7.3 Identification of a Subset to a Point

If A is a subspace of a topological space S, we can define a relation $\sim$ on S by declaring

$$x \sim x \quad \text{for all } x \in S$$

(so the relation is reflexive) and

$$x \sim y \quad \text{for all } x, y \in A.$$

This is an equivalence relation on S. We say that the quotient space $S/\sim$ is obtained from S by *identifying A to a point*.

Example 7.2. Let I be the unit interval $[0,1]$ and $I/\sim$ the quotient space obtained from I by identifying the two points $\{0,1\}$ to a point. Denote by S^1 the unit circle in the complex plane. The function $f\colon I \to S^1$, $f(x) = \exp(2\pi i x)$, assumes the same value at 0 and 1 (Figure 7.2), and so induces a function $\bar{f}\colon I/\sim \to S^1$.

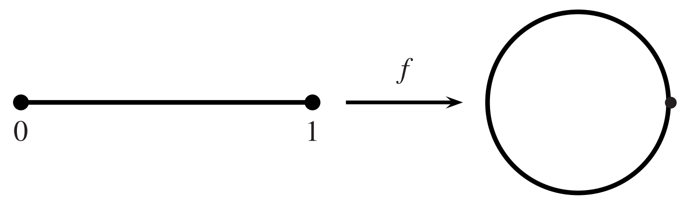

Fig. 7.2. The unit circle as a quotient space of the unit interval.

Proposition 7.3. *The function $\bar{f}\colon I/\sim \to S^1$ is a homeomorphism.*

Proof. Since f is continuous, $\bar{f}$ is also continuous by Proposition 7.1. Clearly, $\bar{f}$ is a bijection. As the continuous image of the compact set I, the quotient $I/\sim$ is compact. Thus, $\bar{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff space S^1. By Corollary A.36, $\bar{f}$ is a homeomorphism. □

7.4 A Necessary Condition for a Hausdorff Quotient

The quotient construction does not in general preserve the Hausdorff property or second countability. Indeed, since every singleton set in a Hausdorff space is closed, if $\pi\colon S \to S/\sim$ is the projection and the quotient $S/\sim$ is Hausdorff, then for any $p \in S$, its image $\{\pi(p)\}$ is closed in $S/\sim$. By the continuity of π, the inverse image $\pi^{-1}(\{\pi(p)\}) = [p]$ is closed in S. This gives a necessary condition for a quotient space to be Hausdorff.

Proposition 7.4. *If the quotient space $S/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S.*

Example. Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $]0,\infty[$ to a point. Then the quotient space $\mathbb{R}/\sim$ is not Hausdorff because the equivalence class $]0,\infty[$ of $\sim$ in $\mathbb{R}$ corresponding to the point $]0,\infty[$ in $\mathbb{R}/\sim$ is not a closed subset of $\mathbb{R}$.

7.5 Open Equivalence Relations

In this section we follow the treatment of Boothby [3] and derive conditions under which a quotient space is Hausdorff or second countable. Recall that a map $f\colon X \to Y$ of topological spaces is *open* if the image of any open set under f is open.

Definition 7.5. An equivalence relation $\sim$ on a topological space S is said to be *open* if the projection map $\pi\colon S \to S/\sim$ is open.

In other words, the equivalence relation $\sim$ on S is open if and only if for every open set U in S, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x\in U} [x]$$

of all points equivalent to some point of U is open.

Example 7.6. The projection map to a quotient space is in general not open. For example, let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1, and $\pi\colon \mathbb{R} \to \mathbb{R}/\sim$ the projection map.

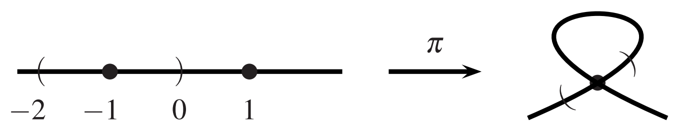

Fig. 7.3. A projection map that is not open.

The map π is open if and only if for every open set V in $\mathbb{R}$, its image $\pi(V)$ is open in $\mathbb{R}/\sim$, which by the definition of the quotient topology means that $\pi^{-1}(\pi(V))$ is open in $\mathbb{R}$. Now let V be the open interval $]-2,0[$ in $\mathbb{R}$. Then

$$\pi^{-1}(\pi(V)) =]-2,0[\, \cup \{1\},$$

which is not open in $\mathbb{R}$ (Figure 7.3). Therefore, the projection map $\pi\colon \mathbb{R} \to \mathbb{R}/\sim$ is not an open map.

