POLAR SYMPLECTIC REPRESENTATIONS

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ABSTRACT. We study polar representations in the sense of Dadok and Kac which are symplectic. We show that such representations are coisotropic and use this fact to give a classification. We also study their moment maps and prove that they separate closed orbits. Our work can also be seen as a specialization of some of the results of Knop on multiplicity free symplectic representations to the polar case.

1. INTRODUCTION

A rational representation of a complex reductive linear algebraic group G on a finite-dimensional complex vector space V is called *polar* if there exists a subspace $c \subset V$ consisting of semisimple elements such that $\dim c = \dim V /\!\!/ G$ (the categorical quotient), and for a dense subset of c, the tangent spaces to the orbits are parallel [DK85]; then it turns out that every closed orbit of G meets c (Prop. 2.2, *ibid*). In this paper we study the class of polar representations which are *symplectic*, namely, preserve a non-degenerate skew-symmetric bilinear form ω on V (polarity of a representation depends only on the identity component, and we assume throughout that all groups are connected). We first prove:

Theorem 1. *A polar symplectic representation is coisotropic.*

Recall that a symplectic representation V of G is *coisotropic* if a generic G-orbit is coisotropic, namely, $(\mathfrak{g} \cdot v)^{\perp_\omega} \subset \mathfrak{g} \cdot v$ where $v \in V$ is generic, $\mathfrak{g}$ denotes the Lie algebra of G and $\perp_\omega$ refers to the symplectic complement. Representations in this class can be characterized by a number of different properties, e.g. the Poisson algebra of invariants $\mathbb{C}[V]^G$ is commutative (cf. [Kno07, p. 224 and Prop. 9.1] and [Los05, Introd.]) ; in particular, they are also called *multiplicity-free* (in the symplectic sense).

Using Theorem 1, we can reduce the classification of polar symplectic representations, up to geometric equivalence, to that of coisotropic representations given in [Kno06]. In

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contrast to the case of coisotropic representations, it turns out that every saturated decomposable polar symplectic representation is a product (see section 2 for unexplained terminology).

Theorem 2. *The saturated indecomposable polar symplectic representations are listed in Tables A and B. Every saturated polar symplectic representation is a product of indecomposable polar symplectic representations.*

G	V	$\dim V/\!\!/G$	*Conditions*
$SO_p \otimes Sp_{2m}$	$\mathbb{C}^p \otimes \mathbb{C}^{2m}$	$\min\{[\frac{p}{2}], m\}$	$m \geq 1,\ p \geq 3$
Sp_{2m}	$\mathbb{C}^{2m}$	0	$m \geq 1$
$Sp_2 \times Spin_7$	$\mathbb{C}^2 \otimes \mathbb{C}^8$	1	–
$SL_2 \times Spin_9$	$\mathbb{C}^2 \otimes \mathbb{C}^{16}$	2	–
$Spin_{11}$	$\mathbb{C}^{32}$	1	–
$Spin_{12}$	$\mathbb{C}^{32}$	1	–
$Spin_{13}$	$\mathbb{C}^{64}$	2	–
SL_2	$S^3(\mathbb{C}^2)$	1	–
SL_6	$\Lambda^3(\mathbb{C}^6)$	1	–
Sp_6	$\Lambda^3(\mathbb{C}^6) \ominus \mathbb{C}^6$	1	–
$SL_2 \times G_2$	$\mathbb{C}^2 \otimes \mathbb{C}^7$	1	–
E_7	$\mathbb{C}^{56}$	1	–

TABLE A: INDECOMPOSABLE POLAR SYMPLECTIC REPRESENTATIONS OF TYPE 1

G	V	$\dim V/\!\!/G$	*Conditions*	$\mathbb{C}^\times$ *essential*
$\mathbb{C}^\times \times SL_m \times SL_n$	$\mathbb{C}^m \otimes \mathbb{C}^n \oplus (\mathbb{C}^m \otimes \mathbb{C}^n)^*$	n	$m \geq n \geq 2$	yes iff $m = n$
GL_n	$\Lambda^2(\mathbb{C}^n) \oplus \Lambda^2(\mathbb{C}^n)^*$	$[\frac{n}{2}]$	$n \geq 4$	yes iff n even
GL_n	$S^2(\mathbb{C}^n) \oplus S^2(\mathbb{C}^n)^*$	n	$n \geq 2$	yes
GL_n	$\mathbb{C}^n \oplus \mathbb{C}^{n*}$	1	$n \geq 1$	yes iff $n = 1$
$\mathbb{C}^\times \times Sp_{2m}$	$\mathbb{C}^{2m} \oplus \mathbb{C}^{2m*}$	1	$m \geq 2$	no
$\mathbb{C}^\times \times SO_m$	$\mathbb{C}^m \oplus \mathbb{C}^{m*}$	2	$m \geq 5$	yes
$\mathbb{C}^\times \times Spin_7$	$\mathbb{C}^8 \oplus \mathbb{C}^{8*}$	2	–	yes
$\mathbb{C}^\times \times Spin_{10}$	$\mathbb{C}^{16} \oplus \mathbb{C}^{16*}$	2	–	no
$\mathbb{C}^\times \times G_2$	$\mathbb{C}^7 \oplus \mathbb{C}^{7*}$	2	–	yes
$\mathbb{C}^\times \times E_6$	$\mathbb{C}^{27} \oplus \mathbb{C}^{27*}$	3	–	yes

TABLE B: INDECOMPOSABLE POLAR SYMPLECTIC REPRESENTATIONS OF TYPE 2

In the last column of Table B, non-essentialness of the center means that its removal does not change the closed orbits; otherwise, the closed orbits change and the representation ceases to be polar.

