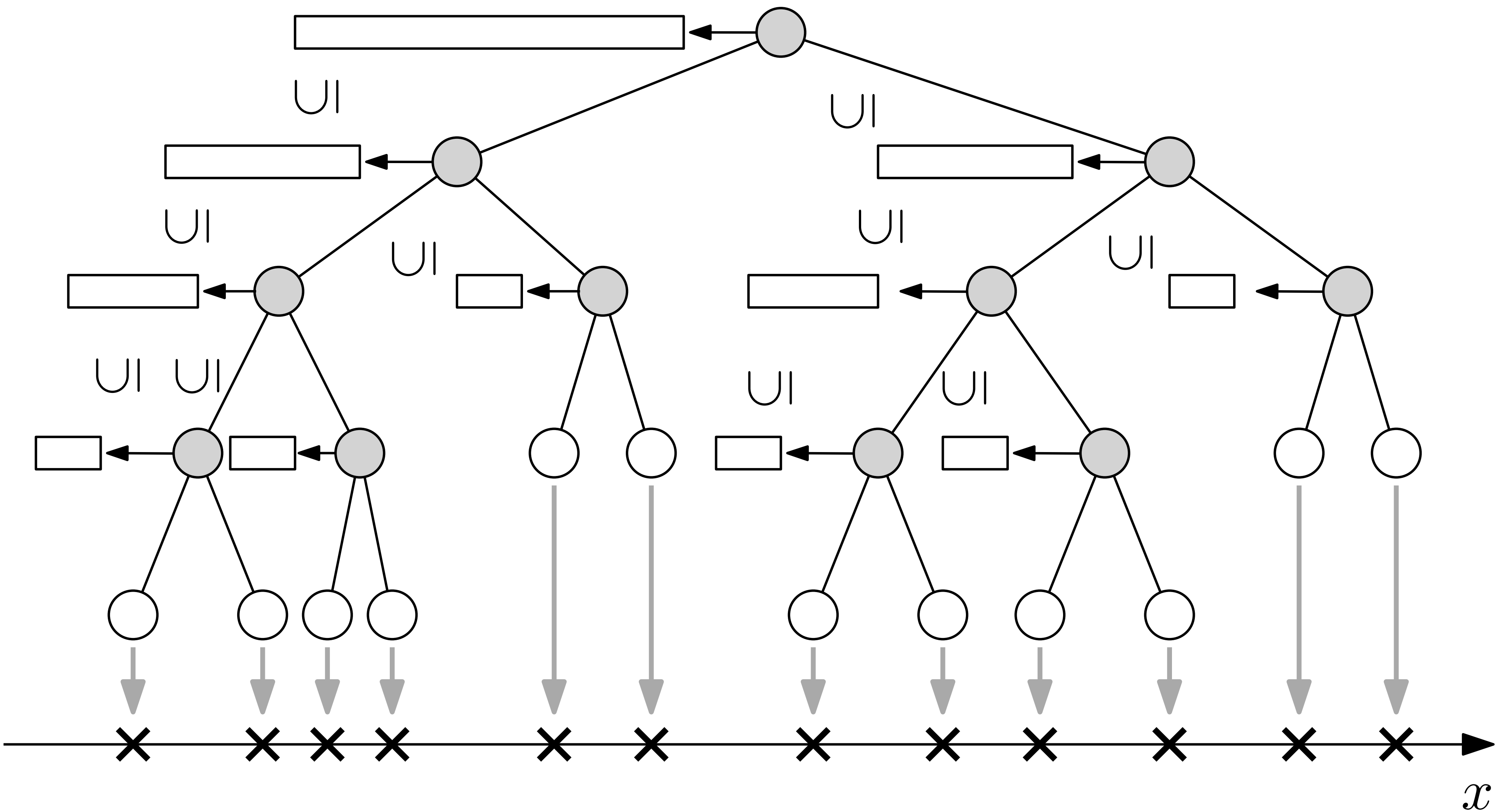

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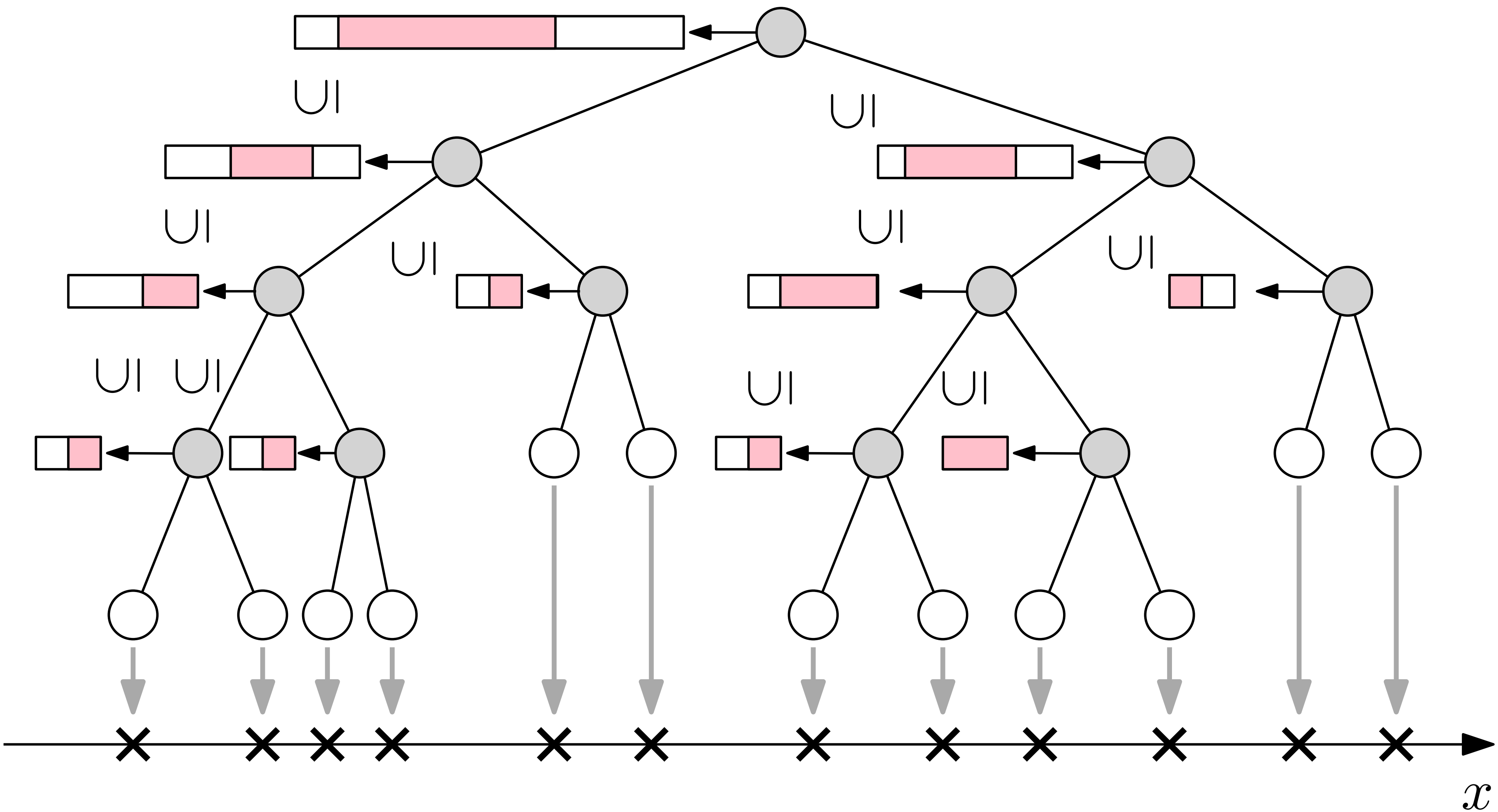
