The sphere packing problem: new developments.

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The sphere packing problem.

• Arrange solid spheres B_r^n of equal radii r in n-dimensional Euclidean space $\mathbb{R}^n$.

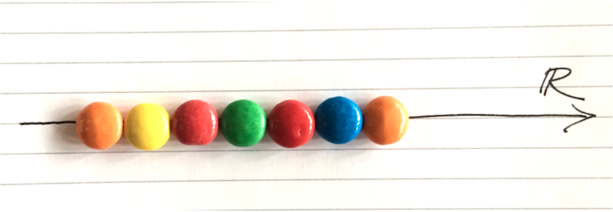
• Find the densest packing: the one which maximizes the volume occupied by the spheres.

In the real line

The highest density of a sphere packing in $\mathbb{R}$ is

$$\Delta_1 = 1.$$

The real line $\mathbb{R}$ can be completely covered by non-overlapping intervals of the same size.

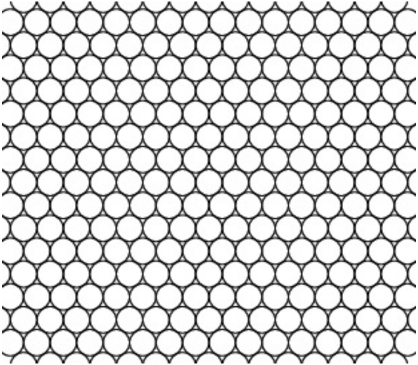


In the plane

The highest density of a sphere packing in $\mathbb{R}^2$ *is*

$$\Delta_2 = \frac{\pi}{\sqrt{12}} = 0.906899\ldots.$$

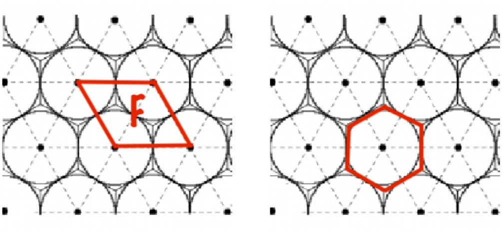
It is achieved by the hexagonal packing:



The spheres have centers on the points of hexagonal lattice

$$\mathrm{A}_2 = \mathrm{span}_{\mathbb{Z}}\{(1,-1,0),(0,1,-1)\}$$

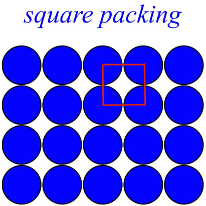
and radii $\mathrm{r} = \sqrt{2}/2$, half the minimum distance between points of the lattice.



Its density is given by

$$\mathrm{Vol}(\mathrm{B}^2_{\sqrt{2}/2})/\mathrm{Vol}(\mathcal{F}) = \frac{\pi}{2\sqrt{3}} \sim 0,906899\ldots.$$

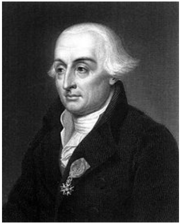
The square packing in the plane:



the spheres have centers on the integral lattice and radius $\mathrm{r} = 1/2$; its density is given by

$$\mathrm{Vol}(\mathrm{B}^2_{1/2})/\mathrm{Vol}(\mathcal{F}) = \frac{\pi}{4} \sim 0,785398\ldots$$

The hexagonal packing was proved to be the densest one among lattice packings by Joseph-Louis Lagrange (1736-1813)



and among arbitrary packings by Axel Thue in 1890.

An irregular packing....

In the 3-dimensional space

Around 1591, explorer Sir Walter Raleigh (left)

asked the English mathematician and astronomer Thomas Harriot (right) to study the best way to stack cannon balls on the decks of his ships.

Thomas Harriot was in correspondence with Johannes Kepler.

In 1611, in his paper *On the six-cornered snowflake*, Kepler formulated the following conjecture:

The highest density of a sphere packing in $\mathbb{R}^3$ is

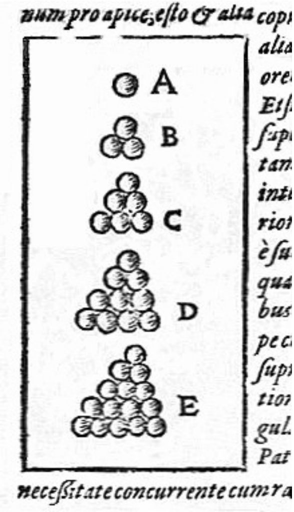
$$\Delta_3 = \frac{\pi}{3\sqrt{2}} = 0.740480489\ldots.$$

It is achieved by infinitely many packings.

In 1831, Carl Friedrich Gauss

proved the Kepler conjecture for lattice packings.