A *symplectic symmetric space* is a symmetric space which is endowed with a symplectic structure invariant by the symmetries. Our interest in them is that the (complexified) isotropy representations of symplectic symmetric spaces provide examples of symplectic θ-groups [Vin76, Kac80], thus, polar symplectic representations. Conversely, it is a natural question to ask which polar symplectic representations come from symplectic symmetric spaces. We say that two symplectic representations are *closed orbit equivalent* if there exists a symplectic isomorphism between the representation spaces mapping closed orbits onto closed orbits (for the sake of comparison, recall that in the orthogonal case all polar representations come from symmetric spaces, up to closed orbit equivalence [Dad85, GG08]). Note that polarity is a property of closed orbit equivalence classes.

Theorem 3. *A polar symplectic representation is closed orbit equivalent to the isotropy representation of a complex semisimple symplectic symmetric space if and only if it is closed orbit equivalent to the complexification of the isotropy representation of a semisimple Hermitian Riemannian symmetric space. In the saturated case, such representations are exactly the products of representations listed in Table B.*

Finally, recall that a symplectic representation (G, V) has a canonical *moment map* $\mu : V \to \mathfrak{g}^*$ (see section 6). Since μ is equivariant, it induces an *invariant moment map* $\psi = \mu/\!\!/G : V/\!\!/G \to \mathfrak{g}^*/\!\!/G$.

Theorem 4. *The moment map of a saturated polar symplectic representation maps closed orbits to closed orbits and separates closed orbits.*

Remark 5. The only saturated indecomposable polar symplectic representation for which the invariant moment map ψ fails to be an isomorphism from $V/\!\!/G$ onto an affine space in $\mathfrak{g}^*/\!\!/G$ is the last one in Table B. Hence, in all the other cases the morphism $\psi^* : \mathbb{C}[\mathfrak{g}^*]^G \to \mathbb{C}[V]^G$ is surjective, that is, all invariants are pull-backs of coadjoint invariants.

Remark 6. In case of type 2 representations, Theorems 1 and 4 reduce to known facts about polar representations of compact Lie groups in the sense of Dadok [Dad85]. Let (K, U) be an orthogonal representation of a compact Lie group K and consider its complexification $(G := K^{\mathbb{C}}, V := U^{\mathbb{C}})$. It is easy to check that (G, V) is polar if and only if (K, U) is polar. Suppose now U admits an invariant complex structure. Then $(G, V = U \oplus U^*)$ is coisotropic if and only (K, U) is multiplicity free (cf. [Kno06, p.532] or [Kno07, Prop. 9.2]) if and only if (K, U) has coisotropic principal K-orbits ([Kno98, Thm. 3.1] and [Vin01, Prop. 12]) so Theorem 1 says that a polar representation of a compact Lie group has coisotropic principal K-orbits (compare [PT02, Thm. 1.1 and Lem. 2.7]). Moreover the moment map μ of (G, V) restricts to the moment map μ_K of (K, U), every G-orbit through a point in U is closed [Bir71], and two different K-orbits in U cannot be contained in the same G-orbit (since $\mathbb{C}[V]^G = \mathbb{R}[U]^K \otimes \mathbb{C}$; see also [Bre93, § 2.2]), so Theorem 4 says that μ_K separates K-orbits (compare [PT02, Cor. 1.5] and [HW90, p. 274]).

2. PRELIMINARIES

We begin by recalling terminology from [Kno07] that will be useful in the sequel. A symplectic representation of G is called *indecomposable* if it is not isomorphic to the sum of two non-trivial symplectic representations of G. A symplectic representation V of G is called of *type 1* if V is irreducible as a G-module, and it is called of *type 2* if $V = U \oplus U^*$ where U is an irreducible G-module not admitting a symplectic structure and the symplectic form on V is given (up to a multiple) by

$$\omega(u_1 + u_1^*, u_2 + u_2^*) = u_1^*(u_2) - u_2^*(u_1).$$

Every indecomposable symplectic representation is either of type 1 or 2. Two symplectic representations are isomorphic as G-modules if and only if they are isomorphic as symplectic representations. Every symplectic representation is a direct sum of finitely many indecomposable symplectic representations, and the summands are unique up to permutation [Kno06, Theorem 2.1].

It is convenient to revisit the result above as follows. Choose a maximal compact subgroup K of G (necessarily connected) and a K-invariant Hermitian inner product h on V. A K-invariant conjugate linear automorphism $\epsilon : V \to V$ is then defined by

$$\omega(u, v) = h(u, \epsilon v)$$

for all $u, v \in V$. Then

$$\begin{aligned} h(u, \epsilon^2 v) &= \omega(u, \epsilon v) = -\omega(\epsilon v, u) = -h(\epsilon v, \epsilon u) \\ &= -\overline{h(\epsilon u, \epsilon v)} = -\overline{\omega(\epsilon u, v)} = \overline{\omega(v, \epsilon u)} = \overline{h(v, \epsilon^2 u)} = h(\epsilon^2 u, v), \end{aligned}$$

so ϵ^2 is a $\mathbb{C}$-linear K-invariant Hermitian endomorphism of V. It also follows from the above that $h(u, \epsilon^2 u) = -||\epsilon u||^2$, so ϵ^2 is negative definite. Now there is a h-orthogonal K-irreducible decomposition $V = \bigoplus V_j$ such that $\epsilon^2|_{V_j} = \lambda_j \mathrm{id}_{V_j}$ for $\lambda_j < 0$ and all j. For each j, either $\omega|_{V_j \times V_j}$ is non-degenerate or it is zero (since ω is K-invariant and V_j is K-irreducible). In the former case, $\epsilon(V_j) = V_j$. In the latter case, $\epsilon(V_j) \perp_h V_j$ and $\epsilon(V_j) = V_j^*$ (since ϵ is conjugate-linear). Hence V is an h-orthogonal direct sum of symplectic representations of *type 1* (V_j is irreducible and anisotropic) and *type 2* ($V_j \oplus V_j^*$, where V_j is irreducible and isotropic). By renormalizing h, we may assume that $\epsilon^2 = -\mathrm{id}_V$; in particular, ϵ becomes an h-isometry.

