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Group automorphisms preserving equivalence classes of unitary representations

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Abstract. We introduce and investigate the notion of a *quasi-complete* group. A group G is quasi-complete if every automorphism $\phi \in \mathrm{Aut}(G)$, with the property that π and $\pi \circ \phi$ are unitarily equivalent for every unitary irreducible representation π of G, is an inner automorphism of G. Our main result is that every connected linear real reductive Lie group is quasi-complete.

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Introduction

The study of the automorphisms group $\mathrm{Aut}(G)$ of a topological group G and of its distinguished normal subgroups raises many interesting questions. In connection with the normal subgroup of inner automorphisms, it was shown in [DG] that any group G can be realized as the outer automorphism group of some group H, i.e. $G \cong \mathrm{Aut}(H)/\mathrm{Inn}(H)$. In another direction, an abstract characterization of inner automorphisms in terms of their extension properties was recently given in [Sc].

Other distinguished normal subgroups of $\mathrm{Aut}(G)$ are the automorphisms $\mathrm{Aut}(G)_C$ preserving the conjugacy classes of G and the automorphisms $\mathrm{Aut}(G)_{\hat{G}}$ preserving the equivalence classes of (continuous) unitary representations of G. These definitions appear in [Bu] for finite groups, but carry over to more general settings. In general, one has the inclusions $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)_{\hat{G}} \subseteq \mathrm{Aut}(G)_C \subseteq \mathrm{Aut}(G)$. On the other hand, if G is an arbitrary compact group, one has that $\mathrm{Aut}(G)_C = \mathrm{Aut}(G)_{\hat{G}}$. The groups all of whose automorphisms are inner are called *complete* and have been extensively studied (cf. [Sz] [Hu]).

In this paper, we consider a notion which generalizes the one of completeness. We call *quasi-complete* the groups for which $\mathrm{Aut}(G)_{\hat{G}} = \mathrm{Inn}(G)$: if $\phi \in \mathrm{Aut}(G)$ is an automorphism with the property that the representations π and $\pi \circ \phi$ are unitarily equivalent for every unitary representation π of G, then ϕ is an inner automorphism of G.

Our interest in quasi-complete groups arose in connection with C^*-algebras. Given a C^*-algebra $\mathscr{A}$ one can consider the crossed product $\mathscr{B}$ of $\mathscr{A}$ by the dual of a compact group G (see [DR]) and study extensions of automorphisms or antiautomorphisms from $\mathscr{A}$ to $\mathscr{B}$. Extensions commuting with the action of G on $\mathscr{B}$ may not exist. If the group G is quasi-complete one can give necessary and sufficient conditions for such extensions to exist. For more details we refer to [CD] or to [BDLR].

Like completeness, the property of being quasi-complete makes sense also for non-compact, locally compact groups. As one can easily deduce from [Pe1] [Pe2], interesting examples of quasi-complete non-complete groups are provided by the non-commutative free groups $\mathbb{F}_s$ over a finite number of generators. Of course, complete groups such as $\mathrm{Aut}(\mathbb{F}_r)$, $\mathrm{Aut}(\mathrm{Aut}(\mathbb{B}_3))$, $\mathrm{Aut}(\mathbb{B}_n)$ $(n \geq 4)$, where $\mathbb{B}_n$ denotes the braid group over n strands, are quasi-complete (see [DF] [DG2]). However direct products of complete groups need not be complete, while direct products of finitely many quasi-complete groups are quasi-complete. In view of the above remarks, our investigation goes beyond the compact case and our main result is the following theorem.

Theorem. *Let G be a connected linear real reductive Lie group. Let $\phi \in \mathrm{Aut}(G)$ be an automorphism with the property that the representations π and $\pi \circ \phi$ are unitarily equivalent, for every unitary representation π of G. Then ϕ is an inner automorphism of G. In short, G is quasi-complete.*

This theorem covers locally compact groups like $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SO}_0(m, n)$, $\mathrm{Sp}(m, n)$, etc. In the course of our investigation we recover the well-known result that an arbitrary compact connected group is quasi-complete. See [Mc] [Wa] [Ha] for further motivations and discussions on the compact case. More generally, we show that the class of quasi-complete compact groups is stable under taking projective limits, and that an arbitrary direct product of compact groups is quasi-complete if and only if every factor is (cf. Sect. 2).

In general, as soon as we relax the connectedness assumption quasi-completeness may fail. An example of a finite group which is not quasi-complete was given by G. C. Wall (see [Hu]). We show that such an example is not isolated but fits into a more general pattern.

From the point of view of abstract DR-duality theory (cf. [DR]), compact connected groups play a distinguished role because an isomorphism of their fusion rings already implies an isomorphism of the underlying groups. In view of possible extensions of duality theory to the non-compact setting our results should supply a large class of examples for which the description of the groups by their duals is somehow "redundancy-free" and thus more accessible.

The paper is organized as follows. In section 1, we give the basic definitions. In section 2, we deal with projective limits and direct products of quasi-complete compact groups. In sections 3–5, we deal with connected linear real reductive groups. In section 3, we reduce to the semisimple case. In section 4, we show that a connected linear real reductive group satisfying the equal-rank condition is quasi-complete. In section 5, using the results about equal-rank groups and parabolic induction, we prove that a connected linear real reductive Lie group is quasi-complete. In section 6 we consider finite groups. We construct a family of finite groups which are not quasi-complete. Wall's counterexample fits into this pattern.

We thank the referee for suggesting the proof of Lemma 3.1 in the present form.

1 Preliminaries

Throughout the paper we denote a Lie group by a roman letter and its Lie algebra by the corresponding gothic letter: if G, H are groups, $\mathfrak{g}, \mathfrak{h}$ are the corresponding Lie algebras. We denote by $\mathfrak{g}^{\mathbb{C}}$ the complexification of a Lie algebra $\mathfrak{g}$. We say that a group G is a real form of a complex group $G^{\mathbb{C}}$ if there exists a conjugation κ of $G^{\mathbb{C}}$ whose fixed point set is G. We write G^0 for the connected component of the identity of a group G.

We denote by $\mathrm{Aut}(G)$ the group of (continuous) automorphisms of G and by $\mathrm{Inn}(G)$ the normal subgroup of the inner automorphisms. For $x \in G$, we denote by Ad_x the inner automorphism $\phi(g) = xgx^{-1}$, $g \in G$. In general, we denote by the same symbol an automorphism of a Lie group, the derived automorphism of its Lie algebra and its extension to the complexified Lie algebra. Since we mainly deal with linear groups, the corresponding actions actually coincide. Let $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ be the quotient group. We say that $\phi \in \mathrm{Out}(G)$, meaning that the image of ϕ in $\mathrm{Out}(G)$ is different from the identity.

We denote by $\hat{G}$ the set of equivalence classes of irreducible unitary representations of G, i.e. the homomorphisms π of G into the unitary linear operators of a Hilbert space V^{π}, such that the resulting map of $G \times V^{\pi}$ into V^{π} is continuous (cf. [Kn1], Ch. 1). Recall that the unitary representations of a non-compact Lie group are generally infinite dimensional. We write $\pi \cong \tau$ meaning that the unitary representations π and τ are unitarily equivalent.

Definition 1.1. A group G is *quasi-complete* if every automorphism $\phi \in \mathrm{Aut}(G)$, with the property that the representations π and $\pi \circ \phi$ are unitarily equivalent for every irreducible unitary representation π of G, is an inner automorphism of G. In short if $\mathrm{Aut}(G)_{\hat{G}} = \mathrm{Inn}(G)$.

The notion of *quasi-complete* group generalizes the one of *complete* group, for which every automorphism is an inner automorphism (and the center is trivial).

Observe that if α is an automorphism of G and $\pi \circ \alpha$ is unitarily equivalent to π for every irreducible unitary representation π of G, then $\pi \circ \alpha$ is unitarily equivalent to π for every unitary representation π of G. The argument goes essentially as follows. If π is not irreducible, it can be disintegrated in a direct integral, with respect to a maximal abelian subalgebra in $\pi(G)'$ (the set of all operators which commute with $\pi(G)$)

$$\pi \cong \int^{\oplus} \pi_{\mu} \, \mathrm{d}\mu,$$

where the representations π_{μ} are irreducible almost everywhere with respect to μ (see [Ma], Ch. II). The fact that such a decomposition is generally not unique is irrelevant. Then $\pi \circ \alpha \cong \int^{\oplus} (\pi_{\mu} \circ \alpha) \, \mathrm{d}\mu$. Since $\pi_{\mu} \cong \pi_{\mu} \circ \alpha$ for almost every μ, there is a unitary U_{μ} such that $\pi_{\mu}(g) U_{\mu} = U_{\mu}(\pi_{\mu} \circ \alpha)(g)$, for all $g \in G$. By the Measurable Cross

Section Theorem ([Ta], Thm. A16) the function $\mu \to U_\mu$ can be chosen to be measurable and $U = \int^{\oplus} U_\mu \, d\mu$ implements the equivalence $\pi \cong \pi \circ \alpha$.

2 Projective limits and direct products of quasi-complete compact groups

In this section we collect some useful results concerning projective limits and direct products, thus showing that the class of quasi-complete groups has natural stability properties.

We briefly recall some facts about "inverse systems" and projective limits, taking as a reference [Du].

