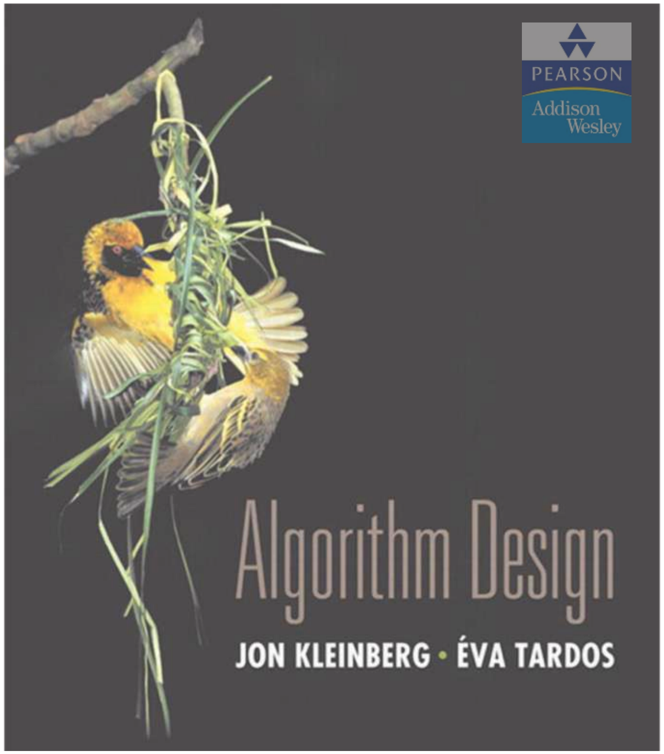


SECTION 11.2

11. APPROXIMATION ALGORITHMS

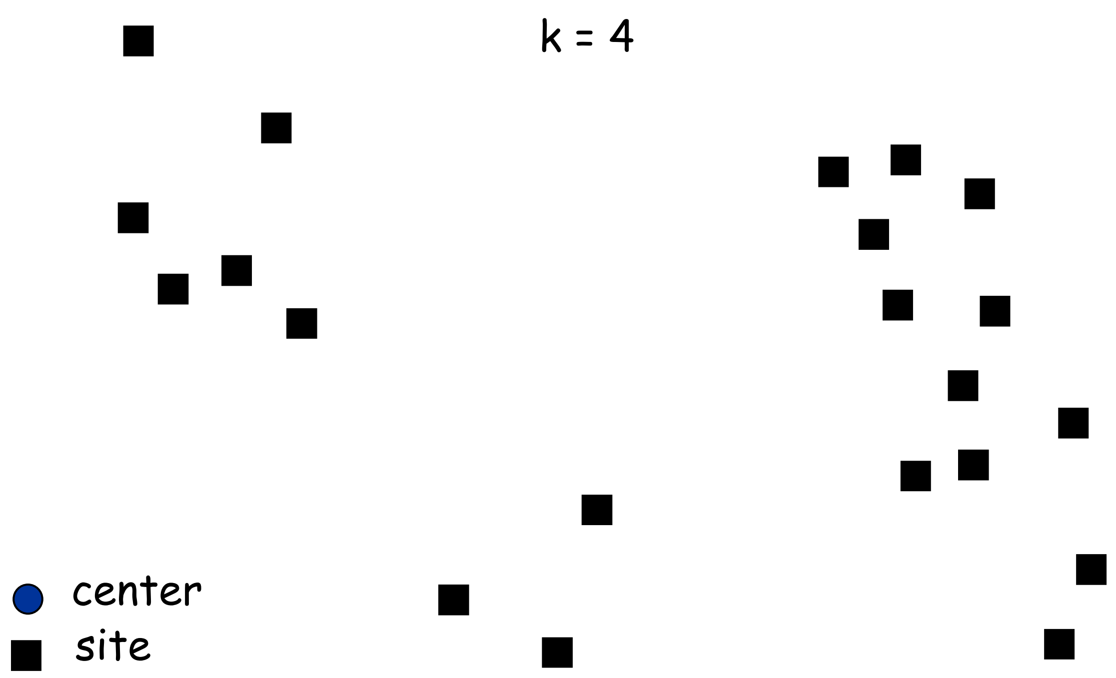
- *load balancing*
- *center selection*

k-Center Problem

k-Center Problem

Input. Set of n sites $s_1, \ldots, s_n$ and integer $k > 0$.

Center selection problem. Select k centers C so that maximum distance from a site to nearest center is minimized.



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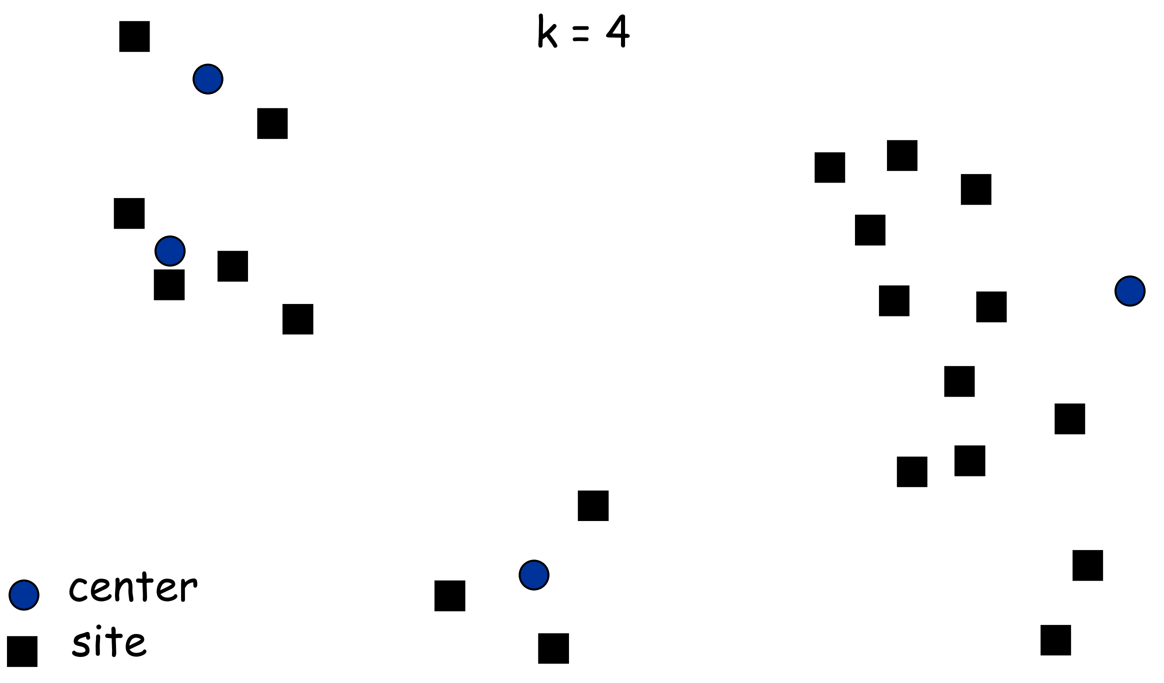

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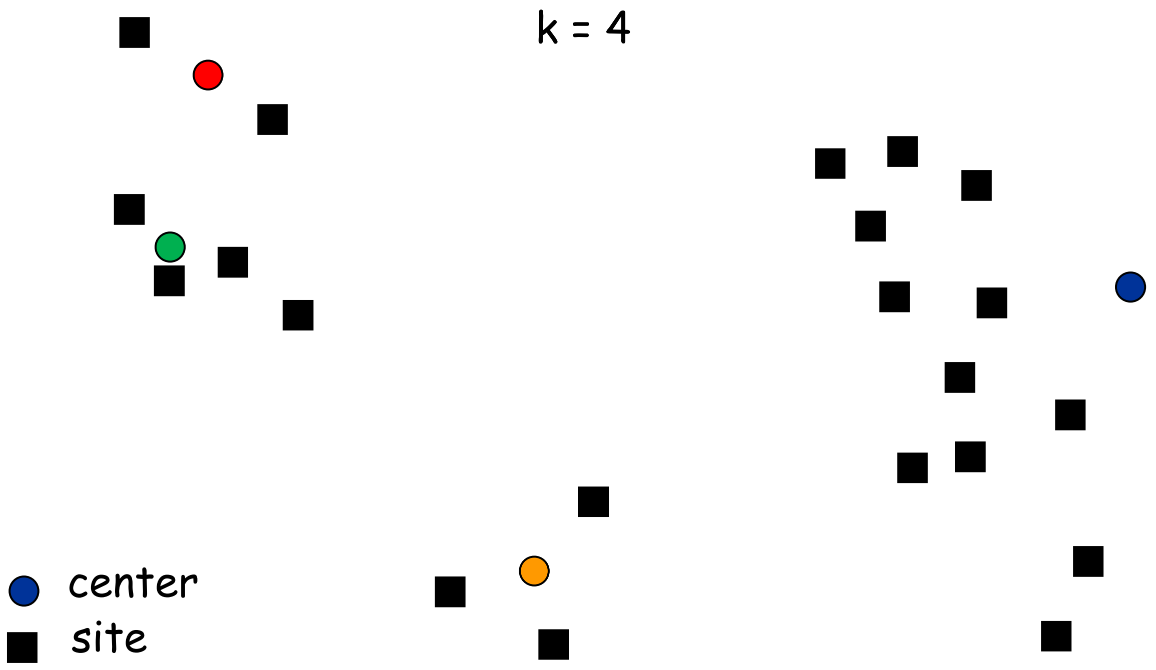

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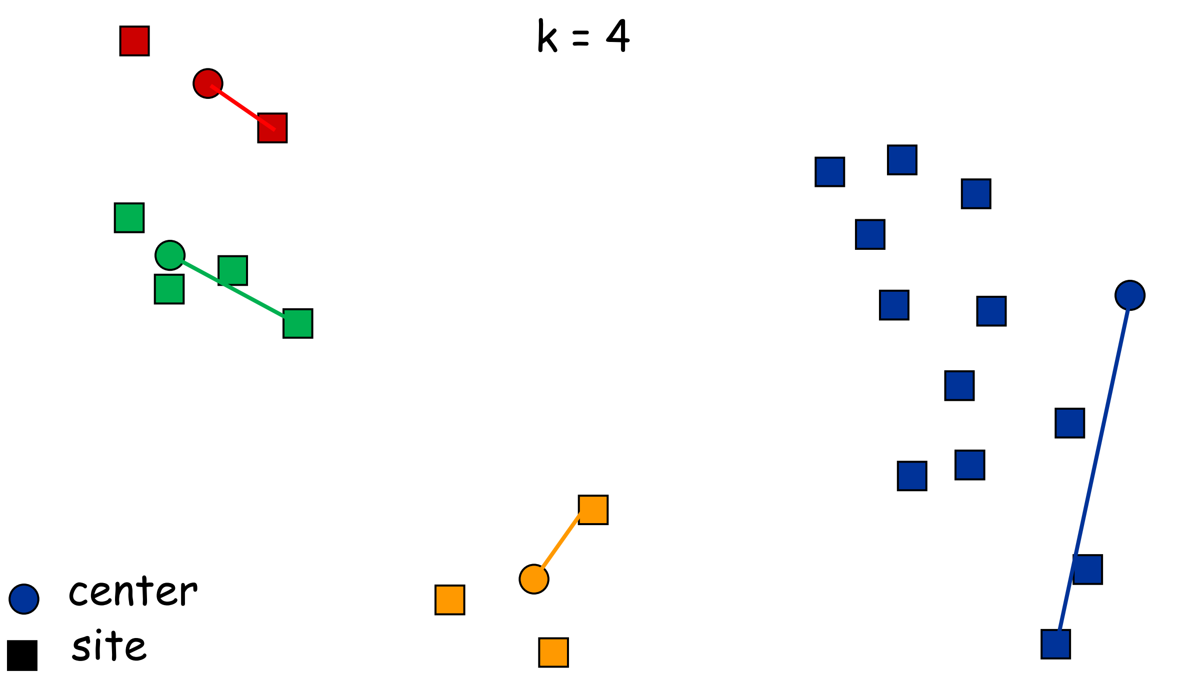

