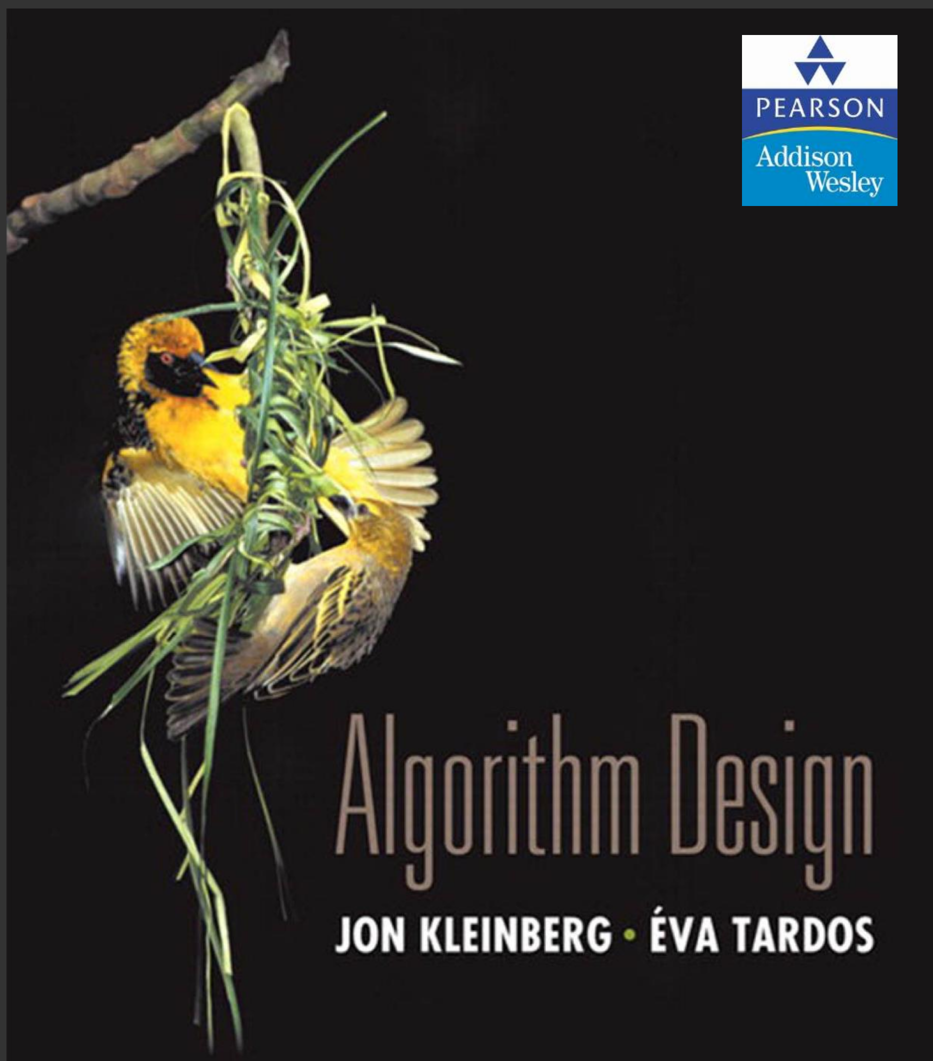
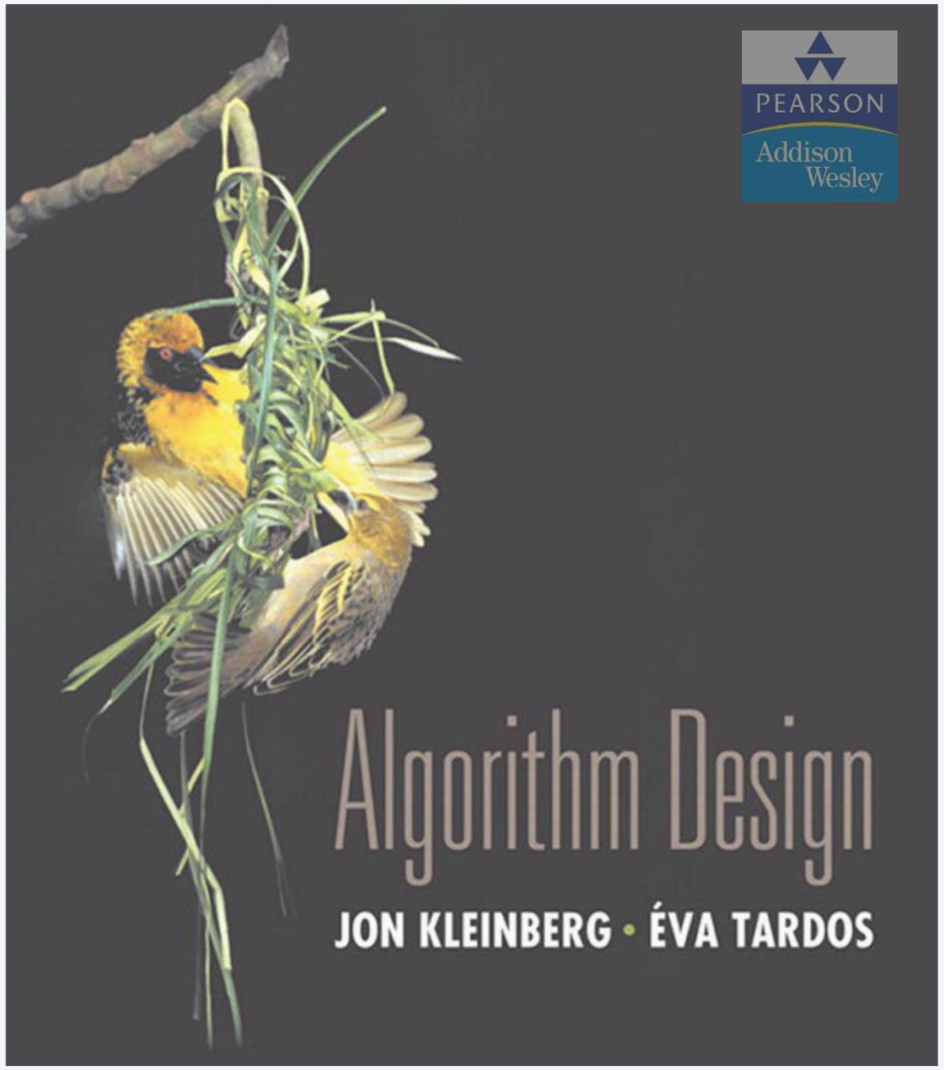


8. INTRACTABILITY I

- *poly-time reductions*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos



SECTION 8.1

8. INTRACTABILITY I

- *poly-time reductions*

Algorithm design patterns and antipatterns

Algorithm design patterns.

