Local Connection Game

Motivations

often built and maintained by self-interested agents

Introduction

- Introduced in [FLMPS'03]
- A LCG is a game that models the creation of networks
- two competing issues: players want
  - to minimize the cost they incur in building the network
  - to ensure that the network provides them with a high quality of service
- Players are nodes that:
  - pay for the links
  - benefit from short paths

[FLMPS'03]:
A. Fabrikant, A. Luthra, E. Maneva, C.H. Papadimitriou, S. Shenker,
On a network creation game, PODC'03

The model

- n players: nodes in a graph to be built
- Strategy for player u: a set of undirected edges that u will build (all incident to u)
- Given a strategy vector S, the constructed network will be G(S)
  - there is the undirected edge (u,v) if it is bought by u or v (or both)
- player u's goal:
  - to make the distance to other nodes small
  - to pay as little as possible

The model

- Each edge costs α
- $dist_{G(S)}(u,v)$: length of a shortest path (in terms of number of edges) between u and v
- Player u aims to minimize its cost:

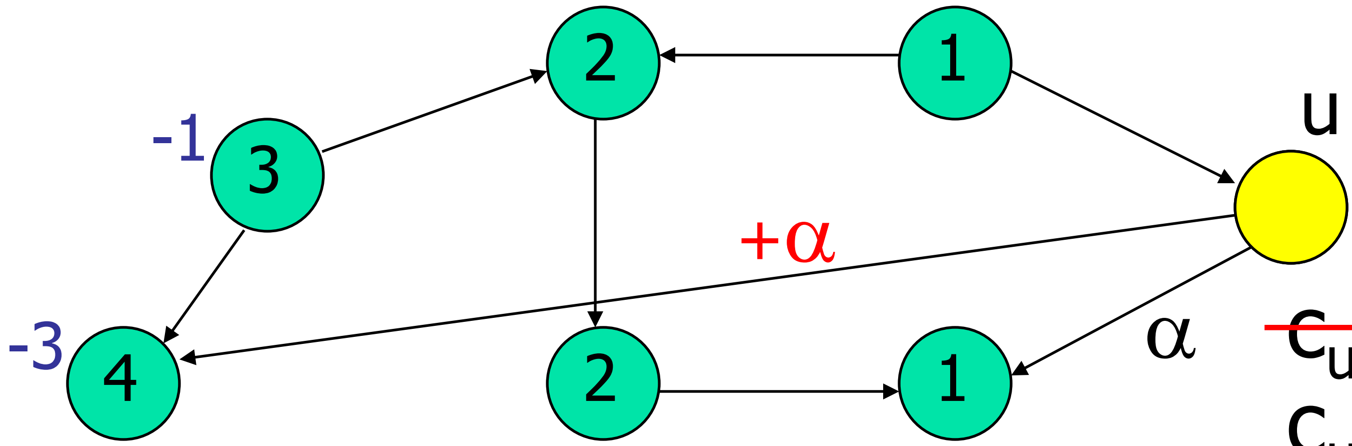
$$cost_u(S) = \alpha n_u + \sum_v dist_{G(S)}(u,v)$$

- n_u: number of edges bought by node u

Remind

- We use Nash equilibrium (NE) as the solution concept
- To evaluate the overall quality of a network, we consider the *social cost*, i.e. the sum of all players' costs
- a network is *optimal* or *socially efficient* if it minimizes the social cost
- A graph $G=(V,E)$ is *stable* (for a value α) if there exists a strategy vector S such that:
  - S is a NE
  - S forms G

Example



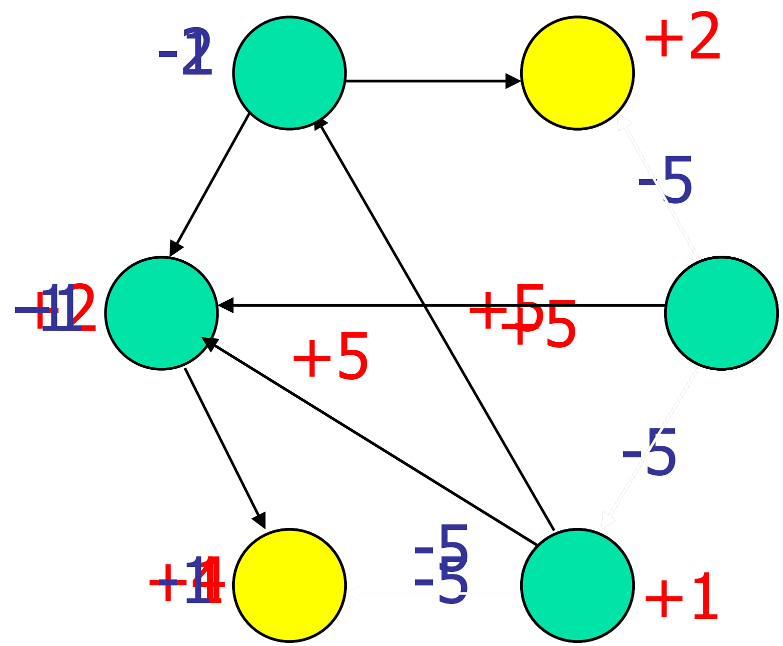
$\cancel{c_u=\alpha+13}$

$c_u=2\alpha+9$

(Convention: arrow *from* the node buying the link)

Example

- Set α=5, and consider:



That's a stable network!

Some simple observations

- In $SC(S)$ each term $dist_{G(S)}(u,v)$ contributes to the overall quality twice
- In a stable network each edge (u,v) is bough at most by one player
- Any stable network must be connected
  - Since the distance $dist(u,v)$ is infinite whenever u and v are not connected

Social cost of a (stable) network $G(S)=(V,E)$:

$$SC(S)=\alpha|E| + \Sigma_{u,v} dist_{G(S)}(u,v)$$

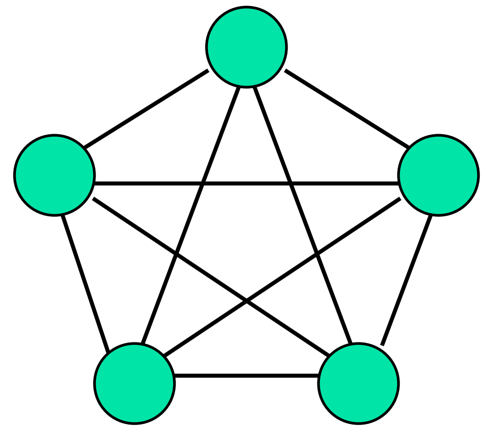
Our goal

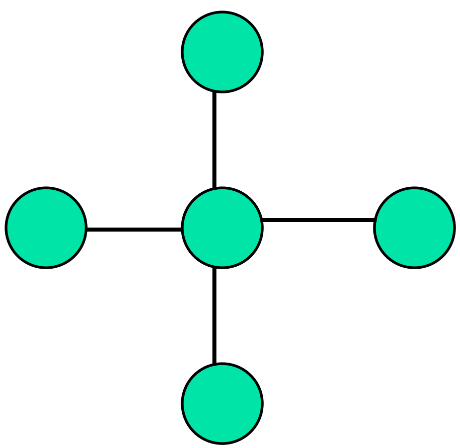
- to bound the efficiency loss resulting from stability
- In particular:
  - To bound the Price of Stability (PoS)
  - To bound the Price of Anarchy (PoA)

How does an optimal network look like?