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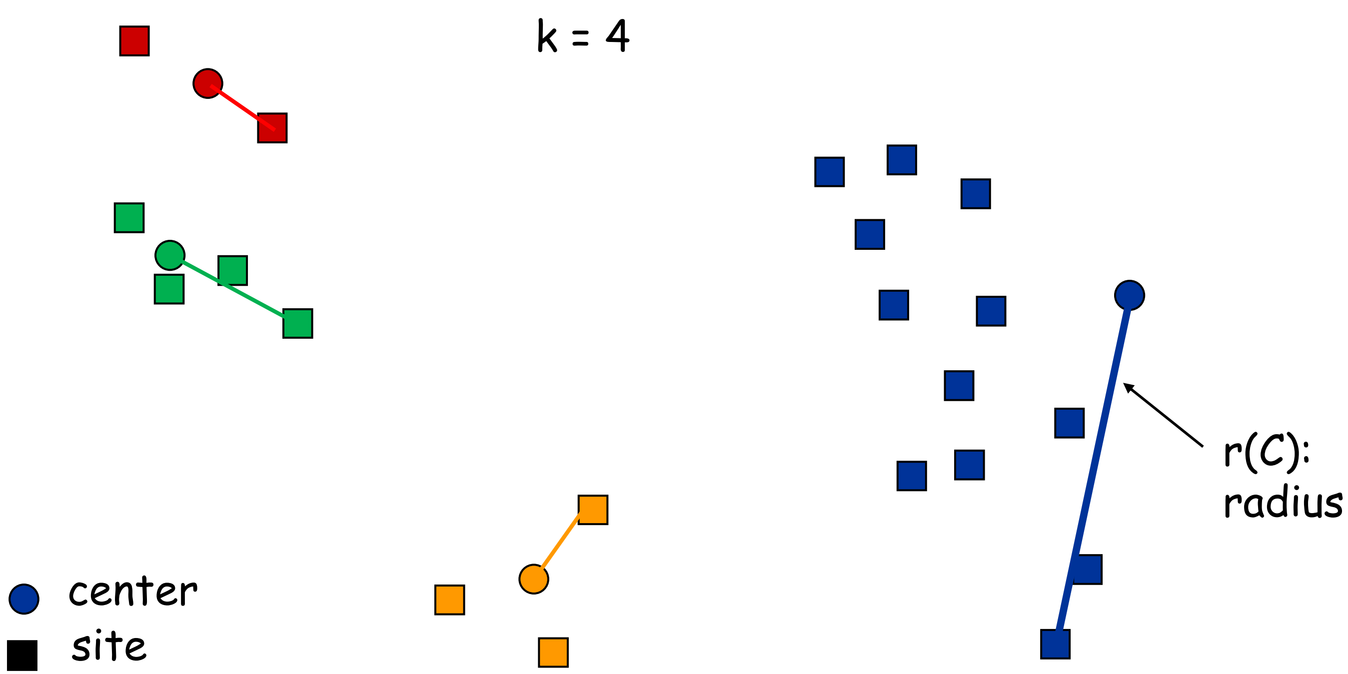

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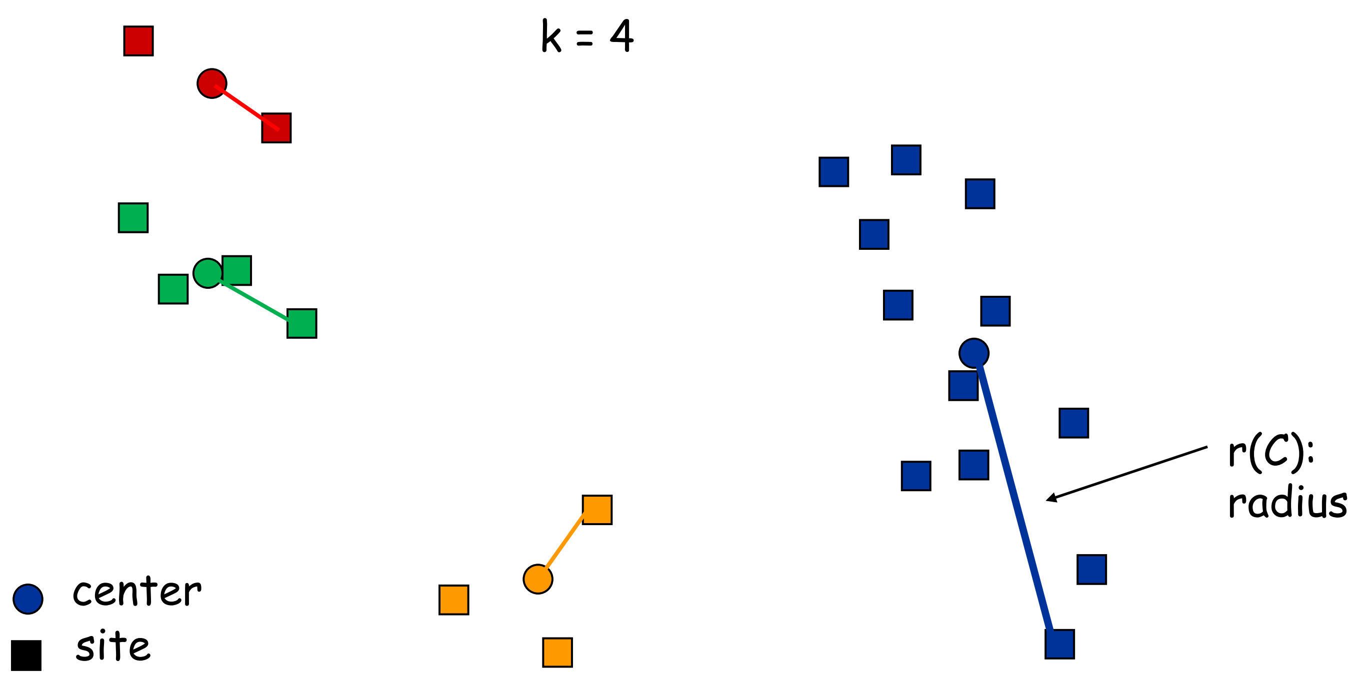

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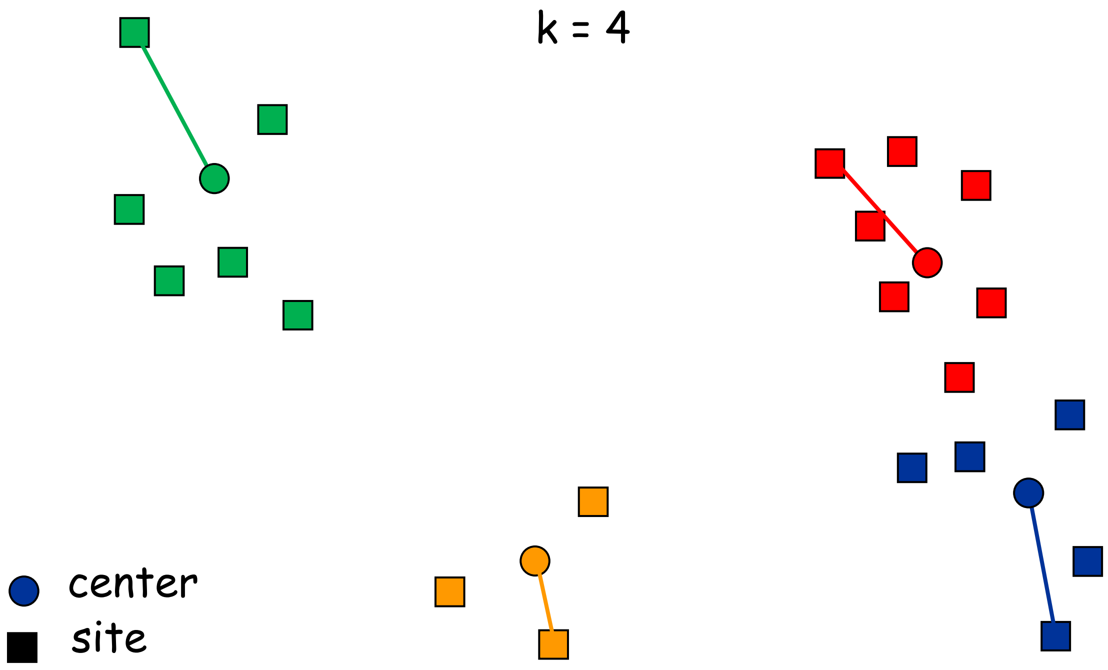

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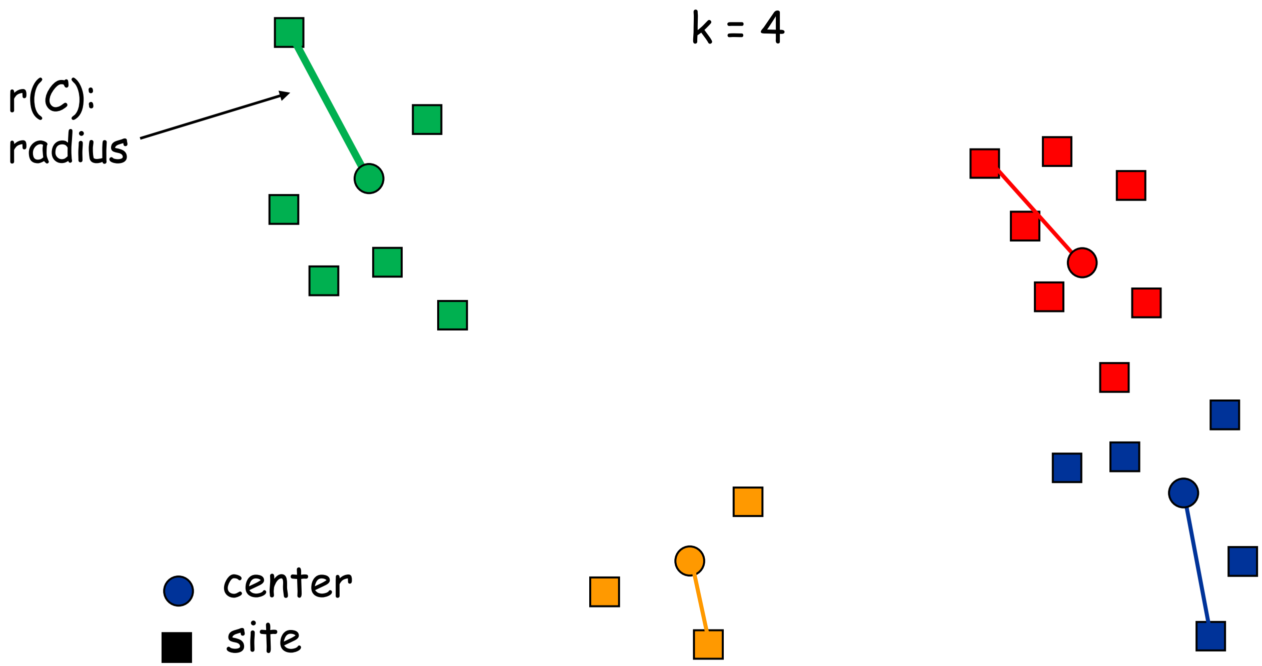

