Some Transitive Linear Actions of Real Simple Lie Groups

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Abstract. In Moskowitz M., and R.Sacksteder, *An extension of the Minkowski-Hlawka theorem*, Mathematika **56** (2010), 203-216, essential use was made of the fact that in its natural linear action the real symplectic group, $\mathrm{Sp}(n,\mathbb{R})$, acts transitively on $\mathbb{R}^{2n}\setminus\{0\}$ (similarly for the theorem of Hlawka itself, $\mathrm{SL}(n,\mathbb{R})$ acts transitively on $\mathbb{R}^n\setminus\{0\}$). This raises the natural question as to whether there are *proper connected* Lie subgroups of either of these groups which also act transitively on $\mathbb{R}^{2n}\setminus\{0\}$, (resp. $\mathbb{R}^n\setminus\{0\}$). Here we determine all the minimal ones. These are $\mathrm{Sp}(n,\mathbb{R})\subseteq \mathrm{SL}(2n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})\subseteq \mathrm{SL}(2n,\mathbb{R})$ acting on $\mathbb{R}^{2n}\setminus\{0\}$; on $\mathbb{R}^{4n}\setminus\{0\}$, they are $\mathrm{Sp}(2n,\mathbb{R})\subseteq \mathrm{SL}(4n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{H})(=\mathrm{SU}^*(2n))\subseteq \mathrm{SL}(4n,\mathbb{R})$.

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1. Introduction

The article [6] is concerned with an extension of the following theorem of Hlawka. Theorem (Minkowski-Hlawka). If $\mathrm{vol}(D)<\zeta(n)$, then there exists a lattice Γ in $\mathbb{R}^n$ of $\mathrm{vol}(\mathbb{R}^n/\Gamma)=1$ with $D\cap\Gamma=\{0\}$.

Here ζ denotes the Riemann zeta function, $\mathbb{R}^n$ takes Lebesgue measure and D is a domain in $\mathbb{R}^n$ star shaped about the origin. Of course, Hlawka's result can be expressed in terms of the group $\mathrm{SL}(n,\mathbb{R})$. Namely, if $\mathrm{vol}(D)<\zeta(n)$, then there exists a $g\in \mathrm{SL}(n,\mathbb{R})$ with $gD\cap\mathbb{Z}^n=\{0\}$ and in this form it was reproved by both Siegel [10] and Weil [11]. In [6] the authors did similarly for the symplectic group. Given a fixed choice of Haar measure for the ambient group, the volume, V_n, of a fundamental domain for the lattice $\mathrm{Sp}(n,\mathbb{Z})$ in $\mathrm{Sp}(n,\mathbb{R})$ was calculated: $V_n=\frac{1}{\sqrt{2}}\prod_{k=1}^{n}\zeta(2k)$, and as a consequence,

1. If $\mathrm{vol}(D)>V_n$, some lattice in $\mathbb{R}^{2n}$ contains a non zero point of D.
2. If $\mathrm{vol}(D)<V_n$, some lattice in $\mathbb{R}^{2n}$ contains only the zero point of D.
3. If D is star shaped about the origin and $\mathrm{vol}(D)<\zeta(2n)V_n$, some lattice in $\mathbb{R}^{2n}$ contains only the zero point of D.

*This paper is dedicated to the memory of Gerhard Hochschild

In that study essential use was made of the fact that in its natural linear action the real symplectic group, $\mathrm{Sp}(n,\mathbb{R})$, acts transitively on $\mathbb{R}^{2n}\setminus\{0\}$ (similarly for the theorem of Hlawka itself, $\mathrm{SL}(n,\mathbb{R})$ acts transitively on $\mathbb{R}^n\setminus\{0\}$). This raises the natural question as to whether there are *proper connected* Lie subgroups, G, of either of these groups which also act transitively on $\mathbb{R}^{2n}\setminus\{0\}$, (resp. $\mathbb{R}^n\setminus\{0\}$).