Let G be a compact group and let A be a maximal collection of pairwise inequivalent finite-dimensional continuous representations of G. For $\alpha \in A$, define $K_\alpha := \ker \alpha$ and $G_\alpha := G/K_\alpha$. Denote by π_α the canonical projection $\pi_\alpha : G \to G_\alpha$, $x \mapsto xK_\alpha$. Observe that G_α is a Lie group, since it is isomorphic with a closed subgroup of $U(\dim(\alpha))$. The set A is a partially ordered directed set when we define $\alpha \leq \beta$ if $K_\beta \subseteq K_\alpha$; This yields a continuous suriective map $f_{\beta\alpha} : G_\beta \to G_\alpha$, $f_{\beta\alpha}(xK_\beta) := xK_\alpha$. For every $\alpha \leq \beta$, we have the following commutative diagram

$$\begin{tikzcd} G \arrow[r, "\pi_\beta"] \arrow[d, "\pi_\alpha"] & G_\beta \arrow[dl, "f_{\beta\alpha}"] \\ G_\alpha & \end{tikzcd} \qquad \pi_\alpha = f_{\beta\alpha} \circ \pi_\beta, \quad \text{for all } \beta > \alpha.$$

In other words, $(G, G_\alpha, \pi_\alpha, f_{\beta\alpha})$ is an "inverse system" and G is the projective limit $G = \varprojlim G_\alpha$ of the G_α. The elements $x \in G$ can be identified with the infinite tuples $\{x_\alpha = \pi_\alpha(x)\}_{\alpha \in A}$ satisfying $x_\alpha = f_{\beta\alpha} x_\beta$, for every $\beta > \alpha$. The group G is connected if and only if all the G_α are connected.

Definition 2.1. A homomorphism between two inverse systems $(X, X_\alpha, \pi_\alpha, f_{\beta\alpha})$ and $(Y, Y_\alpha, \pi'_\alpha, g_{\beta\alpha})$ is a family of maps $\{h_\alpha\}_{\alpha \in A}$, $h_\alpha : X_\alpha \to Y_\alpha$ yielding the following commutative diagram

$$\begin{tikzcd} X_\beta \arrow[r, "h_\beta"] \arrow[d, "f_{\beta\alpha}"] & Y_\beta \arrow[d, "g_{\beta\alpha}"] \\ X_\alpha \arrow[r, "h_\alpha"] & Y_\alpha \end{tikzcd} \qquad g_{\beta\alpha} \circ h_\beta = h_\alpha \circ f_{\beta\alpha}, \quad \text{for all } \beta \geq \alpha.$$

Remark 2.2. Given a homomorphism between two inverse systems $(X, X_\alpha, \pi_\alpha, f_{\beta\alpha})$ and $(Y, Y_\alpha, \pi'_\alpha, g_{\beta\alpha})$, there exists a unique homomorphism $h : X \to Y$ such that $\pi'_\alpha \circ h = h_\alpha \circ \pi_\alpha$, for all $\alpha \in A$ (see [Du]).

The next result follows e.g. from the analysis in [Wa]. For reader's convenience we sketch the arguments.

Theorem 2.3. *Let G be a compact group, $G = \varprojlim G_\alpha$, and assume that every G_α is quasi-complete. Then G is quasi-complete.*

Proof of the theorem. The proof is divided into several elementary steps.

(1) Let $\sigma \in \mathrm{Aut}(G)_{\hat{G}}$. Then σ induces an automorphism $\{\sigma_\alpha\}_{\alpha \in A}$ of the inverse system $(G, G_\alpha, \pi_\alpha, f_{\beta\alpha})$. Of course σ also coincides with the automorphism arising from $\{\sigma_\alpha\}_{\alpha \in A}$ (cf. Remark 2.2).

Proof of (1). Let $\sigma \in \mathrm{Aut}(G)_{\hat{G}}$. Then, for every $\alpha \in A$, one has that $\sigma(K_\alpha) = K_\alpha$ and σ induces an automorphism σ_α of G_α by

$$\sigma_\alpha(\pi_\alpha(x)) = \pi_\alpha(\sigma(x)).$$

One can check that

$$f_{\beta\alpha} \circ \sigma_\beta = \sigma_\alpha \circ f_{\beta\alpha}, \quad \text{for every } \beta \geq \alpha.$$

By Remark 2.2, there exists a $\sigma_\infty \in \mathrm{Aut}(G)$ satisfying $\pi_\alpha \circ \sigma_\infty = \sigma_\alpha \circ \pi_\alpha$, for all α. By the uniqueness, it coincides with σ.

(2) Let $\sigma \in \mathrm{Aut}(G)_{\hat{G}}$. For all $\alpha \in A$, the automorphism σ_α of G_α induced by σ satisfies the condition

$$(*) \qquad \chi_\alpha \cong \chi_\alpha \circ \sigma_\alpha,$$

for every unitary representation χ_α of G_α.

Proof of (2). Given a unitary representation $\chi_\alpha : G_\alpha \to U(\mathbb{C}^N)$, define

$$\tilde{\chi}_\alpha : G \to U(\mathbb{C}^N), \quad \tilde{\chi}_\alpha(g) := \chi_\alpha(\pi_\alpha(g)).$$

Then $\tilde{\chi}_\alpha \in \hat{G}$ and $\ker \tilde{\chi}_\alpha \supset K_\alpha$. Similarly define

$$\widetilde{\chi_\alpha \circ \sigma_\alpha}(g) := \chi_\alpha \circ \sigma_\alpha(\pi_\alpha(g)) = \chi_\alpha \pi_\alpha(\sigma(g)) = \tilde{\chi}_\alpha \circ \sigma(g).$$

Since $\sigma \in \mathrm{Aut}(G)_{\hat{G}}$, the representations $\tilde{\chi}_\alpha$ and $\widetilde{\chi_\alpha \circ \sigma_\alpha} = \tilde{\chi}_\alpha \circ \sigma$ of G are equivalent, for all $\alpha \in A$. This yields

$$\chi_\alpha \circ \sigma_\alpha \cong \chi_\alpha, \quad \forall \alpha \in A.$$

(3) The automorphism $\sigma = \sigma_\infty$ of G is inner.

Proof of (3). Since σ_α is an automorphism of G_α satisfying condition $(*)$, and G_α is assumed to be quasi-complete, for every $\alpha \in A$, σ_α is the inner automorphism of G_α

defined by some $g_\alpha K_\alpha$, with $g_\alpha \in G$. Since G is compact, after passing to a subnet if necessary, one can assume that $g_\alpha \to g$ in G. Then it is straightforward to verify that $\sigma = Ad_g$. This concludes the proof of (3) and of the theorem.

Remark 2.4. An arbitrary compact connected group is a projective limit of compact connected Lie groups (cf. [P]). A straightforward application of the Highest Weight Theorem shows that compact connected Lie groups are quasi-complete. Hence from the above result one recovers in particular the well-known fact that a compact connected group G is quasi-complete (see [Mc] [Wa] [Ha]).

Next we give a precise characterization of direct products of compact groups which are quasi-complete.

Theorem 2.5. *Let $\{G_\iota\}_{\iota \in I}$ be a family of compact groups, parametrized by an arbitrary non-empty set I. Then the direct product $G := \times_{\iota \in I} G_\iota$ is a quasi-complete compact group if and only if every factor G_ι is quasi-complete.*

Proof. Recall that the characters of the irreducible representations of G are exactly those of the form $g = (g_\iota) \to \chi_{\iota_1}(g_{\iota_1}) \dots \chi_{\iota_n}(g_{\iota_n})$, where n ranges over all finite integers, the indices ι_k, for $1 \leq k \leq n$, are all distinct and χ_{ι_k} are the characters of the irreducible representations of G_{ι_k} (see [HR], 27.43).

If $\alpha \in \mathrm{Aut}(G)$ leaves all the characters of G invariant, i.e. $\chi \circ \alpha = \chi$ for all χ, then it is easy to see that it preserves each factor G_ι. Thus, for every $\iota \in I$, the restriction $\alpha|_{G_\iota}$ defines an automorphism α_ι of G_ι such that $\chi_\iota \circ \alpha_\iota = \chi_\iota$ for every irreducible character χ_ι of G_ι. If the G_ι are quasi-complete, it follows that α_ι is inner, i.e. $\alpha_\iota = Ad_{g_\iota^*}$ for some $g_\iota^* \in G_\iota$. Since the G_ι generate G, the automorphism α is completely determined by the α_ι and it is inner: $\alpha = Ad_{g^*}$, where $g^* = (g_\iota^*)$.

In order to show the converse, consider first the case where G is quasi-complete and of the form $G = G_1 \times G_2$. Let α_1 be an automorphism of G_1 with the property that $\rho_1 \circ \alpha_1$ is equivalent to ρ_1 for every unitary representation ρ_1 of G_1. Define $\alpha \in \mathrm{Aut}(G)$ by $\alpha = \alpha_1 \times Id_{G_2}$. Then

$$(\rho_1 \times \rho_2) \circ (\alpha_1 \times Id_{G_2}) = (\rho_1 \circ \alpha_1) \times (\rho_2 \circ Id_{G_2}) \cong \rho_1 \times \rho_2$$

for all unitary representations ρ_1 and ρ_2 of G_1 and G_2 respectively. Since G is quasi-complete, $\alpha_1 \times Id_{G_2}$ is an inner automorphism of G. It follows that there exists $g = (g_1^*, g_2^*) \in G$ such that $Ad_g = \alpha_1 \times Id_{G_2}$. In particular, $\alpha_1 = Ad_{g_1^*}$ and G_1 is quasi-complete. Similarly one obtains that G_2 is quasi-complete. The general case is now immediate.

Remark 2.6. A similar argument (but using directly representations instead of characters) applies to the case where G is the direct product of groups G_ι which are compact for all but finitely many indices ι and locally compact otherwise, so that the resulting G is locally compact. If the G_ι are quasi-complete for all $\iota \in I$ then G is quasi-complete too. Conversely, if such a $G = \times_\iota G_\iota$ is quasi-complete, then every G_ι is, provided that the G_ι are of type I (see [Di], for a definition).

3 Connected linear real reductive groups: reduction to the semisimple case

The main goal of this paper is to prove that connected linear real reductive Lie groups are quasi-complete. In this section we show that it is sufficient to consider the semisimple ones. The result is a consequence of the following general lemma.

Lemma 3.1. *Let G be a locally compact group and suppose that $G = ZH$, where both Z and H are closed characteristic subgroups of G and Z is contained in the center of G. If H is quasi-complete, then so is G.*

Proof. Let ϕ be an automorphism of G such that $\pi \circ \phi \cong \pi$ for all $\pi \in \hat{G}$. Since H and Z are both characteristic, $\phi(H) = H$ and $\phi(Z) = Z$.