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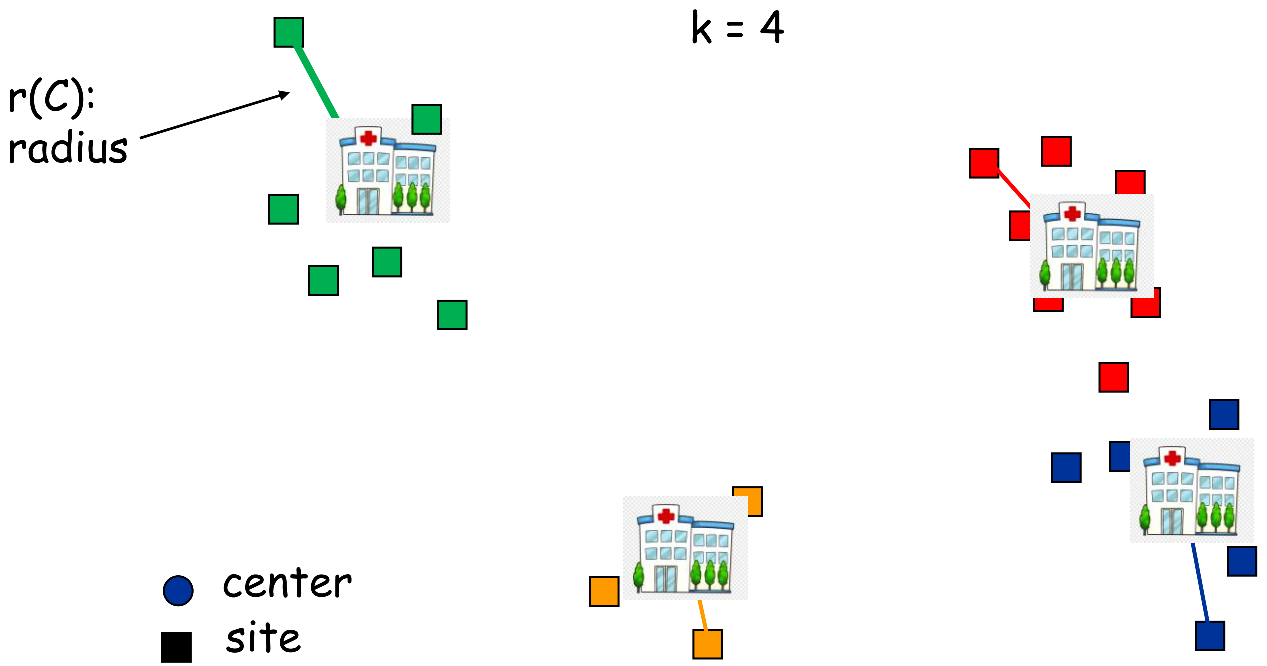

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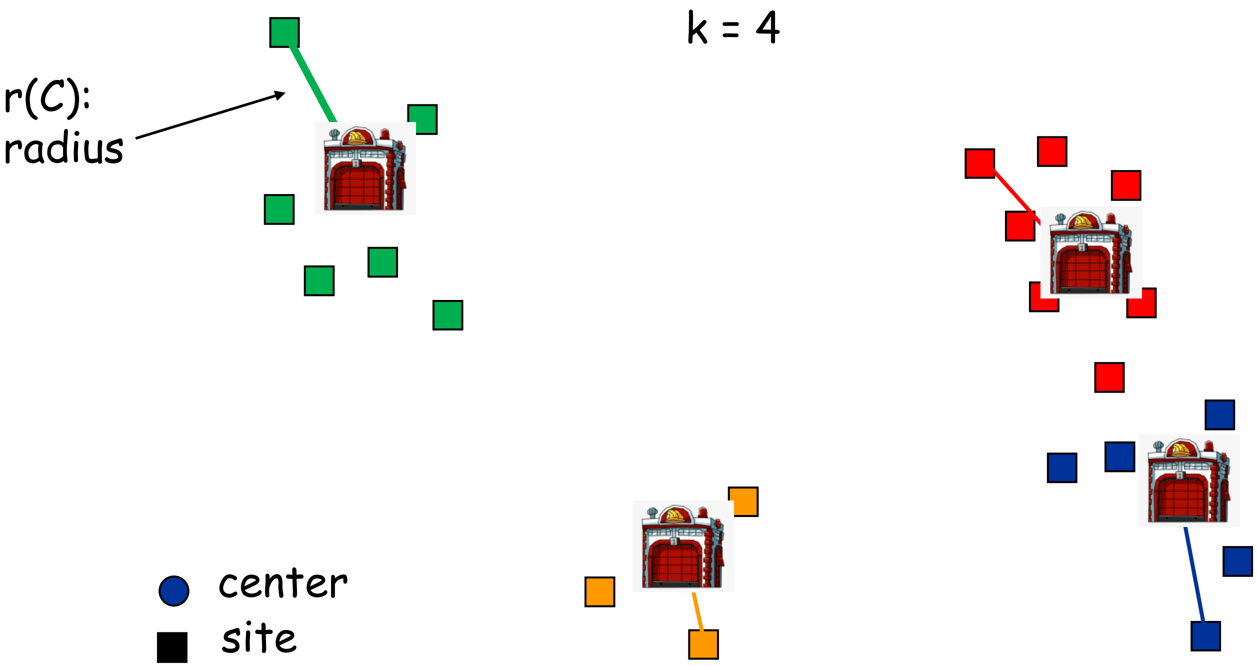


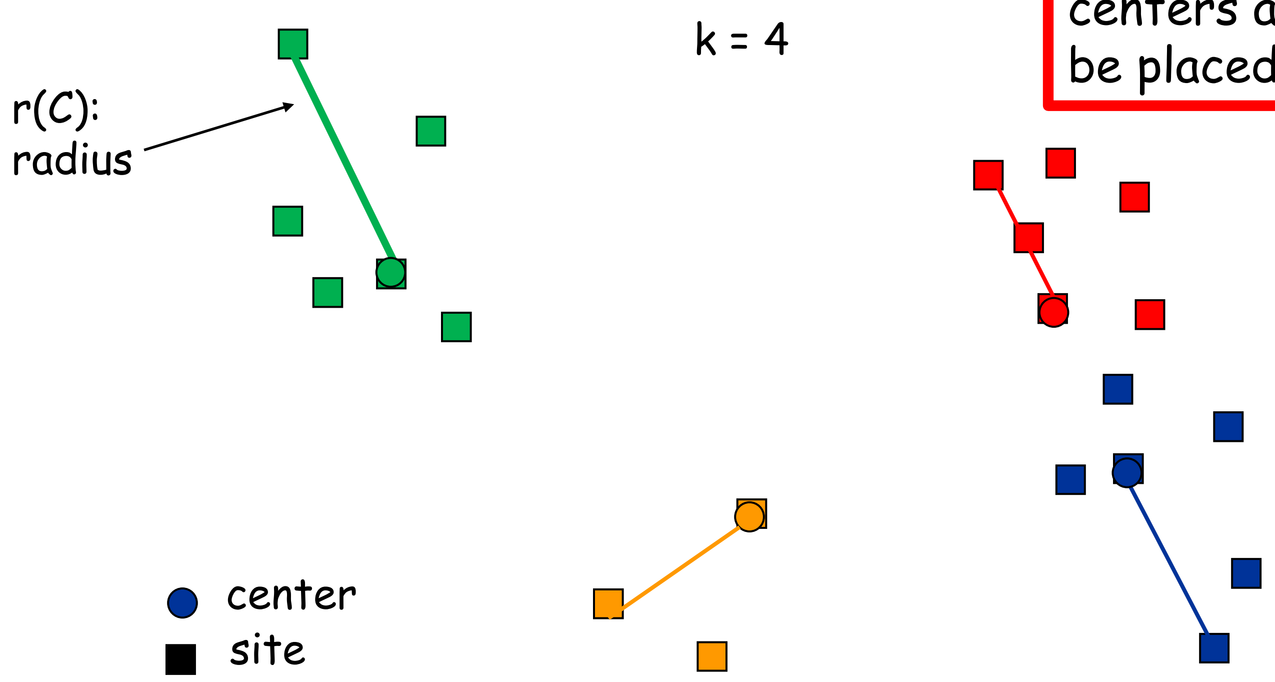
fire station

k-Center Problem

Input. Set of n sites $s_1, \ldots, s_n$ and integer $k > 0$.

Center selection problem. Select k centers C so that maximum distance from a site to nearest center is minimized.

A variant:
centers are constrained to be placed on site locations



k-Center Problem

Input. Set of n sites $s_1, \ldots, s_n$ and integer $k > 0$.

Center selection problem. Select k centers C so that maximum distance from a site to nearest center is minimized.

Notation.

- $dist(x, y)$ = distance between x and y.
- $dist(s_i, C) = \min_{c \in C} dist(s_i, c)$ = distance from s_i to closest center.
- $r(C) = \max_i dist(s_i, C)$ = smallest covering radius.