For $n\geq 2$, $\mathrm{Sp}(n,\mathbb{R})$ is indeed a proper connected Lie subgroup of $\mathrm{SL}(2n,\mathbb{R})$ which acts transitively on $\mathbb{R}^{2n}\setminus\{0\}$. Thus leaving open the case of $\mathrm{SL}(n,\mathbb{R})$, for n odd, and $\mathrm{Sp}(n,\mathbb{R})$, for $2n$ even. However the same argument (see pg. 24 of [1]) showing that $\mathrm{SL}(n,\mathbb{R})$ acts transitively on $\mathbb{R}^n\setminus\{0\}$ also proves $\mathrm{SL}(n,\mathbb{C})$ acts transitively on $\mathbb{C}^n\setminus\{0\}=\mathbb{R}^{2n}\setminus\{0\}$ and $\mathrm{SL}(n,\mathbb{H})$ acts transitively on $\mathbb{H}^n\setminus\{0\}=\mathbb{R}^{4n}\setminus\{0\}$.

Our purpose here is to determine the *minimal* ones, i.e. those which contain no proper connected Lie subgroup with the same property. Namely,

Theorem 1.1. *When n is odd, no connected Lie subgroup of $\mathrm{SL}(n,\mathbb{R})$ can act transitively on $\mathbb{R}^n\setminus\{0\}$. When $n=2k$ is even, with k odd, both $\mathrm{Sp}(k,\mathbb{R})$ and $SL(k,\mathbb{C})$ act transitively on $\mathbb{R}^{2k}\setminus\{0\}$ and they are the minimal ones. When $k=2m$ is even and $n=4m$, both $\mathrm{Sp}(2m,\mathbb{R})\subseteq\mathrm{SL}(4m,\mathbb{R})$ and $\mathrm{SL}(m,\mathbb{H})(=\mathrm{SU}^*(m))\subseteq\mathrm{SL}(4m,\mathbb{R})$ act transitively on $\mathbb{R}^{4m}\setminus\{0\}$ and they are the minimal ones.*

Presumably a similar study as in [6] could be made for $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{H})$.

2. Reduction of the problem.

In this section we reduce the question to the case of a non-compact simple Lie group by proving Theorem 2.2 below.

Let G be any closed connected Lie subgroup of $\mathrm{SL}(n,\mathbb{R})$ acting transitively on $\mathbb{R}^n\setminus\{0\}$. By Proposition 6.4.5 of [2] the Lie algebra of G is reductive, i.e. $\mathfrak{g}=\mathfrak{z}(\mathfrak{g})\oplus[\mathfrak{g},\mathfrak{g}]$, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ and the derived subalgebra, $[\mathfrak{g},\mathfrak{g}]$, is semisimple and so $G=Z(G)_0\cdot[G,G]$, where $Z(G)_0$ is the connected component of the center of G and the derived subgroup, $[G,G]$, is connected and semisimple. Moreover, $Z(G)_0$ acts completely reducibly by [3]. By Mostow's Theorem 6 of [7] (which is equivalent to the Theorem of Section 6) we can assume, which we do from now on, that the Cartan involution of G is the restriction of the usual Cartan involution of $\mathrm{SL}(n,\mathbb{R})$. By a *real reductive* subgroup of $\mathrm{SL}(n,\mathbb{R})$ we always mean a reductive self-conjugate subgroup of $\mathrm{SL}(n,\mathbb{R})$.

Lemma 2.1. *Let G be a connected, non-compact, real reductive Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ and K be a maximal compact subgroup.*[1] *Then G acts transitively on $\mathbb{R}^n\setminus\{0\}$, if and only if K acts transitively on the unit sphere, S^{n-1}.*

Proof. Let $G=KAN=KB$ be an Iwasawa decomposition of G (see [1]). Since B is in real triangular form, let $e_1,\dots e_n$ be the basis of $\mathbb{R}^n$ that puts B into this form consisting of vectors of norm 1. Then $be_1=\lambda(b)e_1$ for all $b\in B$,

[1]Since G is linear, K is actually compact

where λ is a non-trivial element in $\mathrm{Hom}(B, \mathbb{R}_+^\times)$. In particular, $\lambda(b) \not\equiv 1$ on B and $Be_1 = \mathbb{R}_+^\times e_1$. Now it is clear that if K is transitive on the unit sphere then G is transitive on $\mathbb{R}^n \setminus \{0\}$. Conversely, assume G is transitive on $\mathbb{R}^n \setminus \{0\}$. Given an arbitrary unit vector v, there is some $g = kb$ so that $g(e_1) = v$. That is, $kb(e_1) = k(\lambda(b)e_1) = v$. Thus $k(e_1) = \frac{1}{\lambda(b)}v$. Since K preserves the norm, $\frac{1}{\lambda(b)}v$ also has norm 1. Hence $\frac{1}{\lambda(b)} = 1$ and so $k(e_1) = v$. ■

