1. Let $L \subset \mathbf{Z}^2$ be the lattice generated by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.
   (a) Determine the covolume of L. Determine a basis for the dual lattice L^*.
   (b) Determine $\mathrm{covol}(L^*)$.

2. Let $L \subset \mathbf{R}^3$ be the set $\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{Z}^3 : 2x + 3y + 5z \equiv 0 \pmod 7\}$.
   (a) Show that L is a lattice.
   (b) Determine its covolume.

3. Let V be the vector space $\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3 : x_1 + x_2 + x_3 = 0\}$ equipped with the scalar product coming from $\mathbf{R}^3$.
   (a) Show that $L = \mathbf{Z}^3 \cap V$ is a lattice in V.
   (b) Compute the Gram matrix of L. Is L integral? Even?
   (c) Compute the covolume of L. Is L unimodular?

4. Let $n \geq 2$ and put $L = \{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{Z}^n : x_1 + \ldots + x_n \equiv 0 \pmod 2\}$.
   (a) Show that L is a lattice in $\mathbf{R}^n$.
   (b) Compute its covolume.
   (c) Show that for all $\mathbf{v}, \mathbf{w} \in L$ we have $\|\mathbf{v} - \mathbf{w}\|^2 \geq 2$.
   (d) Show that the balls of radius $\sqrt{2}/2$ and with center in L are a sphere packing in $\mathbf{R}^n$. Compute the density.

5. Let n be a natural number. Show that $n! > n^n e^{-n}$.

6. Let $\mathbf{Z}[\sqrt{3}] := \{x = a + b\sqrt{3} \mid a, b \in \mathbf{Z}\}$. Put $x' = a - b\sqrt{3}$. Let

$$L = \{\begin{pmatrix} x \\ x' \end{pmatrix} \in \mathbf{R}^2 : x \in \mathbf{Z}[\sqrt{3}]\}.$$

   (a) Prove that $\mathbf{Z}[\sqrt{3}]$ is a ring.
   (b) Show that L is a lattice in $\mathbf{R}^2$. Draw a picture.
   (c) Compute its covolume.