Given an equivalence relation $\sim$ on S, let R be the subset of $S \times S$ that defines the relation

$$R = \{(x,y) \in S \times S \mid x \sim y\}.$$

We call R the *graph* of the equivalence relation $\sim$.

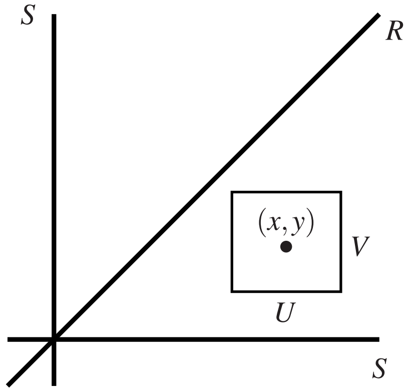

Fig. 7.4. The graph R of an equivalence relation and an open set $U \times V$ disjoint from R.

Theorem 7.7. *Suppose $\sim$ is an open equivalence relation on a topological space S. Then the quotient space $S/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.*

Proof. There is a sequence of equivalent statements:

R is closed in $S \times S$

$\iff$ $(S \times S) - R$ is open in $S \times S$

$\iff$ for every $(x,y) \in S \times S - R$, there is a basic open set $U \times V$ containing (x,y) such that $(U \times V) \cap R = \varnothing$ (Figure 7.4)

$\iff$ for every pair $x \nsim y$ in S, there exist neighborhoods U of x and V of y in S such that no element of U is equivalent to an element of V

$\iff$ for any two points $[x] \neq [y]$ in $S/\sim$, there exist neighborhoods U of x and V of y in S such that $\pi(U) \cap \pi(V) = \varnothing$ in $S/\sim$. $(*)$

We now show that this last statement $(*)$ is equivalent to $S/\sim$ being Hausdorff. First assume $(*)$. Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are disjoint open sets in $S/\sim$ containing $[x]$ and $[y]$ respectively. Therefore, $S/\sim$ is Hausdorff.

Conversely, suppose $S/\sim$ is Hausdorff. Let $[x] \neq [y]$ in $S/\sim$. Then there exist disjoint open sets A and B in $S/\sim$ such that $[x] \in A$ and $[y] \in B$. By the surjectivity of π, we have $A = \pi(\pi^{-1}A)$ and $B = \pi(\pi^{-1}B)$ (see Problem 7.1). Let $U = \pi^{-1}A$ and $V = \pi^{-1}B$. Then $x \in U$, $y \in V$, and $A = \pi(U)$ and $B = \pi(V)$ are disjoint open sets in $S/\sim$. □

If the equivalence relation $\sim$ is equality, then the quotient space $S/\sim$ is S itself and the graph R of $\sim$ is simply the diagonal

$$\Delta = \{(x,x) \in S \times S\}.$$

In this case, Theorem 7.7 becomes the following well-known characterization of a Hausdorff space by its diagonal (cf. Problem A.6).

Corollary 7.8. *A topological space S is Hausdorff if and only if the diagonal Δ in $S \times S$ is closed.*

Theorem 7.9. *Let $\sim$ be an open equivalence relation on a topological space S with projection $\pi\colon S \to S/\!\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $S/\!\sim$.*

Proof. Since π is an open map, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\!\sim$. Let W be an open set in $S/\!\sim$ and $[x] \in W$, $x \in S$. Then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open, there is a basic open set $B \in \mathcal{B}$ such that

$$x \in B \subset \pi^{-1}(W).$$

Then

$$[x] = \pi(x) \in \pi(B) \subset W,$$

which proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\!\sim$. □

Corollary 7.10. *If $\sim$ is an open equivalence relation on a second-countable space S, then the quotient space $S/\!\sim$ is second countable.*

7.6 Real Projective Space

Define an equivalence relation on $\mathbb{R}^{n+1} - \{0\}$ by

$$x \sim y \iff y = tx \text{ for some nonzero real number } t,$$

where x, $y \in \mathbb{R}^{n+1} - \{0\}$. The *real projective space* $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1} - \{0\}$ by this equivalence relation. We denote the equivalence class of a point $(a^0, \ldots, a^n) \in \mathbb{R}^{n+1} - \{0\}$ by $[a^0, \ldots, a^n]$ and let $\pi\colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ be the projection. We call $[a^0, \ldots, a^n]$ *homogeneous coordinates* on $\mathbb{R}P^n$.