Theorem 2.2. *Suppose G is a connected Lie subgroup of $\mathrm{SL}(n, \mathbb{R})$ which acts transitively on $\mathbb{R}^n \setminus \{0\}$ and is minimal with respect to this property. Then G is a non-compact simple Lie group.*

Proof. Let H be a subgroup of $\mathrm{SL}(n, \mathbb{R})$, acting transitively on $\mathbb{R}^n \setminus \{0\}$. As above, we may assume H is a non-compact, real reductive group. By Lemma 2.1, a maximal compact subgroup, K, acts transitively on the sphere S^{n-1}. By [5], Thm. I and Thm. I′, the group K is either simple, or, only when n is even, it is possibly a finite quotient of the product of two compact simple groups K_1 and K_2. When this happens, $K_2 = \mathrm{SO}(2)$ or $\mathrm{SU}(2)$ and K_1 is a simple group acting transitively on S^{n-1}. Also, the subgroup of K corresponding to K_1 under the quotient map acts transitively on S^{n-1} as well.

Let $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}_s$ be the Lie algebra of H, where $\mathfrak{z}(\mathfrak{h})$ is the center and $\mathfrak{h}_s = [\mathfrak{h}, \mathfrak{h}]$ is the derived subalgebra. Recall that such a decomposition is compatible with the Cartan decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup of H.

If K is simple, then $\mathfrak{k} \cap \mathfrak{z}(\mathfrak{h}) = \{0\}$. It follows that there exists a non-compact simple component $\mathfrak{g}$ of $\mathfrak{h}$ with a maximal compact subalgebra equal to $\mathfrak{k}$. Let G be the connected subgroup of H with Lie algebra $\mathfrak{g}$. Then by [5], Thm. I, a maximal compact subgroup of G acts transitively on S^{n-1}. By Lemma 2.1, the simple group G acts transitively on $\mathbb{R}^n \setminus \{0\}$.

Assume now that n is even and the maximal compact subgroup K is not simple. If $n = 2$, then $K = \mathrm{SO}(2)$ and $G = \mathrm{SL}(2, \mathbb{R})$. If $n \geq 4$, then there are the following possibilities for the Lie algebra of H:

(2.a) $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}_s$, with $\mathfrak{h}_s$ simple, $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ maximal compact in $\mathfrak{h}_s$ and $\mathfrak{k} \cap \mathfrak{z}(\mathfrak{h}) = \{0\}$;

(2.b) $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}_s$, with $\mathfrak{h}_s$ simple, $\mathfrak{k}_1$ maximal compact in $\mathfrak{h}_s$ and $\mathfrak{k}_2 \cong \mathbb{R}$ contained in $\mathfrak{z}(\mathfrak{h})$ (this happens for example if $H = \mathrm{Sp}(k, \mathbb{R})$);

(2.c) $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$, with $\mathfrak{h}_i$ simple, $\mathfrak{k}_i$ maximal compact in $\mathfrak{h}_i$, and $\mathfrak{k} \cap \mathfrak{z}(\mathfrak{h}) = \{0\}$ (this happens for example if $H = \mathrm{Sp}(k, 1)$).

We claim that there exists a non-compact simple subgroup $G \subset H$, acting transitively on $\mathbb{R}^n \setminus \{0\}$. In case (2.a) and case (2.b), such group G is the connected subgroup generated by $\mathfrak{h}_s$. In case (2.c), G is the connected subgroup generated by $\mathfrak{h}_1$, for all even $n > 6$. For $n = 6$, the subgroup G is generated either by $\mathfrak{h}_1$ or by $\mathfrak{h}_2$. In all cases, by [5], Thm. I, a maximal compact subgroup of G acts transitively on S^{n-1}, implying that G acts transitively on $\mathbb{R}^n \setminus \{0\}$. ■

The next lemma shows that the Lie algebra $\mathfrak{g}$ and the maximal compact subgroup K uniquely determine G within $\mathrm{SL}(n, \mathbb{R})$.