Let $\pi \in \hat{G}$ and denote by $\mathscr{H}_\pi$ the Hilbert space of π. Then, since π is irreducible and Z is central, $\pi(z)$ is a multiple of the identity for each $z \in Z$. Thus every $\pi(H)$-invariant subspace of $\mathscr{H}_\pi$ is already $\pi(G)$-invariant, whence $\pi|H$ is irreducible. Clearly, $\pi|H \circ \phi|H \cong \pi|H$. Moreover, since every character $\chi \in (Z \cap H)^\wedge$ extends to a character $\chi \in \hat{Z}$ (see [HR]), when π varies in $\hat{G}$, the restrictions $\pi|H$ vary in all $\hat{H}$. Hence, by hypothesis, there exists $h_0 \in H$ such that $\phi(h) = h_0^{-1} h h_0$ for all $h \in H$.

We claim that $\phi|Z = Id_Z$. Since the characters of Z separate the points of Z (see [HR]), it suffices to show that $\chi \circ \phi|Z = \chi$ for all $\chi \in \hat{Z}$. Fix $\chi \in \hat{Z}$. There exists a unitary representation π of G such that $\pi|Z$ is a multiple of χ, that is, $\langle \pi(z)\chi, \eta \rangle = \chi(z)\langle \chi, \eta \rangle$ for all $\xi, \eta \in \mathscr{H}_\pi$. For example, the induced representation $\pi = Ind_Z^G(\chi)$ has this property. Let U be a unitary operator in $\mathscr{H}_\pi$ such that $\pi \circ \phi(x) = U^{-1}\pi(x)U$ for all $x \in G$. Then, for any $\xi \in \mathscr{H}_\pi$ with $\|\xi\| = 1$ and $z \in Z$

$$\begin{aligned} \chi(z) &= \chi(z)\langle U\xi, U\xi \rangle = \langle \pi(z)U\xi, U\xi \rangle \\ &= \langle U^{-1}\pi(z)U\xi, \xi \rangle = \langle (\pi \circ \phi)(z)\xi, \xi \rangle = \chi(\phi(z)). \end{aligned}$$

This proves the above claim. It follows that $\phi(zh) = \phi(z)\phi(h) = zh_0^{-1}hh_0 = h_0^{-1}(zh)h_0$ for all $z \in Z$ and $h \in H$, as required.

Corollary 3.2. *Let G be a connected linear real reductive group. Then G is quasi-complete if the semisimple subgroup G^s is quasi-complete.*

Proof. Let G be a connected linear real reductive Lie group. Then G decomposes as the commuting product

$$G = Z(G)^0 G^s,$$

where $Z(G)$ is the center of G and G^s is the connected semisimple subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. By applying Lemma 3.1 to $Z = Z(G)^0$ and $H = G^s$ we obtain the desired result.

4 The case of equal-rank connected linear real reductive Lie groups

In this section we consider non-compact connected linear real reductive Lie groups G which satisfy the *"equal-rank"* condition $rank(K) = rank(G)$, where K is a maximal compact subgroup of G. We show that they are quasi-complete by using their discrete series representations. The proof generalizes the one for compact connected Lie groups, which is a straightforward application of the Highest Weight Theorem. The reference for this section is [Kn1], Ch. 9, Thm. 9.20.

Fix a maximal compact subgroup K of G and a compact Cartan subgroup T satisfying $T \subset K \subset G$. Denote by $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$ the complexifications of $\mathfrak{g}$ and $\mathfrak{k}$ respectively, and by κ the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. Denote by $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and $\Delta_K = \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ the root systems of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. For $\alpha \in \Delta$, denote by $\mathfrak{g}^{\alpha}$ the corresponding root space in $\mathfrak{g}^{\mathbb{C}}$. Denote by W and W_K the Weyl groups generated by the reflections with respect to the roots in Δ and Δ_K respectively. Both groups act on $i\mathfrak{t}$. Since $W_K \subset W$, each W_K-chamber contains a union of W-chambers.

For a fixed positive system $\Delta^+ \subset \Delta$, a positive system can be chosen in Δ_K so that $\Delta_K^+ = \Delta_K \cap \Delta^+$. In terms of Weyl chambers this corresponds to taking the positive chamber for W inside the positive chamber for W_K. Denote by ρ_{Δ} and ρ_{Δ_K} the half sum of the positive roots in Δ^+ and Δ_K^+ respectively. Let $\lambda \in \mathfrak{t}^*$ be a non-singular weight with respect to Δ, i.e. such that $\langle \lambda, \alpha \rangle \neq 0$, for all $\alpha \in \Delta$. Here $\langle \, , \rangle$ is the restriction of the Killing form of $\mathfrak{g}^{\mathbb{C}}$. We denote by $\Delta_{\lambda}^+ = \{\alpha \in \Delta \,|\, \langle \alpha, \lambda \rangle \geq 0\}$ the positive system in Δ determined by λ and by $\Delta_{K,\lambda}^+$ the compatible positive system in Δ_K. The next lemma generalizes the result which characterizes inner automorphisms of a compact connected Lie group as those leaving a maximal torus pointwise fixed (cf. [Lo], Ch. 6, Thm. 4.5).

Lemma 4.1. *Let G be a connected non-compact linear real semisimple Lie group satisfying the equal rank condition. Let $T \subset K \subset G$ be a compact Cartan subgroup, and let $\phi \in \mathrm{Aut}(G)$ be such that*

$$(4.1) \qquad \phi(x) = x \quad \forall x \in T.$$

Then ϕ is an inner automorphism of G. More precisely, $\phi = Ad_{x_0}$, for some $x_0 \in T$.

Proof. The proof is similar to the proof of [Lo], Ch. 6, Thm. 4.5 and is omitted.

Corollary 4.2. *Let $\phi \in \mathrm{Out}(G)$ be such that $\phi(K) = K$. Then $\phi|K \in \mathrm{Out}(K)$.*

Proof. Assume by contradiction that $\phi|K$ is inner. By [Lo], Ch. 6, Thm. 4.5, there exists a maximal torus $T \subset K$ which is left pointwise fixed by $\phi|K$. Since T is also a maximal torus of G, by Lemma 4.1, ϕ is an inner automorphism of G. This contradicts the assumptions and the lemma follows.

Proposition 4.3. *Let G be a non-compact connected linear real semisimple Lie group satisfying the equal-rank condition. Let $\phi \in \mathrm{Aut}(G)$ be an automorphism with the prop-*

erty that π and $\pi \circ \phi$ are equivalent for every irreducible unitary representation π of G. Then ϕ is an inner automorphism of G. In short, G is quasi-complete.

Proof. Let $T \subset K \subset G$ be a compact Cartan subgroup of G. Modulo $\mathrm{Inn}(G)$, we can assume $\phi(K) = K$, $\phi(T) = T$, and $\phi(i\mathfrak{t}^+) = i\mathfrak{t}^+$, where $i\mathfrak{t}^+$ is a fixed positive Weyl chamber with respect to W_K. Under these assumptions, ϕ induces a map of the dual space $\mathfrak{t}^*$ into itself, given by $\lambda \mapsto \lambda \circ \phi$, for $\lambda \in \mathfrak{t}^*$. Let π_λ be a discrete series representation of G, with Harish-Chandra parameter λ (see [Kn1], Ch. IX, Sect. 7). Recall that $\lambda \in \mathfrak{t}^*$ is a non-singular weight, such that $\lambda + \rho_{\Delta_\lambda^+}$ is analytically integral. The restricted representation $\pi_\lambda|K$ contains with multiplicity one the lowest K-type

$$\Lambda_0 = \lambda + \rho_{\Delta_\lambda^+} - 2\rho_{\Delta_{K,\lambda}^+}$$

and the other K-types are of the form

$$\Lambda' = \Lambda_0 + \sum_{\alpha \in \Delta_\lambda^+} n_\alpha \alpha, \quad n_\alpha \in \mathbb{Z}_{\geq 0}.$$

Consider now the unitary representation $\pi_\lambda \circ \phi$. One has that $\lambda \circ \phi$ is a regular weight with respect to Δ and $\phi(\Delta_\lambda^+) = \Delta_{\lambda\circ\phi}^+$. In particular, $\rho_{\Delta_{\lambda\circ\phi}^+} = \rho_{\Delta_\lambda^+}$ and $\lambda \circ \phi + \phi(\Delta_\lambda^+)$ is analytically integral. Moreover, since $\phi(K) = K$, one also has $\phi(\Delta_{K,\lambda}^+) = \Delta_{K,\lambda\circ\phi}^+$ and $\rho_{\Delta_{K,\lambda\circ\phi}^+} = \rho_{\Delta_{K,\lambda}^+}$. The restricted representation $(\pi_\lambda \circ \phi)\,|\,K$ contains with multiplicity one the K-type

$$\Lambda_0 \circ \phi = \lambda \circ \phi + \rho_{\Delta_\lambda^+} \circ \phi - 2\rho_{\Delta_{K,\lambda}^+} \circ \phi = \lambda \circ \phi + \rho_{\Delta_{\lambda\circ\phi}^+} - 2\rho_{\Delta_{K,\lambda\circ\phi}^+}$$

and for the other K-types we have

$$\Lambda' \circ \phi = \Lambda_0 \circ \phi + \sum_{\alpha \in \Delta_\lambda^+} n_\alpha \alpha \circ \phi = \Lambda_0 \circ \phi + \sum_{\alpha\circ\phi \in \Delta_{\lambda\circ\phi}^+} n_\alpha \alpha \circ \phi, \quad n_\alpha \in \mathbb{Z}_{\geq 0}.$$

In other words, $\Lambda_0 \circ \phi$ is the lowest K-type for $\pi_\lambda \circ \phi$. Suppose now that π_λ and $\pi_\lambda \circ \phi$ are equivalent representations. Then $\pi_\lambda \circ \phi$ is a discrete series representation with Harish-Chandra parameter $\lambda \circ \phi$. In particular, λ and $\lambda \circ \phi$ lie on the same W_K-orbit. On the other hand, since λ and $\lambda \circ \phi$ lie on the same Weyl chamber with respect to W_K and $\phi|K$ is an outer automorphism of K, it follows that

$$(4.2) \qquad \lambda = \lambda \circ \phi.$$

Observe that Harish-Chandra parameters of discrete series representations of G include non-singular weights of the form

$$\mu = \lambda + \sum_{\alpha \in \Delta_\lambda^+} n_\alpha \alpha, \quad n_\alpha \in \mathbb{Z}_{\geq 0},$$

provided that $\Delta^+_\mu = \Delta^+_\lambda$. If π and $\pi \circ \phi$ are equivalent for every unitary representation π of G, and in particular for every discrete series representation π_λ of G, equation (4.2) implies that $\phi|T \equiv Id_T$. By Lemma 4.1, ϕ is an inner automorphism of G and the proposition follows.