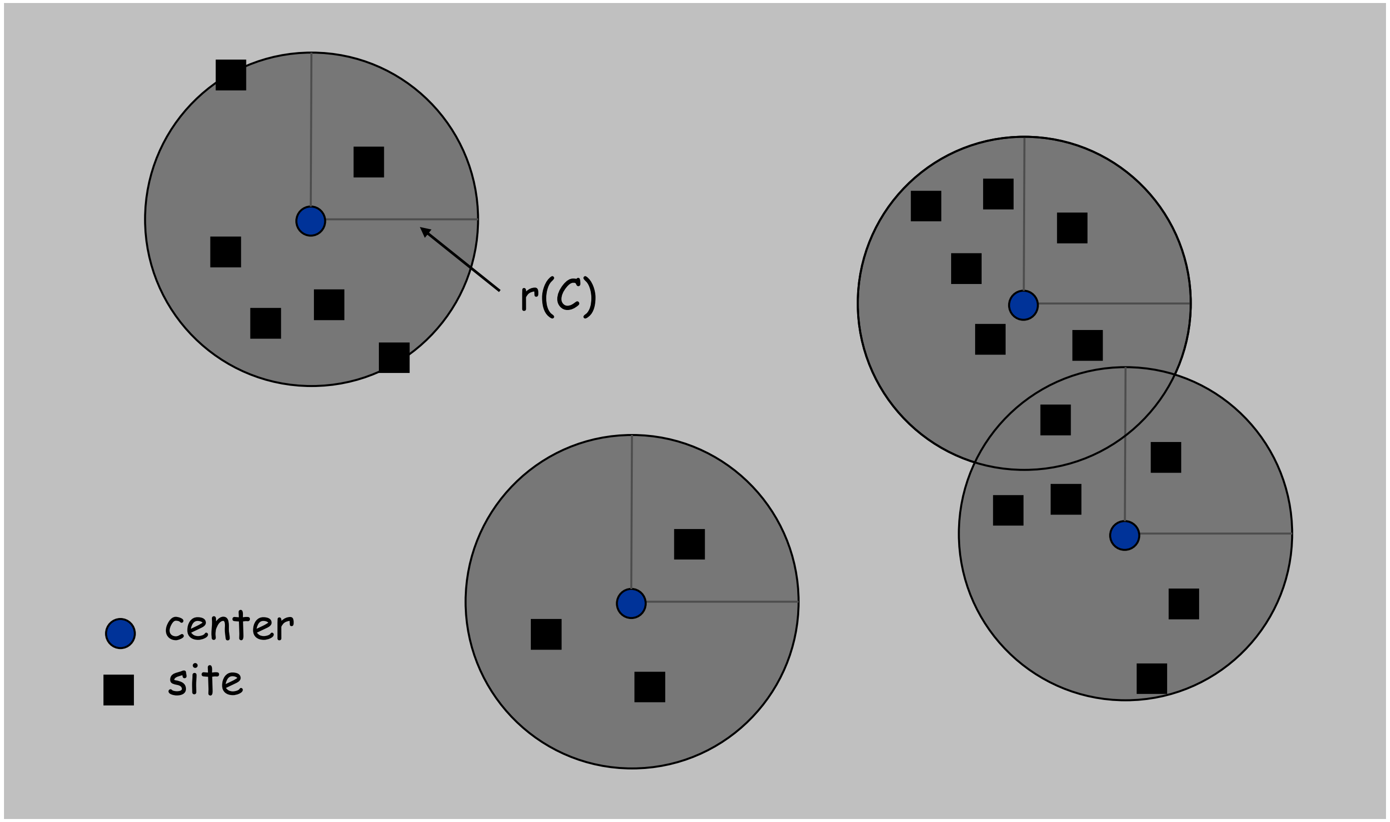
Goal. Find set of centers C that minimizes $r(C)$, subject to $|C| = k$.

Distance function properties.

- $dist(x, x) = 0$ (identity)
- $dist(x, y) = dist(y, x)$ (symmetry)
- $dist(x, y) \leq dist(x, z) + dist(z, y)$ (triangle inequality)

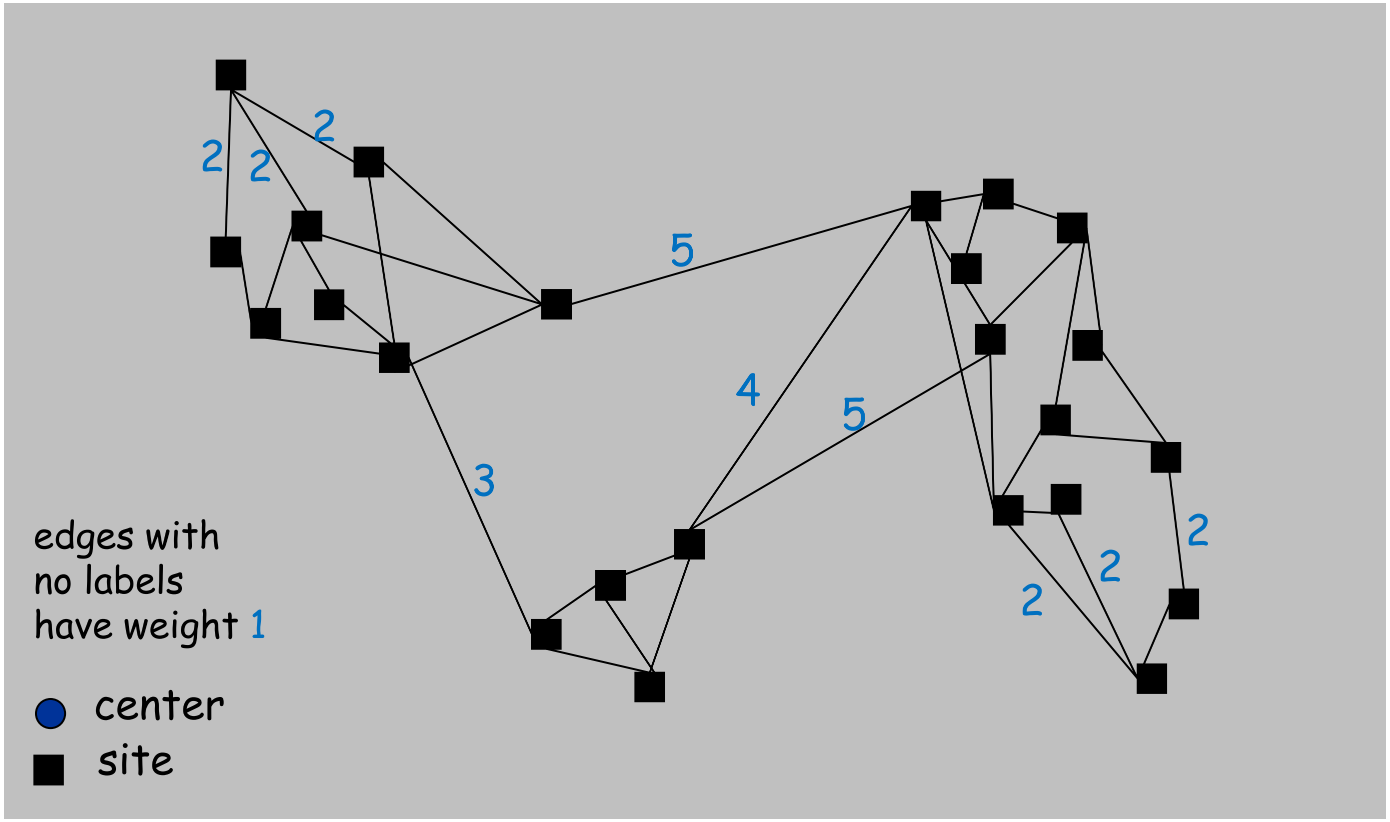
k-Center Problem

Ex: each site is a point in the plane, a center can be any point in the plane, dist(x, y) = Euclidean distance.



k-Center Problem

Ex: each site is a vertex in undirected weighted graph, a center can be any vertex, dist(x, y) = (weighted) distance in G between x and y.



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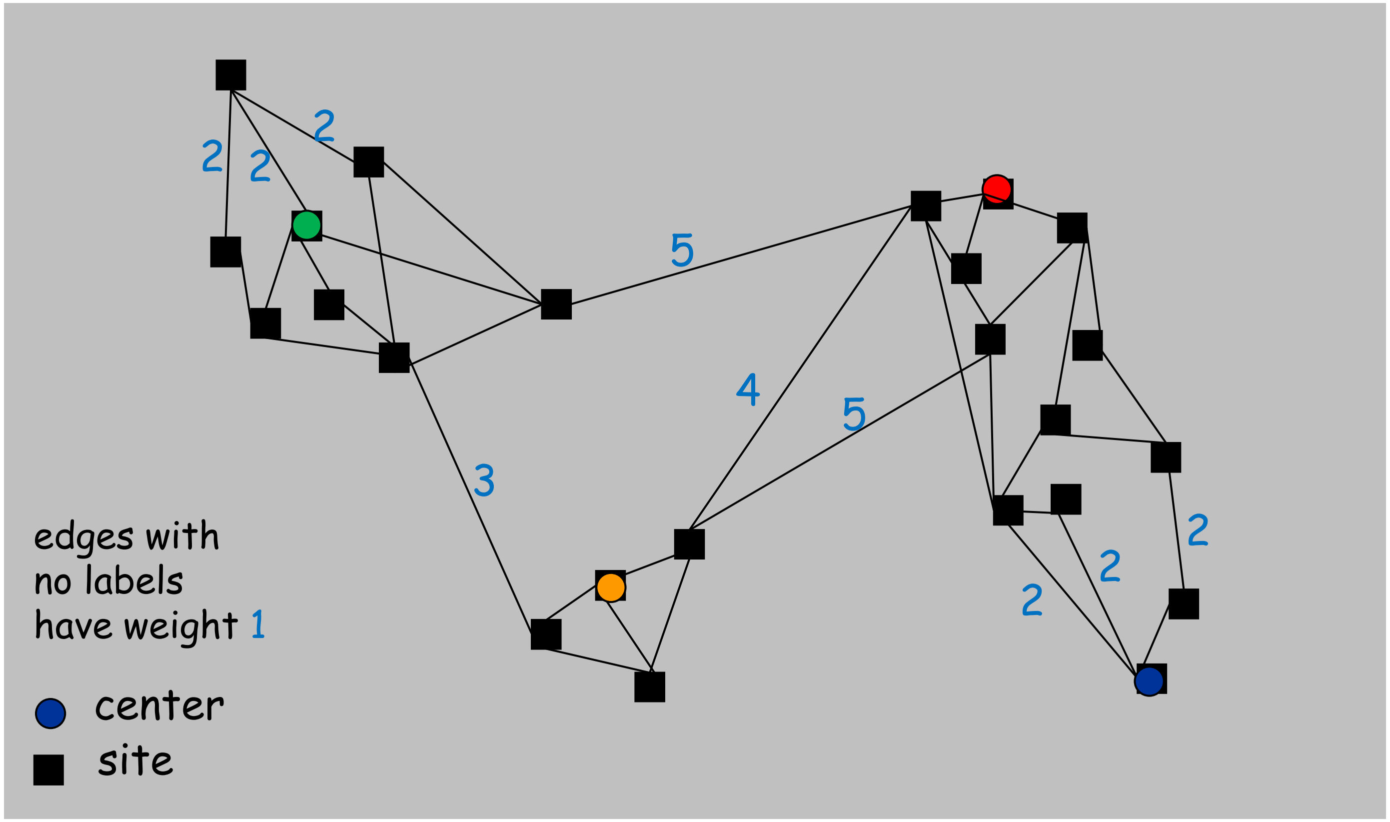