Lemma 2.3. *Let G and H be connected Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$ which are locally isomorphic and K_G and K_H be maximal compact subgroups of each. If K_G and K_H are isomorphic, by say ϕ, then G and H are also isomorphic, by say ψ. By changing K_H via a conjugation by something in H we can arrange for ψ to be an extension of ϕ.*

Proof. Since K_G and K_H are isomorphic they must have the same fundamental groups; $\Pi_1(K_G) = \Pi_1(K_H)$. On the other hand, since K_G is a retract of G and similarly for H we know $\Pi_1(G) \cong \Pi_1(K_G)$ and $\Pi_1(H) \cong \Pi_1(K_H)$ so that $\Pi_1(G) \cong \Pi_1(H)$. Let L be the common universal cover of both G and H, with π_G and π_H the respective covering maps. Then $L/\Pi_1(G) = G$ and $L/\Pi_1(H) = H$ and since $\Pi_1(G) \cong \Pi_1(H)$ it follows that $G \cong H$ (by say ψ).

Now consider the differentials of these isomorphisms $d(\phi) : \mathfrak{k}_G \to \mathfrak{k}_H$ and $d(\psi) : \mathfrak{g} \to \mathfrak{h}$. Since $d(\psi)$ is a Lie algebra isomorphism it takes a maximum compact subalgebra of $\mathfrak{g}$ onto one of $\mathfrak{h}$ and since such things are conjugate we can replace $\mathfrak{k}_H$ by a new maximal compact subalgebra of $\mathfrak{h}$ so that $d(\psi)(\mathfrak{k}_G) = \mathfrak{k}_H$. ■

3. Proof of Theorem 1.

Effective transitive actions of connected compact Lie groups on spheres have been studied and classified by Montgomery-Samelson and Borel. We refer to the list given in [4]:

1. $n = 2, \quad K = \mathrm{SO}(2)$;
2. $n = 2k+1, \quad K = \mathrm{SO}(2k+1)$;

2.a. $n = 7, \quad K = \mathrm{G}_2$;

4. $n = 2k,\ k > 1, \quad K = \mathrm{SO}(2k),\ \mathrm{U}(k),\ \mathrm{SU}(k)$;
5. $n = 4k, \quad K = \mathrm{SO}(4k),\ \mathrm{U}(2k),\ \mathrm{SU}(2k),\ \mathrm{Sp}(k),\ \mathrm{Sp}(k) \cdot \mathrm{S}^1,\ \mathrm{Sp}(k) \cdot \mathrm{Sp}(1)$;

5.a. $n = 16, \quad K = \mathrm{Spin}(9)$;

5.b. $n = 8, \quad K = \mathrm{Spin}(7)$;

with the *only* inclusions:

$\mathrm{G}_2 \subset \mathrm{SO}(7)$;
$\mathrm{SU}(k) \subset \mathrm{U}(k) \subset \mathrm{SO}(2k)$;
$\mathrm{Sp}(k) \subset \mathrm{Sp}(k) \cdot \mathrm{S}^1 \subset \mathrm{Sp}(k) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4k)$;
$\mathrm{Sp}(k) \subset \mathrm{Sp}(k) \cdot \mathrm{S}^1 \subset \mathrm{U}(2k)$;
$\mathrm{SU}(4) \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8)$;
$\mathrm{Spin}(9) \subset \mathrm{SO}(16)$;

The inclusions $\mathrm{SU}(k) \subset \mathrm{U}(k) \subset \mathrm{SO}(2k)$ are given by equivariantly identifying $\mathbb{C}^k$ and $\mathbb{R}^{2k}$ under the standard actions of $\mathrm{U}(k)$ and $\mathrm{SO}(2k)$; the inclusions $\mathrm{Sp}(k) \subset \mathrm{Sp}(k) \cdot \mathrm{S}^1 \subset \mathrm{Sp}(k) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4k)$ are given by equivariantly identifying $\mathbb{H}^k$ and $\mathbb{R}^{4k}$ under the quaternionic representation $\rho_k \otimes \rho_1$ of $\mathrm{Sp}(k) \cdot \mathrm{Sp}(1)$ on $\mathbb{H}^k$

and the standard action of $\mathrm{SO}(4k)$ on $\mathbb{R}^{4k}$, where ρ_k denotes the standard action of $\mathrm{Sp}(k)$ on $\mathbb{H}^k$.