Corollary 4.4. *By Proposition 4.3 and Corollary 3.2, equal-rank connected linear real reductive Lie groups are quasi-complete.*

Remark 4.5. The result of Corollary 4.4 could have also been obtained directly. Discrete series representations ρ of an equal-rank connected linear real reductive Lie group G are parametrized by pairs (λ, χ), where λ is the Harish-Chandra parameter of the dicrete series representation $\rho|G^s$ of G^s, χ is a character of $Z(G)^0$ and on the finite group $Z(G)^0 \cap G^s$ the restriction of $\rho|G^s$ coincides with the scalar character determined by χ. Two discrete series representations $\rho_{\lambda,\chi}$ and $\rho_{\lambda',\chi'}$ are equivalent if and only if $\chi = \chi'$ and the parameters λ, λ' are conjugate by $W_G(\mathfrak{t})$ (see [Kn1], p. 469–470).

5 The case of arbitrary connected linear real reductive Lie groups

In this section, using the results of sections 3 and 4, we prove that an arbitrary connected linear real reductive Lie group G is quasi-complete.

Fix a maximal compact subgroup K of G. Let θ be the Cartan involution of $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Fix a maximally split θ-stable Cartan subalgebra $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{a}_0$, where $\mathfrak{a}_0$ is a maximal abelian subspace in $\mathfrak{p}$ and $\mathfrak{s} \subset \mathfrak{k}$. Denote by $\mathfrak{m}_0 = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_0)$ the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}$. Then $\mathfrak{m}_0$ is a compact reductive subalgebra

$$\mathfrak{m}_0 = \mathfrak{z}(\mathfrak{m}_0) \oplus [\mathfrak{m}_0, \mathfrak{m}_0],$$

with center $\mathfrak{z}(\mathfrak{m}_0)$ and semisimple subalgebra $[\mathfrak{m}_0, \mathfrak{m}_0]$. If $\mathfrak{s}'$ is a Cartan subalgebra of $[\mathfrak{m}_0, \mathfrak{m}_0]$, then

$$\mathfrak{s} = \mathfrak{z}(\mathfrak{m}_0) \oplus \mathfrak{s}',$$

is a Cartan subalgebra of $\mathfrak{m}_0$. Denote by $M_0 = Z_K(\mathfrak{a}_0)$ the centralizer of $\mathfrak{a}_0$ in K. One has that M_0 is a compact reductive group with Lie algebra $\mathfrak{m}_0$, generally disconnected. One can write $M_0 = F \cdot M_0^0$, where $F = K \cap \exp(i\mathfrak{a}_0)$ is a finite abelian group of involutive elements contained in the center of M_0 (see [Kn2], Thm. 7.53, and [Kn1], p. 468). Let $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. Denote by Δ_0 the subsystem of imaginary roots, i.e. the roots in Δ which vanish identically on $\mathfrak{a}_0$. Then

$$\mathfrak{m}_0^{\mathbb{C}} = \mathfrak{s}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta_0} \mathfrak{g}^\alpha \quad \text{and} \quad [\mathfrak{m}_0, \mathfrak{m}_0]^{\mathbb{C}} = (\mathfrak{s}')^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta_0} \mathfrak{g}^\alpha.$$

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_0)$ be the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}_0$. For $\lambda \in \Sigma$, denote by $\mathfrak{g}^\lambda$ the corresponding root space in $\mathfrak{g}$.

Fix a basis of $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{s} \oplus \mathfrak{a}_0$, consisting of a basis of $\mathfrak{a}_0$ followed by a basis of $i\mathfrak{s}$. The lexicographic ordering defines compatible notions of positivity on Σ, Δ_0 and Δ. For roots with non-zero restrictions to $\mathfrak{a}_0$, one has that $\alpha \in \Delta^+$ if and only if $\alpha|\mathfrak{a}_0 \in \Sigma^+$. Similarly, for roots vanishing identically on $\mathfrak{a}_0$, one has that $\alpha \in \Delta^+$ if and only if $\alpha \in \Delta_0^+$. Denote by $\mathfrak{a}_0^+$, $\mathfrak{h}_{\mathbb{R}}^+$ and $i\mathfrak{s}^+$ the corresponding positive Weyl chambers in $\mathfrak{a}_0$, $\mathfrak{h}_{\mathbb{R}}$ and $i\mathfrak{s}$ respectively.

Let P_0 be the minimal parabolic subgroup associated with Σ^+. Then $P_0 = M_0A_0N_0$, where $A_0 = \exp \mathfrak{a}_0$ and $N_0 = \exp \mathfrak{n}_0$, for $\mathfrak{n}_0 = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}^\lambda$. Every parabolic subgroup of G is conjugate to a unique "standard" parabolic subgroup $Q = MAN$ containing P_0. The Lie algebra of Q decomposes as $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{a} \subset \mathfrak{a}_0$, $\mathfrak{n} \subset \mathfrak{n}_0$ and $\mathfrak{m} \supset \mathfrak{m}_0$ (Langlands decomposition). A standard parabolic subgroup $Q = MAN$ is called cuspidal if $\mathfrak{m}$ possesses a compact Cartan subalgebra.

Lemma 5.1. *Let G be a connected non-compact linear real semisimple Lie group. Let $\phi \in \mathrm{Aut}(G)$. Then, modulo $\mathrm{Inn}(G)$, one can assume:*

$$\phi(K) = K \quad \phi(\mathfrak{k}) = \mathfrak{k} \quad \phi(\mathfrak{p}) = \mathfrak{p}$$

$$\phi(\mathfrak{a}_0) = \mathfrak{a}_0 \quad \phi(\mathfrak{a}_0^+) = \mathfrak{a}_0^+ \quad \phi(\mathfrak{s}) = \mathfrak{s} \quad \phi(i\mathfrak{s}^+) = i\mathfrak{s}^+.$$

Proof. Since all maximal compact subgroups in G are conjugate by $\mathrm{Inn}(G)$, one has that $\phi^{-1}(K) = Ad_g(K)$, for some $g \in G$, and $(\phi \circ Ad_g)(K) = K$. In particular, $(\phi \circ Ad_g)(\mathfrak{k}) = \mathfrak{k}$ and $(\phi \circ Ad_g)(\mathfrak{p}) = \mathfrak{p}$. Equivalently, $\phi \circ Ad_g$ commutes with the Cartan involution θ on $\mathfrak{g}$. Since all maximal abelian subspaces $\mathfrak{a}_0 \subset \mathfrak{p}$ are conjugate by $\mathrm{Inn}(K)$, one has that $(\phi \circ Ad_g)^{-1}(\mathfrak{a}_0) = Ad_k(\mathfrak{a}_0)$, for some $k \in K$. Then $(\phi \circ Ad_g \circ Ad_k)(\mathfrak{a}_0) = \mathfrak{a}_0$. Since the Weyl group $W_K(\mathfrak{a}_0)$ acts transitively on the set of Weyl chambers in $\mathfrak{a}_0$, by a similar argument one has $(\phi \circ Ad_g \circ Ad_k \circ Ad_w)(\mathfrak{a}_0^+) = \mathfrak{a}_0^+$, for some $w \in N_K(\mathfrak{a}_0)$. In particular, $(\phi \circ Ad_g \circ Ad_k \circ Ad_w)(M_0) = M_0$ and $(\phi \circ Ad_g \circ Ad_k \circ Ad_w)(M_0^0) = M_0^0$. Since all Cartan subalgebras in m_0 are conjugate by $\mathrm{Inn}(M_0^0)$, there exists $m \in M_0^0$ such that $(\phi \circ Ad_g \circ Ad_k \circ Ad_w \circ Ad_m)(\mathfrak{s}) = \mathfrak{s}$. Finally, since the Weyl group $W_{M_0^0}(\mathfrak{s})$ acts transitively on the set of Weyl chambers in $i\mathfrak{s}$, one also has that $(\phi \circ Ad_g \circ Ad_k \circ Ad_w \circ Ad_m \circ Ad_v)(i\mathfrak{s}^+) = i\mathfrak{s}^+$, for some $v \in W_{M_0^0}(\mathfrak{s})$.