Some notation

K_n: complete graph
with n nodes





A star is a tree
with height at most 1
(when rooted at its
center)

Lemma

Il $\alpha \leq 2$ then the complete graph is an optimal solution, while if $\alpha \geq 2$ then any star is an optimal solution.

proof

Let $G=(V,E)$ be a network with $|E|=m$ edges

$$SC(G) \geq \alpha m + 2m + 2(n(n-1) - 2m) = \underbrace{(\alpha-2)m + 2n(n-1)}_{LB(m)}$$

Notice: $LB(m)$ is equal to $SC(K_n)$ when $m=n(n-1)/2$ and to SC of any star when $m=n-1$

proof

$G=(V,E)$: optimal solution;
$SC(G)=OPT$

$$LB(m)=(\alpha-2)m + 2n(n-1)$$

$$\begin{tikzcd}
 & LB(n-1) = SC \text{ of any star} \\
OPT \geq \min\limits_{m} LB(m) \geq \arrow[ur, "\alpha \geq 2,\ \text{min } m"] \arrow[dr, "\alpha \leq 2,\ \text{max } m"'] & \\
 & LB(n(n-1)/2) = SC(K_n)
\end{tikzcd}$$

Are the complete graph and stars stable?

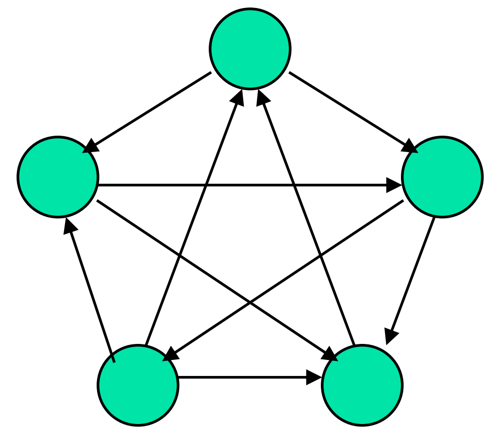
Lemma

If $\alpha \leq 1$ the complete graph is stable, while if $\alpha \geq 1$ then any star is stable.

proof

$\alpha \leq 1$

a node v cannot improve by saving k edges

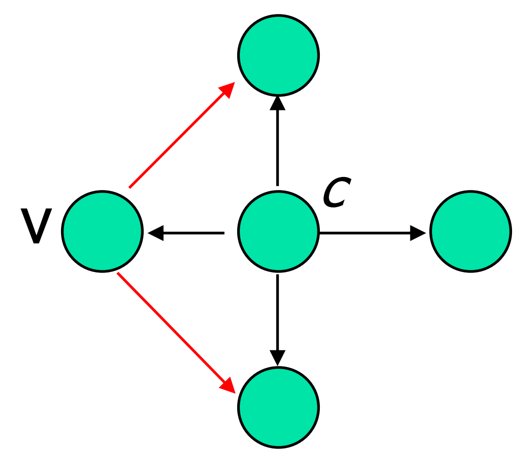


$\alpha \geq 1$

c has no interest to deviate

v buys k more edges…

…pays αk more…
…saves (w.r.t distances) k…



Theorem

For $\alpha \le 1$ and $\alpha \ge 2$ the PoS is 1. For $1<\alpha<2$ the PoS is at most 4/3

proof

$\alpha \le 1$ and $\alpha \ge 2$ …trivial!

$1<\alpha<2$ …K_n is an optimal solution, any star T is stable…

maximized when $\alpha \to 1$

$$\text{PoS} \le \frac{SC(T)}{SC(K_n)} = \frac{(\alpha-2)(n-1) + 2n(n-1)}{\alpha\, n(n-1)/2 + n(n-1)} \le \frac{-1(n-1) + 2n(n-1)}{n(n-1)/2 + n(n-1)}$$

$$= \frac{2n - 1}{(3/2)n} = \frac{4n - 2}{3n} < 4/3$$

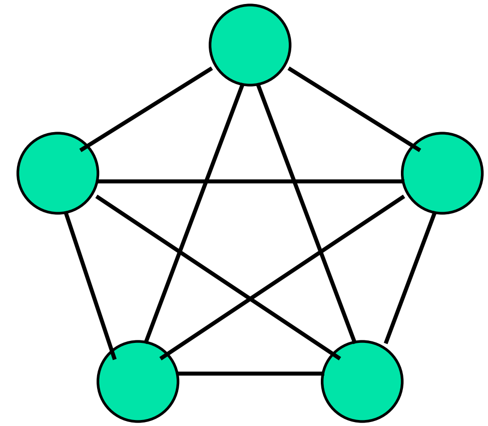
What about price of Anarchy?

…for α<1 the complete graph is the
only stable network,
(try to prove that formally)
hence PoA=1…

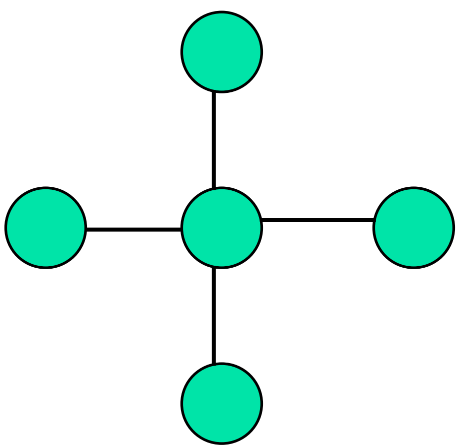
…for larger value of α?