The inclusion $\mathrm{G}_2 \subset \mathrm{SO}(7)$ is given by the 7-dimensional representation of G_2, which is absolutely irreducible (see Samelson [9], Thm.E, pg.140) (a representation of a compact group K on a real vector space V is said to be absolutely irreducible if it remains irreducible over $\mathbb{C}$); the inclusion $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ is given by the 8-dimensional spin representation of $\mathrm{Spin}(7)$. Since $7 = 2 \cdot 3 + 1$ and $3 \not\equiv 1, 2 \bmod 4$ such a representation is absolutely irreducible (see [9], Thm.E, pg.140); the inclusion $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is given by the 16-dimensional spin representation of $\mathrm{Spin}(9)$. Since $9 = 2 \cdot 4 + 1$ and $4 \not\equiv 1, 2 \bmod 4$ such a representation is absolutely irreducible (see [9], Thm.E, pg.140).

Proof. Let $G \subset \mathrm{SL}(n, \mathbb{R})$ be a non-compact simple group acting transitively on $\mathbb{R}^n \setminus \{0\}$. Then by Lemma 2.1, one of its maximal compact subgroups K must appear in the above list. Further, by Lemma 2.3, the group G is completely determined by K and its Lie algebra $\mathfrak{g}$. Now we are left to check which K in the above list is a maximal compact subgroup of some non-compact simple group $G \subset \mathrm{SL}(n, \mathbb{R})$, which in addition, is transitive on $\mathbb{R}^n \setminus \{0\}$.

Observe that if the K-action on $\mathbb{R}^n$ is absolutely irreducible, then $G \neq K^{\mathbb{C}}$ (see Onishchik [8], Thm.1, pg.65).

As we already know, for every integer n the group $\mathrm{SL}(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n \setminus \{0\}$ by its standard representation.

Let $n = 2k + 1$ be odd. We claim there exists no simple group, G, properly contained in $\mathrm{SL}(2k + 1, \mathbb{R})$, which acts transitively on $\mathbb{R}^{2k+1} \setminus \{0\}$.

The group $K = \mathrm{SO}(2k + 1)$ is also a maximal compact subgroup of $G = \mathrm{SO}_0(2k + 1, 1)$, but this group has no linear action on $\mathbb{R}^{2k+1}$. If $k = 3$, the compact group G_2 acts transitively on S^6 via its 7-dimensional fundamental representation. If a non-compact simple group G properly contained in $\mathrm{SL}(7, \mathbb{R})$ were transitive on $\mathbb{R}^7 \setminus \{0\}$, then one of its maximal compact subgroups would satisfy $\mathrm{G}_2 \subset K \subset \mathrm{SO}(7)$ and would act transitively on S^6 as well. Then either $K = \mathrm{G}_2$ and $G = \mathrm{G}_2^{\mathbb{C}}$, or $K = \mathrm{SO}(7)$ and $G = \mathrm{SL}(7, \mathbb{R})$. Since the 7-dimensional fundamental representation of G_2 is absolutely irreducible, by the first observation $G \neq \mathrm{G}_2^{\mathbb{C}}$, and $G = \mathrm{SL}(7, \mathbb{R})$. We conclude, when n is odd, there are no proper subgroups of $\mathrm{SL}(n, \mathbb{R})$ acting transitively on $\mathbb{R}^n \setminus \{0\}$.

Now we turn to even dimensional real vector spaces $\mathbb{R}^{2k}$, $k \geq 1$. Assume first k *odd.* We claim there exists no simple group, G, properly contained in $\mathrm{Sp}(k, \mathbb{R})$, which acts transitively on $\mathbb{R}^{2k} \setminus \{0\}$.

From the compact groups $K = \mathrm{SU}(k)$ and $K = \mathrm{U}(k)$ we get

$$G = \mathrm{SL}(k, \mathbb{C}) \subset \mathrm{SL}(2k, \mathbb{R}), \quad G = \mathrm{GL}(k, \mathbb{C}) \subset \mathrm{GL}(2k, \mathbb{R}),$$

$$G = \mathrm{Sp}(k, \mathbb{R}) \subset \mathrm{SL}(2k, \mathbb{R}).$$

Each of the above non-compact groups acts transitively on $\mathbb{R}^{2k} \setminus \{0\}$ via the standard representation of $\mathrm{GL}(2k, \mathbb{R})$. Both $\mathrm{SL}(k, \mathbb{C})$ and $\mathrm{Sp}(k, \mathbb{R})$ are minimal, $\mathrm{SL}(k, \mathbb{C})$ is the one of smallest dimension. In particular, no proper subgroup of $\mathrm{Sp}(k, \mathbb{R})$ acts transitively on $\mathbb{R}^{2k} \setminus \{0\}$.