Corollary 5.2. *Let $\phi \in \mathrm{Aut}(G)$ be an automorphism of G satisfying the conditions of Lemma 5.1. Then $\phi(\Delta^+) = \Delta^+$. In particular, ϕ induces a (possibly trivial) permutation of the simple roots in Δ^+, commuting with the conjugation action $\kappa(\alpha)(H) := \overline{\alpha(\overline{H})}$, $H \in \mathfrak{h}$, $\alpha \in \Delta$.*

(a) *If such a permutation is non-trivial, then ϕ either induces a non-trivial permutation of the simple restricted roots in Σ^+ or it restricts to an outer automorphism of M_0^0 and of M_0, i.e.*

$$\phi|M_0^0 \in \mathrm{Out}(M_0^0), \quad \phi|M_0 \in \mathrm{Out}(M_0).$$

(b) *If such a permutation is trivial, then* $\phi \in \mathrm{Inn}(\mathfrak{g}^{\mathbb{C}})$ *and* $\phi|\mathfrak{h} = Id_{\mathfrak{h}}$. *As a consequence,* ϕ *preserves all standard* θ*-stable Cartan subalgebras of* $\mathfrak{g}$. *Moreover,* ϕ *preserves all standard parabolic subgroups* Q *containing* P_0 *and respects their Langlands decomposition.*

Proof. Let $\alpha \in \Delta^+$. If $\alpha|\mathfrak{a}_0 \not\equiv 0$, then $\alpha|\mathfrak{a}_0 \in \Sigma^+$. By Lemma 5.1, $\alpha|\mathfrak{a}_0 \circ \phi \in \Sigma^+$ and $\alpha \circ \phi \in \Delta^+$. If $\alpha|\mathfrak{a}_0 \equiv 0$, then $\alpha \in \Delta_0^+$. By Lemma 5.1, $\alpha \circ \phi \in \Delta_0^+$ and in particular $\alpha \circ \phi \in \Delta^+$.

(a) Assume that ϕ induces a non-trivial permutation of the simple roots in Δ^+. Since $\phi(\mathfrak{a}_0) = \mathfrak{a}_0$, one has that $\phi(M_0) = M_0$ and $\phi(M_0^0) = M_0^0$. Also recall that the restrictions of the simple roots in Δ^+ map surjectively onto the simple roots in Σ^+ (cf. [He], p. 585). Suppose that ϕ induces a trivial permutation of the simple roots in Σ^+. Since such roots span $\mathfrak{a}_0^*$, one has that $\phi|\mathfrak{a}_0 \equiv Id_{\mathfrak{a}_0}$. On the other hand, since $\phi|\mathfrak{h}_{\mathbb{R}} \not\equiv Id_{\mathfrak{h}_{\mathbb{R}}}$, it follows that $\phi|\mathfrak{s} \not\equiv Id_{\mathfrak{s}}$. Moreover, $\phi(i\mathfrak{s}^+) = i\mathfrak{s}^+$ implies that $\phi(\Delta_0^+) = \Delta_0^+$. At this point, either ϕ induces a non-trivial permutation of the simple roots in Δ_0^+ or ϕ acts non-trivially on $\mathfrak{z}(\mathfrak{m}_0)$. In both cases, $\phi \in \mathrm{Out}(M_0^0)$. Since ϕ acts non-trivially on M_0^0, while the finite group F centralizes M_0^0, one also has that $\phi \in \mathrm{Out}(M_0)$.

(b) If ϕ induces a trivial permutation of the simple roots in Δ^+, it acts as the identity on the positive Weyl chamber $\mathfrak{h}_{\mathbb{R}}^+$ and on $\mathfrak{h}$. In particular it stabilizes all root spaces $\mathfrak{g}^\alpha$, for $\alpha \in \Delta$, and all θ-stable standard Cartan subalgebras in $\mathfrak{g}$ (see [Su]). Moreover, since ϕ acts as the identity on $\mathfrak{a}_0$, it also acts trivially on the simple restricted roots in Σ^+ and stabilizes the restricted root spaces $\mathfrak{g}^\lambda$, for $\lambda \in \Sigma$. As a consequence, ϕ stabilizes the minimal parabolic subgroup $P_0 = M_0A_0N_0$ associated to Σ^+. The automorphism ϕ also stabilizes all standard parabolic subgroups Q containing P_0 and respects their Langlands decomposition (cf. [Kn1], Ch. 5, Sect. 5).

For every outer automorphism of G, we now construct an irreducible unitary representation π of G with the property that $\pi \circ \phi$ is not unitarily equivalent to π. We do this by using parabolic induction. Let $Q = MAN$ be a standard parabolic subgroup containing P_0. Denote by $\Sigma(\mathfrak{g}, \mathfrak{a})$ the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and by $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ the set of positive roots determined by $\mathfrak{n}$ (i.e. the roots whose root spaces $\mathfrak{g}^\mu$ lie in $\mathfrak{n}$). Define $\rho = \frac{1}{2}\sum_{\Sigma^+(\mathfrak{g},\mathfrak{a})}(\dim \mathfrak{g}^\mu)\mu$.

Let $\sigma : (M, V^\sigma)$ be an irreducible unitary representation of M and $e^{i\nu}$ a unitary character of A, for some real valued linear functional $\nu \in \mathfrak{a}^*$. Consider the following subspace of continuous functions on G with values in V^σ

$$C^{\sigma,\nu} := \{f \in C(G, V^\sigma) \mid f(gman) = \sigma(m)^{-1}e^{-(i\nu+\rho)\log(a)}f(g), man \in MAN, g \in G\},$$

with norm

$$\|f\|^2 := \int_K \|f\|^2_{V^\sigma}$$

and G-action given by

$$g \cdot f(x) := f(g^{-1}x) \quad g \in G, f \in C^{\sigma,\nu}.$$

Denote by $\boldsymbol{H}^{\sigma,\nu}$ the Hilbert space given by the completion of $C^{\sigma,\nu}$ with respect to the above norm and by $\pi^{\sigma,\nu} := Ind_{MAN}^{G}(\sigma \otimes e^{i\nu} \otimes 1)$ the corresponding representation of G on $\boldsymbol{H}^{\sigma,\nu}$. One has that $Ind_{MAN}^{G}(\sigma \otimes e^{i\nu} \otimes 1)$ is a unitary representation of G (see [Kn1], p. 169).

Lemma 5.3. *Let $\phi \in \mathrm{Aut}(G)$ be an automorphism satisfying $\phi(K) = K$. Assume in addition that $\phi(Q) = Q$ and precisely that $\phi(M) = M$, $\phi(A) = A$ and $\phi(N) = N$. Then $Ind_{MAN}^{G}(\sigma \otimes e^{i\nu} \otimes 1) \circ \phi$ is a unitary representation of G, unitarily equivalent to $Ind_{MAN}^{G}((\sigma \circ \phi) \otimes e^{i\nu\circ\phi} \otimes 1)$.*

Proof. Consider the map

$$A : \boldsymbol{H}^{\sigma,\nu} \to \boldsymbol{H}^{\sigma\circ\phi,\nu\circ\phi}, \quad A(f) \mapsto f \circ \phi.$$

The map A is densely defined on $\boldsymbol{H}^{\sigma,\nu}$, with domain containing $C^{\sigma,\nu}$. For every $f \in C^{\sigma,\nu}$, the image $A(f)$ satisfies the functional equation

$$A(f)(xman) = f(\phi(x)\phi(m)\phi(a)\phi(n)) = \sigma(\phi(m^{-1}))e^{-(i\nu+\rho)\log(\phi(a))}f(\phi(x))$$

$$= \sigma(\phi(m^{-1}))e^{-(i\nu\circ\phi+\rho)\log(a)}A(f)(x).$$

In other words, $A(f) \in C^{\sigma\circ\phi,\nu\circ\phi}$, for all $f \in C^{\sigma,\nu}$. Note that we have used the fact that $\phi(\Sigma^{+}(\mathfrak{g},\mathfrak{a})) = \Sigma^{+}(\mathfrak{g},\mathfrak{a})$ and therefore $\rho \circ \phi = \rho$. Moreover, the map A is a densely defined intertwining operator between $\pi^{\sigma,\nu} \circ \phi$ (acting on $\boldsymbol{H}^{\sigma,\nu}$) and $\pi^{\sigma\circ\phi,\nu\circ\phi}$, namely

$$A((\pi^{\sigma,\nu} \circ \phi) \cdot f)(x) = \pi^{\sigma\circ\phi,\nu\circ\phi} \cdot A(f)(x), \quad \forall f \in C^{\sigma,\nu}, \forall x \in G.$$

Indeed, evaluating the left-hand side and the right-hand side of the above equality we get

$$A(f \circ L_{\phi(g)^{-1}})(x) = (f \circ L_{\phi(g)^{-1}} \circ \phi)(x) = f(\phi(g)^{-1}\phi(x)),$$

and

$$(A(f) \circ L_{g^{-1}})(x) = (f \circ \phi \circ L_{g^{-1}})(x) = f(\phi(g^{-1}x)) = f(\phi(g)^{-1}\phi(x)),$$

respectively. Since ϕ preserves the Haar measure on K, one has that $\|A(f)\|^2 = \|f\|^2$. Hence A can be extended to a unitary operator

$$A : \boldsymbol{H}^{\sigma,\nu} \to \boldsymbol{H}^{\sigma\circ\phi,\nu\circ\phi}$$

intertwining $\pi^{\sigma,\nu} \circ \phi$ with $\pi^{\sigma\circ\phi,\nu\circ\phi}$.

Proposition 5.4. *Let G be a non-compact connected linear real semisimple Lie group. Let $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{a}_0$ be a maximally split Cartan subalgebra of $\mathfrak{g}$. Let $\phi \in \mathrm{Aut}(G)$ be an* *automorphism satisfying the conditions of Lemma 5.1 and of Corollary 5.2 (a). Then there exists an irreducible unitary representation $\pi \in \hat{G}$ of the principal series such that $\pi \circ \phi \not\cong \pi$.*

Proof. Suppose that ϕ induces a non-trivial permutation of the simple restricted roots. One can find a regular $\nu \in \mathfrak{a}_0^*$ (i.e. $\langle \nu, \alpha \rangle \neq 0$, for all $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_0)$) such that $\nu \neq \nu \circ \phi$. Then, for an arbitrary $\sigma \in \hat{M}_0$, the induced representation $\pi = Ind_{P_0}^G(\sigma \otimes e^{i\nu} \otimes 1)$ is an irreducible unitary representation of G (see [Kn1], Thm. 7.2). Moreover, by Lemma 5.3 and [Kn1], p. 174, the representation π has the property that $\pi \circ \phi \not\cong \pi$.

