4.8 Huffman Codes

These lecture slides are supplied by Mathijs de Weerd

Data Compression

Q. Given a text that uses ***32*** symbols (26 different letters, space, and some punctuation characters), how can we encode this text in ***bits***?

Q. Some symbols (**e, t, a, o, i, n**) are used far **more often** than others. How can we use this to reduce our encoding?

Q. How do we know when the next symbol begins?

Ex. c(a) = 01
c(b) = 010
c(e) = 1

What is 0101?

Data Compression

Q. Given a text that uses 32 symbols (26 different letters, space, and some punctuation characters), how can we encode this text in bits?
A. We can encode 2^5 different symbols using a **fixed** length of ***5*** bits per symbol. This is called fixed length encoding.

Q. Some symbols ***(e, t, a, o, i, n)*** are used far **more often** than others. How can we use this to reduce our encoding?
A. Encode these characters with **fewer** bits, and the others with **more** bits.

Q. How do we know when the next symbol begins?
A. Use a separation symbol (like the pause in Morse), or make sure that there is no ambiguity by ensuring that no code is a prefix of another one.

Ex. c(a) = 01
c(b) = 010
c(e) = 1

What is 0101?

Prefix Codes

Definition. A prefix code for a set S is a function **c** that maps each $x \in S$ to 1s and 0s in such a way that for $x, y \in S$, $x \neq y$, **c(x)** is not a prefix of $c(y)$.

Ex. c(a) = 11
c(e) = 01
c(k) = 001
c(l) = 10
c(u) = 000

Q. What is the meaning of 1001000001 ?

Suppose **frequencies** are known in a text of 1G:

$f_a=0.4$, $f_e=0.2$, $f_k=0.2$, $f_l=0.1$, $f_u=0.1$

Q. What is the ***size*** of the encoded text?

Prefix Codes

Definition. A prefix code for a set S is a function c that maps each $x \in S$ to 1s and 0s in such a way that for $x, y \in S$, $x \neq y$, $c(x)$ is not a prefix of $c(y)$.

Ex. c(a) = 11
c(e) = 01
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c(u) = 000

Q. What is the meaning of 1001000001 ?
A. "leuk"

Suppose frequencies are known in a text of 1G:
$f_a=0.4$, $f_e=0.2$, $f_k=0.2$, $f_l=0.1$, $f_u=0.1$

Q. What is the size of the encoded text?
A. ***$2^*f_a + 2^*f_e + 3^*f_k + 2^*f_l + 4^*f_u = 2.4G$***

Optimal Prefix Codes

Definition. The average bits per letter of a prefix code ***c*** is the **sum** over all symbols of:

(its frequency) **x** (the number of bits of its encoding):

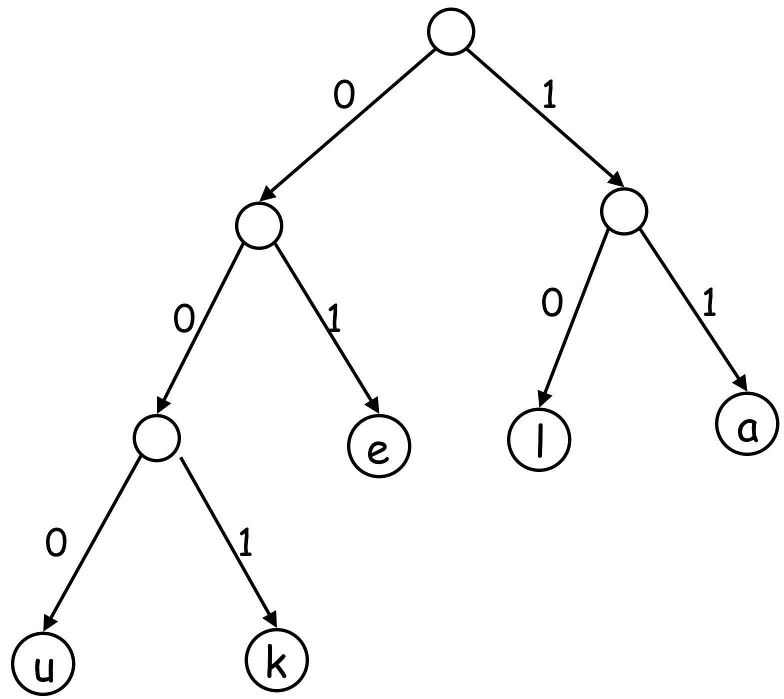
$$ABL(c) = \sum_{x \in S} f_x \cdot |c(x)|$$

GOAL: find a prefix code that is has the ***lowest*** possible ***average bits*** per letter.