Let $\rho_i : \mathfrak{g}_i \to \mathfrak{sp}(V_i)$ for $i = 1, 2$ be two symplectic representations. We say V_1 and V_2 are *(geometrically) equivalent* (resp. *closed orbit equivalent*) if there is a symplectic isomorphism $\varphi : V_1 \to V_2$, inducing an isomorphism $\tilde{\varphi} : \mathfrak{sp}(V_1) \to \mathfrak{sp}(V_2)$, such that $\rho_2(\mathfrak{g}_2) = \tilde{\varphi}(\rho_1(\mathfrak{g}_1))$ (resp. φ maps closed orbits of G_1 onto closed orbits of G_2). The product of ρ_1 and ρ_2 is the algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ acting on $V_1 \oplus V_2$; it is a symplectic representation. A symplectic representation is called *connected* if it is not equivalent to the product of two non-trivial symplectic representations. Of course, it suffices to prove the above theorems for connected representations.

A symplectic representation $\rho : \mathfrak{g} \to \mathfrak{sp}(V)$ is called *saturated* if $\rho[\mathfrak{g}]$ is self-normalizing in $\mathfrak{sp}(V)$. Note that every type 2 representation $U \oplus U^*$ has non-trivial endomorphisms, namely, $\mathfrak{t}^1$ acting by $t \cdot (u, u^*) = (tu, -tu^*)$. We will also use the following notation from [Kno06]. Let U be a representation of a semisimple Lie algebra $\mathfrak{s}$. We denote the type 2 representation of $\mathfrak{g} = \mathfrak{s} + \mathfrak{t}^1$ on $U \oplus U^*$ by $T(U)$. Continuing, if U_1, U_2 are two representations of $\mathfrak{s}$, then $T(U_1) \oplus T(U_2)$ is a representation of $\mathfrak{g} = \mathfrak{s} + \mathfrak{t}^2$.

Remark 7. Let U be a symplectic representation of G with $U \cong U^*$. Then $(G \times SO_2, U \otimes \mathbb{C}^2)$ is isomorphic to $T(U) = U \oplus U$ via $v \otimes e_1 + w \otimes e_2 \mapsto (v + iw, v - iw)$.

Recall that a representation is called *stable* if generic orbits are closed. A representation of the form $U \oplus U^*$ is always stable, since it admits the invariant orthogonal structure given by $\langle u_1 + u_1^*, u_2 + u_2^* \rangle = u_1^*(u_2) + u_2^*(u_1)$ and one can apply [Sch80, Cor. 5.9] or [Lun72, Lun73]. A useful necessary and sufficient condition for the stability of a symplectic representation is that the generic isotropy algebra be reductive [Los05, Thm. 2]. Recall also that the *rank* of a representation V of G is the difference between the dimension of $V /\!\!/ G$ and that of the subspace of fixed points V^G. We next quote two results about polar representations that will be essential to our discussion.

Proposition 8 ([DK85], Thm. 2.4)**.** *Let V be a polar representation of G, let $v \in V$ be semisimple, and set N_v to be the orthocomplement of $\mathfrak{g} \cdot v$ with respect to a K-invariant Hermitian inner product on V, where K is a maximal compact subgroup of G. Then the slice representation of G_v on N_v is polar. Moreover, any Cartan subspace of N_v for the action of G_v is also a Cartan subspace of V for the action of G.*

Proposition 9 ([DK85], Prop. 2.14 and Cor. 2.15)**.** *Let $V = V_1 \oplus V_2$ be a polar representation of G, where V_1 and V_2 are G-invariant. Then:*

(a) *The subrepresentations V_1 and V_2 are polar.*
(b) *If V_1, say, is stable, then every Cartan subspace of V is the direct sum of Cartan subspaces of V_1 and V_2; it follows that rank $(V) =$ rank $(V_1) +$ rank (V_2).*
(c) *Under the assumptions in (b), the set of closed orbits of G in V_2 coincides with the set of closed orbits of G_{v_1}, where $v_1 \in V_1$ is any semisimple point.*

We can now prove:

Proposition 10. *Let $\rho : \mathfrak{g} \to \mathfrak{sp}(V)$ be a polar symplectic representation. Then:*

(a) *The centralizer of $\rho[\mathfrak{g}]$ in $\mathfrak{sp}(V)$ is commutative.*
(b) *Let $\hat{\mathfrak{g}}$ be the normalizer of $\rho[\mathfrak{g}]$ in $\mathfrak{sp}(V)$. Then $(\hat{\mathfrak{g}}, V)$ is saturated and closed orbit equivalent to $(\mathfrak{g}, V)$.*

It follows from (b) that $(\tilde{\mathfrak{g}}, V)$ is polar for every $\rho[\mathfrak{g}] \subset \tilde{\mathfrak{g}} \subset \hat{\mathfrak{g}}$.

Proof. (Compare [Kno06, Prop. 2.2].) Let $V = \bigoplus_i C_i^{n_i}$ be a decomposition into indecomposable symplectic representations where the C_i are mutually non-isomorphic. The centralizer of $\rho[\mathfrak{g}]$ in $\mathfrak{sp}(V)$ is the product of the centralizers of the $C_i^{n_i}$. There are three cases to consider:

1. C_i is of type 1. Then the centralizer is $\mathfrak{so}_{n_i}$.
2a. $C_i = U \oplus U^*$ is of type 2 with $U \ncong U^*$. Then the centralizer is $\mathfrak{gl}_{n_i}$.
2b. $C_i = U \oplus U^*$ is of type 2 with $U \cong U^*$. Then the centralizer is $\mathfrak{sp}_{2n_i}$.