- Greedy.
- Divide and conquer.
- Dynamic programming.
- Duality.
- Reductions.
- Local search.
- Randomization.

Algorithm design antipatterns.

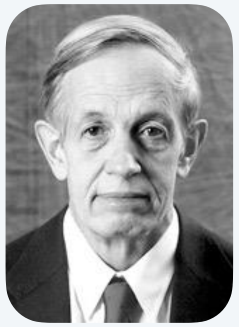
- **NP**-completeness. $O(n^k)$ algorithm unlikely.
- **PSPACE**-completeness. $O(n^k)$ certification algorithm unlikely.
- Undecidability. No algorithm possible.

Classify problems according to computational requirements

Q. Which problems will we be able to solve in practice?

A working definition. Those with poly-time algorithms.

von Neumann (1953)



Nash (1955)

Gödel (1956)

Cobham (1964)

Edmonds (1965)

Rabin (1966)

Turing machine, word RAM, uniform circuits, …

Theory. Definition is broad and robust.

constants tend to be small, e.g., $3n^2$

Practice. Poly-time algorithms scale to huge problems.

Classify problems according to computational requirements

Q. Which problems will we be able to solve in practice?

A working definition. Those with poly-time algorithms.

yes	probably no
shortest path	longest path
min cut	max cut
2-satisfiability	3-satisfiability
planar 4-colorability	planar 3-colorability
bipartite vertex cover	vertex cover
matching	3d-matching
primality testing	factoring
linear programming	integer linear programming

Classify problems

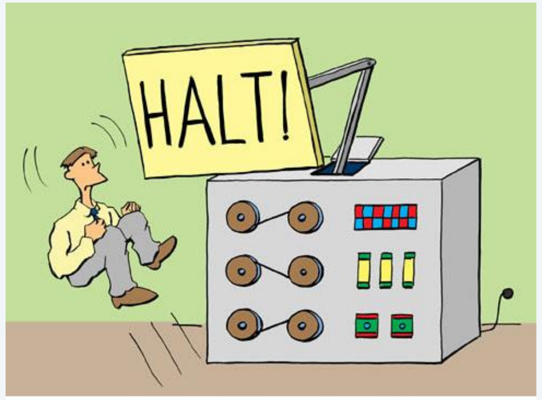
Desiderata. Classify problems according to those that can be solved in polynomial time and those that cannot.

Provably requires exponential time.

- Given a constant-size program, does it halt in at most k steps?
- Given a board position in an n-by-n generalization of checkers, can black guarantee a win?

input size = $c + \log k$

using forced capture rule



Alan designed the perfect computer

Frustrating news. Huge number of fundamental problems have defied classification for decades.

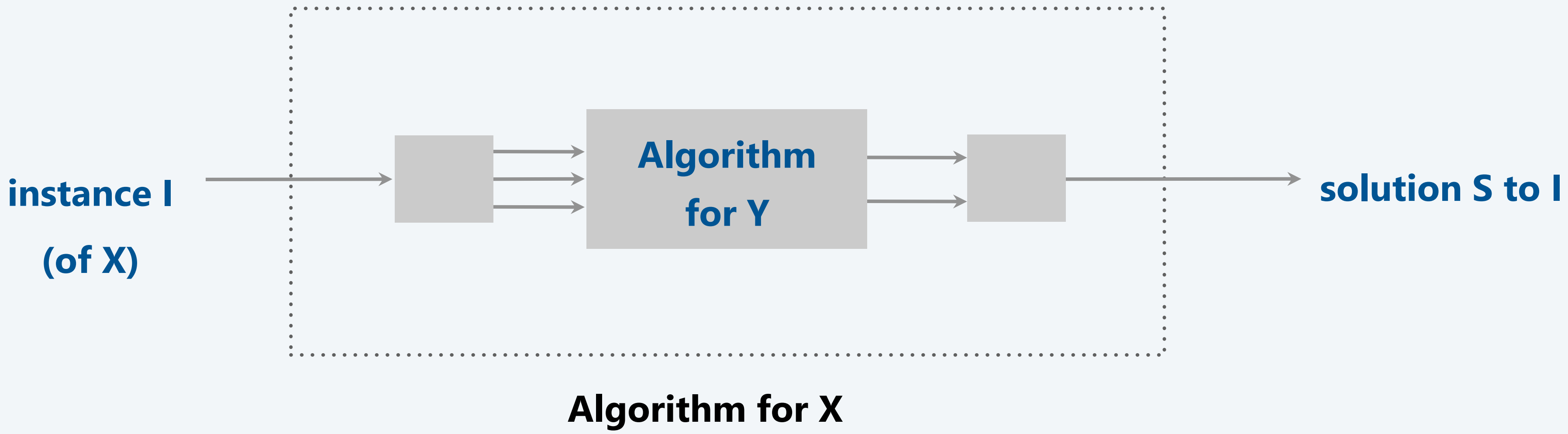
Poly-time reductions

Desiderata′. Suppose we could solve problem Y in polynomial time. What else could we solve in polynomial time?

Reduction. Problem X polynomial-time (Cook) reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

computational model supplemented by special piece of hardware that solves instances of Y in a single step



Poly-time reductions

Desiderata′. Suppose we could solve problem Y in polynomial time. What else could we solve in polynomial time?

Reduction. Problem X polynomial-time (Cook) reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Notation. $X \leq_P Y$.

Note. We pay for time to write down instances of Y sent to oracle $\Rightarrow$ instances of Y must be of polynomial size.

Novice mistake. Confusing $X \leq_P Y$ with $Y \leq_P X$.