We can model a code in a ***binary tree***...

Representing Prefix Codes using Binary Trees

Ex. c(a) = 11
c(e) = 01
c(k) = 001
c(l) = 10
c(u) = 000



Q. How does the tree of a prefix code look?

Representing Prefix Codes using Binary Trees

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c(e) = 01
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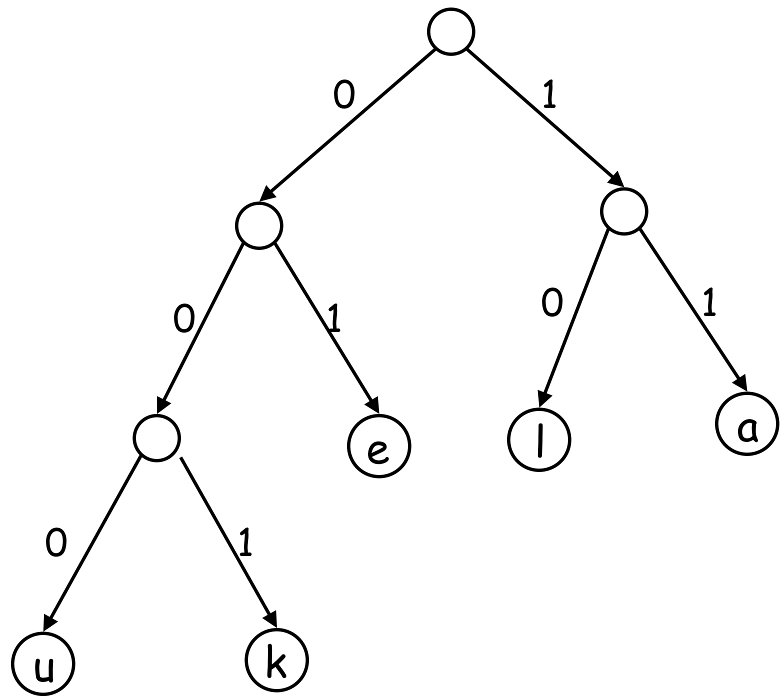