Geometrically, two nonzero points in $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin, so $\mathbb{R}P^n$ can be interpreted as the set of all lines through the origin in $\mathbb{R}^{n+1}$. Each line through the origin in $\mathbb{R}^{n+1}$ meets the unit

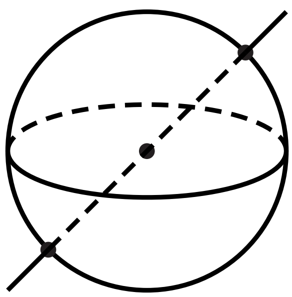

Fig. 7.5. A line through 0 in $\mathbb{R}^3$ corresponds to a pair of antipodal points on S^2.

sphere S^n in a pair of antipodal points, and conversely, a pair of antipodal points on S^n determines a unique line through the origin (Figure 7.5). This suggests that we define an equivalence relation $\sim$ on S^n by identifying antipodal points:

$$x \sim y \iff x = \pm y, \quad x, y \in S^n.$$

We then have a bijection $\mathbb{R}P^n \leftrightarrow S^n/\!\sim$.

Exercise 7.11 (Real projective space as a quotient of a sphere).* For $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, let $\|x\| = \sqrt{\sum_i (x^i)^2}$ be the modulus of x. Prove that the map $f\colon \mathbb{R}^{n+1} - \{0\} \to S^n$ given by

$$f(x) = \frac{x}{\|x\|}$$

induces a homeomorphism $\bar{f}\colon \mathbb{R}P^n \to S^n/\!\sim$. (*Hint*: Find an inverse map

$$\bar{g}\colon S^n/\!\sim \to \mathbb{R}P^n$$

and show that both $\bar{f}$ and $\bar{g}$ are continuous.)

Example 7.12 (*The real projective line* $\mathbb{R}P^1$).

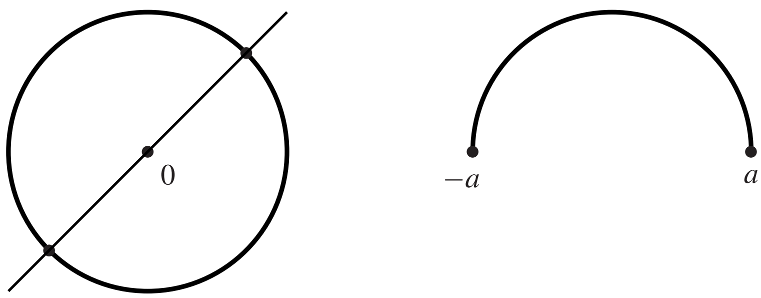

Fig. 7.6. The real projective line $\mathbb{R}P^1$ as the set of lines through 0 in $\mathbb{R}^2$.

Each line through the origin in $\mathbb{R}^2$ meets the unit circle in a pair of antipodal points. By Exercise 7.11, $\mathbb{R}P^1$ is homeomorphic to the quotient $S^1/\!\sim$, which is in turn homeomorphic to the closed upper semicircle with the two endpoints identified (Figure 7.6). Thus, $\mathbb{R}P^1$ is homeomorphic to S^1.

Example 7.13 (*The real projective plane* $\mathbb{R}P^2$). By Exercise 7.11, there is a homeomorphism

$$\mathbb{R}P^2 \simeq S^2/\{\text{antipodal points}\} = S^2/\!\sim.$$

For points not on the equator, each pair of antipodal points contains a unique point in the upper hemisphere. Thus, there is a bijection between $S^2/\!\sim$ and the quotient of the closed upper hemisphere in which each pair of antipodal points on the equator is identified. It is not difficult to show that this bijection is a homeomorphism (see Problem 7.2).