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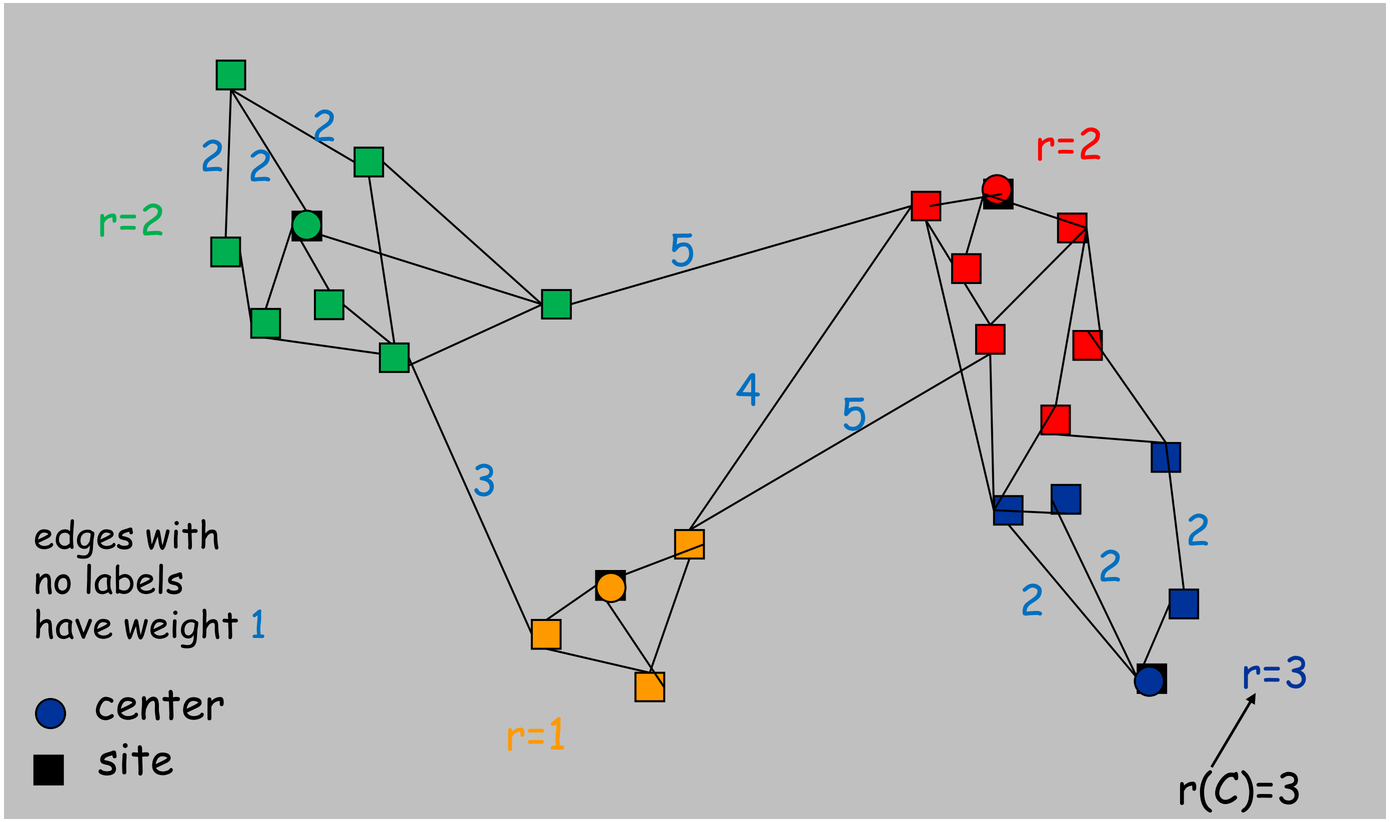


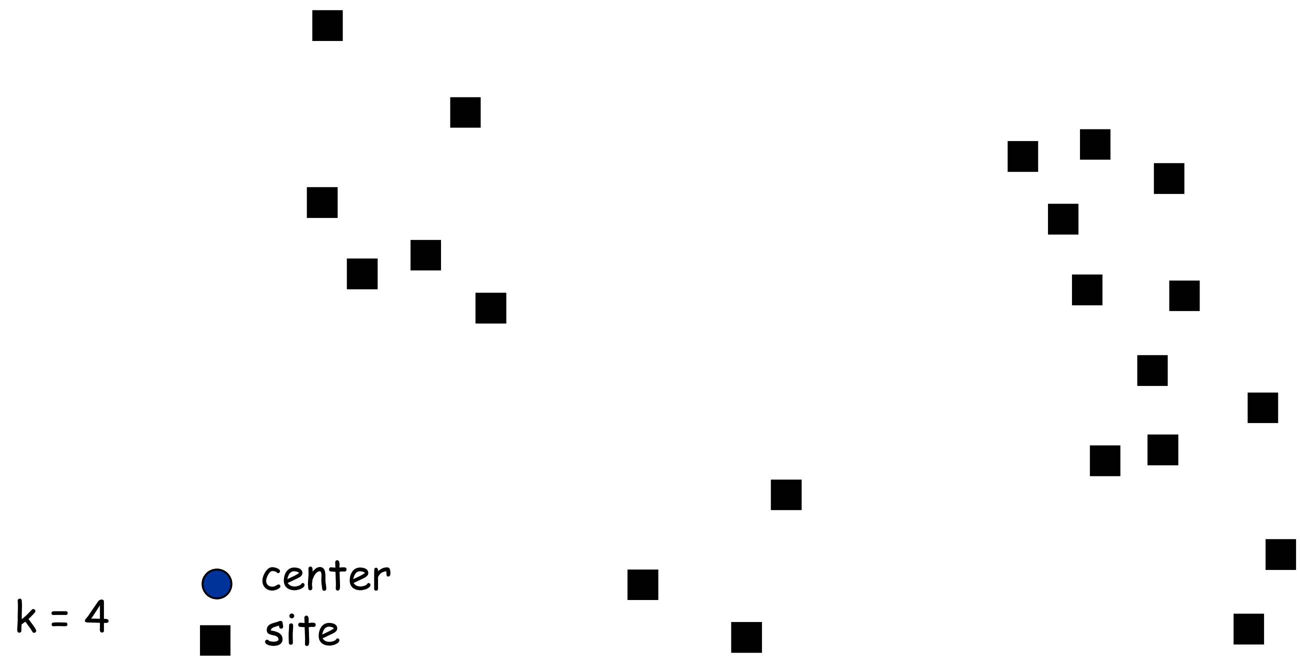
Center Selection: Greedy Algorithm

Greedy algorithm. Repeatedly choose the next center to be the site farthest from any existing center.

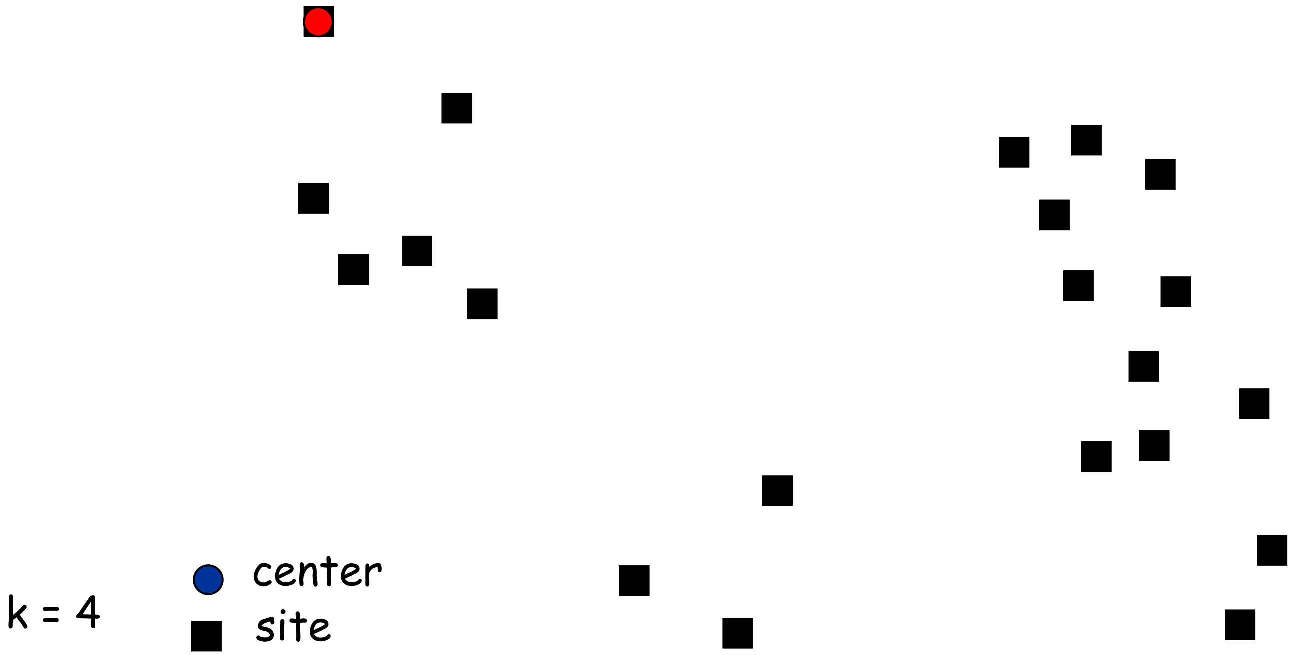
```
Greedy-Center-Selection(k, n, s1,s2,…,sn) {

   C = φ
   repeat k times {
      Select a site si with maximum dist(si, C)
      Add si to C
   }                          site farthest from any center
   return C
}
```

Observation. Upon termination all centers in C are pairwise at least $r(C)$ apart.
Pf. By construction of algorithm.