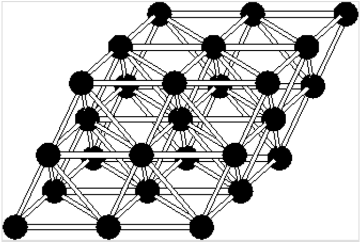
Among lattice packings, the highest density is achieved by the face-centered cubic.



The spheres have centers on the points of the A_3 lattice

$$\mathrm{A}_3 = \mathrm{span}_{\mathbb{Z}}\{(1,-1,0,0),(0,1,-1,0),(0,0,1,-1)\}$$

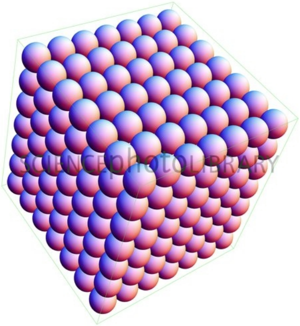
and radius $\mathrm{r} = \sqrt{2}/2$.



Its density is indeed maximal

$$\mathrm{Vol}(\mathrm{B}^3_{\sqrt{2}/2})/\mathrm{Vol}(\mathcal{F}) = \frac{\pi}{2\sqrt{3}} \sim 0,7404\ldots.$$

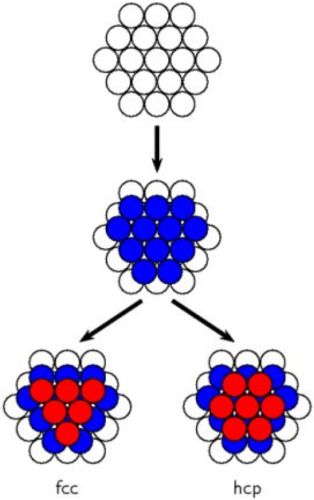
- The cubic packing



the spheres have centers on the integral lattice and radius $\mathrm{r} = 1/2$; its density is given by

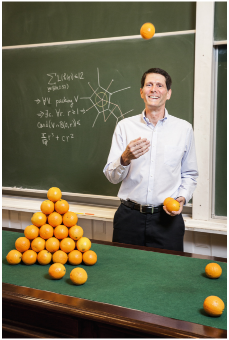
$$\mathrm{Vol}(\mathrm{B}^3_{1/2})/\mathrm{Vol}(\mathcal{F}) = \frac{\pi}{6} \sim 0,5235\ldots$$

The maximal density is also achieved by infinitely many non-lattice packings, obtained by stacking optimal 2-dimensional layers.



For completing the proof of the Kepler conjecture, one must consider also irregular packings. In 1953, Lázló Fejes Tóth reduced the treatment of irregular packings to a finite (big) number of cases.

In 2005, Thomas Hales (Univ. of Michigan)



proved the Kepler conjecture by a computer aided proof.

The sphere packing problem in $\mathbb{R}^n$

How to measure the density of a sphere packing $\mathcal{P}$?

$$\mathcal{P} = \bigcup_{x} B_r^n(x)$$

The local density of the packing is

$$\Delta_{\mathcal{P}}(R) := \frac{\mathrm{Vol}(\mathcal{P} \cap B_R^n(0))}{\mathrm{Vol}(B_R^n(0))}, \quad R > 0$$

The density of the packing is

$$\Delta_{\mathcal{P}} = \limsup_{R \to \infty} \Delta_{\mathcal{P}}(R).$$

- For a lattice Λ packing

$$\Delta_\Lambda = Vol(B_r^n)/Vol(\mathcal{F}), \qquad r = d/2,$$

where $\mathcal{F}$=fundamental domain of the lattice, and d=minimum distance between elements of the lattice.

We want to estimate

$$\Delta_n = \sup_{\mathcal{P}} \Delta_{\mathcal{P}}$$

Until 2016, the problem was solved only up to dimension 3:

$\Delta_1 = 1$

$\Delta_2 = 0,9068\ldots$

$\Delta_3 = 0,7404\ldots$

In each case, the highest density is achieved by a lattice packing.

- Rough lower bound $\Delta_n \geq 2^{-n}$
- Most density upper bounds are "far" from best known densities.