Let H^2 be the closed upper hemisphere

$$H^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\}$$

and let D^2 be the closed unit disk

$$D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

These two spaces are homeomorphic to each other via the continuous map

$$\varphi\colon H^2 \to D^2,$$
$$\varphi(x,y,z) = (x,y),$$

and its inverse

$$\psi\colon D^2 \to H^2,$$
$$\psi(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\right).$$

On H^2, define an equivalence relation $\sim$ by identifying the antipodal points on the equator:

$$(x,y,0) \sim (-x,-y,0), \quad x^2+y^2 = 1.$$

On D^2, define an equivalence relation $\sim$ by identifying the antipodal points on the boundary circle:

$$(x,y) \sim (-x,-y), \quad x^2+y^2 = 1.$$

Then φ and ψ induce homeomorphisms

$$\bar{\varphi}\colon H^2/\!\sim \to D^2/\!\sim, \quad \bar{\psi}\colon D^2/\!\sim \to H^2/\!\sim.$$

In summary, there is a sequence of homeomorphisms

$$\mathbb{R}P^2 \xrightarrow{\sim} S^2/\!\sim \xrightarrow{\sim} H^2/\!\sim \xrightarrow{\sim} D^2/\!\sim$$

that identifies the real projective plane as the quotient of the closed disk D^2 with the antipodal points on its boundary identified. This may be the best way to picture $\mathbb{R}P^2$ (Figure 7.7).

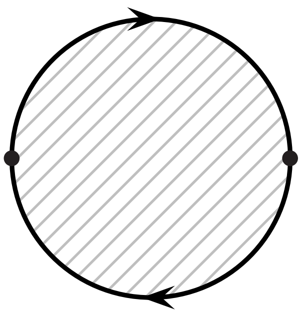

Fig. 7.7. The real projective plane as the quotient of a disk.

The real projective plane $\mathbb{R}P^2$ cannot be embedded as a submanifold of $\mathbb{R}^3$. However, if we allow self-intersection, then we can map $\mathbb{R}P^2$ into $\mathbb{R}^3$ as a cross-cap (Figure 7.8). This map is not one-to-one.

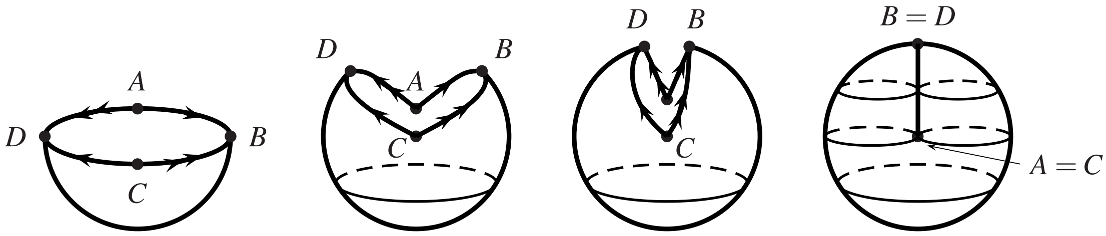

Fig. 7.8. The real projective plane immersed as a cross-cap in $\mathbb{R}^3$.

Proposition 7.14. *The equivalence relation* $\sim$ *on* $\mathbb{R}^{n+1} - \{0\}$ *in the definition of* $\mathbb{R}P^n$ *is an open equivalence relation.*

Proof. For an open set $U \subset \mathbb{R}^{n+1} - \{0\}$, the image $\pi(U)$ is open in $\mathbb{R}P^n$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^{n+1} - \{0\}$. But $\pi^{-1}(\pi(U))$ consists of all nonzero scalar multiples of points of U; that is,

$$\pi^{-1}(\pi(U)) = \bigcup_{t \in \mathbb{R}^\times} tU = \bigcup_{t \in \mathbb{R}^\times} \{tp \mid p \in U\}.$$

Since multiplication by $t \in \mathbb{R}^\times$ is a homeomorphism of $\mathbb{R}^{n+1} - \{0\}$, the set tU is open for any t. Therefore, their union $\bigcup_{t \in \mathbb{R}^\times} tU = \pi^{-1}(\pi(U))$ is also open. □

Corollary 7.15. *The real projective space* $\mathbb{R}P^n$ *is second countable.*

Proof. Apply Corollary 7.10. □

Proposition 7.16. *The real projective space* $\mathbb{R}P^n$ *is Hausdorff.*

Proof. Let $S = \mathbb{R}^{n+1} - \{0\}$ and consider the set

$$R = \{(x,y) \in S \times S \mid y = tx \text{ for some } t \in \mathbb{R}^\times\}.$$