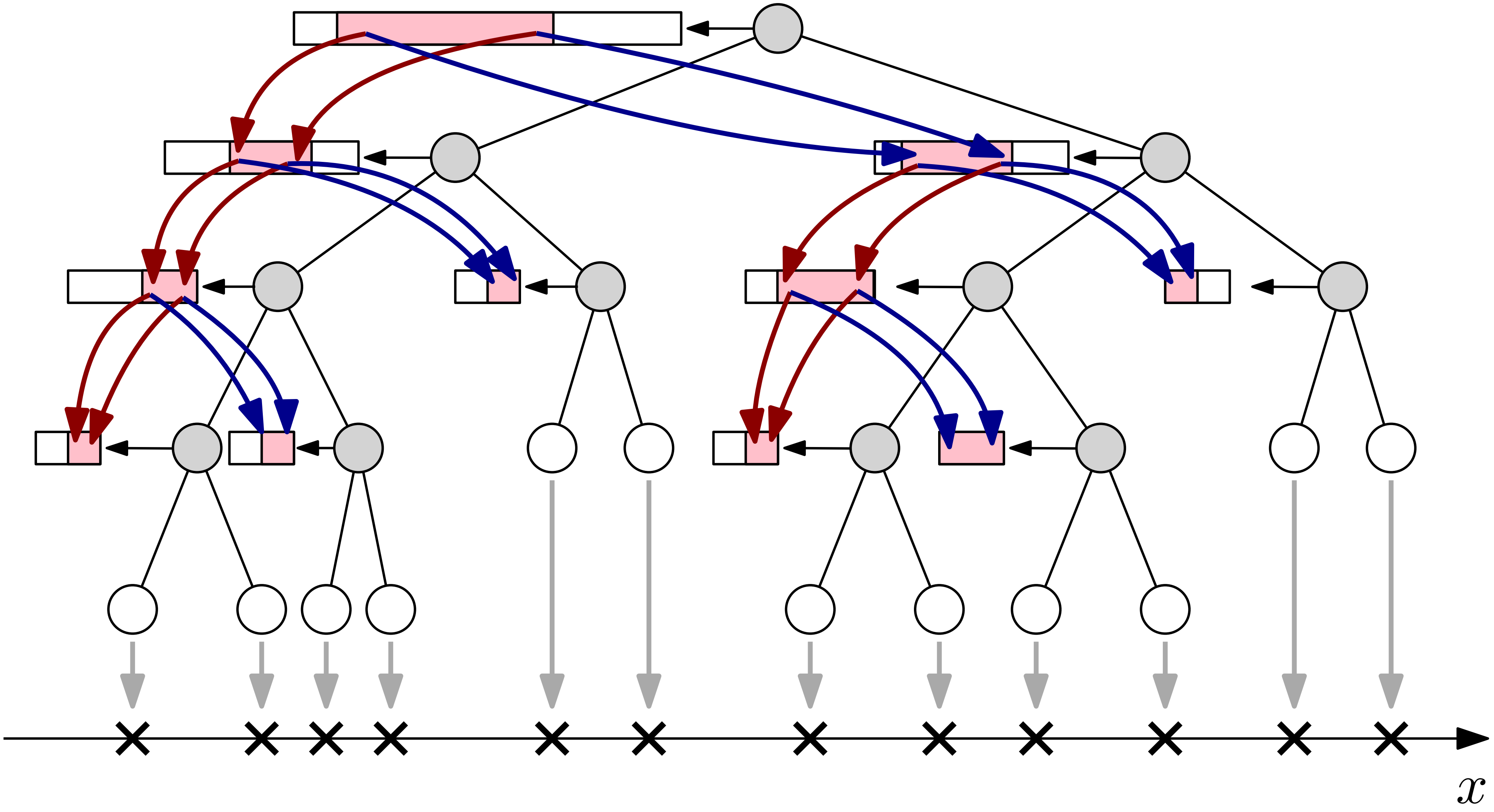


Layered Range Trees, $D = 2$

$\forall$ element y in the 1D range tree of v, store a pointer to the predecessor of y in the 1D range tree of the left/right child of v.