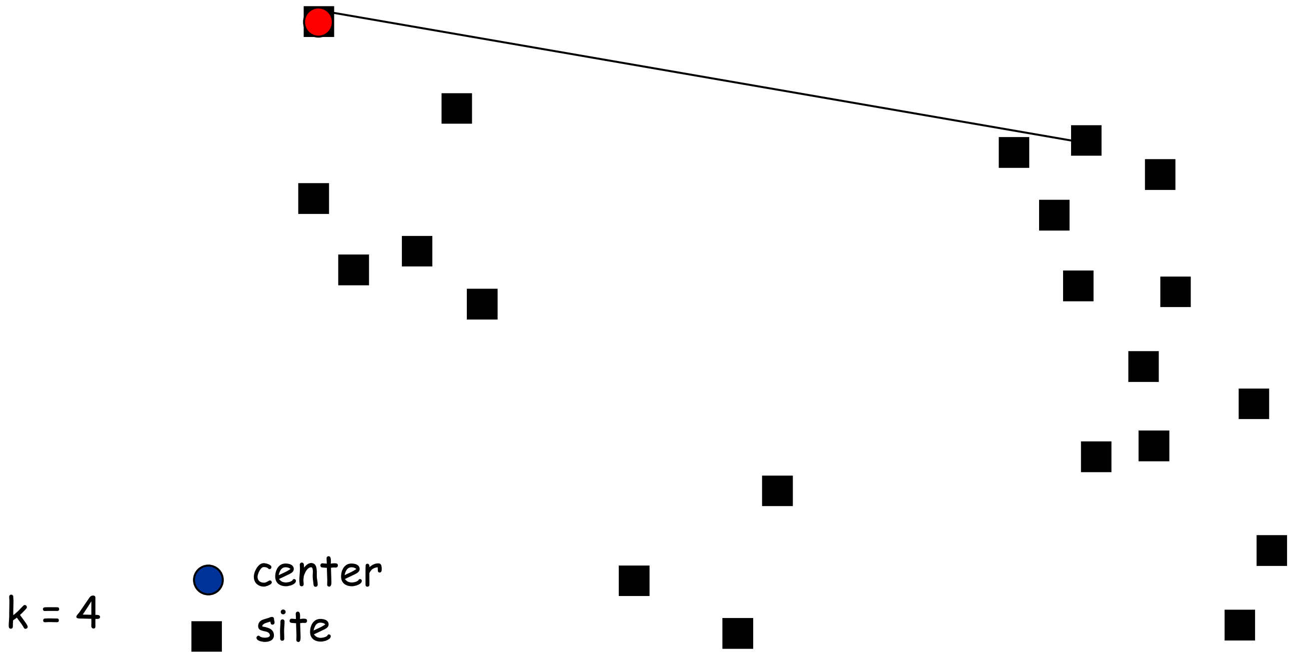
example of execution



example of execution



example of execution



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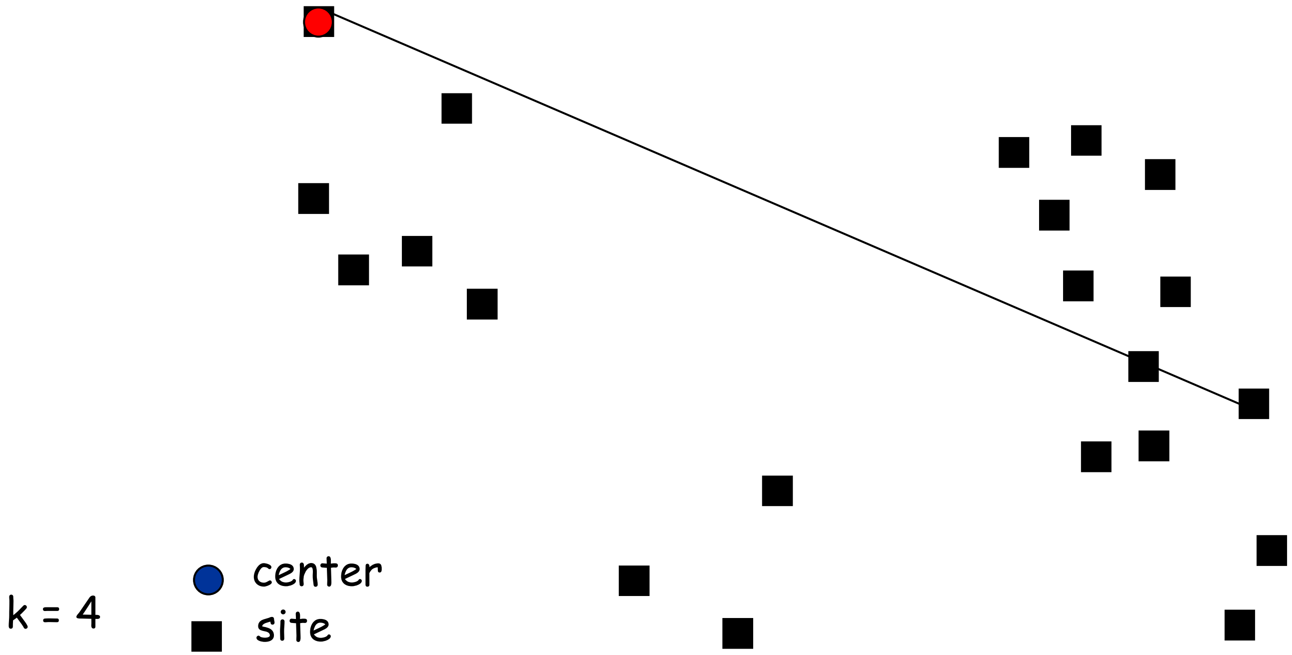

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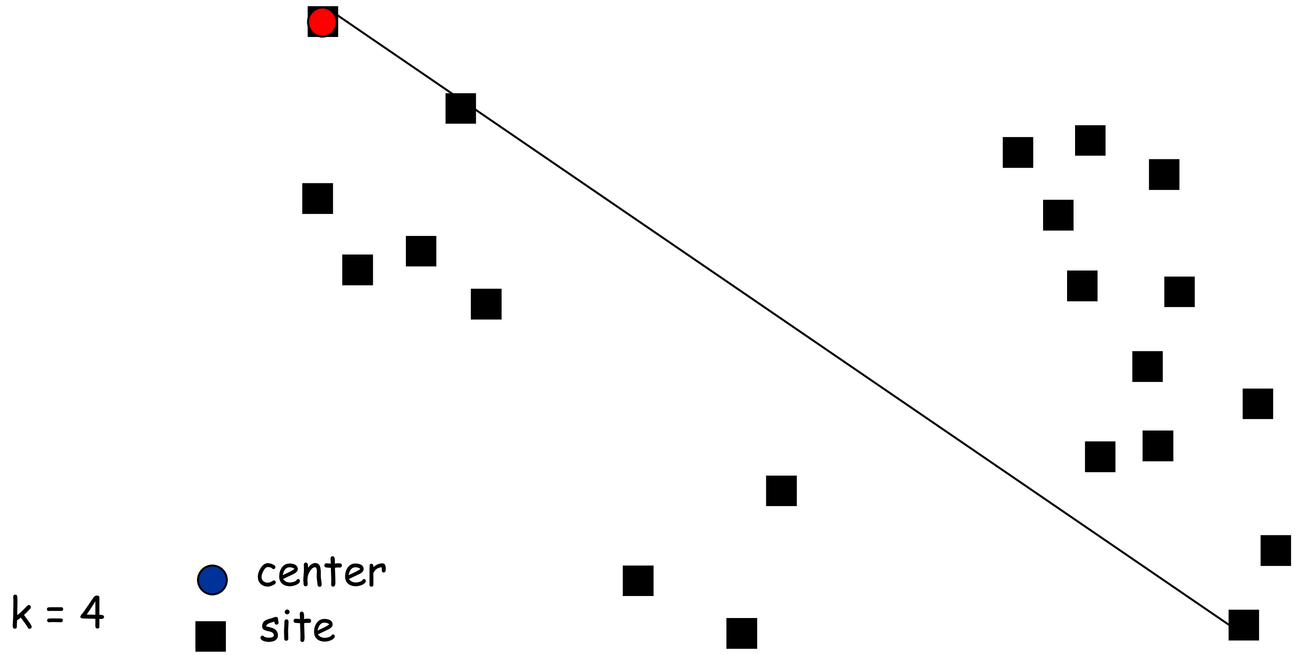

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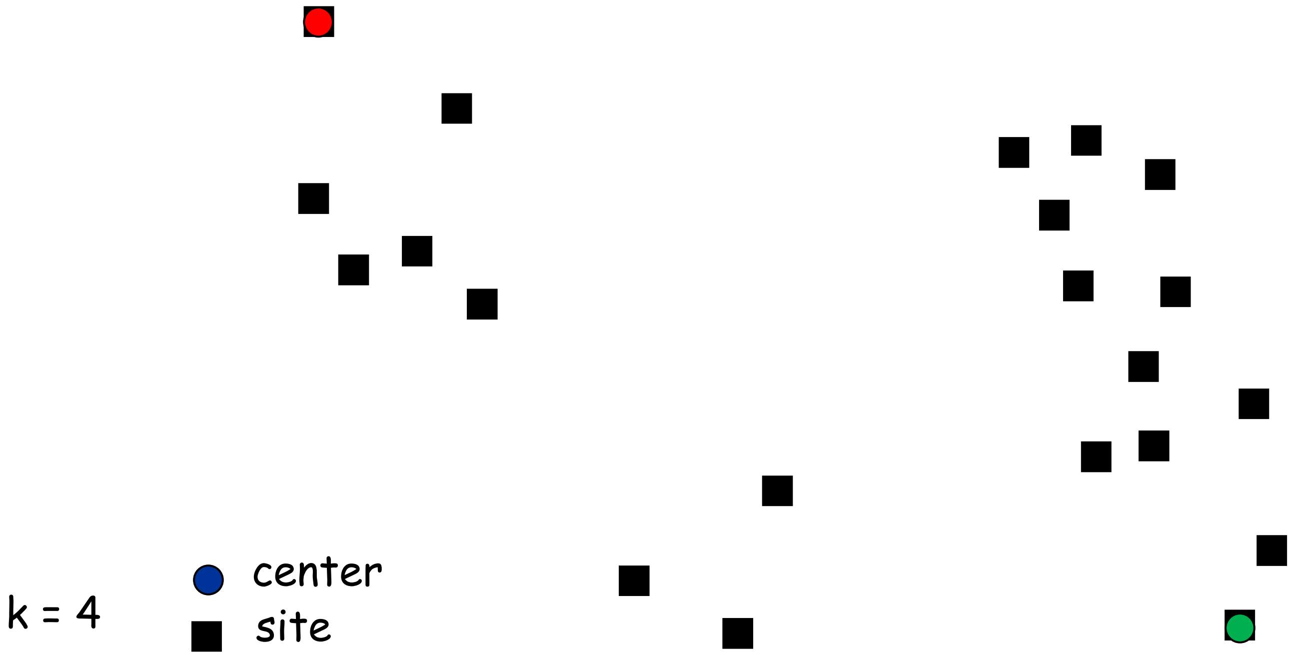

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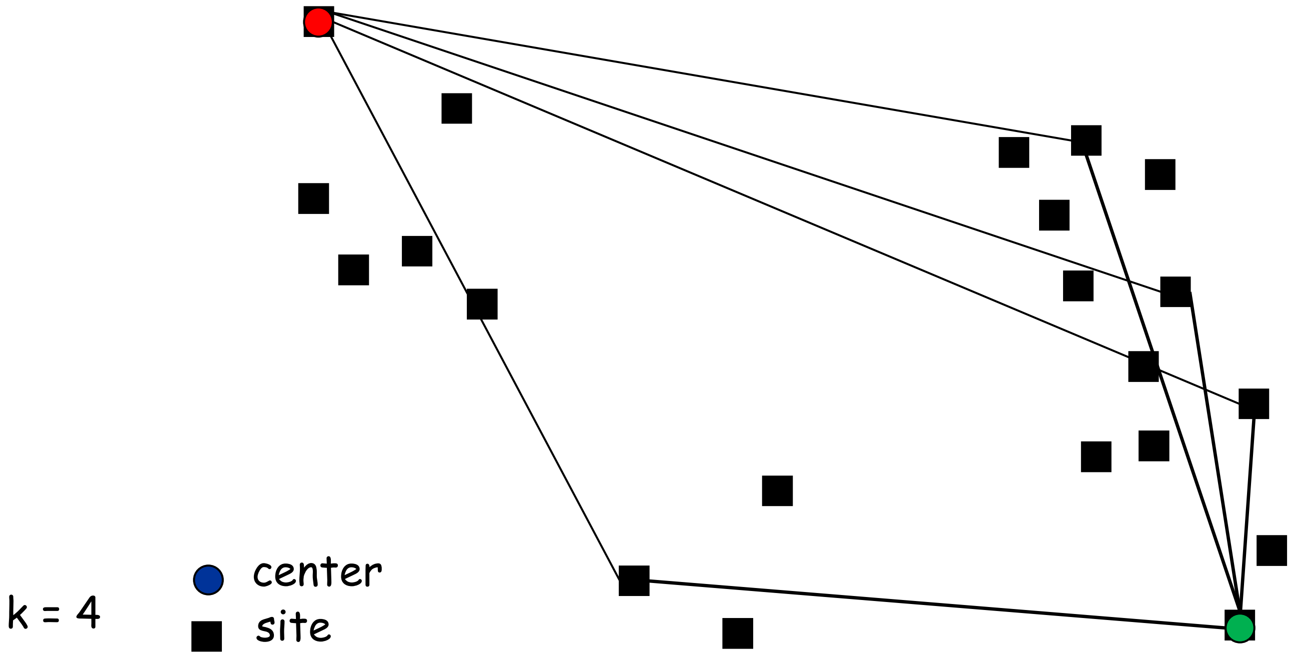