If we write x and y as column vectors, then $[x\ y]$ is an $(n+1) \times 2$ matrix, and R may be characterized as the set of matrices $[x\ y]$ in $S \times S$ of rank ≤ 1. By a standard fact from linear algebra, $\text{rk}[x\ y] \leq 1$ is equivalent to the vanishing of all 2×2 minors of $[x\ y]$ (see Problem B.1). As the zero set of finitely many polynomials, R is a closed subset of $S \times S$. Since $\sim$ is an open equivalence relation on S, and R is closed in $S \times S$, by Theorem 7.7 the quotient $S/\!\sim\, \simeq \mathbb{R}P^n$ is Hausdorff. □

7.7 The Standard C^∞ Atlas on a Real Projective Space

Let $[a^0, \ldots, a^n]$ be homogeneous coordinates on the projective space $\mathbb{R}P^n$. Although a^0 is not a well-defined function on $\mathbb{R}P^n$, the condition $a^0 \neq 0$ is independent of the choice of a representative for $[a^0, \ldots, a^n]$. Hence, the condition $a^0 \neq 0$ makes sense on $\mathbb{R}P^n$, and we may define

$$U_0 := \{[a^0, \dots, a^n] \in \mathbb{R}P^n \mid a^0 \neq 0\}.$$

Similarly, for each $i = 1, \dots, n$, let

$$U_i := \{[a^0, \dots, a^n] \in \mathbb{R}P^n \mid a^i \neq 0\}.$$

Define

$$\phi_0 \colon U_0 \to \mathbb{R}^n$$

by

$$[a^0, \dots, a^n] \mapsto \left(\frac{a^1}{a^0}, \dots, \frac{a^n}{a^0} \right).$$

This map has a continuous inverse

$$(b^1, \dots, b^n) \mapsto [1, b^1, \dots, b^n]$$

and is therefore a homeomorphism. Similarly, there are homeomorphisms for each $i = 1, \dots, n$:

$$\phi_i \colon U_i \to \mathbb{R}^n,$$

$$[a^0, \dots, a^n] \mapsto \left(\frac{a^0}{a^i}, \dots, \widehat{\frac{a^i}{a^i}}, \dots, \frac{a^n}{a^i} \right),$$

where the caret sign $\widehat{\ }$ over a^i/a^i means that that entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with the (U_i, ϕ_i) as charts.

On the intersection $U_0 \cap U_1$, we have $a^0 \neq 0$ and $a^1 \neq 0$, and there are two coordinate systems

$$[a^0, a^1, a^2, \dots, a^n]$$

$$\phi_0 \swarrow \qquad \searrow \phi_1$$

$$\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \dots, \frac{a^n}{a^0} \right) \qquad\qquad \left(\frac{a^0}{a^1}, \frac{a^2}{a^1}, \dots, \frac{a^n}{a^1} \right).$$

We will refer to the coordinate functions on U_0 as $x^1, \dots, x^n$, and the coordinate functions on U_1 as $y^1, \dots, y^n$. On U_0,

$$x^i = \frac{a^i}{a^0}, \quad i = 1, \dots, n,$$

and on U_1,

$$y^1 = \frac{a^0}{a^1}, \quad y^2 = \frac{a^2}{a^1}, \quad \dots, \quad y^n = \frac{a^n}{a^1}.$$

Then on $U_0 \cap U_1$,

$$y^1 = \frac{1}{x^1}, \quad y^2 = \frac{x^2}{x^1}, \quad y^3 = \frac{x^3}{x^1}, \quad \dots, \quad y^n = \frac{x^n}{x^1},$$

so

$$(\phi_1 \circ \phi_0^{-1})(x) = \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \frac{x^3}{x^1}, \dots, \frac{x^n}{x^1}\right).$$

This is a C^∞ function because $x^1 \neq 0$ on $\phi_0(U_0 \cap U_1)$. On any other $U_i \cap U_j$ an analogous formula holds. Therefore, the collection $\{(U_i, \phi_i)\}_{i=0,\dots,n}$ is a C^∞ atlas for $\mathbb{R}P^n$, called the *standard atlas*. This concludes the proof that $\mathbb{R}P^n$ is a C^∞ manifold.

Problems

7.1. Image of the inverse image of a map
Let $f\colon X \to Y$ be a map of sets, and let $B \subset Y$. Prove that $f(f^{-1}(B)) = B \cap f(X)$. Therefore, if f is surjective, then $f(f^{-1}(B)) = B$.

