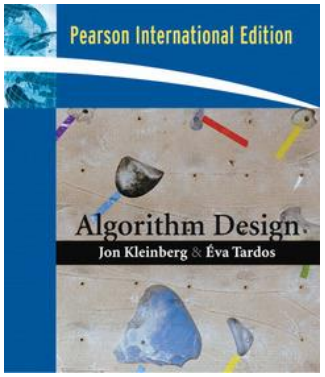
# Hash tables

## A randomized implementation of dictionaries

reference
(Chapter 13.6)

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Design and Analysis of Algorithms
(MIT opencourseware)
Lecture 8

https://ocw.mit.edu/courses/6-046j-design-and-analysis-of-algorithms-spring-2015/resources/lecture-8-randomization-universal-perfect-hashing/

## The dictionary problem:

Given a universe U of possible elements, maintain a subset $S \subseteq U$ subject to the following operations:

- make-dictionary(): Initialize an empty dictionary.
- insert(u): Add element $u \in U$ to S.
- delete(u): Delete u from S, if u is currently in S.
- look-up(u): Determine whether u is in S.

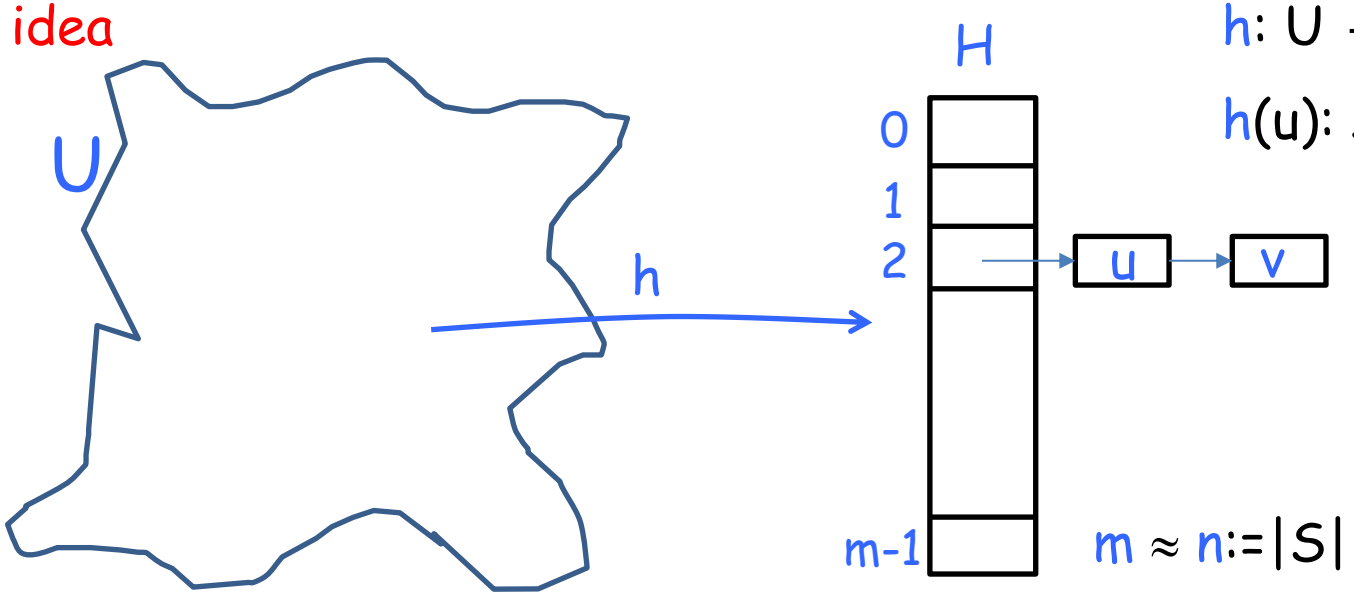
Challenge: Universe U can be extremely large so defining an array of size |U| is infeasible.

A solution: balanced (e.g. AVL) trees
- $O(|S|)$ space
- $O(\log |S|)$ time per operation

hash tables:

- $O(|S|)$ space
- $O(1)$ expected amortized time per operation

idea




h: U ⟶ {0,1,…,m-1}

h(u): slot where u is put

m ≈ n:=|S|

collision: when h(u) = h(v) but u ≠ v.

H[i]: linked list of all elements that h maps to slot i
(hashing with chaining)

Insert/Delete/Lookup of u:

- compute h(u)
- insert/delete/search u by scanning list H[h(u)]

goal: find a function h that "spreads out" elements

# choosing a good hash function

obs: for any deterministic hash function one can find a set S where all elements of S are mapped to the same slot

➡ $\Theta(n)$ time per operation

idea: use randomization

obvious approach: for each u, choose h(u) uniformly at random

look-up(u): ...where did we put u?

we have to maintain the set of pairs $\{(u,h(u)): u \in S\}$

that's the dictionary problem!

maybe I can use a hash table

# universal hashing

A family $\mathcal{H}$ of hash functions is universal if

for each distinct $u,v \in U$ $\quad \Pr_{h \in \mathcal{H}}(h(u)=h(v)) \leq 1/m$

**Theorem**

Let $\mathcal{H}$ be a family of universal hash functions. Let $S \subseteq U$ of n elements. Let $u \in S$. Pick a random function h from $\mathcal{H}$, and let X be the random variable counting the number of elements of S mapped to h(u). Then $E[X] \leq 1+n/m$