Suppose now that ϕ restricts to an outer automorphism of M_0^0 and of M_0. Let σ be an irreducible unitary representation of M_0^0 such that $\sigma \circ \phi | M_0^0 \not\cong \sigma$. Such a representation exists because compact connected Lie groups are quasi-complete. Let χ be the character determined by the restriction of σ to $Z(M_0^0)^0$. We can extend σ from M_0^0 to M_0, by extending χ from $F \cap Z(M_0^0)^0$ to F. In this way, we get a representation of M_0 which satisfies the condition $\sigma \circ \phi | M_0 \not\cong \sigma$. For every regular $\nu \in \mathfrak{a}_0^*$, the induced representation $\pi = Ind_{P_0}^G(\sigma \otimes e^{i\nu} \otimes 1)$ is an irreducible unitary representation of G (by [Kn1], Thm. 7.2) with the property that $\pi \circ \phi \not\cong \pi$ (by Lemma 5.3 and [Kn1], p. 174).

Lemma 5.5. *Let G be a connected non-compact linear real semisimple Lie group. Let $\phi \in \mathrm{Aut}(G)$ be an automorphism satisfying the conditions of Lemma 5.1 and of Corollary 5.2 (b). If $\phi \in \mathrm{Out}(G)$, then $\phi | K \in \mathrm{Out}(K)$.*

Proof. First we show that it is sufficient to prove Lemma 5.5 for a real simple group G with simply connected complexification $G^{\mathbb{C}}$. Embed G in a complexification $G^{\mathbb{C}}$, as the connected component of the identity of a real form of $G^{\mathbb{C}}$. Let κ be the corresponding conjugation of $G^{\mathbb{C}}$. Denote by $\tilde{G}^{\mathbb{C}}$ the universal covering group of $G^{\mathbb{C}}$ and by $\check{G}$ the real form of $\tilde{G}^{\mathbb{C}}$ with respect to the lifted conjugation $\tilde{\kappa}$. The group $\check{G}$ is always connected and $G = \check{G}/\Gamma$, for some central subgroup $\Gamma \subset \check{G}$. Let ϕ be an automorphism of G satisfying the conditions of Lemma 5.1 and of Corollary 5.2 (b). It follows that ϕ is the restriction to G of an inner automorphism of $G^{\mathbb{C}}$ commuting with κ, namely $\phi = Ad_z$, for some $z \in G^{\mathbb{C}}$. In the same way, there exists $\tilde{z} \in \tilde{G}^{\mathbb{C}}$ such that $\tilde{\phi} = Ad_{\tilde{z}}$ is an inner automorphism of $\tilde{G}^{\mathbb{C}}$ preserving $\check{G}$ and which is the lifting of ϕ. If $\tilde{\phi} \in \mathrm{Inn}(\check{G})$, then also $\phi \in \mathrm{Inn}(G)$. Recall that a simply connected complex semisimple Lie group decomposes as the direct product

$$\tilde{G}^{\mathbb{C}} = \tilde{G}_1^{\mathbb{C}} \times \cdots \times \tilde{G}_n^{\mathbb{C}},$$

where each $\tilde{G}_i^{\mathbb{C}}$ is a simply connected complex simple group. Likewise, a real form $\check{G}$ of $\tilde{G}^{\mathbb{C}}$ decomposes as the direct product of real forms of the following two types:

- $\check{G}_i \hookrightarrow \tilde{G}_i^{\mathbb{C}}$, with $\check{G}_i$ real simple and $\tilde{G}_i^{\mathbb{C}}$ complex simple;
- $\check{G}_i = \tilde{G}_j^{\mathbb{C}} \hookrightarrow \tilde{G}_j^{\mathbb{C}} \times \overline{\tilde{G}^{\mathbb{C}}}_j$, with $\tilde{G}_j^{\mathbb{C}}$ complex simple embedded in $\tilde{G}_j^{\mathbb{C}} \times \overline{\tilde{G}^{\mathbb{C}}}_j$ as the diagonal subgroup. Here $\overline{\tilde{G}^{\mathbb{C}}}_j$ denotes a copy of $\tilde{G}_j^{\mathbb{C}}$ with the opposite complex structure.

By the above remarks, G can be assumed to be a real simple group with simply connected complexification $G^{\mathbb{C}}$. If G satisfies the equal-rank condition, the lemma follows from Corollary 4.2. Suppose now that G is a simple group with $rank(K) < rank(G)$. Assume by contradiction that $\phi|K = Ad_z|K = Ad_k$, for some $k \in K$. Then $k^{-1}z \in Z_{G^{\mathbb{C}}}(K)$. We claim that $Z_{G^{\mathbb{C}}}(K) = Z(G^{\mathbb{C}})$. The inclusion $Z(G^{\mathbb{C}}) \subset Z_{G^{\mathbb{C}}}(K)$ is obvious. For the opposite inclusion let $x \in Z_{G^{\mathbb{C}}}(K)$ and let k be an arbitrary regular elliptic element in G (one such element exists). If x centralizes k, it centralizes a θ-stable maximally compact Cartan subgroup H of G containing k. Let $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{h}_{\mathfrak{p}}$ be the Cartan decomposition of the Lie algebra of H. Then x centralizes $K, \mathfrak{h}_{\mathfrak{p}}$ and $\bigcup_{k \in K} Ad_k \mathfrak{h}_{\mathfrak{p}}$. Since G is simple and Ad_K acts irreducibly on $\mathfrak{p}$, the element x centralizes $\mathfrak{p}$ and also $G = K \exp \mathfrak{p}$. Therefore $x \in Z_{G^{\mathbb{C}}}(G) = Z(G^{\mathbb{C}})$ and the claim is proved. It follows that $k^{-1}z \in Z_{G^{\mathbb{C}}}(K) = Z(G^{\mathbb{C}})$ and $\phi = Ad_z = Ad_k$, contradicting the assumption that $\phi \in \mathrm{Out}(G)$.

Let $Q = MAN$ be a maximal standard cuspidal parabolic subgroup of G containing P_0. Unless G is equal-rank, M is a proper subgroup of G. If $Q = P_0$, then $\mathfrak{g}$ has a unique conjugacy class of Cartan subalgebras and $M = M_0$ is a connected compact reductive group. If $Q \neq P_0$, then M is a non-compact linear real reductive group, satisfying the equal-rank condition. Moreover, M is non-abelian, with non-trivial semisimple part, and is generally disconnected: $M = M^0 \cdot F$, where $F \subset K$ is a finite abelian group (cf. [Kn1], p. 468–469, [Kn2], Prop. 7.87). If $\mathfrak{t} \subset \mathfrak{m} \cap \mathfrak{k}$ is a compact Cartan subalgebra of $\mathfrak{m}$, then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}$. Since $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ contains no real roots, it follows that $F \cap Z(M) = 1$ (see [Kn1], p. 468). Since $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{k}$, there is an inclusion of Weyl groups $W_{M^0}(\mathfrak{t}) \subset W_M(\mathfrak{t}) \subset W_K(\mathfrak{t})$. Finally observe that by Corollary 5.2 (b), the compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{m}$ can be assumed ϕ-stable.

Corollary 5.6. *Let Q be a standard maximal cuspidal parabolic subgroup of G. Under the assumptions of Lemma 5.5, if $\phi \in \mathrm{Out}(G)$ then $\phi|M \in \mathrm{Out}(M)$ and $\phi|M^0 \in \mathrm{Out}(M^0)$.*

Proof. By Lemma 5.5, if $\phi \in \mathrm{Out}(G)$, then $\phi|K \in \mathrm{Out}(K)$. Since $\phi|K \in \mathrm{Out}(K)$, one has that $\phi|\mathfrak{t} \not\equiv Id_{\mathfrak{t}}$, by [Lo], Ch. 6, Thm. 4.5. Moreover, the action of $\phi|\mathfrak{t}$ does not coincide with the action of any Weyl group element $w \in W_K(\mathfrak{t})$ (otherwise $Ad_{w^{-1}} \circ \phi|\mathfrak{t} \equiv Id_{\mathfrak{t}}$ and $\phi|K$ would be inner on K, by [Lo], Ch. 6, Thm. 4.5). In particular, the action of $\phi|\mathfrak{t}$ does not coincide with the action of any Weyl group element in $W_M(\mathfrak{t})$ nor in $W_{M^0}(\mathfrak{t})$. Therefore $\phi|M \in \mathrm{Out}(M)$ and $\phi|M^0 \in \mathrm{Out}(M^0)$.

Proposition 5.7. *Let G be a connected non-compact linear real semisimple Lie group, not equal-rank. Let $\phi \in \mathrm{Aut}(G)$ be an automorphism satisfying the conditions of Lemma 5.1 and of Corollary 5.2 (b). Then there exists an irreducible unitary representation $\pi \in \hat{G}$ such that $\pi \circ \phi \not\cong \pi$.*

Proof. Let $Q = MAN$ be a maximal cuspidal parabolic subgroup. By Corollary 5.2 (b) we have that $\phi(M) = M$ and by Corollary 5.6 that $\phi|M \in \mathrm{Out}(M)$. Assume first that $Q = P_0$. Since compact connected groups are quasi-complete, there exists $\sigma \in \hat{M}$ such that $\sigma \circ \phi|M \not\cong \sigma$. For an arbitrary regular $\nu \in \mathfrak{a}_0^*$, the induced representation $\pi = Ind_{P_0}^G(\sigma \otimes e^{i\nu} \otimes 1)$ is an irreducible unitary principal series representation of G (by [Kn1], Thm. 7.2) with the required properties: by Lemma 5.3, $\pi \circ \phi$ is equivalent to $Ind_{P_0}^G(\sigma \circ \phi \otimes e^{i\nu} \otimes 1)$, and by [Kn1], p. 174, one has that $\pi \circ \phi \not\cong \pi$.