Q. How does the **tree** of a **prefix code** look?

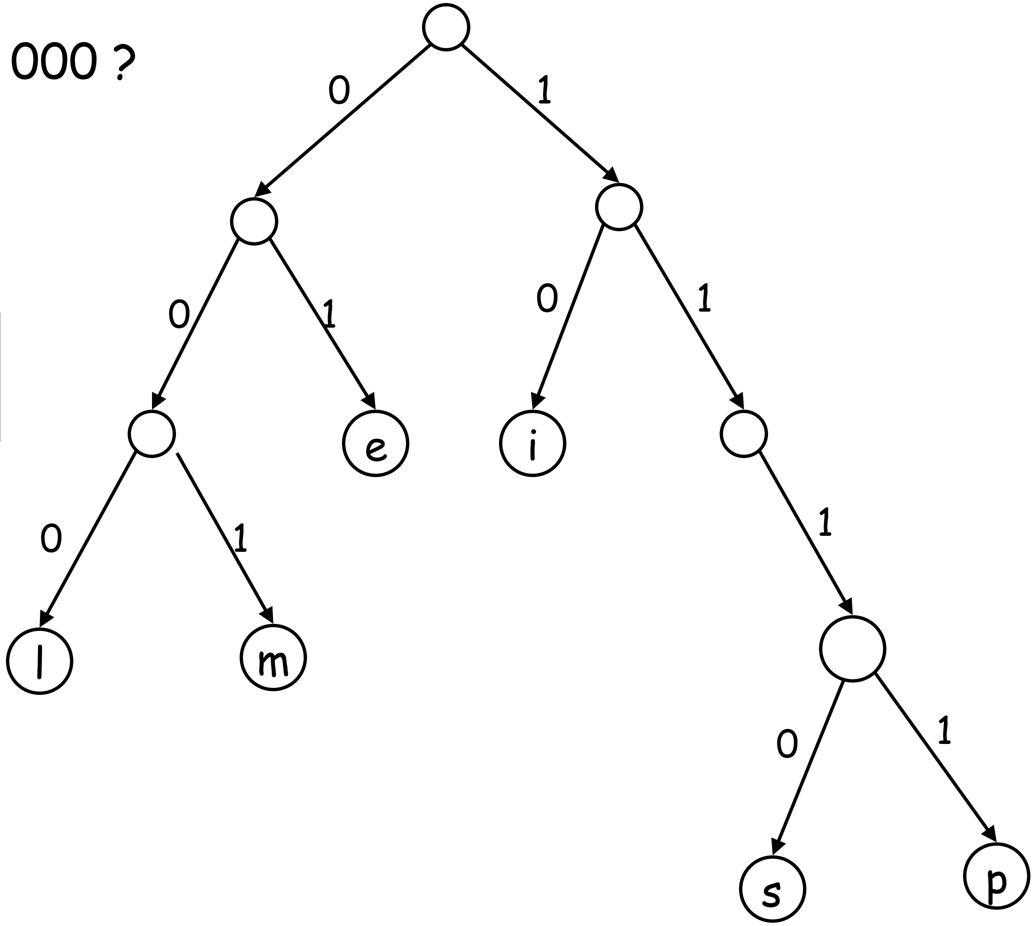
A. Only the ***leaves*** have a ***label***.

Proof. An encoding of ***x*** is a prefix of an encoding of ***y*** **iff** the path of ***x*** is a prefix of the path of ***y***.