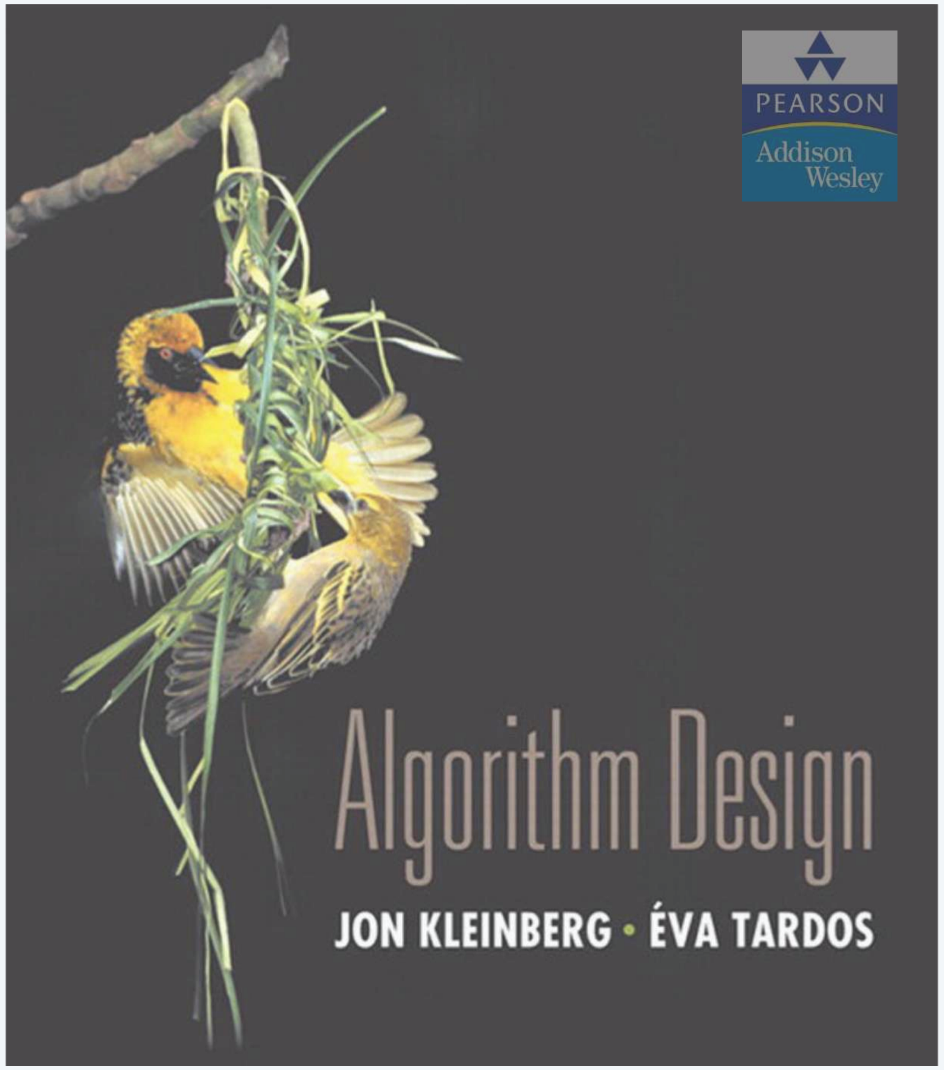
Poly-time reductions

Design algorithms. If $X \leq_P Y$ and Y can be solved in polynomial time, then X can be solved in polynomial time.

Establish intractability. If $X \leq_P Y$ and X cannot be solved in polynomial time, then Y cannot be solved in polynomial time.

Establish equivalence. If both $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$. In this case, X can be solved in polynomial time iff Y can be.

Bottom line. Reductions classify problems according to relative difficulty.



SECTION 8.1

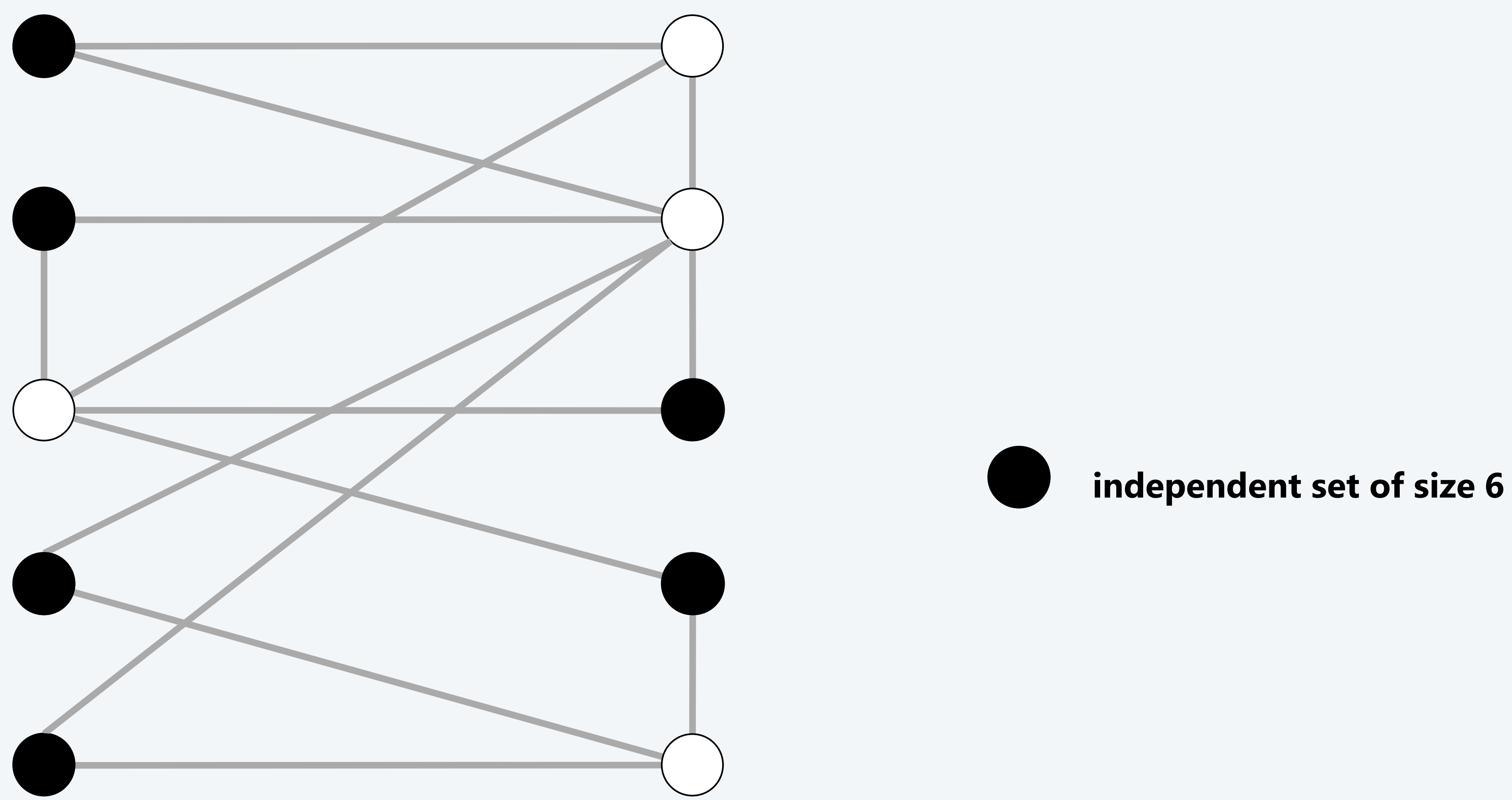
8. INTRACTABILITY I

- *packing and covering problems*

Independent set

INDEPENDENT-SET. Given a graph $G = (V, E)$ and an integer k, is there a subset of k (or more) vertices such that no two are adjacent?