Some more notation

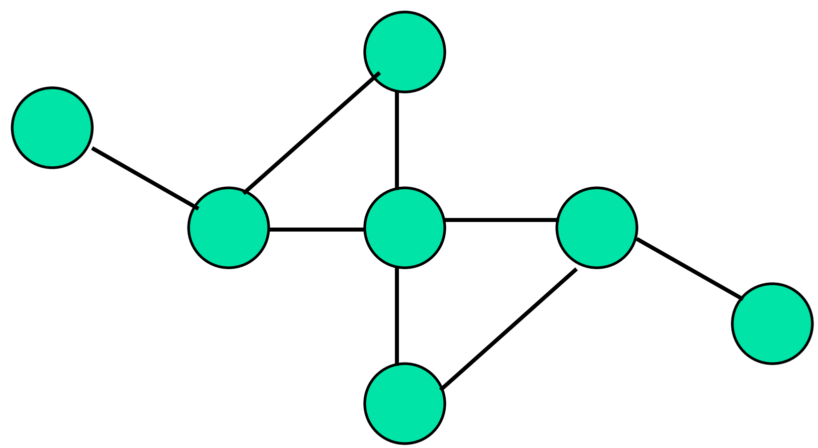
The diameter of a graph G is the maximum distance between any two nodes



diam=1



diam=2

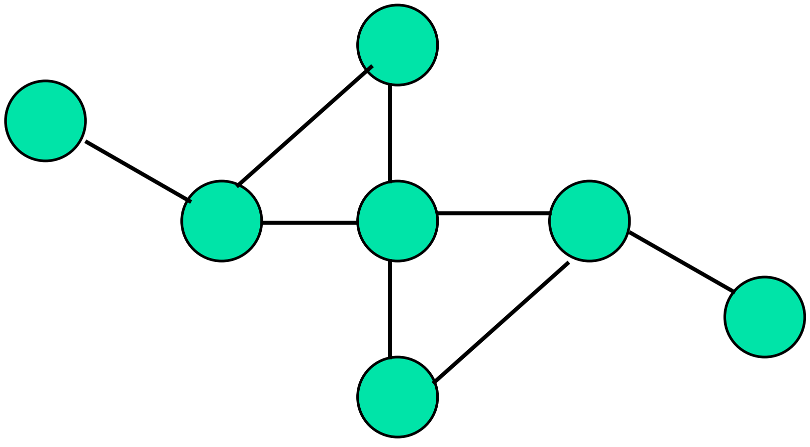


diam=4

Some more notation

An edge e is a cut edge of a graph $G=(V,E)$ if $G-e$ is disconnected

$G-e=(V,E\setminus\{e\})$



A simple property:
Any graph has at most n-1 cut edges

Theorem

The PoA is at most $O(\sqrt{\alpha})$.

proof

It follows from the following lemmas:

Lemma 1

The diameter of any stable network is at most $2\sqrt{\alpha}+1$.

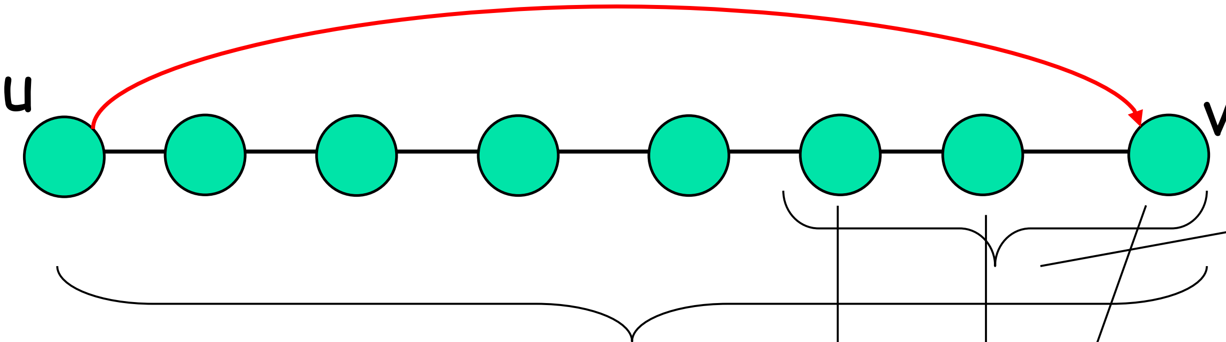
Lemma 2

The SC of any stable network with diameter d is at most $O(d)$ times the optimum SC.

proof of Lemma 1

G: stable network

Consider a shortest path in G between two nodes u and v



k vertices reduce their distance from u

$2k \leq dist_G(u,v) \leq 2k+1$ for some k

from $\geq 2k$ to 1 $\rightarrow$ $\geq 2k-1$

from $\geq 2k-1$ to 2 $\rightarrow$ $\geq 2k-3$

⋮

from $\geq k+1$ to k $\rightarrow$ ≥ 1

$$\sum_{i=0}^{k-1}(2i+1)=k^2$$

…since G is stable:

$\Rightarrow$ $\alpha \geq k^2$ $\Rightarrow$ $k \leq \sqrt{\alpha}$

$\Rightarrow$ $dist_G(u,v) \leq 2\sqrt{\alpha} + 1$

■

Lemma 2

> The SC of any stable network $G=(V,E)$ with diameter d is at most $O(d)$ times the optimum SC.

idea of the proof (we'll formally prove it later)

$OPT \geq \alpha\,(n-1) + n(n-1)$ $\Rightarrow$ $OPT \geq \alpha\,(n-1)$, $OPT = \Omega(n^2)$

$$SC(G)= \Sigma_{u,v} d_G(u,v) + \alpha\,|E|$$

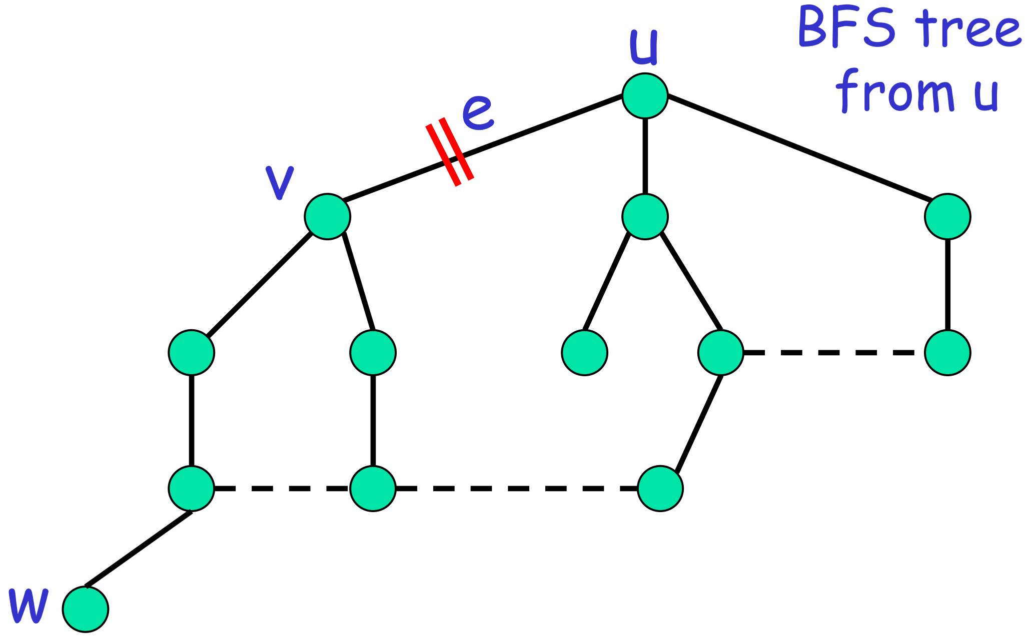
$$=\underbrace{\Sigma_{u,v} d_G(u,v)}_{O(d\ n^2)\ =\ O(d)\ OPT} + \overbrace{\alpha\,\underbrace{|E_{cut}|}_{\leq(n-1)}}^{\leq OPT} + \overbrace{\alpha\,\underbrace{|E_{non\text{-}cut}|}_{O(n^2 d/\alpha)}}^{O(d)\ OPT} = O(d)\ OPT$$