7.2. Real projective plane
Let H^2 be the closed upper hemisphere in the unit sphere S^2, and let $i\colon H^2 \to S^2$ be the inclusion map. In the notation of Example 7.13, prove that the induced map $f\colon H^2/{\sim} \to S^2/{\sim}$ is a homeomorphism. (*Hint*: Imitate Proposition 7.3.)

7.3. Closedness of the diagonal of a Hausdorff space
Deduce Theorem 7.7 from Corollary 7.8. (*Hint*: To prove that if $S/{\sim}$ is Hausdorff, then the graph R of $\sim$ is closed in $S \times S$, use the continuity of the projection map $\pi\colon S \to S/{\sim}$. To prove the reverse implication, use the openness of π.)

7.4.* Quotient of a sphere with antipodal points identified
Let S^n be the unit sphere centered at the origin in $\mathbb{R}^{n+1}$. Define an equivalence relation $\sim$ on S^n by identifying antipodal points:

$$x \sim y \iff x = \pm y, \quad x, y \in S^n.$$

(a) Show that $\sim$ is an open equivalence relation.
(b) Apply Theorem 7.7 and Corollary 7.8 to prove that the quotient space $S^n/{\sim}$ is Hausdorff, without making use of the homeomorphism $\mathbb{R}P^n \simeq S^n/{\sim}$.

7.5.* Orbit space of a continuous group action
Suppose a right action of a topological group G on a topological space S is continuous; this simply means that the map $S \times G \to S$ describing the action is continuous. Define two points x, y of S to be equivalent if they are in the same orbit; i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the *orbit space* of the action. Prove that the projection map $\pi\colon S \to S/G$ is an open map. (This problem generalizes Proposition 7.14, in which $G = R^\times = \mathbb{R} - \{0\}$ and $S = \mathbb{R}^{n+1} - \{0\}$. Because $\mathbb{R}^\times$ is commutative, a left $\mathbb{R}^\times$-action becomes a right $\mathbb{R}^\times$-action if scalar multiplication is written on the right.)

7.6. Quotient of $\mathbb{R}$ by $2\pi\mathbb{Z}$
Let the additive group $2\pi\mathbb{Z}$ act on $\mathbb{R}$ on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that the orbit space $R/2\pi\mathbb{Z}$ is a smooth manifold.

7.7. The circle as a quotient space

(a) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha=1}^2$ be the atlas of the circle S^1 in Example 5.7, and let $\bar{\phi}_\alpha$ be the map ϕ_α followed by the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. On $U_1 \cap U_2 = A \amalg B$, since ϕ_1 and ϕ_2 differ by an integer multiple of 2π, $\bar{\phi}_1 = \bar{\phi}_2$. Therefore, $\bar{\phi}_1$ and $\bar{\phi}_2$ piece together to give a well-defined map $\bar{\phi}\colon S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Prove that $\bar{\phi}$ is C^∞.

(b) The complex exponential $\mathbb{R} \to S^1$, $t \mapsto e^{it}$, is constant on each orbit of the action of $2\pi\mathbb{Z}$ on $\mathbb{R}$. Therefore, there is an induced map $F\colon \mathbb{R}/2\pi\mathbb{Z} \to S^1$, $F([t]) = e^{it}$. Prove that F is C^∞.

(c) Prove that $F\colon \mathbb{R}/2\pi\mathbb{Z} \to S^1$ is a diffeomorphism.

7.8. The Grassmannian $G(k,n)$

The Grassmannian $G(k,n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $a_1, \ldots, a_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix $A = [a_1 \ \cdots \ a_k]$ of rank k, where the *rank* of a matrix A, denoted by $\operatorname{rk} A$, is defined to be the number of linearly independent columns of A. This matrix is called a *matrix representative* of the k-plane. (For properties of the rank, see the problems in Appendix B.)

Two bases $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k, \mathbb{R})$ such that

$$b_j = \sum_i a_i g_{ij}, \quad 1 \le i, j \le k.$$

In matrix notation, $B = Ag$.

Let $F(k,n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$, and $\sim$ the equivalence relation

$$A \sim B \quad \text{iff} \quad \text{there is a matrix } g \in \mathrm{GL}(k,\mathbb{R}) \text{ such that } B = Ag.$$

In the notation of Problem B.3, $F(k,n)$ is the set $D_{\max}$ in $\mathbb{R}^{n\times k}$ and is therefore an open subset. There is a bijection between $G(k,n)$ and the quotient space $F(k,n)/\sim$. We give the Grassmannian $G(k,n)$ the quotient topology on $F(k,n)/\sim$.

