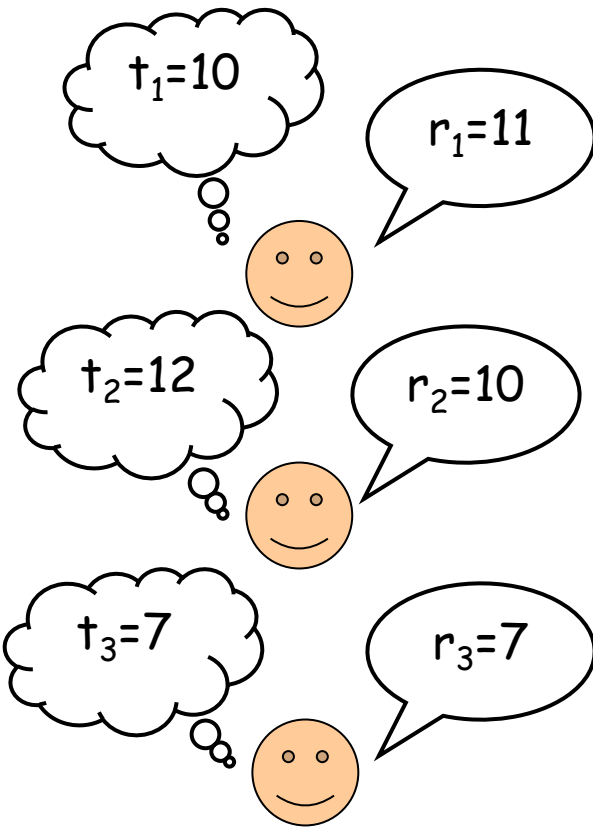




# Combinatorial Auction

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# A single item auction

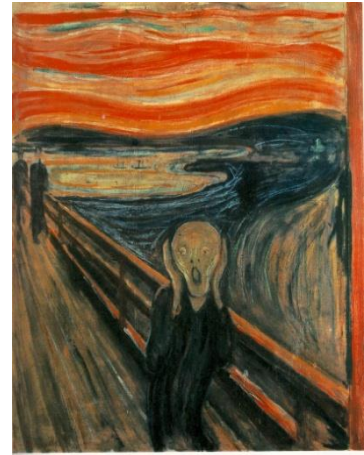


$r_i$  is the amount of money player  $i$  bids (in a sealed envelope) for the painting

$t_i$  is the **maximum** amount of money player  $i$  is willing to pay for the painting

If player  $i$  wins and has to pay  $p$   
its utility is  $u_i = t_i - p$

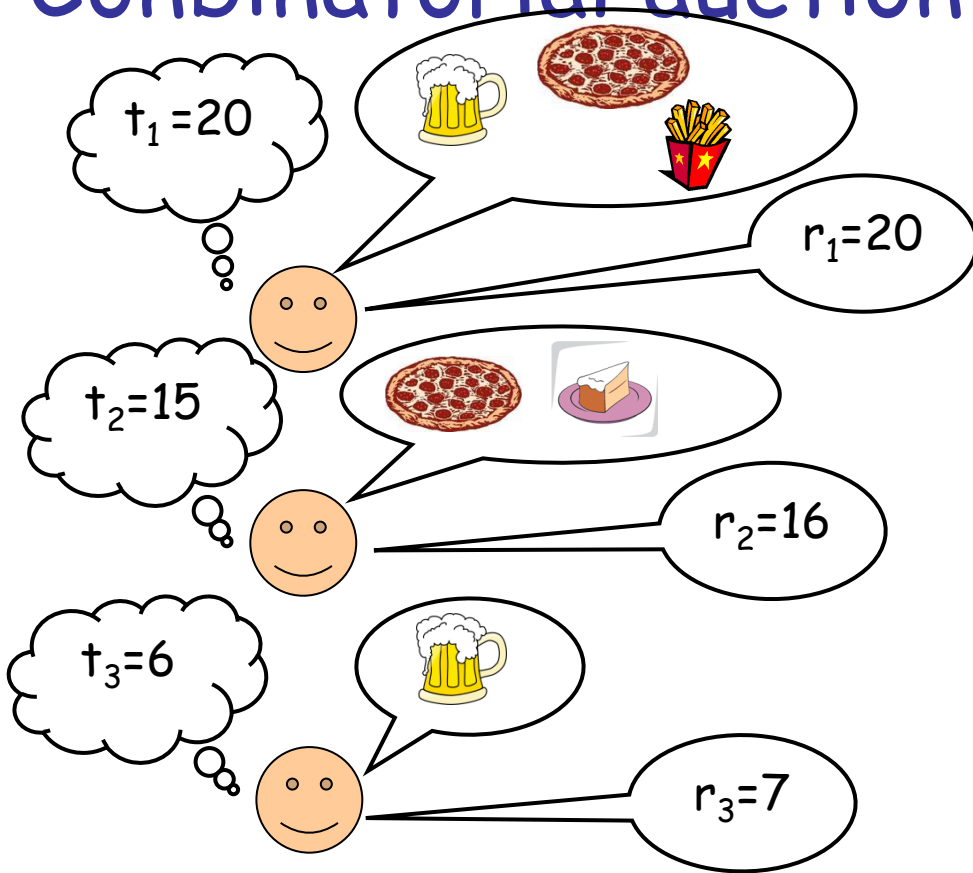
**Social-choice function:**  
the winner should be the guy **having in mind** the highest value for the painting