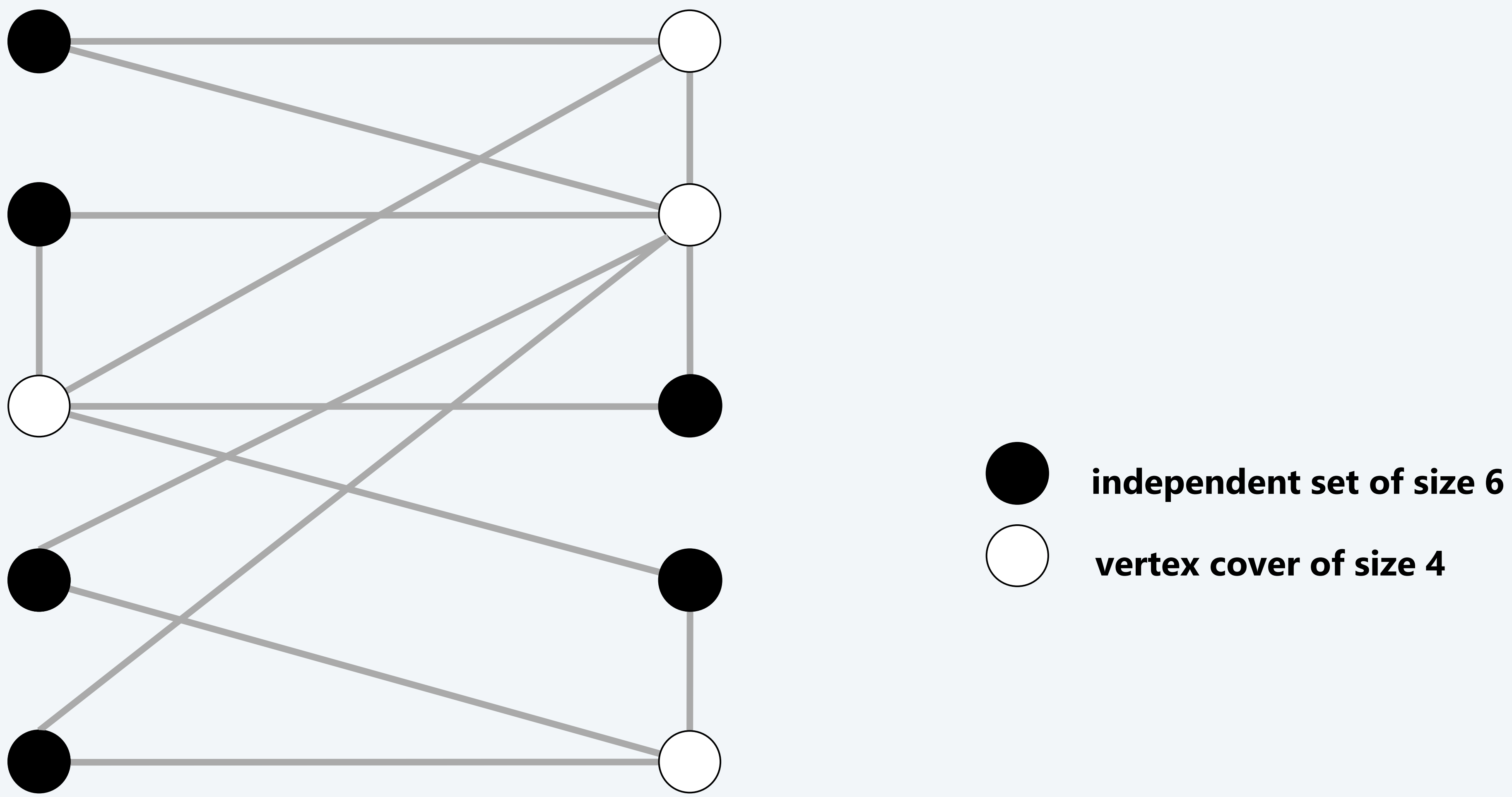
Ex. Is there an independent set of size ≥ 6 ?
Ex. Is there an independent set of size ≥ 7 ?



Vertex cover

VERTEX-COVER. Given a graph $G = (V, E)$ and an integer k, is there a subset of k (or fewer) vertices such that each edge is incident to at least one vertex in the subset?

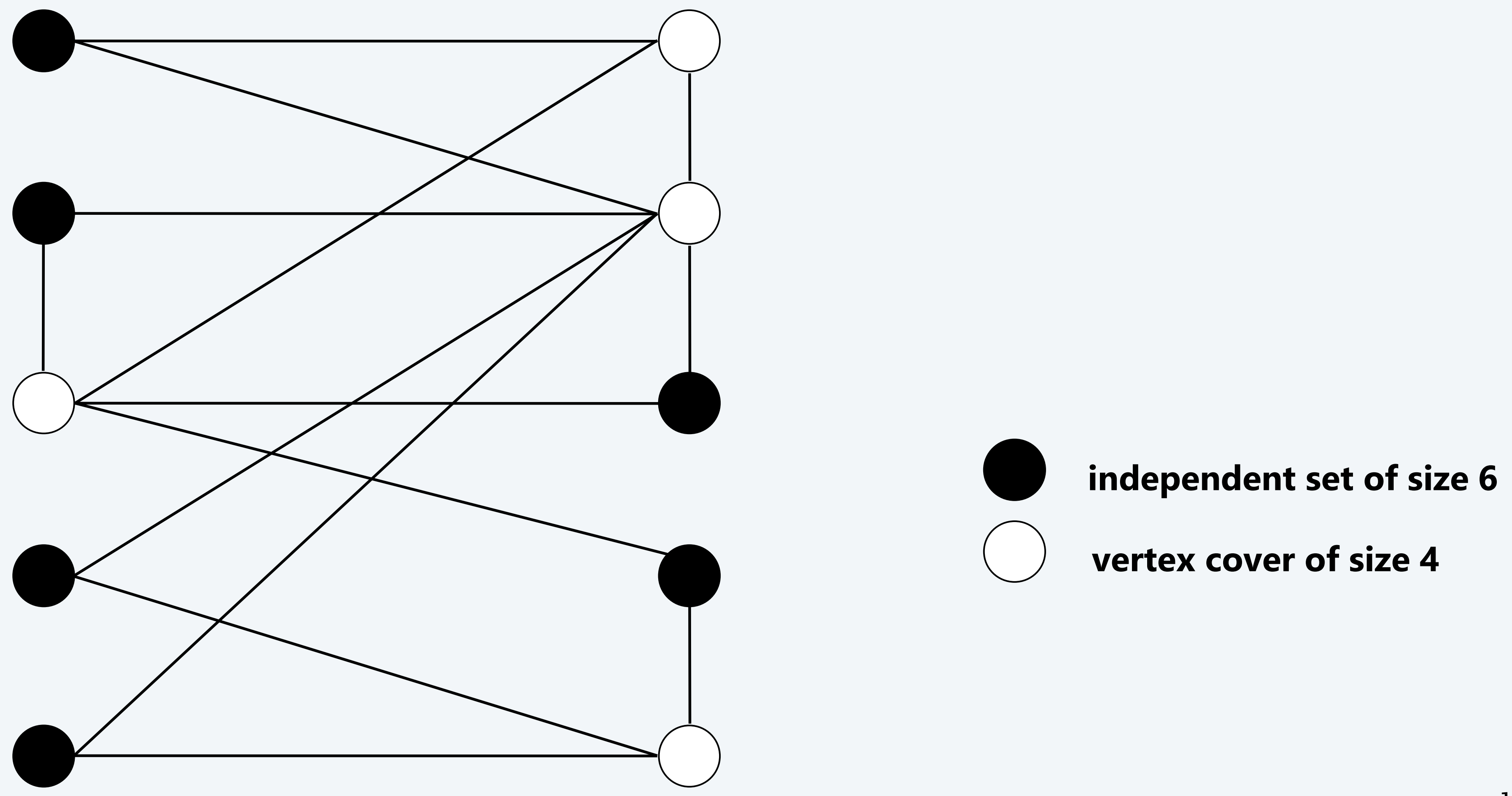
Ex. Is there a vertex cover of size ≤ 4 ?
Ex. Is there a vertex cover of size ≤ 3 ?



Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.



Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.

$\Rightarrow$

- Let S be any independent set of size k.
- $V - S$ is of size $n - k$.
- Consider an arbitrary edge $(u, v) \in E$.
- S independent $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
  $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
- Thus, $V - S$ covers (u, v). ▪

Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.

$\Leftarrow$

- Let $V - S$ be any vertex cover of size $n - k$.
- S is of size k.
- Consider an arbitrary edge $(u, v) \in E$.
- $V - S$ is a vertex cover $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
  $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
- Thus, S is an independent set. ▪

Set cover

SET-COVER. Given a set U of elements, a collection S of subsets of U, and an integer k, are there $\leq k$ of these subsets whose union is equal to U?

Sample application.

- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The i^{th} piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $S_f = \{ 1, 2, 6, 7 \}$

$k = 2$

a set cover instance

One more example

Given the universe U = { 1, 2, 3, 4, 5, 6, 7 } and the following sets, which is the minimum size of a set cover?

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 1, 4, 6 \}$

$S_b = \{ 1, 6, 7 \}$

$S_c = \{ 1, 2, 3, 6 \}$

$S_d = \{ 1, 3, 5, 7 \}$

$S_e = \{ 2, 6, 7 \}$

$S_f = \{ 3, 4, 5 \}$

minimum size: 3

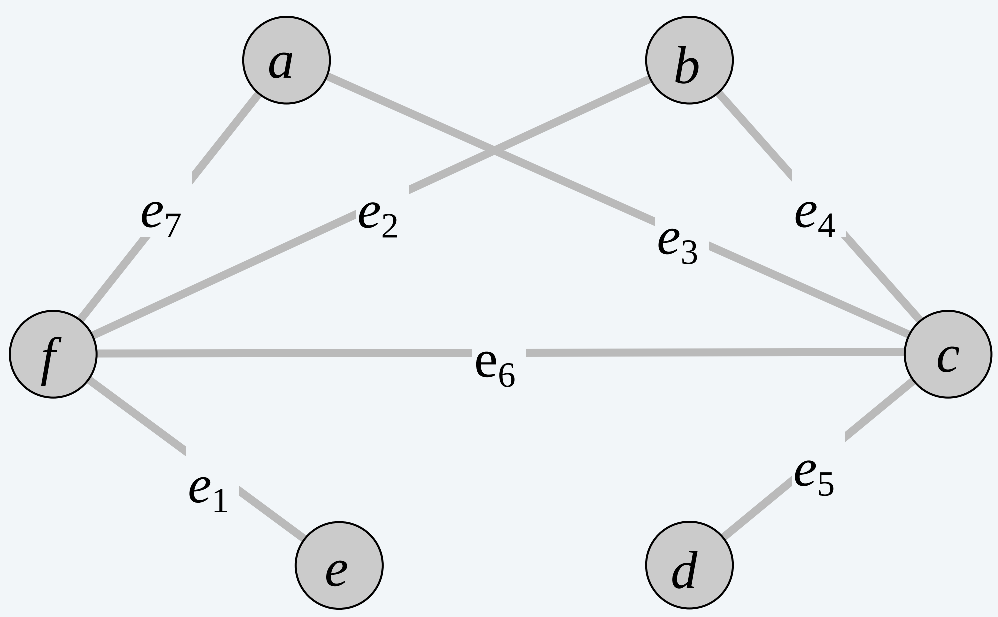
Vertex cover reduces to set cover

Theorem. VERTEX-COVER $\leq_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance $G = (V, E)$ and k, we construct a SET-COVER instance (U, S, k) that has a set cover of size k iff G has a vertex cover of size k.

Construction.

- Universe $U = E$.
- Include one subset for each node $v \in V$: $S_v = \{e \in E : e \text{ incident to } v\}$.



vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad$ $S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad$ $S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

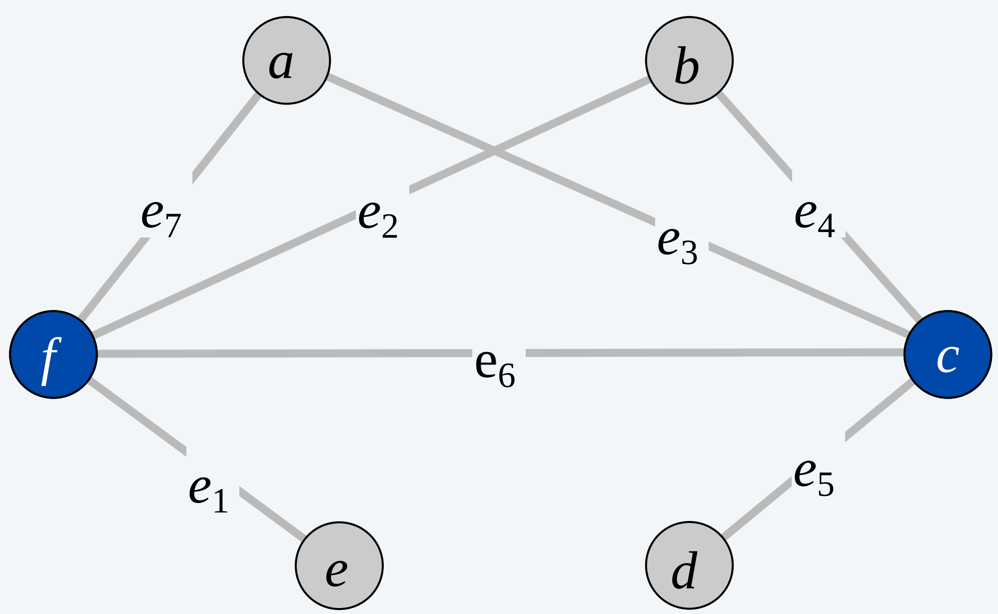
Vertex cover reduces to set cover

Lemma. $G = (V, E)$ contains a vertex cover of size k iff (U, S, k) contains a set cover of size k.