**proof**

for each $s \in S$ $\quad X_s$ r. v. $= \begin{cases} 1 & \text{if } h(s)=h(u) \\ 0 & \text{otherwise} \end{cases}$ $\qquad X = \sum_{s \in S} X_s$

$$E[X] = E\left[\sum_{s \in S} X_s\right] = \sum_{s \in S} E[X_s] = \sum_{s \in S} Pr(h(s)=h(u))$$

$$= 1 + \sum_{s \in S \setminus \{u\}} Pr(h(s)=h(u)) \leq 1 + n/m$$

notice: $m=\Theta(n)$  expected $O(1)$ time per operation

# designing a universal family of hash functions

always exists [Chebyshev 1850]

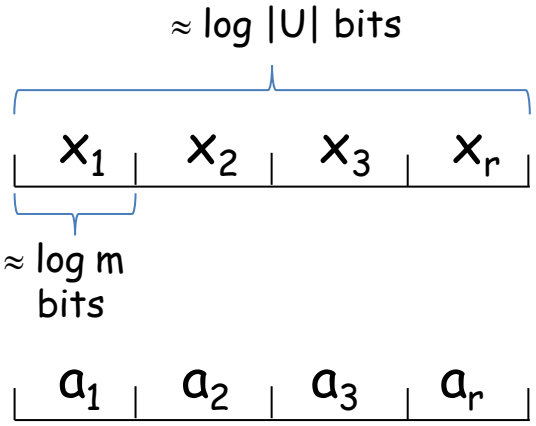
**Table size:** choose m as a prime number such that $n \le m \le 2n$

**Integer encoding:** Identify each element $x \in U$ with a base-m integer of r digits: $x = (x_1, x_2, \ldots, x_r)$, $x_i \in \{0,1,\ldots,m-1\}$.

**Hash function:**

given $a \in U$, $a = (a_1, a_2, \ldots, a_r)$

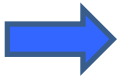
$$h_a(x) = \left( \sum_{i=1}^{r} a_i x_i \right) \bmod m$$




**hash function family:** $\mathcal{H} = \{h_a : a \in U\}$

**word RAM model:**

- manipulating $O(1)$ machine words takes $O(1)$ time
- every object of interest fits in a machine word



- storing $h_a$ requires just storing a single value, a (1 machine word)
- computing $h_a(x)$ takes $O(1)$ time

## Theorem

$\mathcal{H} = \{h_a : a \in U\}$ is universal

## proof

Let $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ be two distinct elements of U. We need to show that $Pr[h_a(x) = h_a(y)] \leq 1/m$.

since $x \neq y$, there exists an integer j such that $x_j \neq y_j$.

we have $h_a(x) = h_a(y)$ iff

$$a_j \underbrace{(y_j - x_j)}_{z} = \underbrace{\sum_{i \neq j} a_i (x_i - y_i)}_{\alpha} \bmod m$$

we can assume a was chosen uniformly at random by first selecting all coordinates $a_i$ where $i \neq j$, then selecting $a_j$ at random. Thus, we can assume $a_i$ is fixed for all coordinates $i \neq j$.

since m is prime & $z \neq 0$, z has a multiplicative inverse $z^{-1}$, i.e. $z\, z^{-1} = 1 \bmod m$

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since m is prime & $z \neq 0$, z has a multiplicative inverse $z^{-1}$, i.e. $z\, z^{-1} = 1 \mod m$

$\Rightarrow$ $\Pr[h_a(x) = h_a(y)] \leq 1/m$.

# another universal hash family

choose a prime $p \geq |U|$ (once)

Hash function:

given $a,b \in U$,

$$h_{ab}(x) = [(ax+b) \bmod p\ ] \bmod m$$

hash function family: $\mathcal{H} = \{h_{ab} : a,b \in U\}$

# how to (dynamically) choose the table size

notice: S changes over time and we want to use $O(|S|)$ space

parameters:

- n: # of elements currently in the table, i.e. $n=|S|$;
- N: virtual size of the table
- m: actual size of the table (a prime number between N and 2N)

doubling/halving technique:

- init $n=N=1$;
- whenever $n>N$:
  - $N:=2N$
  - choose a new m
  - re-hash all items (in $O(n)$ time)
- whenever $n<N/4$:
  - $N:=N/2$
  - choose a new m
  - re-hash all items (in $O(n)$ time)

$O(1)$ amortized time per insertion/deletion

# What's next?

randomization: an amazing algorithmic tool

- hash functions/tables
- sampling
- sketches
- ...

it allows to solve efficiently many practical important problems:

counting distinct elements in a stream:

given a stream of $m$ element $x_1, x_2, \ldots x_m$ where each $x_i \in U$, return the number $d$ of distinct elements

simple solution: $O(d)$ space or $O(|U|)$ space

probabilistic counters: $O(\log d)$ space

fining similar items

Given $N$ items, find pairs of them whose similarity is above a give threshold

main challenge: $N$ is huge and a $\Theta(N^2)$-time solution is infeasible

LSH technique: solves the problem in $O(N \text{ polylog } N)$ time and space

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if interested:

Algorithms for Big Data

(prof. Clementi)

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