The mechanism tells to players:

- (1) How the item will be allocated (i.e., who will be the **winner**), depending on the received bids
- (2) The payment the winner has to return, as a function of the received bids

# Combinatorial auction



Each player wants a bundle of objects

$t_i$ : value player  $i$  is willing to pay for its bundle

if player  $i$  gets the bundle at price  $p$  his utility is  $u_i = t_i - p$



the mechanism decides the set of winners and the corresponding payments

$F = \{ W \subseteq \{1, \dots, N\} : \text{winners in } W \text{ are compatible} \}$

# Combinatorial Auction (CA) problem - single-minded case

- Input:
  - $n$  buyers,  $m$  indivisible objects
  - each buyer  $i$ :
    - Wants a subset  $S_i$  of the objects
    - has a value  $t_i$  for  $S_i$
- Solution:
  - $W \subseteq \{1, \dots, n\}$ , such that for every  $i, j \in W$ , with  $i \neq j$ ,  $S_i \cap S_j = \emptyset$
- Measure (to maximize):
  - Total value of  $W$ :  $\sum_{i \in W} t_i$



# CA game

- each buyer  $i$  is selfish
- Only buyer  $i$  knows  $t_i$  (while  $S_i$  is public)
- We want to compute a “good” solution w.r.t. the true values
- We do it by designing a mechanism
- Our mechanism:
  - Asks each buyer to report its value  $v_i$
  - Computes a solution using an output algorithm  $g(\cdot)$
  - takes payments  $p_i$  from buyer  $i$  using some payment function  $p$



# More formally

- Type of agent buyer  $i$ :
  - $t_i$ : value of  $S_i$
  - Intuition:  $t_i$  is the maximum value buyer  $i$  is willing to pay for  $S_i$
- Buyer  $i$ 's valuation of  $W \in F$ :
  - $v_i(t_i, W) = t_i$  if  $i \in W$ , 0 otherwise
- SCF: a good allocation of the objects w.r.t. the true values



# How to design a truthful mechanism for the problem?

Notice that:  
the (true) total value of a feasible  $W$  is:

$$\sum_{i \in W} \mathbf{t}_i = \sum_i v_i(\mathbf{t}_i, W)$$

the problem is **utilitarian**!

...VCG mechanisms apply



# VCG mechanism

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- $M = \langle g(r), p(x) \rangle$ :

- $g(r)$ :  $x^* = \arg \max_{x \in F} \sum_j v_j(r_j, x)$

- $p_i(r)$ : for each  $i$ :

$$p_i(r) = \sum_{j \neq i} v_j(r_j, g(r_{-i})) - \sum_{j \neq i} v_j(r_j, x^*)$$

$g(r)$  has to compute an optimal solution...