Assume now $k = 2m$ *even*. We claim there exists no simple group G, properly contained in $\mathrm{Sp}(2m, \mathbb{R})$, which acts transitively on $\mathbb{R}^{4m} \setminus \{0\}$. In this case there are additional compact groups acting transitively on the sphere S^{4m-1}.

For $K = \mathrm{Sp}(m) \subset \mathrm{SU}(2m)$, we get $G = \mathrm{SU}^*(m)$ acting transitively on $\mathbb{R}^{4m} \setminus \{0\}$. We have the inclusions

$$\mathrm{SU}^*(m) \subset \mathrm{SL}(2m, \mathbb{C}) \subset \mathrm{SL}(4m, \mathbb{R}), \quad \mathrm{GL}(2m, \mathbb{C}) \subset \mathrm{GL}(4m, \mathbb{R}),$$

$$\mathrm{Sp}(2m, \mathbb{R}) \subset \mathrm{SL}(4m, \mathbb{R}),$$

where each of the above non-compact groups acts transitively on $\mathbb{R}^{4m} \setminus \{0\}$. Both $\mathrm{SU}^*(m) = \mathrm{SL}(m, \mathbb{H})$ and $\mathrm{Sp}(2m, \mathbb{R})$ are minimal, $\mathrm{SU}^*(m) = \mathrm{SL}(m, \mathbb{H})$ is the one of smallest dimension. In particular, no proper subgroup of $\mathrm{Sp}(2m, \mathbb{R})$ acts transitively on $\mathbb{R}^{4m} \setminus \{0\}$.

It remains to show no other groups, G, act transitively on $\mathbb{R}^{4m} \setminus \{0\}$. Consider $\mathrm{Sp}(m) \subset \mathrm{Sp}(m) \cdot \mathrm{S}^1 \subset \mathrm{Sp}(m) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4m)$. Since $K = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ is a maximal compact subgroup of $G = \mathrm{Sp}(m, 1)$ and G does not act on $\mathbb{R}^{4m}$, we get nothing new from these cases.

For the transitive actions of $\mathrm{Spin}(7)$ and $\mathrm{Spin}(9)$ on the spheres S^7 and S^{15}, respectively, we argue as in the case of G_2 on S^6. If a simple group $G = K \exp \mathfrak{p}$, properly contained in $\mathrm{SL}(8, \mathbb{R})$ (resp. in $\mathrm{SL}(16, \mathbb{R})$), were transitive on $\mathbb{R}^8 \setminus \{0\}$ (resp. $\mathbb{R}^{16} \setminus \{0\}$), then one of its maximal compact subgroups would satisfy $\mathrm{Spin}(7) \subset K \subset \mathrm{SO}(8)$ (resp. $\mathrm{Spin}(9) \subset K \subset \mathrm{SO}(16)$). If $K = \mathrm{Spin}(7)$ (resp. $K = \mathrm{Spin}(9)$), then $G = \mathrm{Spin}(7, \mathbb{C})$ (resp. $G = \mathrm{Spin}(9, \mathbb{C})$, or $G = \mathrm{F}_{4(-20)}$). This is impossible because $\mathrm{Spin}(7, \mathbb{C})$ has no 8-dimensional real representations (resp. $\mathrm{Spin}(16, \mathbb{C})$ and $\mathrm{F}_4^{\mathbb{C}}$ have no 16-dimensional real representation).

We conclude the discussion by remarking that $\mathrm{U}(2m)$ is also a maximal compact subgroup of $\mathrm{SO}^*(4m)$, which does not act on $\mathbb{R}^{4m}$, that $\mathrm{Sp}(4)$ is also a maximal compact subgroup of $\mathrm{E\,I}$, which does not act on $\mathbb{R}^8$, that $\mathrm{SO}(16)$ is also a maximal compact subgroup of $\mathrm{E\,VIII}$, and $\mathrm{SU}(8)$ is a maximal compact subgroup of $\mathrm{E\,V}$, which do not act on $\mathbb{R}^{16} = \mathbb{C}^8$. Since we checked all compact groups acting transitively on spheres, the proof of the theorem is complete. ∎

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