A component C_i of type 1 has multiplicity $n_i \leq 2$ since C_i^2 is stable and we can apply Proposition 9(c) to C_i^3. The same corollary yields that $n_i \leq 1$ in case 2a since C_i is stable in that case, and that components of type 2b cannot occur since U is stable in that case. This proves part (a).

It follows from (a) and Remark 7 that we can write $V = W \oplus U \oplus U^*$, where $W = W_1 \oplus \cdots \oplus W_r$, $U = U_1 \oplus \cdots \oplus U_s$, the W_i are indecomposable of type 1, and either the $U_j \oplus U_j^*$ are indecomposable of type 2 or U_j is of type 1; moreover, the W_i (resp. $U_j \oplus U_j^*$) are pairwise non-isomorphic, and $\hat{\mathfrak{g}} = \mathfrak{t}^s + \mathfrak{g} = \mathfrak{t}^s \oplus \mathfrak{g}'$, where $\mathfrak{g}'$ is the derived algebra of $\mathfrak{g}$.

Since $U_j \oplus U_j^*$ is always a stable representation and $(\mathfrak{g}, U \oplus U^*)$ is polar (Proposition 9(a)), the latter representation is closed orbit equivalent to a product of polar representations $\bigoplus_{j=1}^s (\mathfrak{g}_j, U_j \oplus U_j^*)$ [GG08, Lem. 5]. This is the complexification of $\bigoplus_{j=1}^s (\mathfrak{k}_j, U_j)$, where $\mathfrak{k}_j$ is a maximal compact subalgebra of $\mathfrak{g}_j$. Now $(\mathfrak{k}_j, U_j)$ is real polar irreducible with an invariant complex structure, so it is orbit equivalent to the action of its normalizer $(\hat{\mathfrak{k}}_j, U_j)$ according to [Dad85, p. 129]; note that $\hat{\mathfrak{k}}_j = \mathfrak{u}_1 \oplus \mathfrak{k}_j$ if $\mathfrak{k}_j$ is not self-normalizing. Passing to the complexifications we deduce that $(\mathfrak{g}_j, U_j \oplus U_j^*)$ is closed orbit equivalent to the action of its normalizer $(\hat{\mathfrak{g}}_j, T(U_j))$ [GG08, Lem. 8], where $\hat{\mathfrak{g}}_j = \mathfrak{t}^1 \oplus \mathfrak{g}_j$ if $\mathfrak{g}_j$ is not self-normalizing, and hence $(\mathfrak{g}, U \oplus U^*)$ is closed orbit equivalent to $(\hat{\mathfrak{g}}, T(U))$.

Since $U \oplus U^*$ is stable, every Cartan subspace c for $(\mathfrak{g}, V)$ is of the form $c = c' \oplus c''$ where c' is a Cartan subspace of $(\mathfrak{g}, W)$ and c'' is a Cartan subspace of $(\mathfrak{g}, U \oplus U^*)$ (Proposition 9(b)). Let $v \in c$ be arbitrary and write $v = w + u$ where $w \in c'$ and $u \in c''$. By the above, given $X \in \mathfrak{t}^s$, there is $Y \in \mathfrak{g}$ such that $X \cdot u = Y \cdot u$. Since $(\mathfrak{g}, U \oplus U^*)$ is stable, there is $Z \in \mathfrak{g}_u$ such that $Z \cdot w = Y \cdot w$ (Proposition 9(c)). Now

$$\begin{aligned} X \cdot v &= X \cdot u + X \cdot w = X \cdot u = Y \cdot u \\ &= Y \cdot v - Y \cdot w = Y \cdot v - Z \cdot w = Y \cdot v - Z \cdot v = (Y - Z) \cdot v \end{aligned}$$

where $Y - Z \in \mathfrak{g}$, which proves that the G and $\hat{G}$-orbits through v coincide. Since every closed $\hat{G}$-orbit contains a closed G-orbit, we are done. □

3. Polar symplectic representations are coisotropic

In this section we prove Theorem 1. Let G be a connected complex reductive linear algebraic group and let V be a rational representation. Assume (G, V) is symplectic, i.e. there exists a G-invariant non-degenerate skew-symmetric bilinear form ω on V.

We begin with an interesting observation which will not be needed later.

Proposition 11. *If (G, V) is polar symplectic without trivial components then every Cartan subspace is isotropic.*

Proof. Let $c \subset V$ be a Cartan subspace. The restriction $\omega|_{c\times c}$ is $W(c)$-invariant, where $W(c) = N_G(c)/Z_G(c)$ is the Weyl group of (G, V) with respect to c, and $W(c)$ is generated by unitary reflections [DK85, Lem. 2.7 and Th. 2.10]. For $w \in W(c)$, a vector u in the fixed point set c^w of w, and a w-eigenvector $v \in c$ transversal to c^w, we have $\omega(u, v) = \omega(w \cdot u, w \cdot v) = e^{\frac{2\pi i}{q}}\omega(u, v)$ for some positive integer $q \neq 1$, thus $\omega(c^w, v) = 0$. We deduce that $v \in \ker \omega|_{c\times c}$. Since a basis of c can be constructed which consists of such eigenvectors of reflections (otherwise c has a non-zero $W(c)$-fixed subspace which implies that V has a non-zero G-fixed subspace [DK85, Lem. 2.11 and Prop. 2.13]), this shows that the restriction of ω to c is null. $\square$

3.1. **Knop reduction.** Denote by $\mathfrak{g}$ the Lie algebra of G and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a system of positive roots $\Delta^+ \subset \Delta$. For each $\alpha \in \Delta$, the corresponding coroot is denoted by $\alpha^\vee$. The weight system of V is denoted by Λ. A weight $\lambda \in \Lambda$ is called:

(i) *extremal* or *highest* if $\alpha \in \Delta$ and $\langle\lambda|\alpha^\vee\rangle > 0$ implies $\lambda + \alpha \notin \Lambda$;
(ii) *toroidal* if $\langle\lambda|\alpha^\vee\rangle = 0$ for all $\alpha \in \Delta$;
(iii) *singular* if it is extremal and $2\lambda \in \Delta$ and the multiplicity of λ is one.