Pf. $\Rightarrow$ Let $X \subseteq V$ be a vertex cover of size k in G.

- Then $Y = \{ S_v : v \in X \}$ is a set cover of size k. ▪

"yes" instances of VERTEX-COVER are solved correctly



vertex cover instance (k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $S_f = \{ 1, 2, 6, 7 \}$

set cover instance (k = 2)

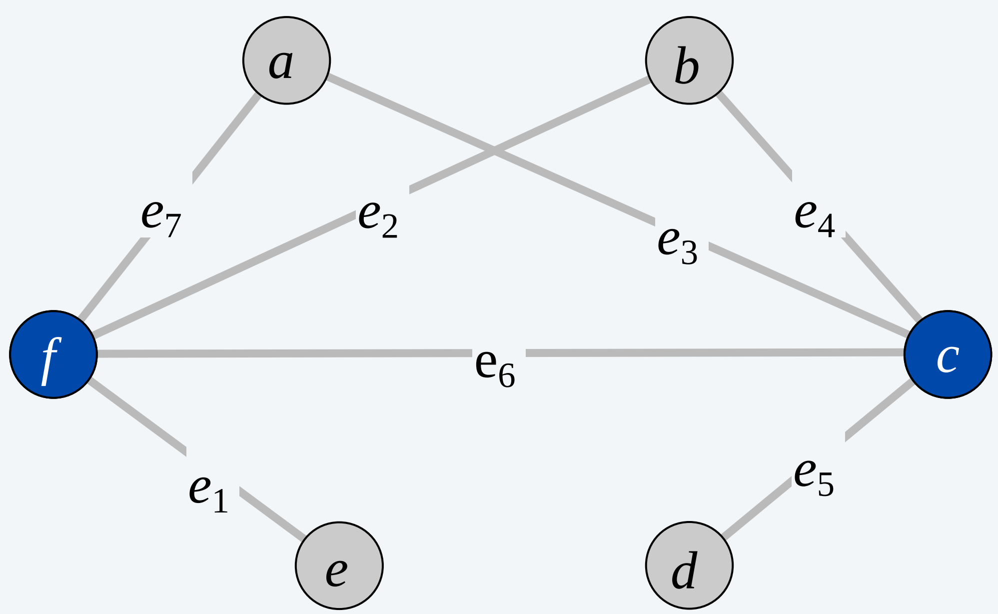
Vertex cover reduces to set cover

Lemma. $G = (V, E)$ contains a vertex cover of size k iff (U, S, k) contains a set cover of size k.

Pf. $\Leftarrow$ Let $Y \subseteq S$ be a set cover of size k in (U, S, k).

- Then $X = \{ v : S_v \in Y \}$ is a vertex cover of size k in G. ▪

"no" instances of VERTEX-COVER are solved correctly



vertex cover instance (k = 2)

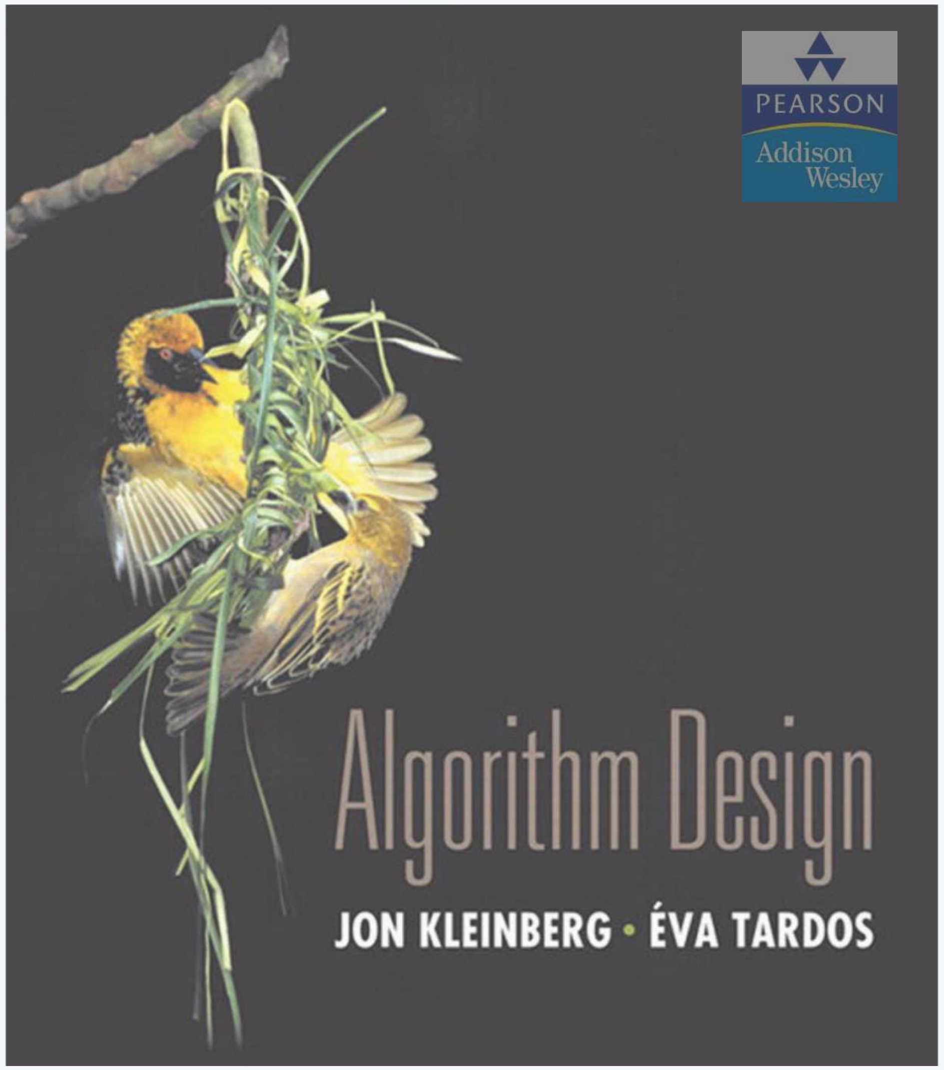
$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad S_f = \{ 1, 2, 6, 7 \}$

set cover instance (k = 2)



SECTION 8.2

8. INTRACTABILITY I

- *constraint satisfaction problems*

Satisfiability

Literal. A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause. A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form (CNF). A propositional formula Φ that is a conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT. Given a CNF formula Φ, does it have a satisfying truth assignment?
3-SAT. SAT where each clause contains exactly 3 literals
(and each literal corresponds to a different variable).