Representing Prefix Codes using Binary Trees

Q. What is the meaning of
1110 10 001 1111 01 000 ?

$$ABL(T) = \sum_{x \in S} f_x \cdot \text{depth}_T(x)$$

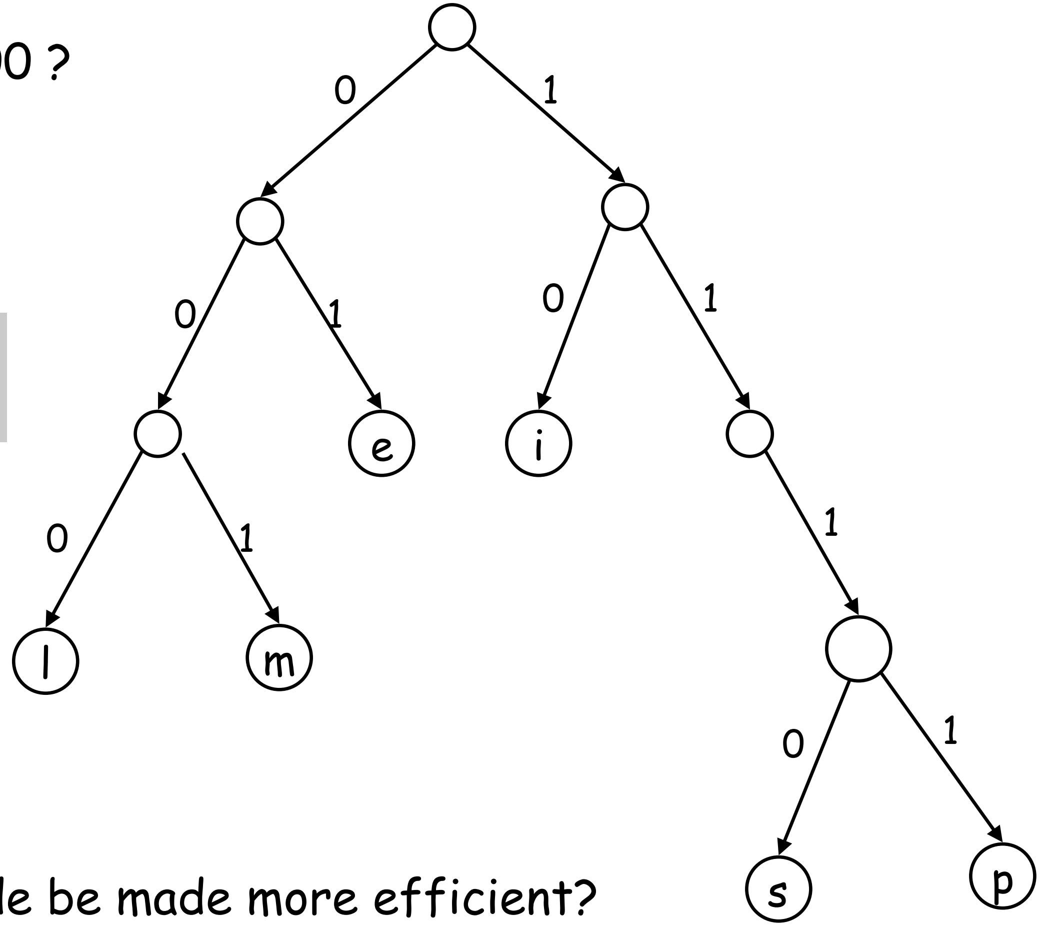


Representing Prefix Codes using Binary Trees

Q. What is the meaning of
111010001111101000 ?

A. "simpel"

$$ABL(T) = \sum_{x \in S} f_x \cdot \text{depth}_T(x)$$



Q. How can this prefix code be made more efficient?

Representing Prefix Codes using Binary Trees

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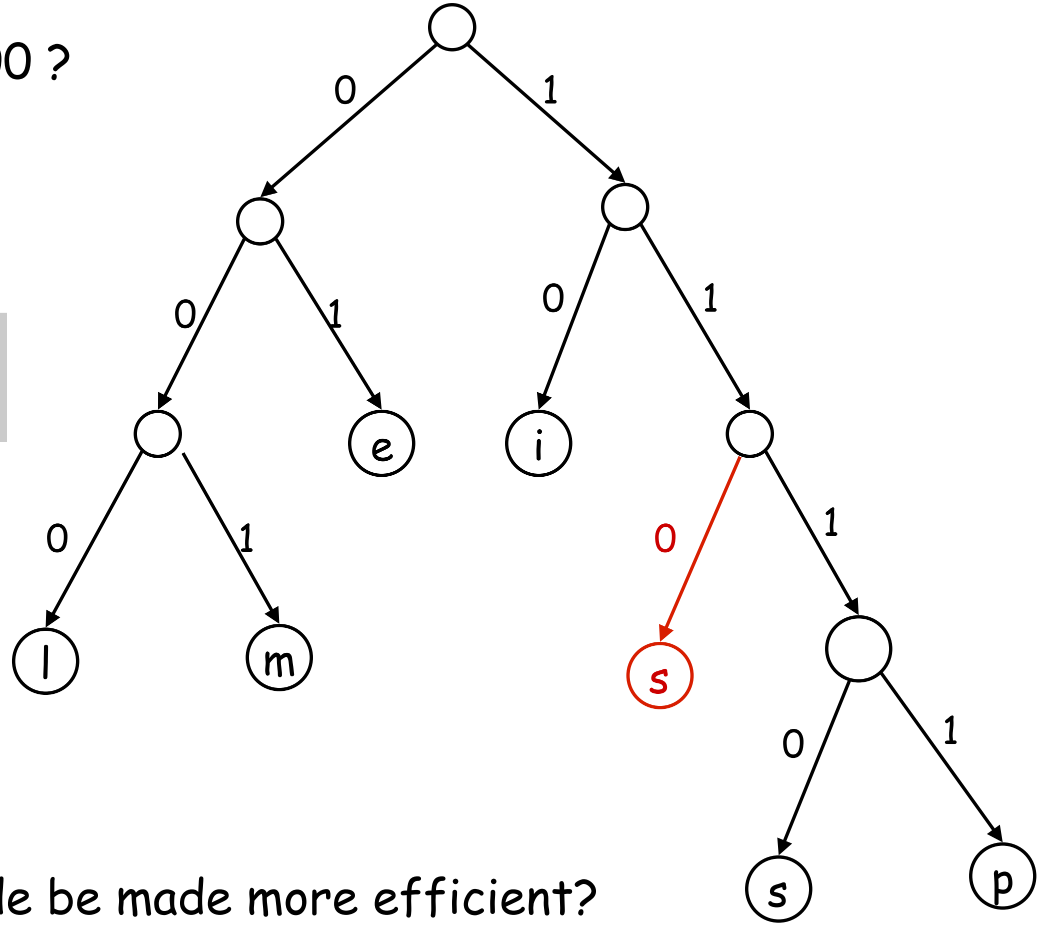