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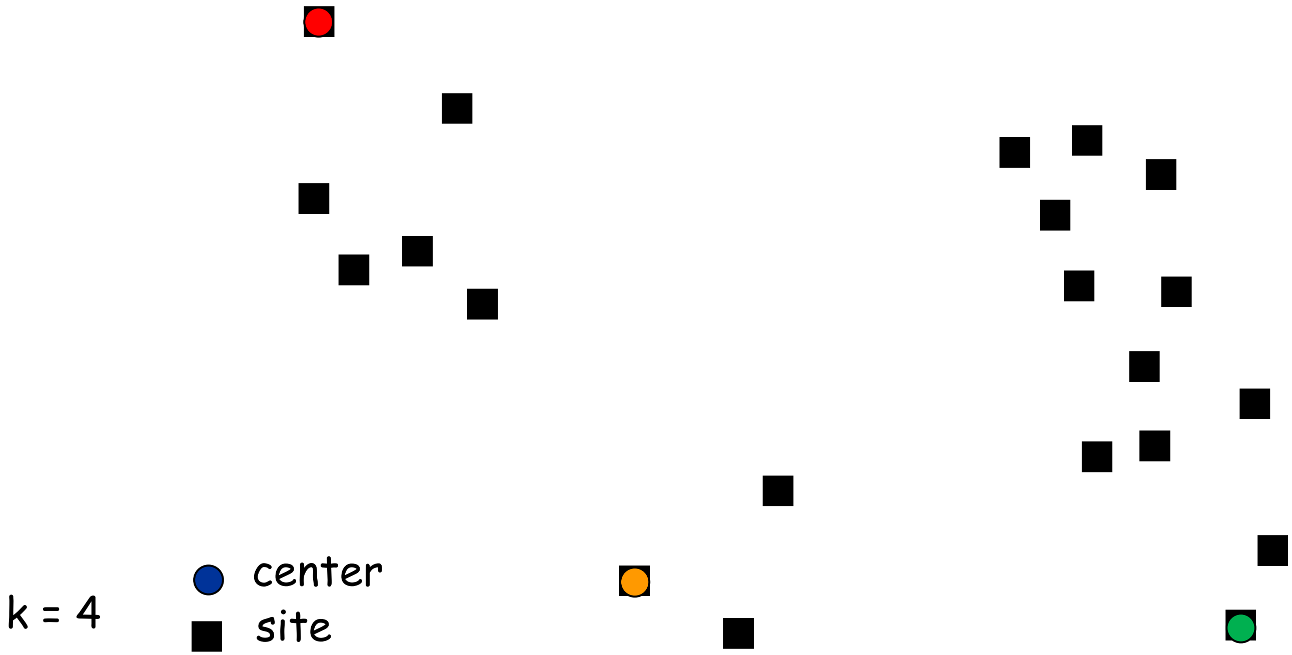

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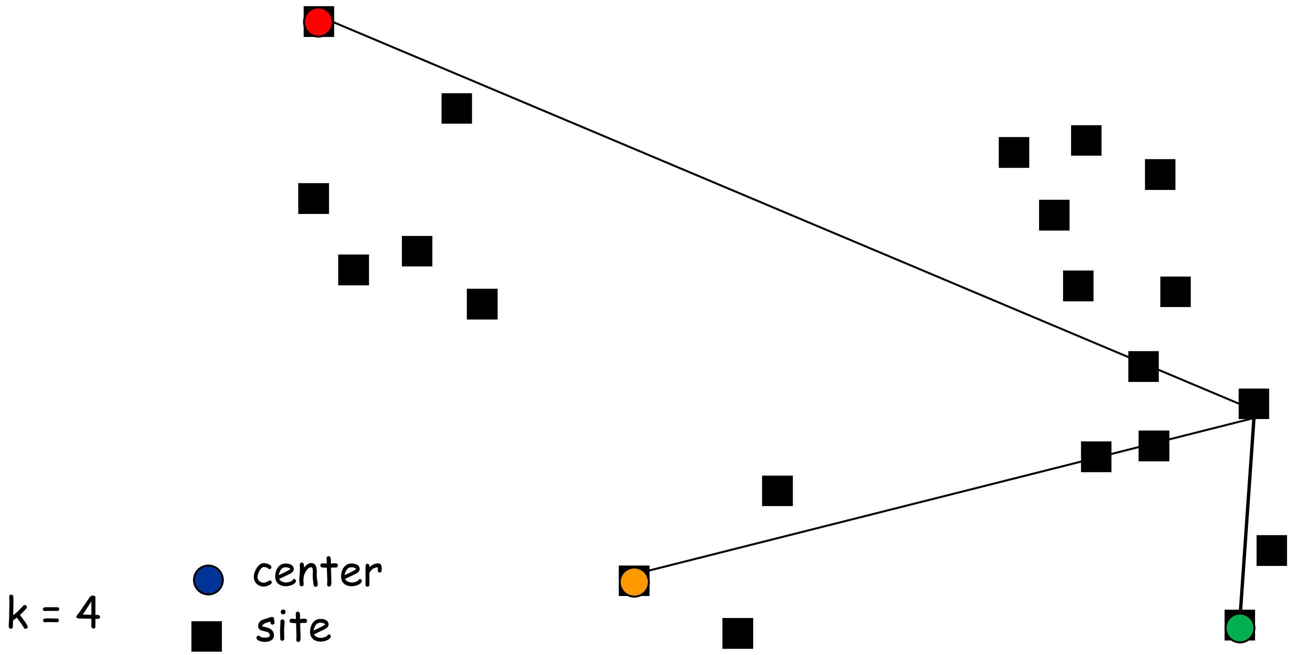
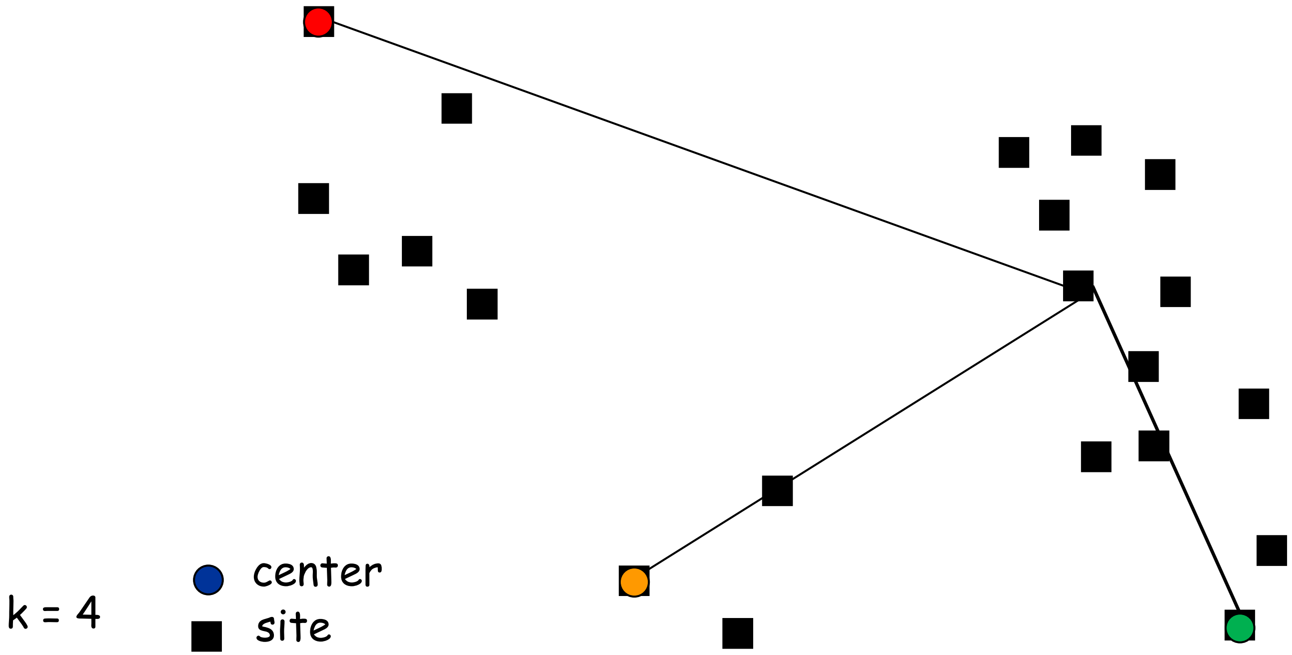


example of execution



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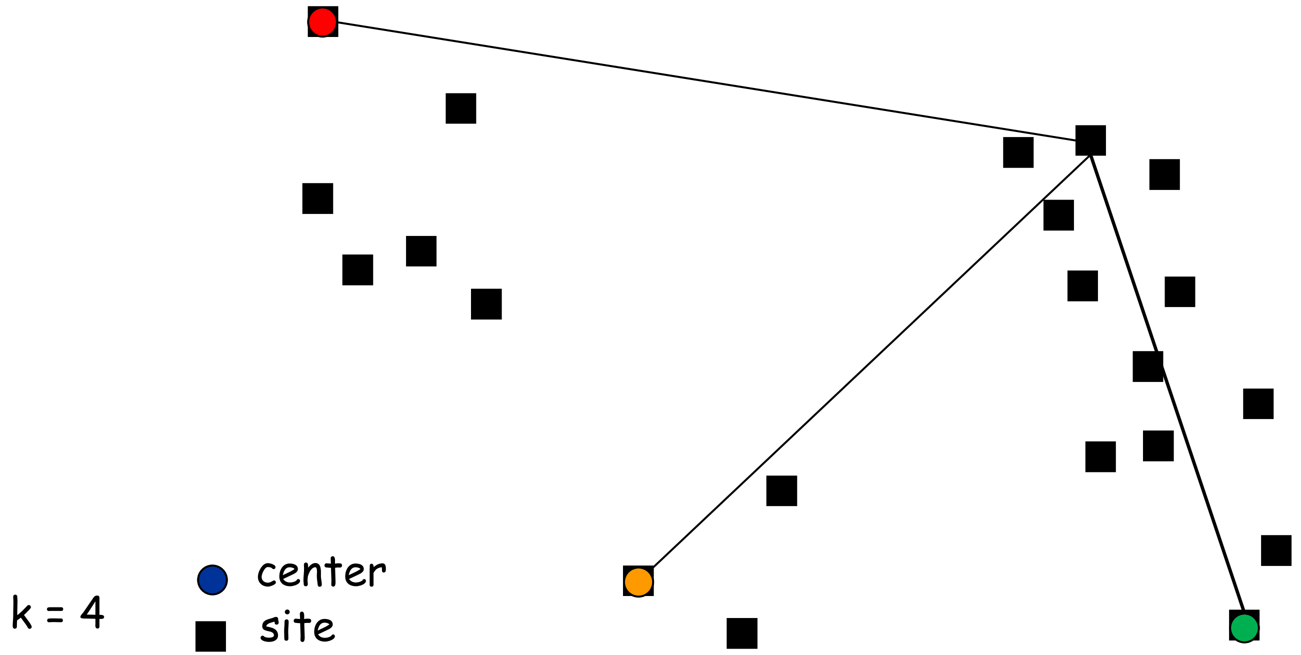