...can we do that?

# Theorem

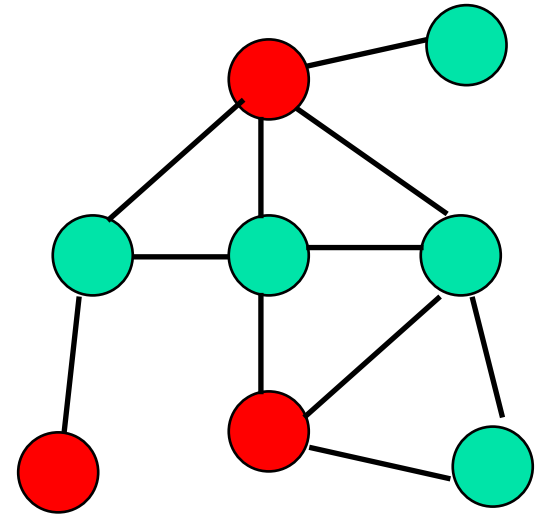
Approximating CA problem within a factor better than  $m^{1/2-\varepsilon}$  is NP-hard, for any fixed  $\varepsilon > 0$ .

proof

Reduction from maximum independent set problem

# Maximum Independent Set (IS) problem

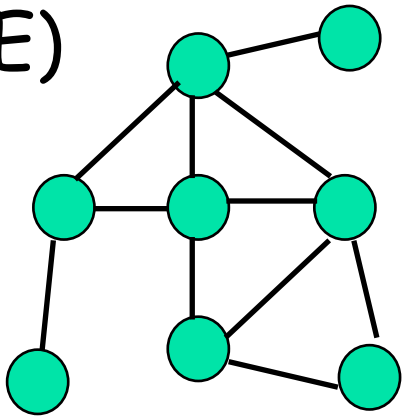
- Input:
  - a graph  $G=(V,E)$
- Solution:
  - $U \subseteq V$ , such that no two vertices in  $U$  are jointed by an edge
- Measure:
  - Cardinality of  $U$



Theorem (J. Håstad, 2002)

Approximating IS problem within a factor better than  $n^{1-\varepsilon}$  is NP-hard, for any fixed  $\varepsilon > 0$ .

$G=(V,E)$



# the reduction

each **edge** is an **object**

each **node**  $i$  is a buyer with:

$S_i$ : set of edges incident to  $i$

$t_i=1$

CA instance has a solution of total value  $\geq k$  **if and only if** there is an IS of size  $\geq k$

A solution of value  $k$  for the instance of CA with  $\text{Opt}_{CA}/k \leq m^{\frac{1}{2}-\varepsilon}$  for some  $\varepsilon > 0$

would imply

A solution of value  $k$  for the instance of IS and hence:

$$\text{Opt}_{IS}/k = \text{Opt}_{CA}/k \leq m^{\frac{1}{2}-\varepsilon} \leq n^{1-2\varepsilon}$$

since  $m \leq n^2$



# How to design a truthful mechanism for the problem?

Notice that:  
the (true) total value of a feasible  $W$  is:

$$\sum_i v_i(\dagger_i, W)$$

the problem is **utilitarian**!

...but a VCG mechanism is not computable  
in polynomial time!

what can we do?

...fortunately, our problem is **one parameter**!



---

A problem is **binary demand (BD)** if

1.  $a_i$ 's type is a **single parameter**  $t_i \in \mathbb{R}$
2.  $a_i$ 's valuation is of the form:

$$v_i(t_i, o) = t_i w_i(o),$$

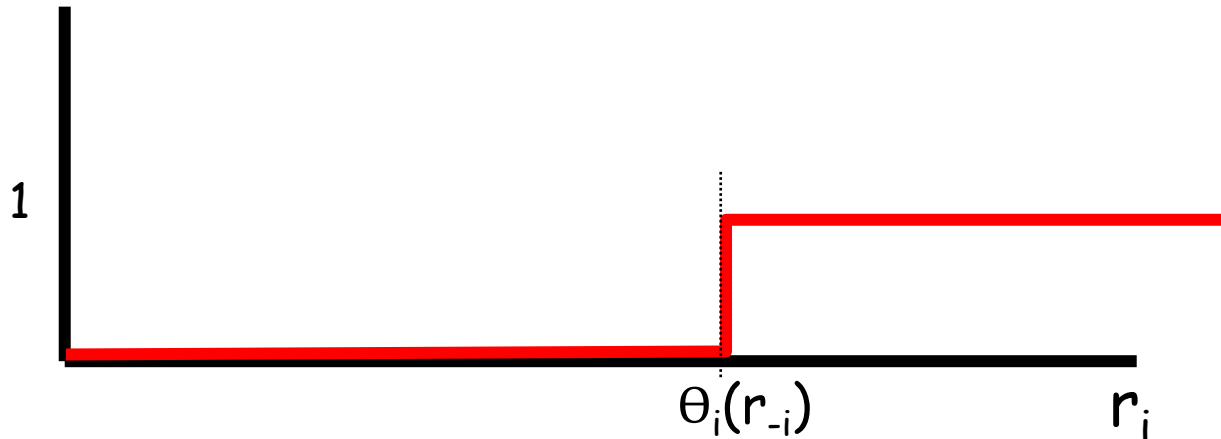
$w_i(o) \in \{0, 1\}$  **work load** for  $a_i$  in  $o$

when  $w_i(o) = 1$  we'll say that  $a_i$  is  
**selected** in  $o$

# Definition

An algorithm  $g()$  for a maximization BD problem is **monotone** if

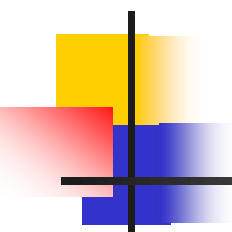
$\forall$  agent  $a_i$ , and for every  $r_{-i} = (r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_N)$ ,  $w_i(g(r_{-i}, r_i))$  is of the form:



$\theta_i(r_{-i}) \in \mathbb{R} \cup \{+\infty\}$ : **threshold**

**payment** from  $a_i$  is:

$$p_i(r) = \theta_i(r_{-i})$$

- 
- 
- Our goal: to design a mechanism satisfying:
    1.  $g(\cdot)$  is monotone
    2. Solution returned by  $g(\cdot)$  is a “good” solution, i.e. an approximated solution
    3.  $g(\cdot)$  and  $p(\cdot)$  computable in polynomial time

# A greedy $\sqrt{m}$ -approximation algorithm

1. reorder (and rename) the bids such that

$$v_1/\sqrt{|S_1|} \geq v_2/\sqrt{|S_2|} \geq \dots \geq v_n/\sqrt{|S_n|}$$

2.  $W \leftarrow \emptyset; X \leftarrow \emptyset$

3. for  $i=1$  to  $n$  do

1. if  $S_i \cap X = \emptyset$  then  $W \leftarrow W \cup \{i\}; X \leftarrow X \cup S_i$

4. return  $W$

## Lemma

The algorithm  $g(\ )$  is monotone

proof

It suffices to prove that, for any selected **agent**  $i$ , we have that  $i$  is still selected when it raises its bid

$$v_1/\sqrt{|S_1|} \geq \dots \geq v_i/\sqrt{|S_i|} \geq \dots \geq v_n/\sqrt{|S_n|}$$

Increasing  $v_i$  can only move **bidder**  $i$  up in the greedy order, making it easier to win





# Computing the payments

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...we have to compute for each selected bidder  $i$   
its threshold value

How much can bidder  $i$  decrease  
its bid before being non-  
selected?