A submodule U of V generated by a highest weight vector is called *singular* if U is an anisotropic subspace of V and $G \to Sp(U)$ is surjective. Note that if λ is an extremal weight of V and $2\lambda \in \Delta$, then we can always find a highest weight vector for λ that generates a singular submodule of V; however, in case the multiplicity of λ is bigger than one, one can also find a highest weight vector that generates an isotropic, hence non-singular submodule [Kno07, Remarks, p. 228].

A symplectic representation is called *terminal* if all of its highest weights are either toroidal or singular. Equivalently, a symplectic representation is terminal if every highest weight vector generates either a one-dimensional module or a singular submodule. Such a representation (G, V) decomposes as

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_s, \qquad G = G_0 \times Sp(V_1) \times \cdots \times Sp(V_s) \tag{3·1}$$

where $V_0 = \oplus_{i=1}^m(\mathbb{C}_{\lambda_i} \oplus \mathbb{C}_{-\lambda_i})$ is a direct sum of 1-dimensional G_0-modules [Kno07, Proposition 4.1]. A terminal symplectic representation is coisotropic if and only if the set of weights $\{\lambda_1, \ldots, \lambda_m\}$ is linearly independent [Kno06, Theorem 3.1].

Knop reduction is a finite algorithm which, for a given symplectic representation, outputs a terminal symplectic representation. Each intermediate step of the algorithm produces a representation which is coisotropic if and only if the original representation was coisotropic. Since coisotropicity is easy to check for terminal representations, this provides an effective criterion to decide whether a given symplectic representation is coisotropic or not. Indeed it was used in [Kno06] to classify coisotropic symplectic representations.

3.2. **The step in the algorithm.** Let (G, V) be a non-terminal symplectic representation. Choose an extremal weight $\lambda \in \Lambda$ which is neither toroidal nor singular. Put $P = \{\alpha \in \Delta | \langle \lambda | \alpha^\vee \rangle > 0\}$ and $Q = \lambda - P$ as multisets (i.e. sets with multiplicities), and

$$\Delta' = \Delta \setminus (P \cup -P), \qquad \Lambda' = \Lambda \setminus (Q \cup -Q).$$

The choice of λ ensures that Δ' is the root system of a reductive Lie algebra $\mathfrak{l}$ (namely, a Levi subalgebra of the stabilizer of the line through a highest weight vector of λ), and Λ' is a weight system of a symplectic representation S of $\mathfrak{l}$.

Theorem 12 ([Kno07], Thm. 8.4 and Prop. 9.1)**.** $(\mathfrak{g}, V)$ *is coisotropic if and only if so is* $(\mathfrak{l}, S)$.

We remark that the multiset Λ_t of toroidal weights is invariant under multiplication by -1 and contains zero with even multiplicity, and its cardinality strictly increases in passing from $(\mathfrak{g}, V)$ to $(\mathfrak{l}, S)$. In fact, λ is non-toroidal for $(\mathfrak{g}, V)$ but it is toroidal for $(\mathfrak{l}, S)$; moreover, for all $\alpha \in \Delta^+$ with $\langle \lambda | \alpha^\vee \rangle \neq 0$, we have that $\mu = \lambda - \alpha$ is non-toroidal for $(\mathfrak{g}, V)$ (since $\sum_{\beta \in \Delta^+} \langle \lambda | \beta^\vee \rangle$ is an odd number [Sam90, Prop. G, p. 142]), so any toroidal weight μ of $(\mathfrak{g}, V)$ has weight space contained in S, and thus μ is also toroidal for $(\mathfrak{l}, S)$. By [Kno07, Theorem 8.4], $\dim V /\!\!/ G = \dim S /\!\!/ L$. Now the integer

$$m(\mathfrak{g}, V) := \dim V /\!\!/ G - \frac{1}{2} \# \Lambda_t \geq 0$$

strictly decreases under each step in Knop reduction. The algorithm stops after finitely many steps, necessarily at a terminal representation. The final value of $m(\mathfrak{g}, V)$ equals zero if and only if the terminal representation is coisotropic [Kno06, Theorem 3.1].

3.3. **Relation to slice representations.** Suppose λ is a highest weight which is neither toroidal nor singular. Take a corresponding highest weight vector v_λ of unit length that generates a non-singular submodule. Consider:

$v_{-\lambda} = \epsilon(v_\lambda)$:	lowest weight vector, so that $\omega(v_\lambda, v_{-\lambda}) = 1$
$\mathfrak{p}$:	stabilizer of $\mathbb{C}\, v_\lambda$ (parabolic subalgebra of $\mathfrak{g}$)
$\mathfrak{p}_u$:	unipotent radical of $\mathfrak{p}$
$\mathfrak{l}$:	Levi subalgebra of $\mathfrak{p}$, so that $\mathfrak{p} = \mathfrak{l} + \mathfrak{p}_u$
$\mathfrak{p}^- = \mathfrak{l} + \mathfrak{p}_u^-$:	opposite parabolic subalgebra