that's the tricky bound

Proposition 1

Let G be a network with diameter d, and let e=(u,v) be a non-cut edge. Then in G-e, every node w increases its distance from u by at most 2d

proof



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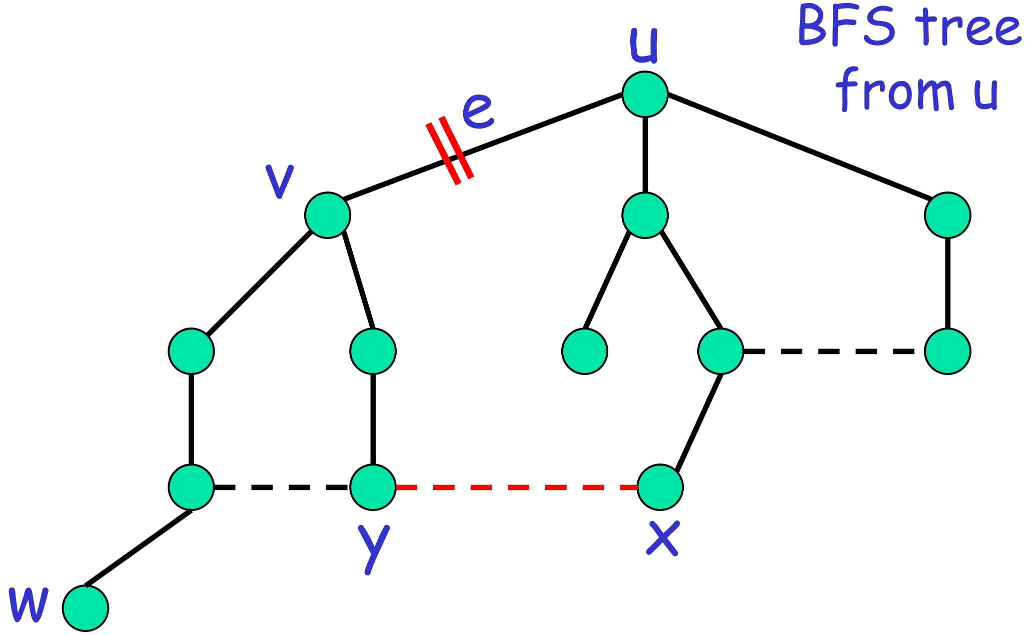

(x,y): any edge crossing the cut induced by the removal of e

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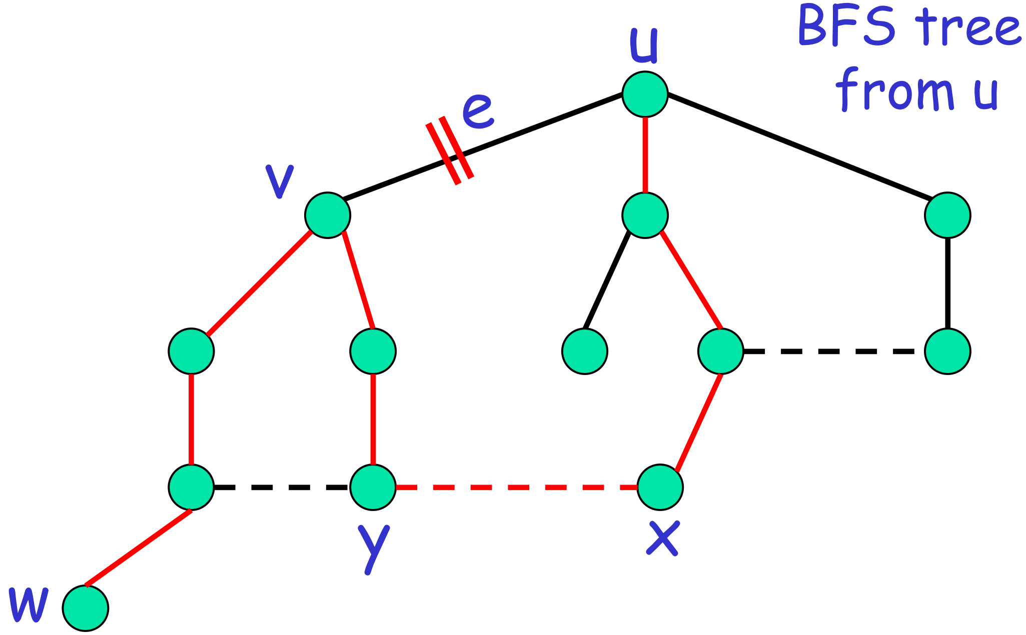


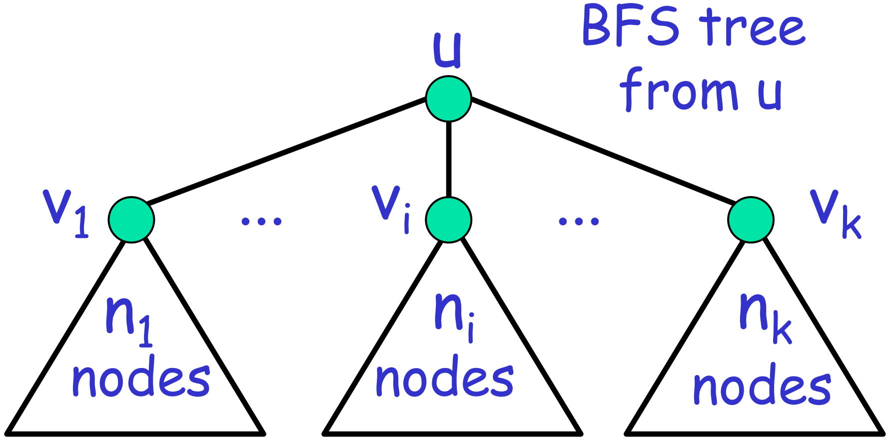
(x,y):
any edge crossing the cut induced by the removal of e

$$d_{G-e}(u,w) \leq \underbrace{d_G(u,x)}_{\leq d} + 1 + \underbrace{d_G(y,v)}_{\leq d} + \underbrace{d_G(v,w)}_{= d_G(u,w)-1} \leq d_G(u,w) + 2d$$