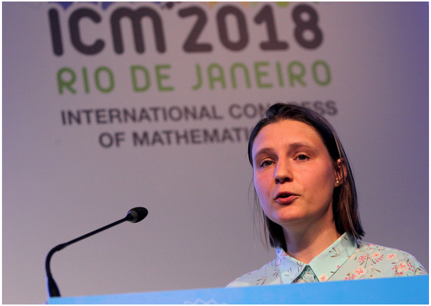
n	highest dens. known	lattice	Cohn-Elkies' u.b.
1	1	$\mathbb{Z}$	1
2	0,906899...	A_2	0,906899...
3	0,740480...	A_3	0,77974...
4	0,6168...	D_4	0,64774...
5	0,4652...	D_5	0,5249...
6	0,3729...	E_6	0,4176...
7	0,2952...	E_7	0,3274...
8	0,253669507...	E_8	0,253669508...
9	0,1457...	Λ_9	0,1945...
10	0,099...	—	0,1479...

- Sphere packings in $\mathbb{R}^n$ become less and less dense as $n \to \infty$.

The volume of a ball of radius r in $\mathbb{R}^n$ tends to zero for $n \to \infty$.

$$Vol(B_r^n) = \frac{\pi^{n/2}}{(n/2)!} r^n, \qquad (n/2)! = \Gamma(n/2+1).$$

In 2016, Maryna Viazovska



solved the sphere packing problem in $\mathbb{R}^8$

$$\mathbb{R}^8: \qquad \Delta_8 = \frac{\pi^4}{384} \sim 0,25367, \qquad E_8\text{-lattice packing}$$

and in $\mathbb{R}^{24}$, with H. Cohn, A, Kumar, S. D. Miller, D. Radchenko,

$$\mathbb{R}^{24}: \qquad \Delta_{24} = \frac{\pi^{12}}{12!} \sim 0,001929, \qquad \text{Leech-lattice packing.}$$

The starting point of Viazovska's proof is the following theorem:

Theorem. (H. Cohn, N. Elkies, 2001) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a Schwartz function and* $r \in \mathbb{R}_{>0}$ *such that*

- $f(0) = \hat{f}(0) > 0$,
- $\hat{f}(y) \geq 0$, *for all* $y \in \mathbb{R}^n$,
- $f(x) \leq 0$, *for* $|x| \geq r$.

Then

$$\Delta_n \leq Vol(B^n_{r/2}).$$

For $n = 1$, the function

$$f(x) = (1 - |x|)\chi_{[-1,1]}(x),$$

with Fourier transform $\hat{f}(y) = \left(\frac{\sin(\pi y)}{\pi y}\right)^2$ proves that $\Delta_1 = 1$.

- $f(0) = \hat{f}(0) = 1 > 0$;
- $f(x) \leq 0$, for all $|x| \geq 1$;
- $\hat{f}(y) \geq 0$, for all $y \in \mathbb{R}$.

Its higher dimensional analogue produces the trivial bound $\Delta_n \leq 1$.

Another function which proves $\Delta_1 = 1$:

$$f(x) = \frac{1}{1-x^2}\left(\frac{\sin(\pi x)}{\pi x}\right)^2$$

$$\hat{f}(y) = (1-|y|) + \frac{\sin(2\pi|y|)}{2\pi}, \qquad |y| < 1,$$
$$\hat{f}(y) = 0, \qquad |y| \geq 1.$$

• Several upper bounds for sphere packing densities have been improved by Cohn-Elkies by constructing appropriate functions satisfying their theorem for some r.

• In dimension 8 and 24, the Cohn-Elkies bounds are very close to the density of the E_8 lattice and the Leech lattice, respectively.

• Finding a function with optimal r is a major problem.

How does Cohn-Elkies's theorem imply a bound on Δ_n?

• Densest packings can be approximated by "periodic packings" (spheres are centered on the union of finitely many translates of a lattice).

• For "periodic packings", the theorem follows from the Poisson summation formula.

• We sketch the proof for lattice packings.

If Λ is a lattice in $\mathbb{R}^n$ and $\Lambda^* = \{x \in \mathbb{R}^n \mid \langle x, \lambda \rangle \in \mathbb{Z},\ \forall \lambda \in \Lambda\}$, then

$$\sum_{x \in \Lambda} f(x) = \frac{1}{Vol(\mathcal{F})} \sum_{y \in \Lambda^*} \hat{f}(y).$$

If Λ is a lattice with minimum distance r

$$f(0) + \underbrace{\sum_{x \in \Lambda \setminus \{0\}} f(x)}_{\leq 0} = \frac{1}{Vol(\mathcal{F})} (\hat{f}(0) + \underbrace{\sum_{x \in \Lambda^* \setminus \{0\}} \hat{f}(x)}_{\geq 0})$$

$$\Rightarrow \quad f(0) \geq \frac{1}{Vol(\mathcal{F})} \hat{f}(0).$$

$$\Rightarrow \quad Vol(\mathcal{F}) \geq \frac{\hat{f}(0)}{f(0)} = 1$$

$$\Delta_\Lambda = Vol(B^n_{r/2}) / Vol(\mathcal{F}) \leq Vol(B^n_{r/2}).$$

The lattice E_8 is generated over $\mathbb{Z}$ by the vectors

$$\frac{1}{2}(e_1+e_8)-\frac{1}{2}(e_2+e_3+e_4+e_5+e_6+e_7),\ e_1+e_2,$$
$$e_2-e_1,\ e_3-e_2,\ e_4-e_3,\ e_5-e_4,\ e_6-e_5,\ e_7-e_6,$$

where e_i denotes the i^{th} vector of the canonical basis of $\mathbb{R}^8$.