Q. How can this prefix code be made more efficient?

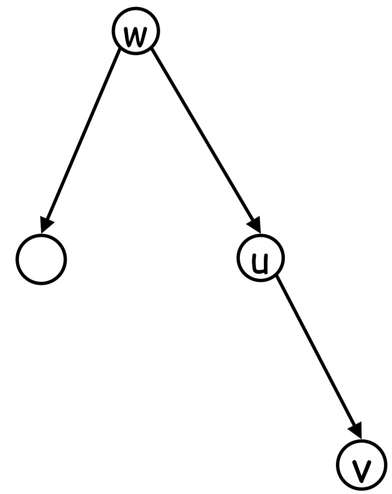
A. Change encoding of p and s to a shorter one.

This tree is now full.

Representing Prefix Codes using Binary Trees

Definition. A tree is full if every node that is not a leaf has two children.

Claim. The binary tree corresponding to an optimal prefix code is **full**.
Pf.

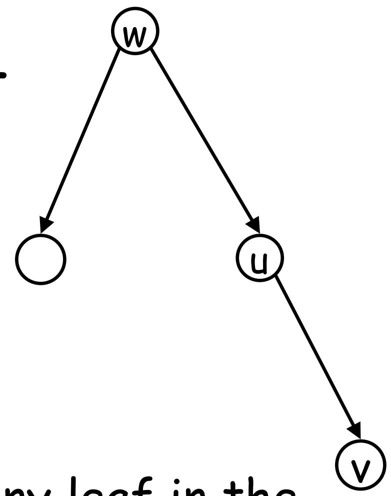


Representing Prefix Codes using Binary Trees

Definition. A tree is full if every node that is not a leaf has two children.

Claim. The binary tree corresponding to the optimal prefix code is full.
Proof. (by contradiction)

- Suppose T is binary tree of optimal prefix code and is not full.
- This means there is a node ***u*** with only one child ***v***.
- **Case 1**: ***u*** is the root; delete ***u*** and use ***v*** as the **root**
- **Case 2:** ***u*** is not the root
  - let ***w*** be the parent of u
  - delete ***u*** and make ***v*** be a child of ***w*** in place of ***u***



- In both cases the number of bits needed to encode any leaf in the subtree of ***v*** is **decreased**. The rest of the tree is not affected.
- Clearly this new tree ***T'*** has a smaller ABL than ***T***. Contradiction.

Optimal Prefix Codes: False Start

Q. Where should letters be placed with a high frequency in the tree of an optimal prefix code ?

Optimal Prefix Codes: False Start

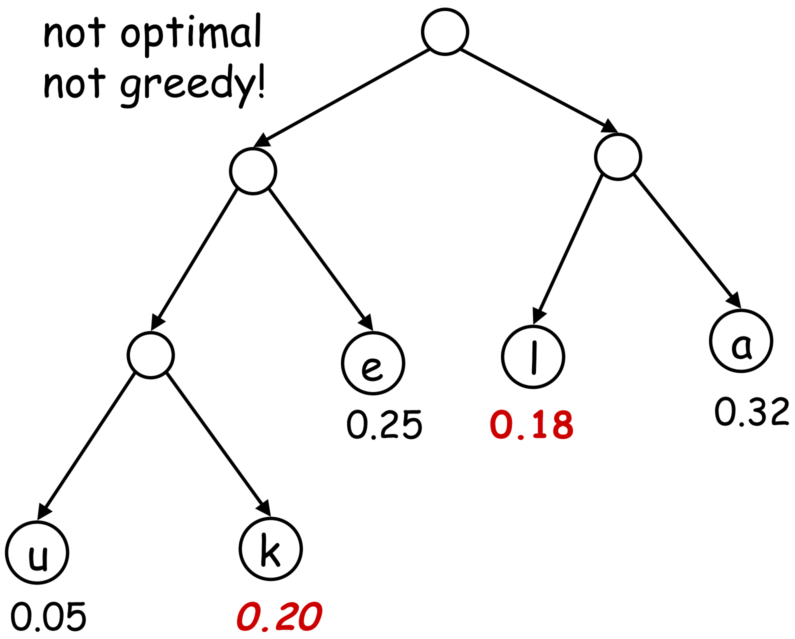
Q. Where in the tree of an optimal prefix code should letters be placed with a high frequency?