(a) Show that $\sim$ is an open equivalence relation. (*Hint*: Either mimic the proof of Proposition 7.14 or apply Problem 7.5.)

(b) Prove that the Grassmannian $G(k,n)$ is second countable. (*Hint*: Apply Corollary 7.10.)

(c) Let $S = F(k,n)$. Prove that the graph R in $S \times S$ of the equivalence relation $\sim$ is closed. (*Hint*: Two matrices $A = [a_1 \ \cdots \ a_k]$ and $B = [b_1 \ \cdots \ b_k]$ in $F(k,n)$ are equivalent if and only if every column of B is a linear combination of the columns of A if and only if $\operatorname{rk}[A\ B] \le k$ if and only if all $(k+1) \times (k+1)$ minors of $[A\ B]$ are zero.)

(d) Prove that the Grassmannian $G(k,n)$ is Hausdorff. (*Hint*: Mimic the proof of Proposition 7.16.)

Next we want to find a C^∞ atlas on the Grassmannian $G(k,n)$. For simplicity, we specialize to $G(2,4)$. For any 4×2 matrix A, let A_{ij} be the 2×2 submatrix consisting of its ith row and jth row. Define

$$V_{ij} = \{A \in F(2,4) \mid A_{ij} \text{ is nonsingular}\}.$$

Because the complement of V_{ij} in $F(2,4)$ is defined by the vanishing of $\det A_{ij}$, we conclude that V_{ij} is an open subset of $F(2,4)$.

(e) Prove that if $A \in V_{ij}$, then $Ag \in V_{ij}$ for any nonsingular matrix $g \in \mathrm{GL}(2,\mathbb{R})$.

Define $U_{ij} = V_{ij}/\!\sim$. Since $\sim$ is an open equivalence relation, $U_{ij} = V_{ij}/\!\sim$ is an open subset of $G(2,4)$.

For $A \in V_{12}$,

$$A \sim AA_{12}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}.$$

This shows that the matrix representatives of a 2-plane in U_{12} have a canonical form B in which B_{12} is the identity matrix.

(f) Show that the map $\tilde{\phi}_{12} \colon V_{12} \to \mathbb{R}^{2\times 2}$,

$$\tilde{\phi}_{12}(A) = A_{34}A_{12}^{-1},$$

induces a homeomorphism $\phi_{12} \colon U_{12} \to \mathbb{R}^{2\times 2}$.

(g) Define similarly homeomorphisms $\phi_{ij} \colon U_{ij} \to \mathbb{R}^{2\times 2}$. Compute $\phi_{12} \circ \phi_{23}^{-1}$, and show that it is C^∞.

(h) Show that $\{U_{ij} \mid 1 \le i < j \le 4\}$ is an open cover of $G(2,4)$ and that $G(2,4)$ is a smooth manifold.

Similar consideration shows that $F(k,n)$ has an open cover $\{V_I\}$, where I is a strictly ascending multi-index $1 \le i_1 < \cdots < i_k \le n$. For $A \in F(k,n)$, let A_I be the $k \times k$ submatrix of A consisting of i_1th, ..., i_kth rows of A. Define

$$V_I = \{A \in G(k,n) \mid \det A_I \neq 0\}.$$

Next define $\tilde{\phi}_I \colon V_I \to \mathbb{R}^{(n-k)\times k}$ by

$$\tilde{\phi}_I(A) = (AA_I^{-1})_{I'},$$

where $(\ )_{I'}$ denotes the $(n-k)\times k$ submatrix obtained from the complement I' of the multi-index I. Let $U_I = V_I/\!\sim$. Then $\tilde{\phi}$ induces a homeomorphism $\phi \colon U_I \to \mathbb{R}^{(n-k)\times k}$. It is not difficult to show that $\{(U_I, \phi_I)\}$ is a C^∞ atlas for $G(k,n)$. Therefore the Grassmannian $G(k,n)$ is a C^∞ manifold of dimension $k(n-k)$.

7.9.* Compactness of real projective space

Show that the real projective space $\mathbb{R}P^n$ is compact. (*Hint*: Use Exercise 7.11.)