Proposition 2

Let G be a stable network, and let F be the set of Non-cut edges paid for by a node u. Then $|F| \leq (n-1)2d/\alpha$

proof



$k=|F|$

if u removes (u,v_i) saves α and its distance cost increses by at most $2d\, n_i$ (Prop. 1)

since G is stable:

$\alpha \leq 2d\, n_i$

by summing up for all i

$$k\,\alpha \leq 2d\sum_{i=1}^{k} n_i \leq 2d\,(n-1)$$

$\Rightarrow$

$$k \leq (n-1)\, 2d/\alpha$$

Lemma 2

The SC of any stable network $G=(V,E)$ with diameter d is at most $O(d)$ times the optimum SC.

proof

$OPT \geq \alpha\,(n-1) + n(n-1)$

$$SC(G)= \underbrace{\Sigma_{u,v} d_G(u,v)}_{\leq d n(n-1) \leq d\, OPT} + \alpha\,|E| \leq d\, OPT + 2d\, OPT = 3d\, OPT$$

$$\alpha|E| = \alpha\underbrace{|E_{cut}|}_{\leq (n-1)} + \alpha\underbrace{|E_{non\text{-}cut}|}_{\substack{\leq n(n-1)2d/\alpha \\ \text{Prop. 2}}} \leq \alpha(n-1)+n(n-1)2d \leq 2d\, OPT$$

Theorem

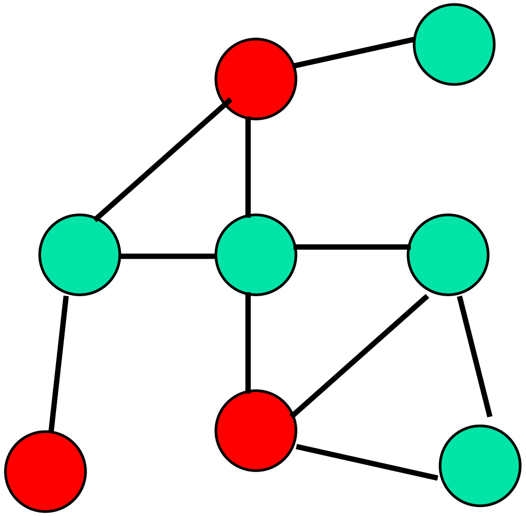
It is NP-hard, given the strategies of the other agents, to compute the best response of a given player.

proof

Reduction from dominating set problem

Dominating Set (DS) problem

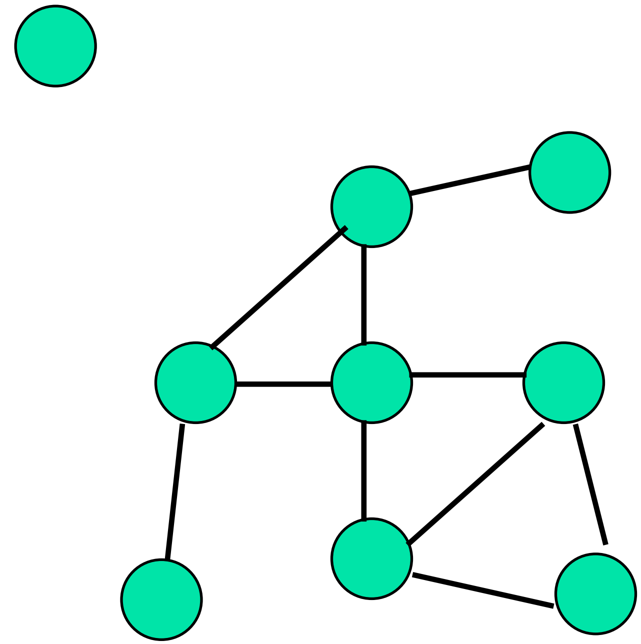
- Input:
  - a graph $G=(V,E)$
- Solution:
  - $U \subseteq V$, such that for every $v \in V-U$, there is $u \in U$ with $(u,v) \in E$
- Measure:
  - Cardinality of U



the reduction

$1<\alpha<2$

player i

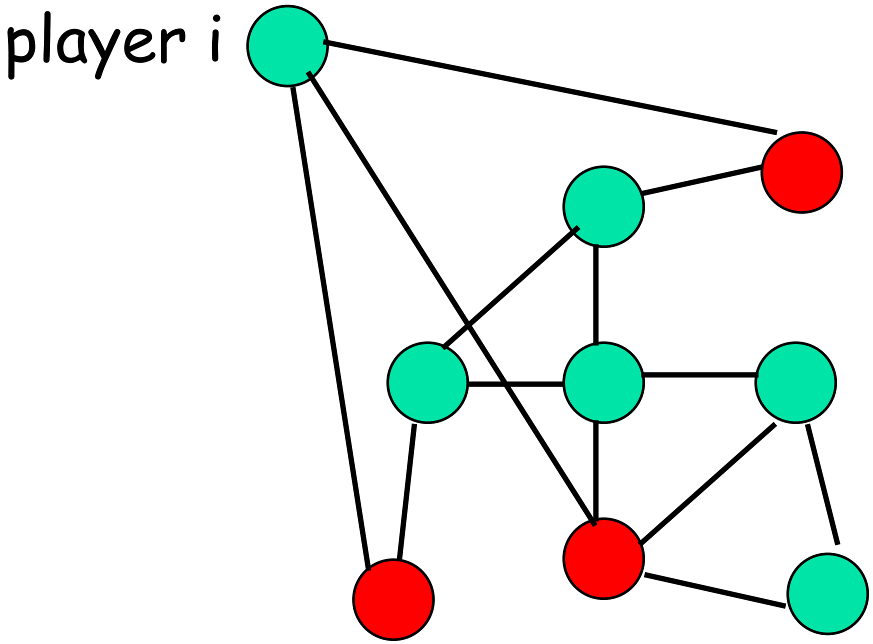


$G=(V,E) = G(S_{-i})$

Player i has a strategy yielding a cost $\leq \alpha k+2n-k$ if and only if there is a DS of size $\leq k$