Assume now that $Q \neq P_0$. We claim that there exists a unitary representation ρ in the discrete series of M with the property that $\rho \circ \phi|M \not\cong \rho$. (see [Kn1], Ch. XII, Sect. 8 for the appropriate definitions). Start with a unitary representation ρ^0 in the discrete series of M^0, with the property that $\rho^0 \circ \phi|M^0 \not\cong \rho^0$. Such a representation exists because equal-rank connected linear real reductive groups are quasi-complete (cf. Proposition 4.3, Corollary 4.4, Remark 4.5). Recall that discrete series representations ρ^0 of M^0 are parametrized by pairs (λ, χ), where λ is the Harish-Chandra parameter of the discrete series representation $\rho|(M^0)^s$ of $(M^0)^s$, χ is a character of $Z(M^0)^0$ and on the finite group $Z(M^0)^0 \cap (M^0)^s$ the restriction of $\rho|(M^0)^s$ coincides with the scalar character determined by χ. Hence, there exists $\rho^0 = \rho^0_{\lambda,\chi}$ such that either $\chi \circ \phi \neq \chi$ or $\lambda \circ \phi$ does not belong to the $W_{M^0}(\mathfrak{t})$-orbit of λ. Since M^0 has finite index in M and the action of $\phi|\mathfrak{t}$ does not coincide with the action of any Weyl group element in $W_M(\mathfrak{t})$, one can actually choose $\rho^0_{\lambda,\chi}$ such that either $\chi \circ \phi \neq \chi$ or $\lambda \circ \phi$ does not belong to the $W_M(\mathfrak{t})$-orbit of λ.

Consider next the discrete series representation of M given by $\rho_{\lambda,\chi} := Ind_{M^0}^M(\rho^0_{\lambda,\chi})$. We claim that $\rho_{\lambda,\chi} \circ \phi \not\cong \rho_{\lambda,\chi}$. Observe that $\rho_{\lambda,\chi} \circ \phi$ is equivalent to $\rho_{\lambda\circ\phi,\chi\circ\phi} = Ind_{M^0}^M(\rho^0_{\lambda\circ\phi,\chi\circ\phi})$. On the other hand, by [Kn1], Prop. 12.32, two discrete series representations $\rho_{\lambda,\chi}$ and $\rho_{\lambda',\chi'}$ of M are equivalent if and only if $\chi' = \chi$ and $\lambda' = w\lambda$, for some $w \in W_M(\mathfrak{t})$. So the claim follows.

Finally, consider the induced representation $\pi = Ind_Q^G(\rho_{\lambda,\chi} \otimes e^{i\nu} \otimes 1)$. For an arbitrary regular $\nu \in \mathfrak{a}^*$, one has that π is an irreducible unitary tempered representation of G (cf. [Kn1], Thm. 14.15). Moreover, since λ is a regular weight, π is induced from "non-degenerate" data ([Kn1], p. 611). By Lemma 5.3 and [Kn1], Thm. 14.91, π has the property that $\pi \circ \phi \not\cong \pi$.

By the results of the previous sections, Corollary 3.2, Proposition 5.4 and Proposition 5.7, we finally obtain our main result.

Theorem 5.8. *Let G be a connected linear real reductive Lie group. Let $\phi \in \mathrm{Aut}(G)$ be an automorphism with the property that the representations π and $\pi \circ \phi$ are unitarily equivalent, for every irreducible unitary representation π of G. Then ϕ is an inner automorphism of G. In short, G is quasi-complete.*

6 The case of finite groups

Finite abelian groups do not admit any non-trivial inner automorphisms. They are quasi-complete, since characters separate points. On the other hand, arbitrary finite groups are not necessarily quasi-complete. An example of a finite group which is not quasi-complete was given by G. C. Wall (cf. [Hu], p. 22). In this section we show that this counterexample is not isolated, but fits into a more general pattern.

We recall some elementary notions from *group cohomology* (cf. [Se], Ch. VII).

Definition 6.1. Let G be a group, and let A be a G-module. A *1-cocycle* is a function $\sigma : G \to A$ satisfying the condition

$$\sigma(g \cdot g') = \sigma(g) + g\sigma(g'), \quad \text{for all } g, g' \in G.$$

The 1-cocycles form a group denoted by $Z^1(G, A)$.

Definition 6.2. A *1-coboundary* is a function $\sigma : G \to A$ of the form

$$\sigma(g) = g \cdot a - a,$$

for some fixed $a \in A$ independent of $g \in G$. The 1-coboundaries form a subgroup of the group of 1-cocycles. It is denoted by $B^1(G, A)$.

The first cohomology group of G with values in A is the quotient group

$$H^1(G, A) = Z^1(G, A)/B^1(G, A).$$

To any exact sequence of G-modules

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0 \tag{6.1}$$

there is associated an exact cohomology sequence

$$0 \to A^G \to B^G \to C^G \xrightarrow{\delta} H^1(G, A) \to H^1(G, B) \to H^1(G, C) \to \cdots,$$

where A^G, B^G, C^G denote the subgroups of G-invariant elements of A, B, C. The connecting homomorphism $\delta : C^G \to H^1(G, A)$ is defined in the following way. Let $c \in C^G$. By the exactness of (6.1), one has $c = \psi(b)$, for some $b \in B$. Since c is G-invariant, we have that $g \cdot b - b \in \ker(\psi) = \mathrm{Im}(\phi)$ for all $g \in G$. Then define $\delta(c)$ to be the 1-cocycle $G \to A$ that maps $g \in G$ to the unique $a \in A$ for which $\phi(a) = g \cdot b - b$.

Our examples of finite groups Γ that are not quasicomplete, are constructed as follows. Let R be a finite commutative local ring with identity 1, maximal ideal $\mathfrak{M}$ and residue field $\mathbb{F} = R/\mathfrak{M}$ (see [AM]). Consider the multiplicative subgroup of the units R^*

$$U := \{u \in R, u \in 1 + \mathfrak{M}\}.$$

The additive group R is a U-module by multiplication. Now define

$$\Gamma := \begin{pmatrix} 1 & R \\ 0 & U \end{pmatrix} = \left\{ \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} : x \in R, u \in U \right\}.$$

Then Γ is a finite group. It is non-abelian when $\mathfrak{M} \neq 0$ or, equivalently, when R is not a field. One can easily check that the map $H^1(U,R) \to \mathrm{Out}(\Gamma)$ given by $\sigma \mapsto f_\sigma$ where

$$f_\sigma \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} = \begin{pmatrix} 1 & x + \sigma(u) \\ 0 & u \end{pmatrix},$$

is a well defined injective homomorphism. Indeed, f_σ is an automorphism of Γ if and only if σ is a 1-cocycle and it is inner if and only if σ is a 1-coboundary.

Next we show that for certain choices of the ring R and the 1-cocycle σ, the automorphism f_σ preserves conjugacy classes.

Definition 6.3. A finite commutative local ring R with maximal ideal $\mathfrak{M}$ and residue field $\mathbb{F} = R/\mathfrak{M}$ is called *Gorenstein* if the ideal

$$\mathfrak{m} = \mathrm{Ann}(\mathfrak{M}) = \{x \in R \mid xy = 0, \text{ for all } y \in \mathfrak{M}\}$$

is the *unique* minimal ideal of R.

When R is Gorenstein, the ideal $\mathfrak{m}$ is by minimality a one-dimensional $\mathbb{F}$-vector space.

Consider the exact sequence of U-modules

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$$

and the associated cohomology sequence

$$\cdots \to (R/\mathfrak{m})^U \to H^1(U,\mathfrak{m}) \to H^1(U,R) \to \cdots.$$

Proposition 6.4. *If R is a finite local Gorenstein ring, then the 1-cocycles σ in the image of $H^1(U,\mathfrak{m}) \to H^1(U,R)$ have the property that $f_\sigma(g)$ is conjugate to g, for all $g \in \Gamma$.*

Proof. Let $g = \left(\begin{smallmatrix} 1 & x \\ 0 & u \end{smallmatrix}\right) \in \Gamma$ and let σ be a 1-cocycle in the image of the map $H^1(U,\mathfrak{m}) \to H^1(U,R)$. If $u = 1$, then $\sigma(u) = 0$ and $f_\sigma(g) = g$. If $u \neq 1$, then the minimal ideal $\mathfrak{m}$ is contained in $(u-1)R$ and in particular $\sigma(u) = (u-1)y$, for some $y \in R$. Then

$$f_\sigma \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} = \begin{pmatrix} 1 & x + (u-1)y \\ 0 & u \end{pmatrix} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}$$

as required.

It follows from Proposition 6.4 that the group Γ associated to a finite local Gorenstein ring is not quasi-complete whenever the image of the natural map

(6.2) $\quad H^1(U, \mathfrak{m}) \to H^1(U, R)$

is not zero. The next proposition gives a characterization of the local Gorenstein rings for which this map is zero. For any prime number p, let $\mathbb{F}_p$ denote the finite field with p elements. If R is a ring and M_1, M_2 are R-modules, denote by $Hom_R(M_1, M_2)$ the group of homomorphisms between M_1, M_2 commuting with the R-action.