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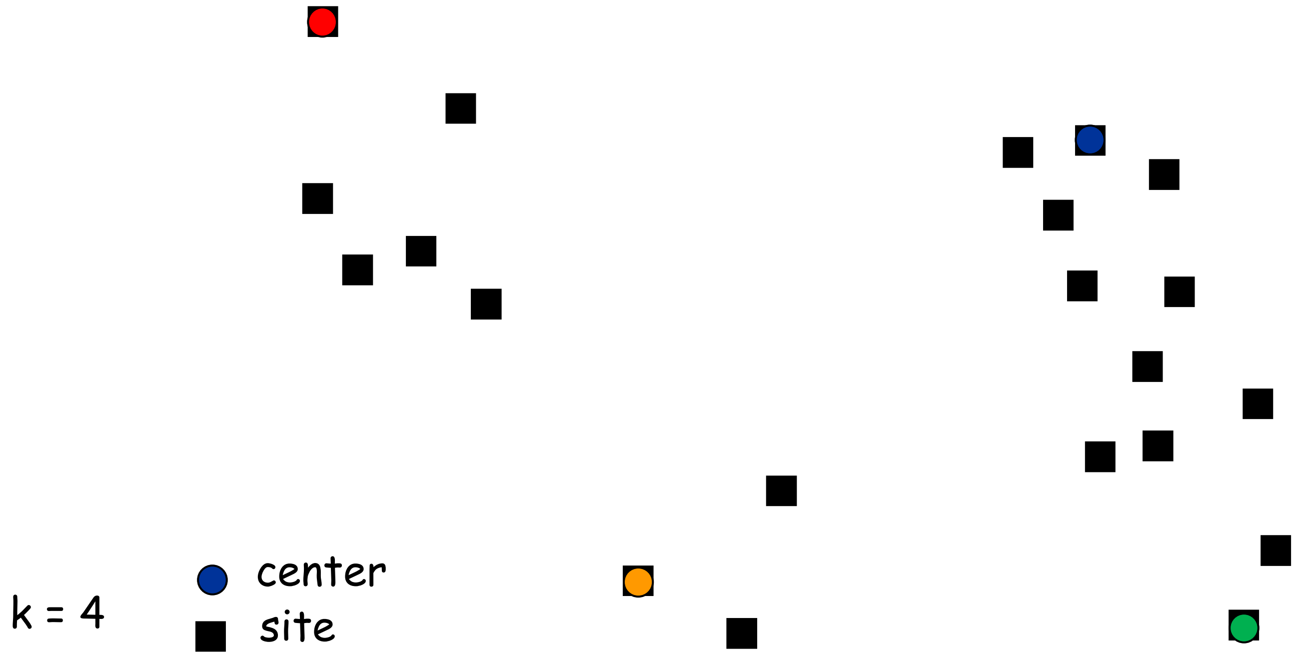

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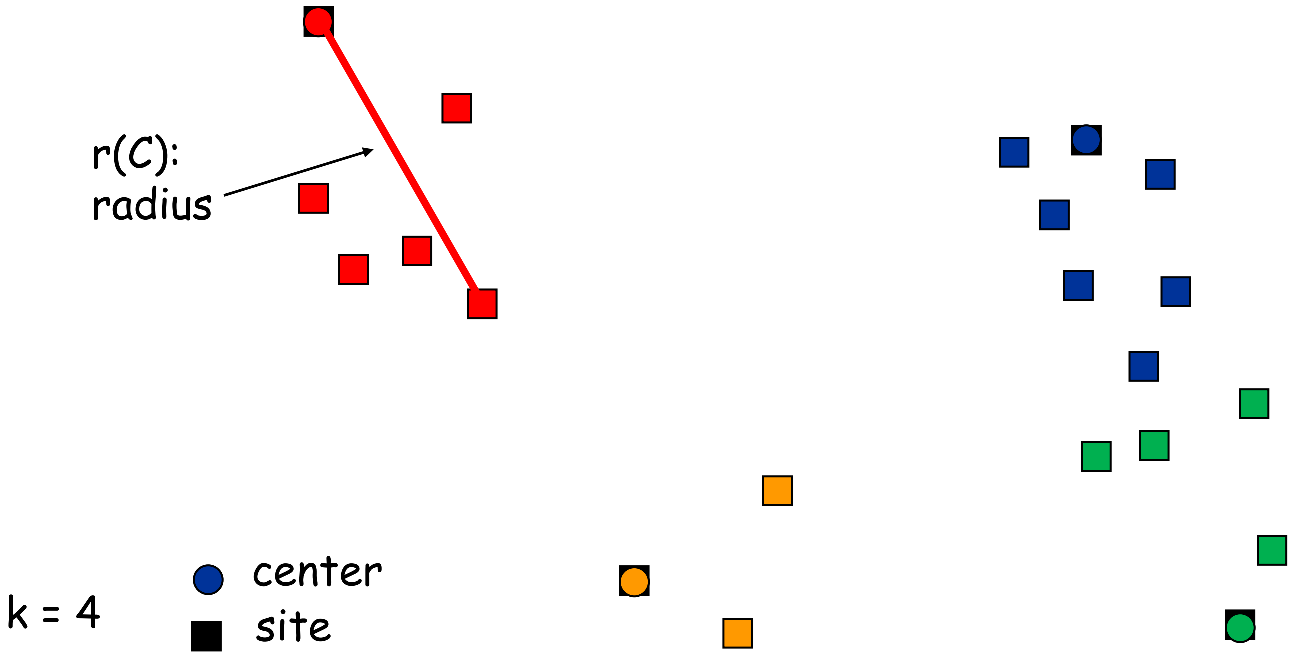


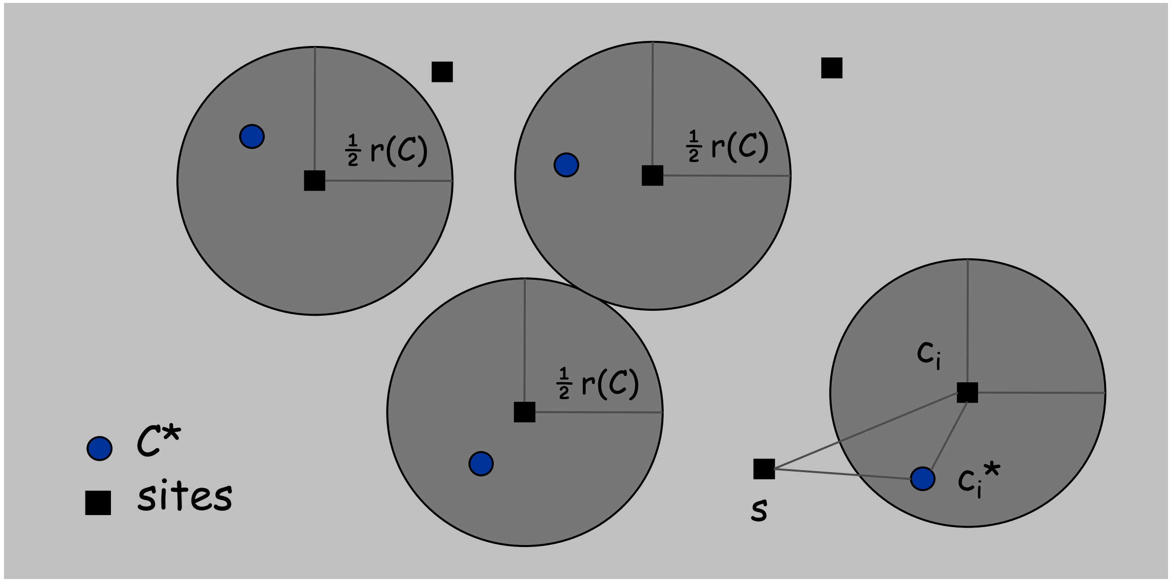
Center Selection: Analysis of Greedy Algorithm

balls are disjoint since all centers in C are pairwise at distance at least $r(C)$

Theorem. Let C^* be an optimal set of centers. Then $r(C) \le 2r(C^*)$.

Pf. (by contradiction) Assume $r(C^*) < \frac{1}{2} r(C)$.

- For each site c_i in C, consider ball of radius $\frac{1}{2} r(C)$ around it.
- Exactly one c_i^* in each ball;
  - each ball with center $c_i \in C$ must contain a center in C^* (otherwise $\text{dist}(c_i, C^*) \ge \frac{1}{2} r(C) > r(C^*)$);
  - balls are disjoint and $|C|=|C^*|$.



Center Selection: Analysis of Greedy Algorithm

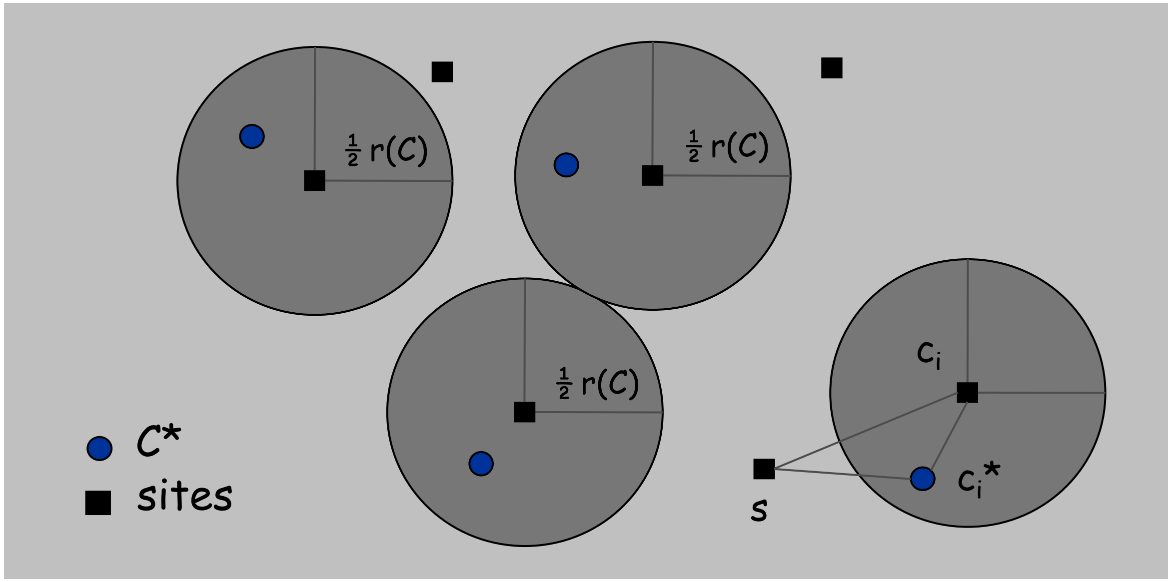
Theorem. Let C^* be an optimal set of centers. Then $r(C) \le 2r(C^*)$.

Pf. (by contradiction) Assume $r(C^*) < \frac{1}{2} r(C)$.

- For each site c_i in C, consider ball of radius $\frac{1}{2} r(C)$ around it.
- Exactly one c_i^* in each ball; let c_i be the site paired with c_i^*.
- Consider any site s and its closest center c_i^* in C^*.
- $\text{dist}(s, C) \le \text{dist}(s, c_i) \le \text{dist}(s, c_i^*) + \text{dist}(c_i^*, c_i) \le 2r(C^*)$.
- Thus $r(C) \le 2r(C^*)$. ▪

Δ-inequality

$\le r(C^*)$ since c_i^* is closest center



Center Selection

Theorem. Let C^* be an optimal set of centers. Then $r(C) \leq 2r(C^*)$.

Theorem. Greedy algorithm is a 2-approximation for center selection problem.

Remark. Greedy algorithm always places centers at sites, but is still within a factor of 2 of best solution that is allowed to place centers anywhere.

e.g., points in the plane

Question. Is there hope of a better approximation?

…very unlikely:

Theorem. Unless P = NP, there no ρ-approximation for center-selection problem for any $\rho < 2$.