the reduction

$1<\alpha<2$



$G=(V,E) = G(S_{-i})$

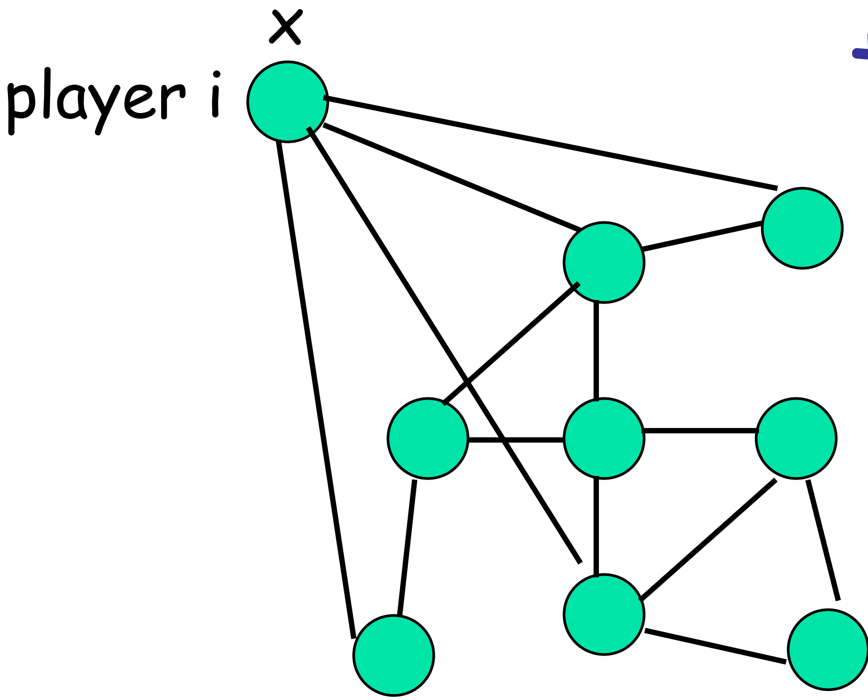
($\Leftarrow$)

easy: given a dominating set U of size k, player i buys edges incident to the nodes in U

Cost for i is $\alpha k+2(n-k)+k = \alpha k+2n-k$

$1<\alpha<2$

the reduction



$G=(V,E) = G(S_{-i})$

($\Rightarrow$)

Let S_i be a strategy giving a cost $\leq \alpha k+2n-k$

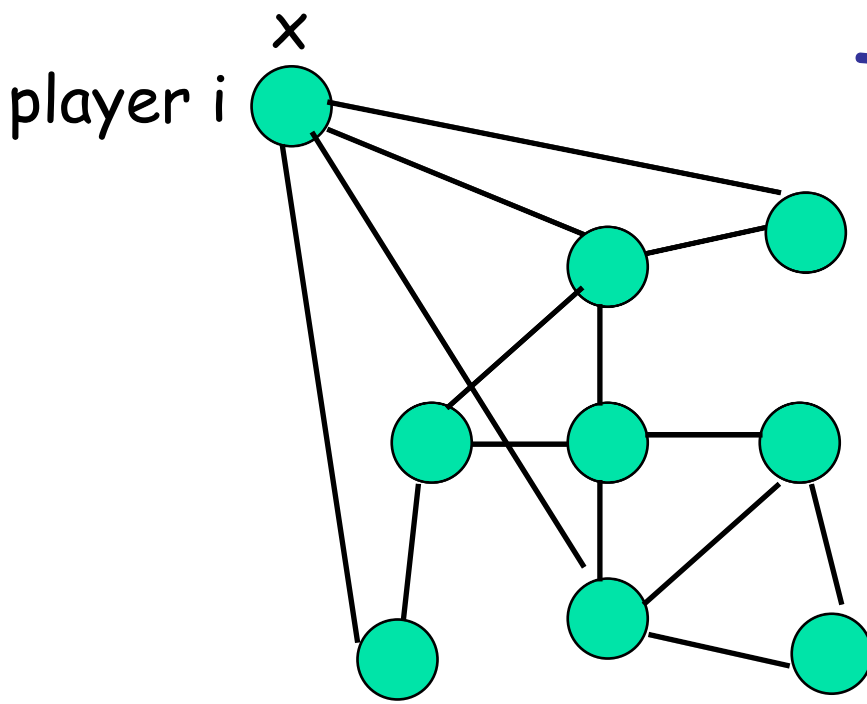
Modify S_i as follows:

repeat:

if there is a node v with distance ≥3 from x in G(S), then add edge (x,v) to S_i (this decreases the cost)

$1<\alpha<2$

the reduction



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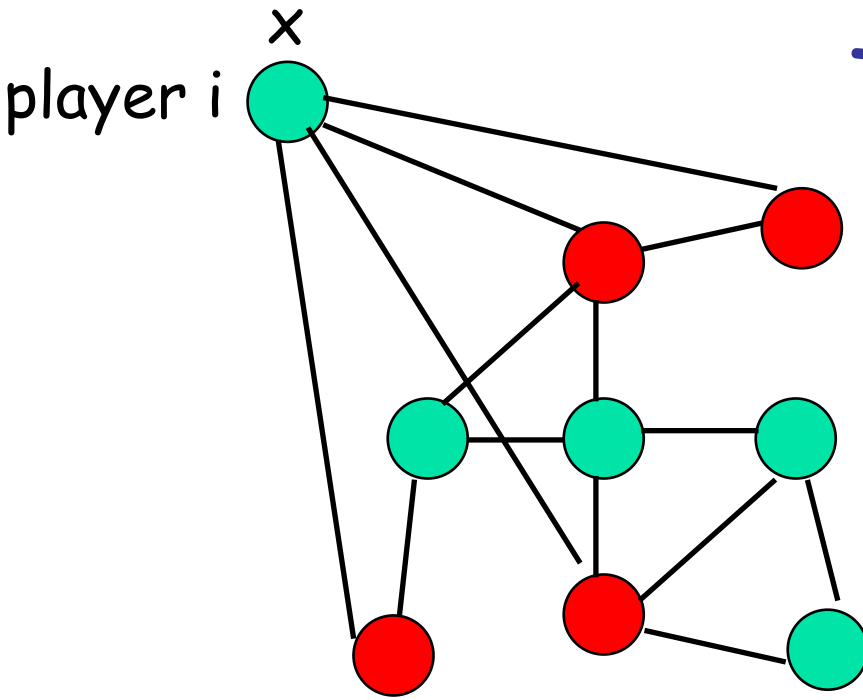
Finally, every node has distance either 1 or 2 from x