A. Near the top! Use recursive structure of trees.

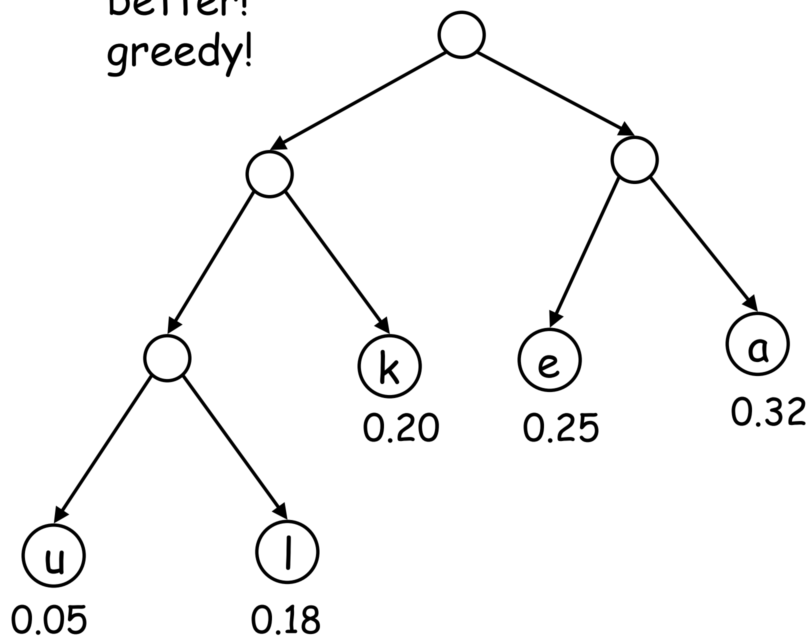
Greedy template. Create tree top-down, split S into two sets S_1 and S_2 with (almost) ***equal frequencies***. Recursively build tree for S_1 and S_2.

[Shannon-Fano, 1949] $f_a=0.32$, $f_e=0.25$, $f_k=0.20$, $f_l=0.18$, $f_u=0.05$

S-F is
not optimal
not greedy!



better!
greedy!



Optimal Prefix Codes: Huffman Encoding

Observation 1. ***Lowest frequency items*** should be at the ***lowest level*** in tree of optimal prefix code.

Observation 2. For $n > 1$, the lowest level always contains **at least *two leaves*** (optimal trees are **full**!).

Observation 3. **The order** in which items appear in a level ***does not*** matter.

Claim 1. There is an optimal prefix code with tree T^* where the two lowest-frequency letters are assigned to leaves that are brothers in T^*.

Huffman Code

Greedy template. [Huffman, 1952]

Create tree bottom-up.

a) Make ***two leaves*** for ***two lowest-frequency*** letters **y** and ***z***.

b) Recursively build tree for the rest using a **meta-letter** for ***yz***.

Optimal Prefix Codes: Huffman Encoding

```
Huffman(S) {
   if |S|=2 {
      return tree with root and 2 leaves
   } else {
      let y and z be lowest-frequency letters in S
      S' = S
      remove y and z from S'
      insert new letter ω in S' with fω=fy+fz
      T' = Huffman(S')
      T = add two children y and z to leaf ω from T'
      return T
   }
}
```

Q. What is the ***time complexity***?

Optimal Prefix Codes: Huffman Encoding

```
Huffman(S) {
   if |S|=2 {
      return tree with root and 2 leaves
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      T = add two children y and z to leaf ω from T'
      return T
   }
}
```

Q. What is the time complexity?

A. $T(n) = T(n-1) + O(n)$ ---> $O(n^2)$

Q. How to implement finding ***lowest-frequency letters*** efficiently?

A. Use ***priority queue*** for ***S***: $T(n) = T(n-1) + O(\log n)$ --> $O(n \log n)$

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.
Pf. by induction, based on optimality of T' (y and z removed, ω added) (see next page)

Claim. ABL(T')=ABL(T)-f_ω
Pf.