Proposition 6.5. *Let R be a finite local Gorenstein ring with maximal ideal $\mathfrak{M}$ and minimal ideal $\mathfrak{m}$. Then the natural map $H^1(U, \mathfrak{m}) \to H^1(U, R)$ is zero if and only if $R/\mathfrak{M} = \mathbb{F}_p$ and $(1 + \mathfrak{M})^p = 1 + \mathfrak{M}^2$.*

Proof. The map $H^1(U, \mathfrak{m}) \to H^1(U, R)$ is zero if and only if the map $(R/\mathfrak{m})^U \to H^1(U, \mathfrak{m})$ is surjective. Recall that the connecting homomorphism $\delta : (R/\mathfrak{m})^U \to H^1(U, \mathfrak{m})$ maps x to the 1-cocycle given by $u \mapsto (u-1)x$ for all $u \in U$. We write δ as the composite of several homomorphisms.

$$\begin{tikzcd}
(R/\mathfrak{m})^U \arrow[r, "\cong"] \arrow[d, "\delta"] & \mathrm{Ann}(\mathfrak{M}^2)/\mathfrak{m} \arrow[r, hook, "\gamma"] & Hom_R(\mathfrak{M}/\mathfrak{M}^2, \mathfrak{m}) & Hom_{\mathbb{Z}}(\mathfrak{M}/\mathfrak{M}^2, \mathfrak{m}) \arrow[l, phantom, "\supset"] \arrow[l, "i"'] \arrow[dl, "\cong"] \\
H^1(U, \mathfrak{m}) = Hom_{\mathbb{Z}}(1 + \mathfrak{M}, \mathfrak{m}) & & Hom_{\mathbb{Z}}((1 + \mathfrak{M})/(1 + \mathfrak{M}^2), \mathfrak{m}) \arrow[ll, hook', "j"'] &
\end{tikzcd}$$

Here the homomorphism $(R/\mathfrak{m})^U \to \mathrm{Ann}(\mathfrak{M}^2)/\mathfrak{m}$ is the inverse of the map induced by the inclusion of $\mathrm{Ann}(\mathfrak{M}^2)$ in R. The map γ sends x to the homomorphism given by $y \mapsto xy$, for all $y \in \mathfrak{M}/\mathfrak{M}^2$. It is injective. The diagonal homomorphism maps an additive map f to the multiplicative map given by $y \mapsto f(y-1)$. It is easily seen to be an isomorphism. Finally, note that $U = 1 + \mathfrak{M}$ and that $H^1(U, \mathfrak{m}) = Hom_{\mathbb{Z}}(1 + \mathfrak{M}, \mathfrak{m})$ because the action of U on $\mathfrak{m}$ is trivial.

It follows that δ is surjective if and only if the three maps γ, i and j are surjective. The map γ is a bijection. Indeed, since R is Gorenstein, the functor $Hom_R(-, R)$ is exact. Applying it to the exact sequence $0 \to \mathfrak{M}/\mathfrak{M}^2 \to R/\mathfrak{M}^2 \to R/\mathfrak{M} \to 0$, we obtain the exact sequence

$$0 \to \mathfrak{m} \to \mathrm{Ann}(\mathfrak{M}^2) \xrightarrow{\gamma'} Hom_R(\mathfrak{M}/\mathfrak{M}^2, \mathfrak{m}) \to 0,$$

with γ' inducing γ. Here we used the fact that for every ideal $I \subset R$, the map $Hom_R(R/I, R) \to \mathrm{Ann}(I)$ given by $f \mapsto f(1)$ is an isomorphism and the fact that the image of any homomomorphism $f : \mathfrak{M}/\mathfrak{M}^2 \to R$ is automatically contained in $\mathrm{Ann}(\mathfrak{M}) = \mathfrak{m}$, so that $Hom_R(\mathfrak{M}/\mathfrak{M}^2, R) = Hom_R(\mathfrak{M}/\mathfrak{M}^2, \mathfrak{m})$. We conclude that γ is bijective. The inclusion map i is a bijection if and only if *any* homomorphism $\mathfrak{M}/\mathfrak{M}^2 \to \mathfrak{m}$ is automatically R-linear. Since both $\mathfrak{M}/\mathfrak{M}^2$ and $\mathfrak{m}$ are $\mathbb{F}$-vector spaces, this happens precisely when $\mathbb{F}$ is equal to the prime field $\mathbb{F}_p$. The map j is surjective if and only if every homomorphism $f : 1 + \mathfrak{M} \to \mathfrak{m} \cong \mathbb{F}$ has $1 + \mathfrak{M}^2$ in its kernel. Since $\mathfrak{m} \cong \mathbb{F}$ is killed by p, this happens precisely when $1 + \mathfrak{M}^2 \subset (1 + \mathfrak{M})^p$.

This proves the proposition.

Remark 6.6. Finally we observe that there are plenty of finite local Gorenstein rings: $\mathbb{Z}/p^n\mathbb{Z}$, for p prime, is a finite local Gorenstein ring with minimal ideal $\mathfrak{m}$ generated by p^{n-1}. If R is a finite local Gorenstein ring and $\phi \in R[X]$ is a monic polynomial for which $R[X]/(\phi)$ is local, then the ring $R[X]/(\phi)$ is Gorenstein as well. Examples of finite groups Γ which are not quasi-complete arise whenever at least one of the two conditions in Proposition 6.5 is not fulfilled.

An example where the first condition is not satisfied, (i.e. the residue field $R/\mathfrak{M}$ is not a prime field) is given by $R = \mathbb{F}_4[X]/(X^2)$. In this case, $\mathfrak{M} = \mathfrak{m} = (X)$ and $U = 1 + (X)$. The corresponding group Γ has cardinality $|\Gamma| = 64$. In Wall's example the second condition is not fulfilled: $(1 + \mathfrak{M})^p \neq 1 + \mathfrak{M}^2$.

Example 6.7. Wall's example corresponds to the ring $R = \mathbb{Z}/8\mathbb{Z}$ with maximal ideal $\mathfrak{M} = (2)$ and residue field of characteristic $p = 2$. Since $(1 + \mathfrak{M})^p$ is trivial, while $1 + \mathfrak{M}^2$ is cyclic of order 2, the second condition of Prop. 6.5 is not satisfied. Therefore the corresponding group Γ is not quasi-complete.

Explicitly, Γ is the 32 element group given by

$$\Gamma = \left\{ \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} : x \in \mathbb{Z}/8\mathbb{Z}, u \in \{1, 3, 5, 7\} \right\}.$$

The map $\sigma : U \to R$ given by $\sigma(u) := \frac{u^2-1}{2}$ is a non-trivial 1-cocycle in $H^1(U, R)$. Since $\frac{u^2-1}{2}$ is divisible by 4, it lies in the image of the map (6.2). The corresponding map

$$f_\sigma \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} := \begin{pmatrix} 1 & x + \sigma(u) \\ 0 & u \end{pmatrix}$$

is an involutive outer automorphism of Γ which preserves every conjugacy class of Γ.

References

[AM] Atiyah M., McDonald I.: Introduction to Commutative Algebra. Addison-Wesley, 1969

[BDLR] Buchholz D., Doplicher S., Longo R., Roberts J. E.: Extension of automorphisms and gauge symmetries. Comm. Math. Phys. **155** (1993), 123–134

[B] Bourbaki N.: Algèbre. Masson, 1981

[Bu] Burnside W.: The theory of groups of finite order, 2nd ed. Dover, New York 1995

[CD] Conti R., D'Antoni C.: Extension of antiautomorphisms and PCT-symmetry. Rev. Math. Phys. **12** (2000), 725–738

[DF] Dyer J. L., Formanek E.: The automorphism group of a free group is complete. J. London Math. Soc. (2) **11** (1975), 181–190

[DG] Dugas M., Göbel R.: Outer automorphisms of groups. Illinois J. Math. **35** (1991), 27–46

[DG2] Dyer J. L., Grossman E. K.: The automorphism groups of the braid groups. Amer. J. Math. **103** (1981), 1151–1169

[Di] Dixmier J.: Les C^*-algebres et leur représentations. Gauthier-Villars, Paris 1969

[DR] Doplicher S., Roberts J. E.: Endomorphisms of C^*-algebras, cross products and duality for compact groups. Ann. Math. **130** (1989), 75–117
[Du] Dugundji J.: Topology. Allyn & Bacon Inc., Boston 1966
[E] Eisenbud D.: Commutative Algebra. Graduate Text in Mathematics 150. Springer, 1995
[Ha] Handelman D.: Representation rings as invariants for compact groups and limit ratio theorems for them. Intern. J. Math. **4** (1993), 59–88
[He] Helgason S.: Differential Geometry, Lie groups, and Symmetric Spaces. Academic Press, 1978
[HR] Hewitt E., Ross K. A.: Abstract Harmonic Analysis, vol. I, II. Springer, 1970
[Ho] Hochschild G.: The structure of Lie groups. Holden-Day, 1965
[Hu] Huppert B.: Endliche Gruppen I. Grundlehren der mathematischen Wissenschaften 134. Springer, 1979
[Kn1] Knapp A. W.: Representation theory of semisimple Lie groups. Princeton Mathematical Series 36. Princeton University Press, 1986
[Kn2] Knapp A. W.: Lie groups beyond an introduction. Progress in Mathematics 140. Birkhäuser, 1986
[Lo] Loos O.: Symmetric spaces, Vol. I, Vol. II. Mathematics Lecture Notes Series. Benjamin, 1969
[Ma] Mackey G. W.: The theory of unitary group representations. Univ. of Chicago Press, Chicago 1976
[Mc] McMullen J. R.: On the dual object of a compact connected group. Math. Z. **185** (1984), 539–552
[Pe1] Pensavalle C.: Spherical principal series and generators of a free group. Boll. UMI **(7) 8-B** (1994), 57–86
[Pe2] Pensavalle C.: Anisotropic principal series and generators of a free group. Bull. Austral. Math. Soc. **49** (1994), 205–217
[P] Price J.: Lie groups and compact groups. London Mathematical Society Lecture Notes 25. Cambridge University Press, 1977
[Sc] Schupp P. E.: A characterization of inner automorphisms. Proc. Amer. Math. Soc. **101** (1987), 226–228
[Se] Serre J. P.: Corps locaux. Actualités Scientifiques et Industrielles 1296. Hermann, Paris 1968
[Su] Sugiura M.: Conjugate classes of Cartan subalgebras in real semisimple Lie algebras. J. Math. Society Japan **11** (1959), 374–434
[Sz] Suzuki M.: Group Theory I. Grundlehren der Mathematischen Wissenschaften 247. Springer, 1980
[Ta] Takesaki M.: Theory of operator algebras I. Springer, 1979
[Wa] Wassermann A.: Ergodic actions of compact groups on operator algebras II. Classification of full multiplicity ergodic actions. Canad. J. Math. **40** (1988), 1482–1527

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