$1<\alpha<2$

the reduction



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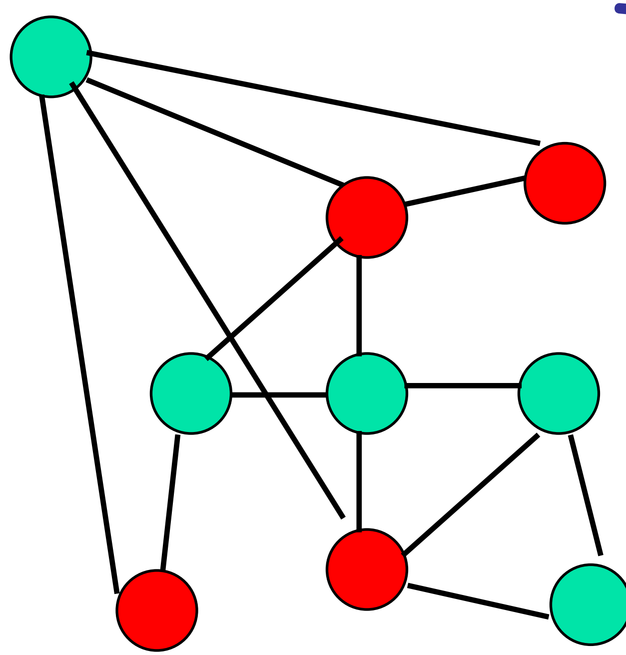
Finally, every node has distance either 1 or 2 from x

Let U be the set of nodes at distance 1 from x…

$1<\alpha<2$

the reduction



$G=(V,E) = G(S_{-i})$

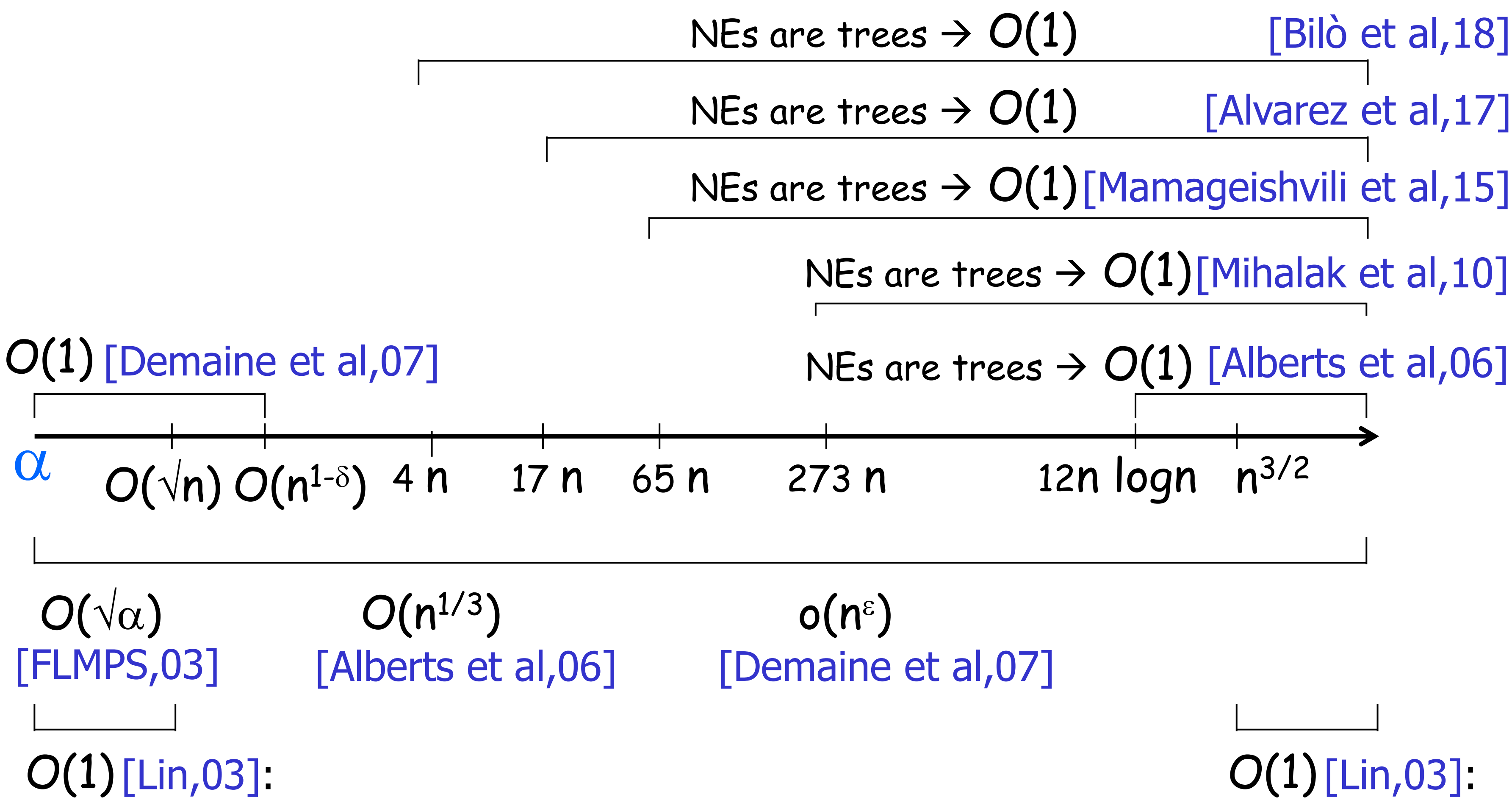
($\Rightarrow$)

…U is a dominating set of the original graph G

We have $cost_i(S)= \alpha|U|+2n-|U| \leq \alpha k+2n-k$

$|U| \leq k$

PoA as function of α: state of the art



PoA as function of α: state of the art

Colleague, remember to mention that the right bound is 4n-13

NEs are trees → O(1) [Bilò et al,18]

NEs are trees → O(1) [Alvarez et al,17]

NEs are trees → O(1) [Mamageishvili et al,15]

NEs are trees → O(1) [Mihalak et al,10]

O(1) [Demaine et al,07]

NEs are trees → O(1) [Alberts et al,06]

α: $O(\sqrt{n})$, $O(n^{1-\delta})$, 4 n, 17 n, 65 n, 273 n, 12n logn, $n^{3/2}$

$O(\sqrt{\alpha})$ [FLMPS,03]

$O(n^{1/3})$ [Alberts et al,06]

$o(n^{\varepsilon})$ [Demaine et al,07]

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α $O(\sqrt{n})$ $O(n^{1-\delta})$ $4n-13$ $17n$ $65n$ $273n$ $12n \log n$ $n^{3/2}$

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D. Bilò, P. Lenzner, On the Tree Conjecture for the Network Creation Game, STACS'18