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

yes instance: x_1 = true, x_2 = true, x_3 = false, x_4 = false

Key application. Electronic design automation (EDA).

Satisfiability is hard

Scientific hypothesis. There does not exist a poly-time algorithm for 3-SAT.

P vs. NP. This hypothesis is equivalent to **P ≠ NP** conjecture.

3-satisfiability reduces to independent set

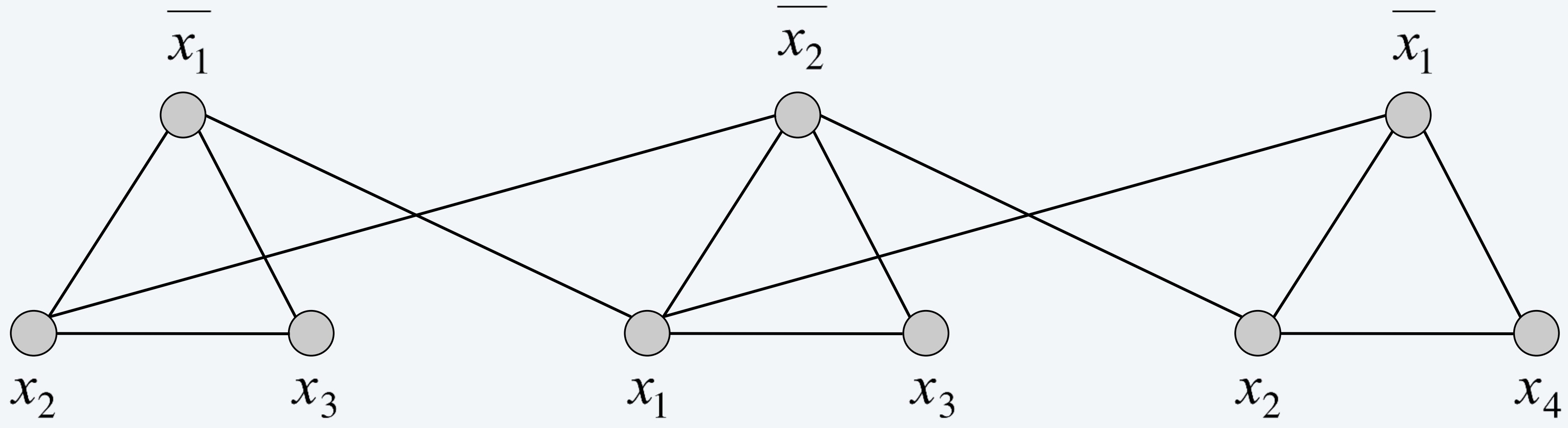
Theorem. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Construction.

- G contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3-satisfiability reduces to independent set

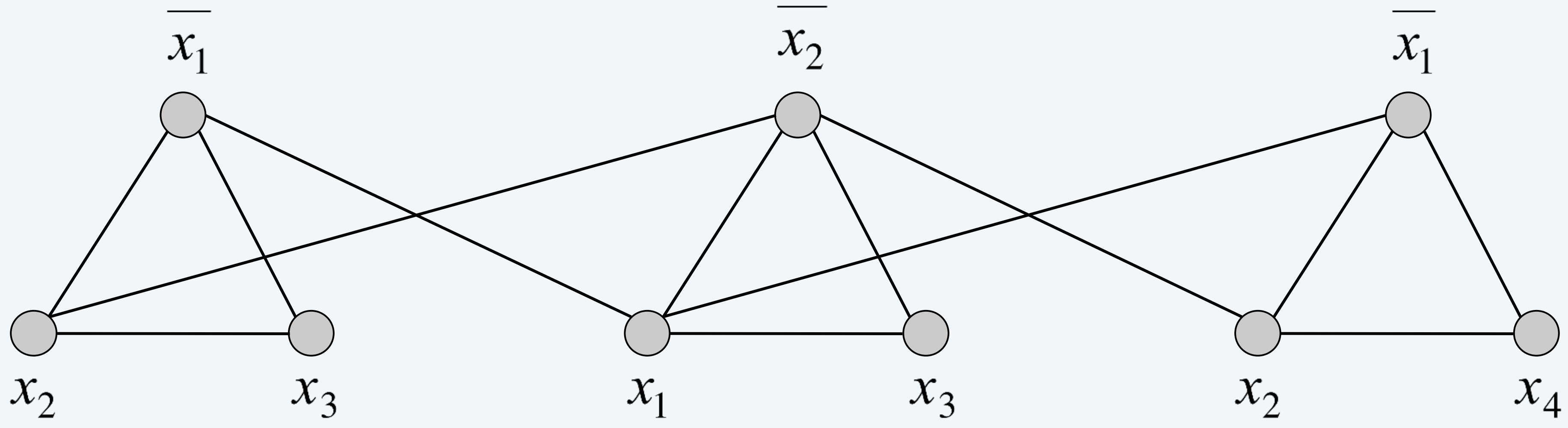
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Rightarrow$ Consider any satisfying assignment for Φ.

- Select one true literal from each clause/triangle.
- This is an independent set of size $k = |\Phi|$. ▪

"yes" instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3-satisfiability reduces to independent set

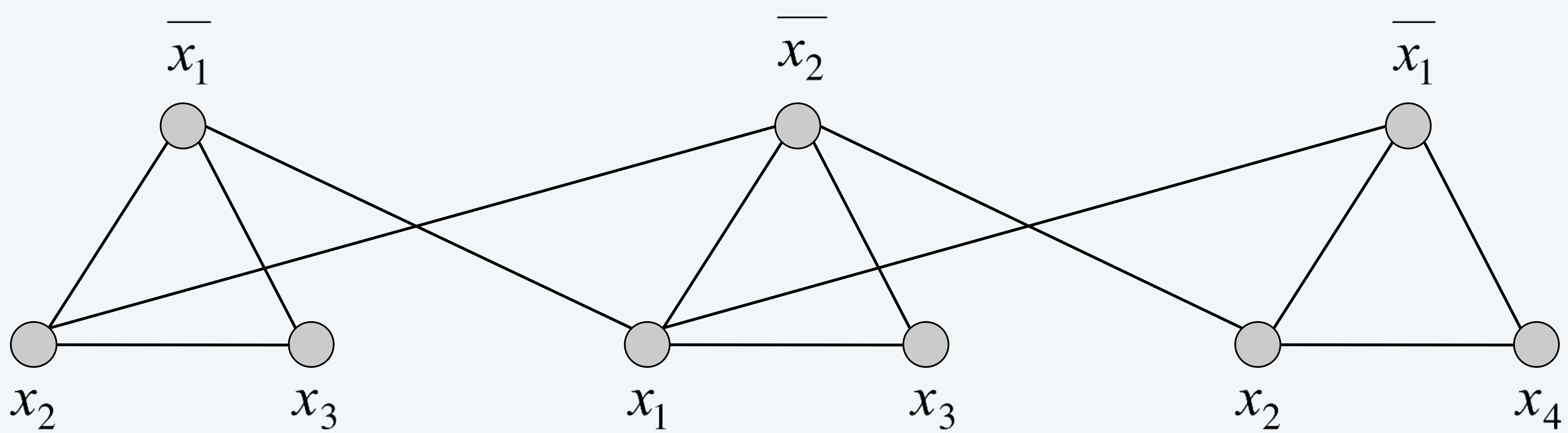
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Leftarrow$ Let S be independent set of size k.