Then $\mathbb{C}\, v_\lambda \oplus \mathfrak{p}_u^- v_\lambda$ and $\mathbb{C}\, v_{-\lambda} \oplus \mathfrak{p}_u v_{-\lambda}$ are isotropic subspaces of V [Kno07, Lem. 3.2]. There are also decompositions

$$V = \mathfrak{p}_u v_{-\lambda} \oplus (\mathfrak{p}_u^- v_\lambda)^{\perp_\omega} = \mathfrak{p}_u^- v_\lambda \oplus (\mathfrak{p}_u v_{-\lambda})^{\perp_\omega}, \tag{3·2}$$

see [Kno07, Lem. 3.3]. We claim these decompositions are h-orthogonal. In fact, for $\xi \in \mathfrak{p}_u$, $u \in (\mathfrak{p}_u^- v_\lambda)^{\perp_\omega}$,

$$h(\xi v_{-\lambda}, u) = -\omega(\xi v_{-\lambda}, \epsilon u) = \overline{\omega(\epsilon \xi v_{-\lambda}, u)} = -\overline{\omega(\tilde{\xi} v_\lambda, u)} = 0$$

where $\tilde{\xi} = \epsilon^{-1} \xi \epsilon \in \mathfrak{p}_u^-$; similarly for the second decomposition in (3·2). Since v_λ generates a non-singular submodule, $\mathfrak{p}_u v_{-\lambda} \cap \mathfrak{p}_u^- v_\lambda = \{0\}$. Also [Kno07, Eq. (3.3)] or [Kno06, Thm. 3.2]

$$S = (\mathfrak{p}_u v_{-\lambda})^{\perp_\omega} \cap (\mathfrak{p}_u^- v_\lambda)^{\perp_\omega}. \tag{3·3}$$

Note that

$$\mathfrak{p} = \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \lambda | \alpha^\vee \rangle = 0}} \mathfrak{g}_{-\alpha}$$

and

$$\mathfrak{l} = \mathfrak{h} + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \lambda | \alpha^\vee \rangle = 0}} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) = \mathfrak{l}' + \mathfrak{z}$$

where $\mathfrak{l}'$ is the derived subalgebra of $\mathfrak{l}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}$.

Let $v = v_\lambda + v_{-\lambda}$. Then v is a semisimple point [DK85, Proposition 1.2]. The tangent space to the G-orbit through v is

$$\mathfrak{g} \cdot v = \mathfrak{p}_u v_{-\lambda} \oplus \mathfrak{p}_u^- v_\lambda \oplus \mathbb{C}(v_\lambda - v_{-\lambda}). \tag{3·4}$$

Consider the isotropy subalgebra

$$\mathfrak{g}_v = \underbrace{\ker \lambda}_{\subset \mathfrak{h}} + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \lambda | \alpha^\vee \rangle = 0}} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}).$$

Since $\mathfrak{l}'$ is generated by the $\mathfrak{g}_\alpha$ satisfying $\langle \lambda | \alpha^\vee \rangle = 0$, we have $\mathfrak{l}' \subset \mathfrak{g}_v$. Now we can find a complementary line to $\mathfrak{g}_v$ in $\mathfrak{l}$ contained in $\mathfrak{z}$. Hence there is a direct sum of ideals

$$\mathfrak{l} = \mathfrak{g}_v \oplus \mathfrak{t}^1. \tag{3·5}$$

Assume now our symplectic representation (G, V) is in addition polar. It is interesting to relate Knop reduction with respect to the weight λ to the slice representation at v. The slice representation $(\mathfrak{g}_v, N_v)$ is also polar, where N_v is the h-orthocomplement of $\mathfrak{g} \cdot v$ in V, in view of Proposition 8. Write $S = \mathbb{C}\, v_\lambda \oplus \mathbb{C}\, v_{-\lambda} \oplus S_\circ$, h-orthogonal decomposition. Then $S_\circ$ is a symplectic subspace of S and $S_\circ \oplus \mathbb{C}\, v = N_v$ (due to (3·2), (3·3) and (3·4)). Since $\mathfrak{g}_v$ acts trivially on $\mathbb{C}\, v_\lambda \oplus \mathbb{C}\, v_{-\lambda}$, the subspace $S_\circ$ is $\mathfrak{g}_v$-invariant and $(\mathfrak{g}_v, S_\circ)$ is polar and thus also $(\mathfrak{g}_v, S)$ is polar. It follows from (3·5) and Proposition 10(b) that $(\mathfrak{l}, S)$ is polar. Since $\mathfrak{l} \cdot v = \mathbb{C}(v_\lambda - v_{-\lambda})$, any Cartan subspace c of $(\mathfrak{l}, S)$ passing through v satisfies $c \perp_h v_\lambda - v_\lambda$, so that $c \subset N_v$. It follows from Proposition 8 that c is a Cartan subspace of $(\mathfrak{g}, V)$. We have proved:

Proposition 13. *If $(\mathfrak{g}, V)$ is a non-terminal polar symplectic representation, then any Knop reduction $(\mathfrak{l}, S)$ is also a polar symplectic representation. Moreover, any Cartan subspace of $(\mathfrak{l}, S)$ is a Cartan subspace of $(\mathfrak{g}, V)$.*

3.4. **End of the proof of Theorem 1.** We apply induction on $m(\mathfrak{g}, V)$. By the above, $m(\mathfrak{l}, S) < m(\mathfrak{g}, V)$ and $(\mathfrak{l}, S)$ is polar, so the induction step implies that $(\mathfrak{l}, S)$ is coisotropic. Hence $(\mathfrak{g}, V)$ is coisotropic by Theorem 12. The initial case of the induction is the case of a terminal representation.

Lemma 14. *A terminal representation is polar if and only if it is coisotropic.*

Proof. Let (G, V) be a terminal representation as in (3·1). It has already been remarked that (G, V) is coisotropic if and only if the weights $\lambda_1, \ldots, \lambda_m$ are linearly independent. We check that the latter condition is equivalent to the polarity of (G, V).