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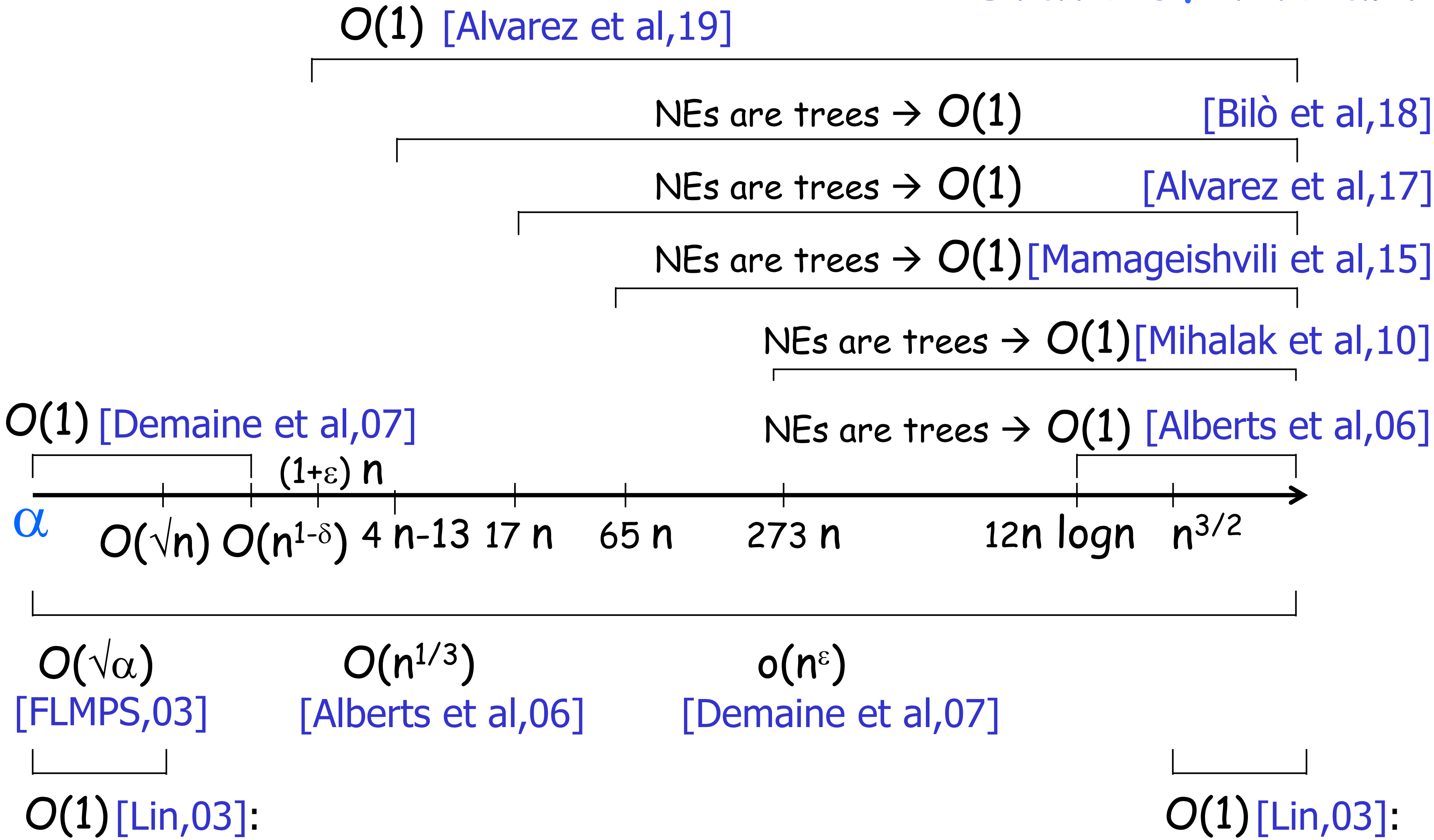

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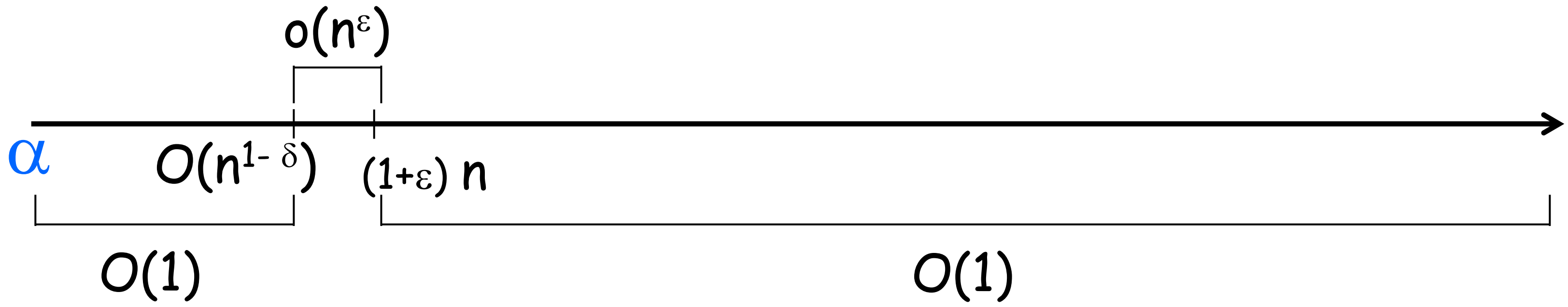

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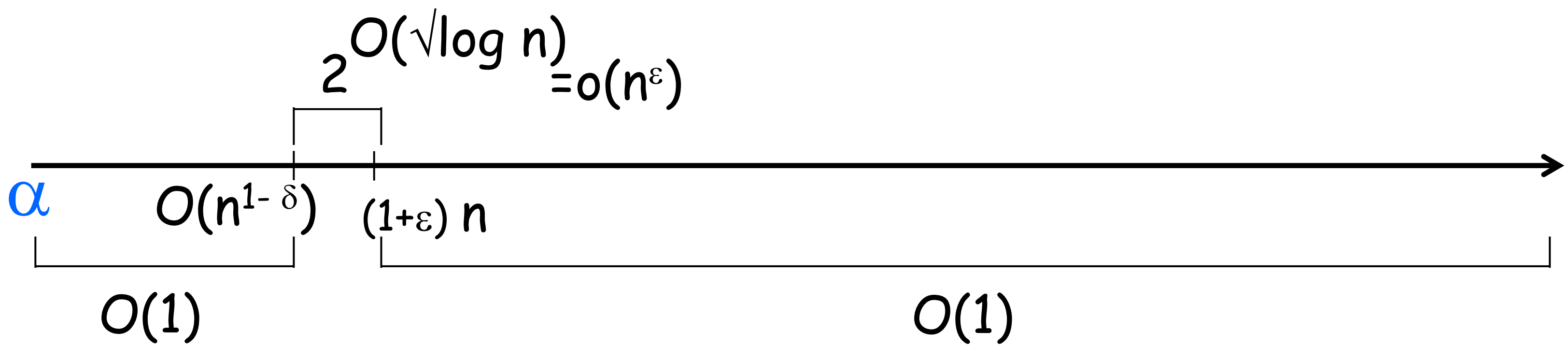

Open: is POA always constant?