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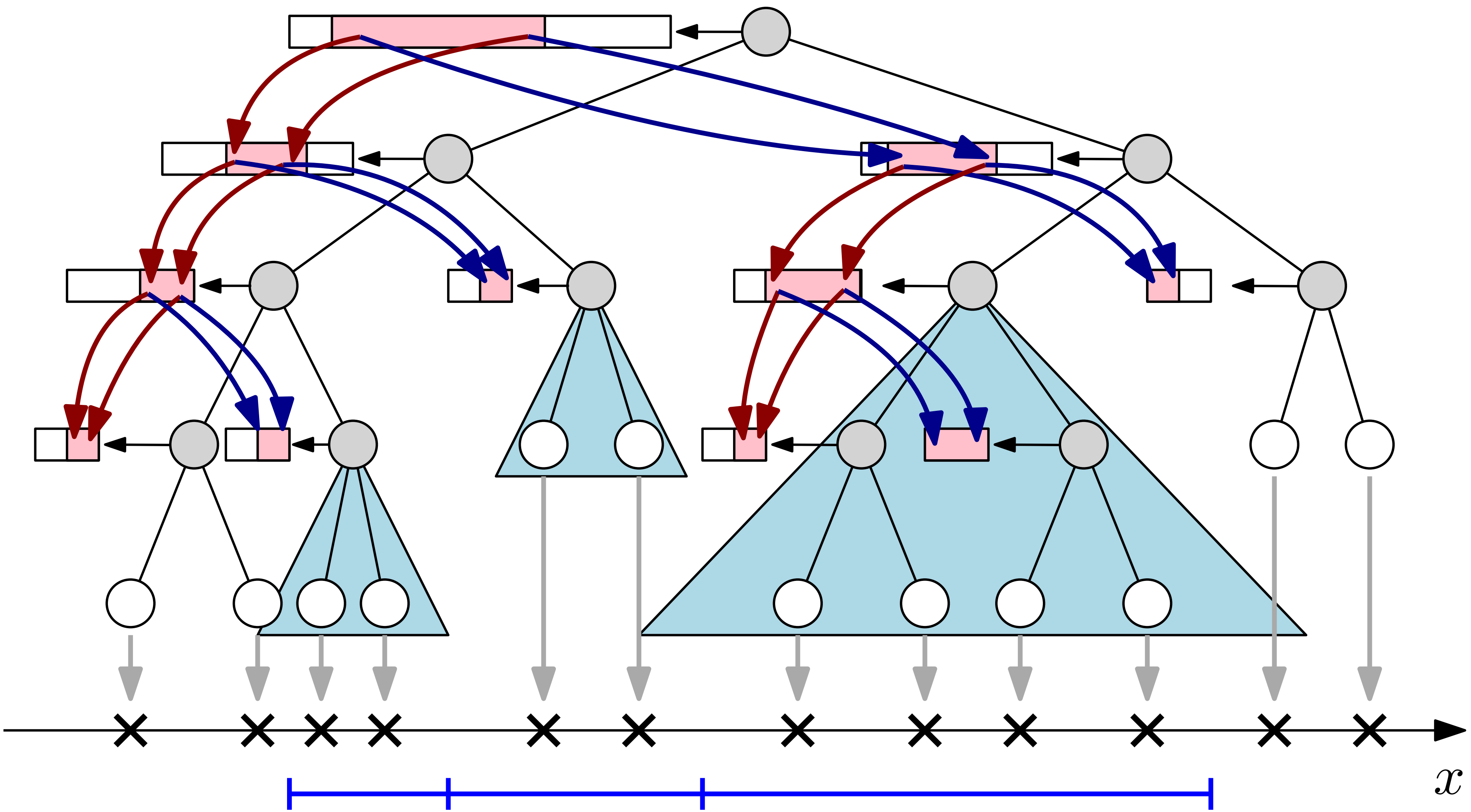

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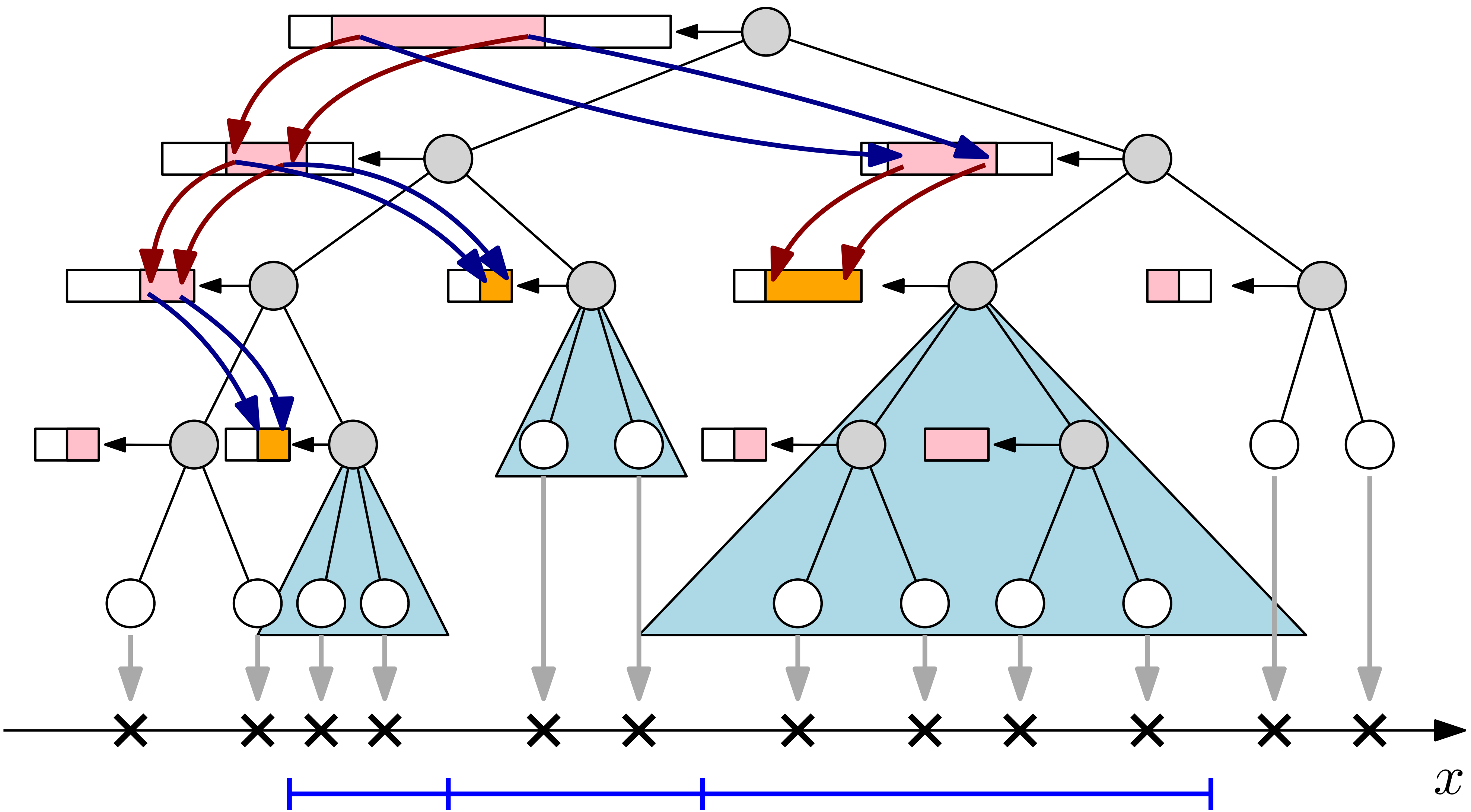