Since $(Sp(V_i), V_i)$ has no non-trivial closed orbits, we may assume $V_i = 0$ for $i = 1, \ldots, s$ and G is a complex torus $\mathbb{T}^k = (\mathbb{C}^\times)^k$ acting effectively on $V = V_0$. Effectiveness says that $k \leq m$ and equality holds if and only if the weights are l.i.. If $k = m$ then the representation is the m-fold product of the standard representation $(\mathbb{T}^1, \mathbb{C} \oplus \mathbb{C}^*)$ and hence polar. Conversely, assume the representation is polar. Let $v_1 = v_{\lambda_1} + v_{-\lambda_1} = (1, 1) \in \mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{-\lambda_1}$. Then G_{v_1} is the subtorus $\mathbb{T}^{k-1}$ defined by $\lambda_1 = 0$, and the normal space N_{v_1} equals $\oplus_{i=2}^m (\mathbb{C}_{\lambda_i} \oplus \mathbb{C}_{-\lambda_i})$ plus a one-dimensional trivial component. The slice representation at v_1 is polar and effective, thus by induction $k - 1 = m - 1$ and hence $k = m$, as we wished. □

Remark 15. In the situation of Lemma 14, a Cartan subspace is given by $\langle v_{\lambda_1} + v_{-\lambda_1}, \ldots, v_{\lambda_m} + v_{-\lambda_m} \rangle$.

4. The classification

In this section, we prove Theorem 2. Due to Theorem 1, a polar symplectic representation is coisotropic, so we will extract the list of saturated polar symplectic representations from the lists of saturated coisotropic representations given, up to geometric equivalence, by [Kno06, Thms. 2.4, 2.5 and 2.6].

Suppose V is a saturated indecomposable polar symplectic representation of $\mathfrak{g}$. If it is of type 1, then it is listed in [Kno06, Table 1]. Representations in this table with $\dim V /\!\!/ G \leq 1$ are trivially polar, so we run through the other cases. Some representations with $\dim V /\!\!/ G = 2$ are already discussed in [DK85, p. 512 and 523]. We finish this case by referring to [Lit89, Tabelle, p.199 and p.201], where irreducible polar representations of connected semisimple Lie groups are classified (see also discussion in [Lit89, p. 208]). We obtain our Table A.

Suppose now V is of type 2. Then it is listed in [Kno06, Table 2]. In this case $V = U \oplus U^*$ and $(\mathfrak{g}, V)$ is the complexification of $(\mathfrak{k}, U)$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ and U is a real irreducible polar representation with an invariant complex structure. Therefore we can refer to the classification of irreducible polar representations of compact connected Lie groups [Dad85, EH99]. We obtain our Table B.

We will complete the proof of the theorem by showing that every saturated decomposable polar symplectic representation is a product, namely, connected saturated decomposable polar symplectic representations do not exist. An $\mathfrak{sl}_2$-*link* is an $\mathfrak{sl}_2$-factor of $\mathfrak{g}$ which acts effectively on at least two components of V. All connected saturated decomposable coisotropic representations without $\mathfrak{sl}_2$-links are listed in [Kno06, Tables 11, 12 and 22], and we will see shortly that none of these is polar. Indeed due to Proposition 9(a), we need only examine the representations in tables 11, 12 and 22 whose irreducible components are all polar. The only unstable representations in Table A are $(Sp_{2m}, \mathbb{C}^{2m})$ for all $m \geq 1$ and $(SO_p \times Sp_{2m}, \mathbb{C}^p \otimes \mathbb{C}^{2m})$ where $3 \leq p < 2m$ and p is odd, while all representations in Table B are stable. Now all representations in Tables 12 and 22 have both components polar and at least one component stable, and we check that the rank condition of Proposition 9(b) is violated by all of them. The same argument applies to the representations of Table 11, but $\langle 11.13 \rangle$ which has a non-polar component and therefore is not polar, and the two sub-cases not having stable components of $\langle 11.11 \rangle$ and $\langle 11.14 \rangle$, which are discussed in Lemmata 16 and 17.

We borrow more notation from [Kno06] (cf. (2.4), p. 538). The line under the $\oplus$-sign below means that the algebras immediately to the left and to the right are being identified and the resulting algebra is acting diagonally.

Lemma 16. *$\mathfrak{so}_p \otimes \mathfrak{sp}_{2m} \underline{\oplus} \mathfrak{sp}_{2m}$ is not polar for $3 \leq p < 2m$ and p odd.*

Proof. We will use Proposition 13. The Lie algebra is $\mathfrak{so}_p + \mathfrak{sp}_{2m}$ and the representation space is $V_1 \oplus V_2$, where $V_1 = \mathbb{C}^p \otimes \mathbb{C}^{2m}$ and $V_2 = \mathbb{C}^{2m}$. By performing Knop reduction with respect to a highest weight vector of V_1 and proceeding by induction, we may assume $p = 3$ and $m \geq 2$. A further step of Knop's algorithm yields

$$\mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{-\lambda_1} \oplus \mathbb{C}_{\epsilon'_1} \oplus \mathbb{C}_{-\epsilon'_1} \oplus \mathfrak{sp}_{2m-2} \underline{\oplus} \mathfrak{sp}_{2m-2}$$

where $\lambda_1 = 2\epsilon_1 + \epsilon'_1$. This representation is polar [DK85, p. 522] with Cartan subspace $c = c_0 \oplus c_1 \oplus c_2$, where c_0, c_1 and c_2 are one-dimensional Cartan subspaces for $\mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{-\lambda_1}$, $\mathbb{C}_{\epsilon'_1} \oplus \mathbb{C}_{-\epsilon'_1}$ and $\mathfrak{sp}_{2m-2} \underline{\oplus} \mathfrak{sp}_{2m-2}$, respectively. If the given representation were polar, then it would have c as a Cartan subspace. Since $\mathbb{C}_{\epsilon'_1} \oplus \mathbb{C}_{-\epsilon'_1} \subset V_2$ and V_2 contains no non-zero semisimple points, this is not possible. $\square$