# Computing payment $p_i$

Consider the greedy order without  $i$

$$v_1/\sqrt{|S_1|} \geq \dots \geq \cancel{v_i/\sqrt{|S_i|}} \geq \dots \geq v_n/\sqrt{|S_n|}$$

↑  
index  $j$

Use the greedy algorithm to find the smallest index  $j$  (if any) such that:

1.  $j$  is selected
2.  $S_j \cap S_i \neq \emptyset$

$$p_i = v_j \sqrt{|S_i|} / \sqrt{|S_j|}$$
$$p_i = 0 \text{ if } j \text{ doesn't exist}$$

## Lemma

Let  $OPT$  be an optimal solution for CA problem, and let  $W$  be the solution computed by the algorithm, then

$$\sum_{i \in OPT} v_i \leq \sqrt{m} \sum_{i \in W} v_i$$

proof

$$\forall i \in W \quad OPT_i = \{j \in OPT : j \geq i \text{ and } S_j \cap S_i \neq \emptyset\}$$

since  $\bigcup_{i \in W} OPT_i = OPT$  it suffices to prove:  $\sum_{j \in OPT_i} v_j \leq \sqrt{m} v_i \quad \forall i \in W$

$$\sum_{j \in OPT} v_j \leq \sum_{i \in W} \sum_{j \in OPT_i} v_j \leq \sum_{i \in W} \sqrt{m} v_i \leq \sqrt{m} \sum_{i \in W} v_i$$

## Lemma

Let  $OPT$  be an optimal solution for CA problem, and let  $W$  be the solution computed by the algorithm, then

$$\sum_{i \in OPT} v_i \leq \sqrt{m} \sum_{i \in W} v_i$$

proof

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crucial observation  
for greedy order we have

$$v_j \leq \frac{v_i \sqrt{|S_j|}}{\sqrt{|S_i|}} \quad \forall j \in OPT_i$$

proof

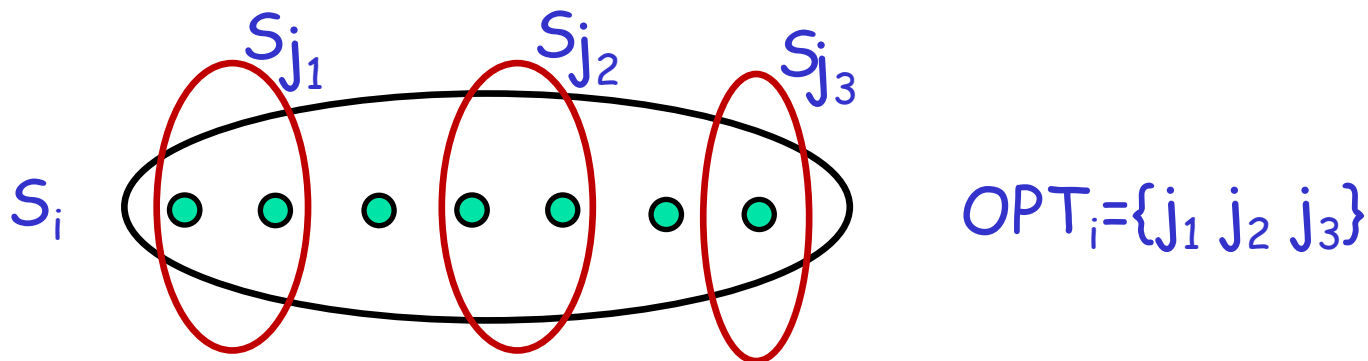
$\forall i \in W$

$$\sum_{j \in \text{OPT}_i} v_j \leq \frac{v_i}{\sqrt{|S_i|}} \sum_{j \in \text{OPT}_i} \sqrt{|S_j|} \leq \sqrt{m} v_i$$

we can bound

Cauchy-Schwarz  
inequality

$$\sum_{j \in \text{OPT}_i} \sqrt{|S_j|} \leq \underbrace{\sqrt{|\text{OPT}_i|}}_{\leq |S_i|} \underbrace{\sqrt{\sum_{j \in \text{OPT}_i} |S_j|}}_{\leq m} \leq \sqrt{|S_i|} \sqrt{m}$$





# Cauchy-Schwarz inequality

$$\left( \sum_{i=1}^n x_i y_i \right) \leq \left( \sum_{i=1}^n x_i^2 \right)^{1/2} \left( \sum_{i=1}^n y_i^2 \right)^{1/2}.$$

...in our case...

$$n = |OPT_i| \quad \begin{array}{l} x_j = 1 \\ y_j = \sqrt{|S_j|} \end{array} \quad \text{for } j=1, \dots, |OPT_i|$$