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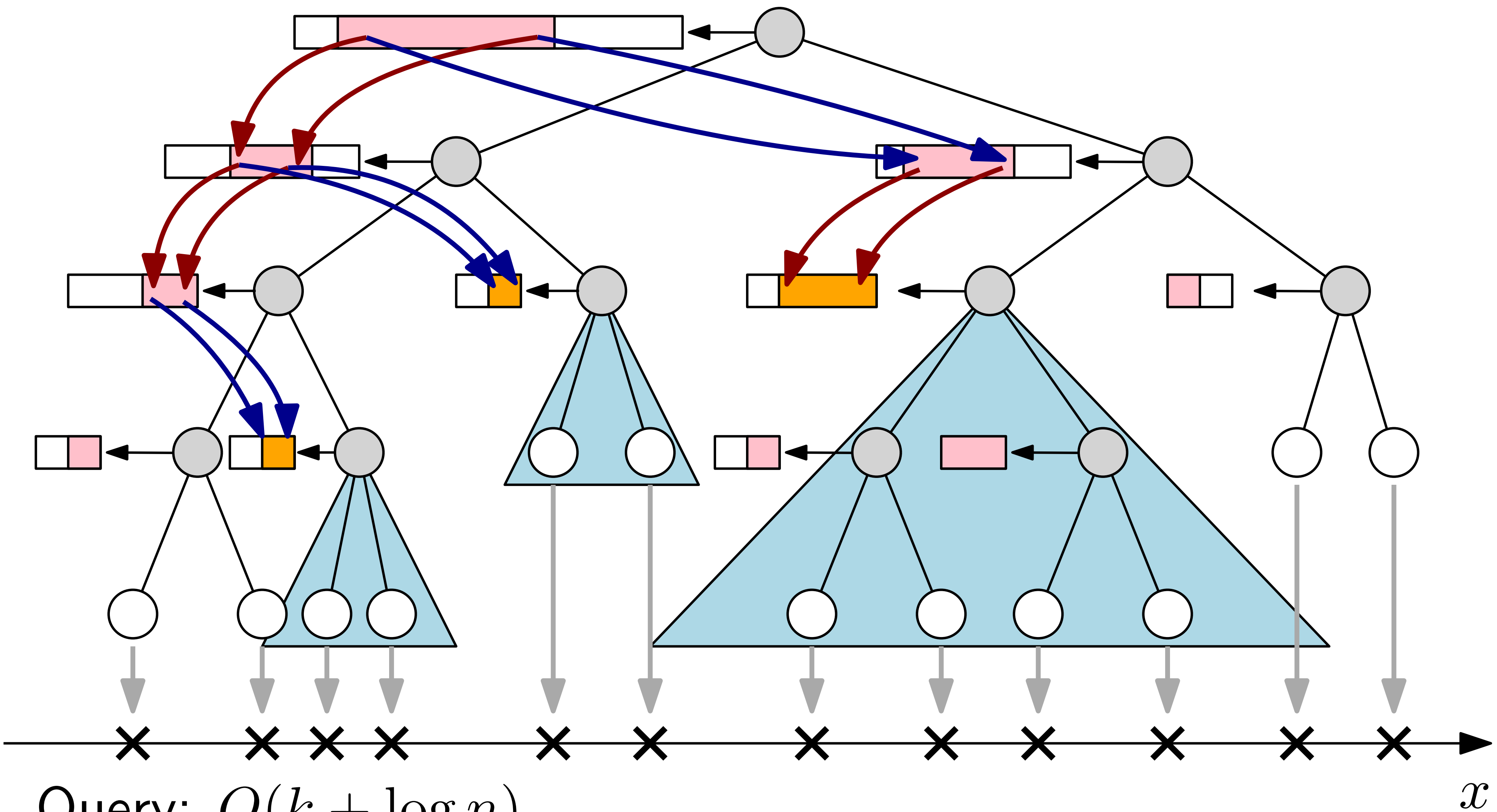

Query: $O(k + \log n)$

Recap

D	Size	Preprocessing Time	Query Time	Notes
1	$O(n)$	$O(n \log n)$	$O(\log n + k)$	
2	$O(n \log n)$	$O(n \log n)$	$O(\log^2 n + k)$	
> 2	$O(n \log^{D-1} n)$	$O(n \log^{D-1} n)$	$O(\log^D n + k)$	

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Can be made dynamic (supports point insertion / deletion) in $O(\log^D n)$ amortized time per update.