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.

Proof. by induction, based on optimality of T' (y and z removed, ω added)

(see next page)

Claim. **ABL(T') = ABL(T) - f_ω**

Proof.

$$\begin{aligned}
\text{ABL}(T) &= \sum_{x\in S} f_x \cdot \text{depth}_T(x) \\
&= f_y \cdot \text{depth}_T(y) + f_z \cdot \text{depth}_T(z) + \sum_{x\in S, x\neq y,z} f_x \cdot \text{depth}_T(x) \\
&= \left(f_y + f_z\right) \cdot \left(1 + \text{depth}_T(\omega)\right) + \sum_{x\in S, x\neq y,z} f_x \cdot \text{depth}_T(x) \\
&= f_\omega \cdot \left(1 + \text{depth}_T(\omega)\right) + \sum_{x\in S, x\neq y,z} f_x \cdot \text{depth}_T(x) \\
&= f_\omega + \sum_{x\in S'} f_x \cdot \text{depth}_{T'}(x) \\
&= f_\omega + \text{ABL}(T')
\end{aligned}$$

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.

Prooff. (by induction over n=|S|)

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.

Pf. (by induction over n=|S|)

Base: For ***n=2*** there is no ***shorter*** code than root and two leaves.

Hypothesis: Suppose Huffman tree ***T'*** for ***S'*** of size ***n-1*** with ω instead of ***y*** and ***z*** is optimal.

Step: (by contradiction)

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.

Pf. (by induction)

Base: For n=2 there is no shorter code than root and two leaves.

Hypothesis: Suppose Huffman tree T' for S' of size n-1 with ω instead of y and z is optimal. (IH)

Step: (by contradiction)

- *Idea of proof:*
  - Suppose other tree Z of size n is better.
  - Delete lowest frequency items y and z from Z creating Z'
  - Z' cannot be better than T' by IH.

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.

Pf. (by induction)

Base: For n=2 there is no shorter code than root and two leaves.

Hypothesis: Suppose Huffman tree ***T'*** for ***S'*** with ω instead of ***y*** and ***z*** is **optimal**. (Inductive Hyp.)

Step: (by contradiction)

- Suppose Huffman tree ***T*** for ***S*** is not optimal.
- So there is some tree ***Z*** such that **ABL(Z) < ABL(T)**.
- Then there is also a tree ***Z*** for which leaves ***y*** and ***z*** exist that are **brothers** and have the **lowest frequency** (see Claim 1).
- Let ***Z'*** be ***Z*** with ***y*** and ***z*** deleted, and their **former parent** labeled ω.
- Similar ***T'*** is derived from ***S'*** in our algorithm.
- We know that **ABL(Z')=ABL(Z)-f_ω**, as well as **ABL(T')=ABL(T)-f_ω**.
- But also ***ABL(Z) < ABL(T)*** **-->** **ABL(Z') < ABL(T')**.
- Contradiction with IH.

Steps of the Proof

Step: (by contradiction)

- Suppose Huffman tree T for S is not optimal.
- So there is some tree Z such that ABL(Z) < ABL(T).

- Then there is also a tree Z for which leaves ***y*** and ***z*** exist that are brothers and have the lowest frequency (see Obs. 1-2: fullness!).

- Let Z' be Z with y and z deleted, and their former parent labeled ω.
- Similar T' is derived from S' in our algorithm.

- We know that ABL(Z')=ABL(Z)-f_ω, as well as ABL(T')=ABL(T)-f_ω.

- But also (Absurd Hyp) ABL(Z) < ABL(T), so ABL(Z') < ABL(T').
- Contradiction with IND IH.