- S must contain exactly one node in each triangle.
- Set these literals to *true* (and remaining literals consistently).
- All clauses in Φ are satisfied. ▪

"no" instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

Review

Basic reduction strategies.

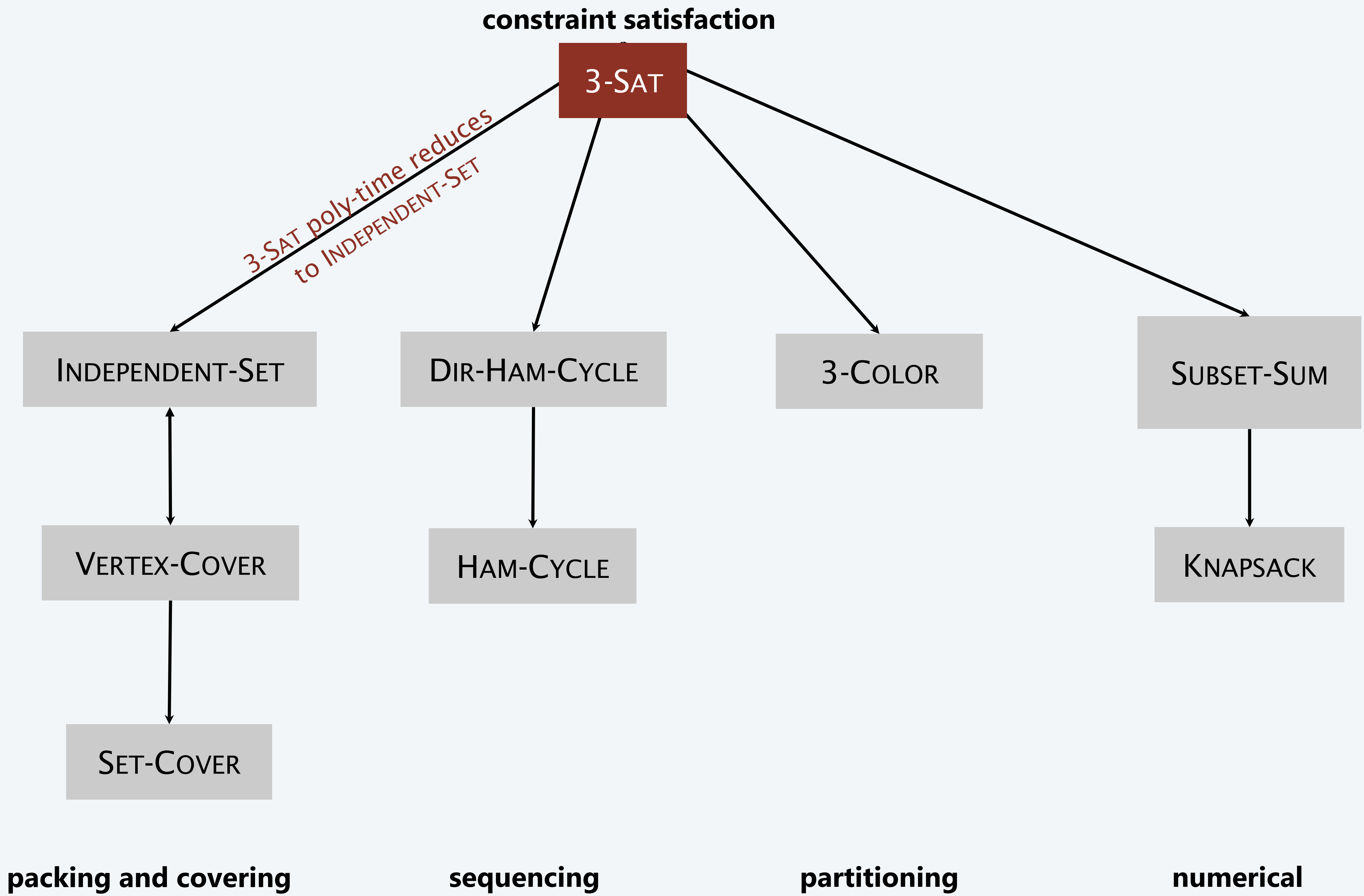
- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.

Pf idea. Compose the two algorithms.

Ex. 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

Poly-time reductions



DECISION, SEARCH, AND OPTIMIZATION PROBLEMS

Decision problem. Does there exist a vertex cover of size $\leq k$?

Search problem. Find a vertex cover of size $\leq k$.

Optimization problem. Find a vertex cover of minimum size.

Goal. Show that all three problems poly-time reduce to one another.

SEARCH PROBLEMS VS. DECISION PROBLEMS

VERTEX-COVER. Does there exist a vertex cover of size $\leq k$?

FIND-VERTEX-COVER. Find a vertex cover of size $\leq k$.

Theorem. VERTEX-COVER $\equiv_P$ FIND-VERTEX-COVER.

Pf. $\leq_P$ Decision problem is a special case of search problem. ▪

Pf. $\geq_P$

To find a vertex cover of size $\leq k$:

- Determine if there exists a vertex cover of size $\leq k$.
- Find a vertex v such that $G - \{v\}$ has a vertex cover of size $\leq k - 1$. (any vertex in any vertex cover of size $\leq k$ will have this property)
- Include v in the vertex cover.
- Recursively find a vertex cover of size $\leq k - 1$ in $G - \{v\}$. ▪

delete v and all incident edges

OPTIMIZATION PROBLEMS VS. SEARCH PROBLEMS

FIND-VERTEX-COVER. Find a vertex cover of size $\le k$.

FIND-MIN-VERTEX-COVER. Find a vertex cover of minimum size.

Theorem. FIND-VERTEX-COVER $\equiv_P$ FIND-MIN-VERTEX-COVER.

Pf. $\le_P$ Search problem is a special case of optimization problem. ▪

Pf. $\ge_P$ To find vertex cover of minimum size:

- Binary search (or linear search) for size k^* of min vertex cover.
- Solve search problem for given k^*. ▪