- It is unimodular: $Vol(\mathcal{F})=1$.
- It is selfdual: $E_8 \cong E_8^*$.
- It has minimum distance $d=\sqrt{2}$.
- The density of the associated sphere packing is

$$\Delta_{E_8}=Vol\left(B^8_{\sqrt{2}/2}\right)=\frac{\pi^4}{384}.$$

If there exists a Schwartz function f as in the Cohn-Elkies theorem, with $r = \sqrt{2}$, then

$$\Delta_8 \le Vol\left(B^8_{\sqrt{2}/2}\right) = \Delta_{E_8}.$$

$$\Rightarrow \quad \Delta_8 = \Delta_{E_8} = \frac{\pi^4}{384}.$$

Remark.

• f may be taken radial;

• assumptions of the Cohn-Elkies theorem $+$ unimodularity of E_8 $+$ Poisson summation formula imply

$$f(x) = \widehat{f}(x) = 0, \quad \forall x \in E_8 \setminus \{0\}.$$

- Does one such function f exists?

Viazovska constructed f by using modular forms:

$$f(x) = \sin^2\left(\frac{\pi|x|^2}{2}\right)\int_0^{+\infty}\left(t^2\varphi\left(\frac{i}{t}\right) + \psi(it)\right)e^{-\pi t|x|^2}\,dt, \quad |x| > \sqrt{2},$$

where

$$\varphi = \frac{4\pi(E_2E_4 - E_6)^2}{5(E_6^2 - E_4^3)}$$

and ψ is a rational function in the Θ-series of the lattice $\mathbb{Z}$.

For $k > 2$ even, E_k denotes the Eisenstein series:
for $q = e^{2\pi iz}$, $z \in \mathcal{H}^+$ and $\sigma_k(n) = \sum_{d|n} d^k$,

$$E_2(z) = 1 - 24 \sum_{\mathfrak{n} \geq 1} \sigma_1(n) q^n$$

$$E_4(z) = 1 + 240 \sum_{\mathfrak{n} \geq 1} \sigma_3(n) q^n$$

$$E_6(z) = 1 - 504 \sum_{\mathfrak{n} \geq 1} \sigma_5(n) q^n$$

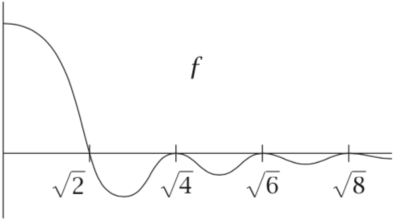
and

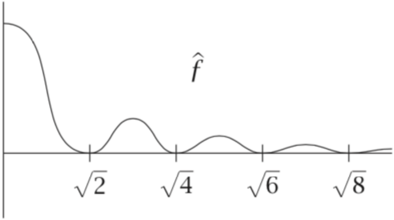
$$\Theta(z) = \sum_{x \in \mathbb{Z}} e^{\pi i |x|^2 z}.$$

The Fourier transform of f is

$$\widehat{f}(y) = \sin^2\left(\frac{\pi|y|^2}{2}\right) \int_0^{+\infty} \left(t^2 \varphi\left(\frac{i}{t}\right) - \psi(it)\right) e^{-\pi t |y|^2} dt.$$

The radial functions f and $\hat{f}$.





(pictures from H. Cohn, Notices AMS, 2017)

• The arguments used for the 8-dimensional case were generalized to solve the 24-dimensional case.

• The E_8-lattice and the Leech lattice sphere packings are the only periodic packings with maximal density in $\mathbb{R}^8$ and $\mathbb{R}^{24}$, respectively.

Final remarks

- The sphere packing problem in $\mathbb{R}^n$ is open for $n \neq 1,\ 2,\ 3,\ 8,\ 24$.

- Possibly the next case to be solved:

the lattice packing D_4 is the candidate for densest packing in $\mathbb{R}^4$, where

$$D_4 = span_{\mathbb{Z}}\{e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_3 + e_4\}.$$

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THANK YOU!