Lemma 17. *$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5 \underline{\oplus} \mathfrak{sp}_4$ is not polar for $m \geq 3$.*

Proof. This representation has rank 4. Knop reduction with respect to a highest weight λ_1 of the first summand yields

$$\mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{-\lambda_1} \oplus \mathfrak{sp}_{2m-2} \otimes S^2\mathfrak{sl}_2 \underline{\oplus} T(\mathfrak{sl}_2).$$

Consider the last two summands, namely, $\langle S.13\rangle + \langle S.10\rangle$ in [Kno06, Table S]. This is not polar, since its rank is 3, $T(\mathfrak{sl}_2)$ is stable of rank 1, and $\mathfrak{sp}_{2m-2} \otimes S^2\mathfrak{sl}_2$ has rank 1, so we can apply Proposition 9(b). □

We finish the proof by considering connected saturated decomposable coisotropic representations with $\mathfrak{sl}_2$-links. According to [Kno06, Thm. 2.6] they are obtained by taking any collection of representations from Table S (*ibid*) and identifying any number of disjoint pairs of underlined $\mathfrak{sl}_2$'s, except that not allowed is the identification of the two $\mathfrak{sl}_2$'s of $\langle S.1\rangle$ and the combination of $\langle S.9\rangle$ with itself. Again by Proposition 9(a), we need only consider entries in Table S which are polar; for convenience, we list them below, and make some remarks which will be useful in the sequel.

	$(\mathfrak{g}, V)$	
$\langle S.1\rangle$	$\underline{\mathfrak{sl}}_2 \otimes \mathfrak{sp}_{2m} \otimes \underline{\mathfrak{sl}}_2$	$m \geq 1$
$\langle S.3\rangle$	$\mathfrak{so}_n \otimes \underline{\mathfrak{sl}}_2$	$n \geq 3$
$\langle S.5\rangle$	$\mathfrak{spin}_9 \otimes \underline{\mathfrak{sl}}_2$	
$\langle S.7\rangle$	$\mathfrak{spin}_7 \otimes \underline{\mathfrak{sl}}_2$	
$\langle S.9\rangle$	$\underline{\mathfrak{sl}}_2$	
$\langle S.10\rangle$	$T(\underline{\mathfrak{sl}}_2)$	
$\langle S.11\rangle$	$T(\mathfrak{sl}_m \otimes \underline{\mathfrak{sl}}_2)$	$m \geq 2$
$\langle S.13\rangle$	$\mathfrak{sp}_{2m} \otimes S^2\underline{\mathfrak{sl}}_2$	$m \geq 1$
$\langle S.16\rangle$	$\mathfrak{g}_2 \otimes \underline{\mathfrak{sl}}_2$	

TABLE S′: POLAR REPRESENTATIONS IN [Kno06, Table S]

In Table S′, all representations but $\langle S.9\rangle$ contain nonzero semisimple points, and the only unstable representations are $\langle S.9\rangle$ and $\langle S.13\rangle$ with $m \geq 2$. Suppose V is different from $\langle S.9\rangle$ and $\langle S.13\rangle$ with $m \geq 1$. Then $V = W \otimes \mathbb{C}^2$, where $\mathfrak{g} = \mathfrak{h} + \mathfrak{sl}_2$, W is a representation of $\mathfrak{h}$ and $\mathbb{C}^2 = \langle e_1, e_2\rangle$ is acted on by $\mathfrak{sl}_2$; also W is irreducible and non-singular if V is not $\langle S.10\rangle$, $\langle S.11\rangle$, and $W = U \oplus U^*$ otherwise. Let λ be an extremal weight of V, choose weight vectors $v_{\pm\lambda}$ such that $\omega(v_\lambda, v_{-\lambda}) = 1$ and put

$$v = v_\lambda + v_{-\lambda}. \tag{4·1}$$

Then $v = w \otimes e_1 + w^- \otimes e_2$ where $w, w^- \in W$ and $\mathfrak{g}_v = \mathfrak{h}_v + \mathfrak{t}^1$. Finally, note that

$$w \otimes e_2 \in \mathfrak{sl}_2 \cdot v \subset \mathfrak{g} \cdot v, \quad \text{but} \quad w \otimes e_2 \notin (\mathfrak{h} + \mathfrak{t}^1) \cdot v. \tag{4·2}$$

Suppose now V is $\langle S.13\rangle$. Then $V = \mathbb{C}^{2m} \otimes \mathbb{C}^3$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{sl}_2$, where $\mathfrak{h} = \mathfrak{sp}_{2m}$. The isotropy algebra at $v = v_\lambda + v_{-\lambda}$ is $\mathfrak{g}_v = \mathfrak{h}_v + \mathfrak{t}^1$, where $\mathfrak{h}_v = \mathfrak{sp}_{2m-2}$. Last, the restriction of V to $\mathfrak{h} + \mathfrak{t}^1$ is $T(\mathfrak{sp}_{2m})\underline{\oplus}\mathfrak{sp}_{2m}$.

Lemma 18. *The combination of $\langle S.9\rangle$ with any other representation in Table S′ is not polar.*

Proof. Suppose the assertion is not true. Consider such a polar representation $V = V_1 \oplus V_2$ where $V_2 = \mathbb{C}^2$ is acted on by $\mathfrak{sl}_2$, and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{sl}_2$. Due to Proposition 13, Knop reduction with respect to the highest weight λ_1 of V_1 yields a polar representation $(\mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{-\lambda_1} \oplus U) \oplus (\mathbb{C}_{\epsilon_1} \oplus \mathbb{C}_{-\epsilon_1})$, where U is a subspace of V_1, $\pm 2\epsilon_1$ are the roots of $\mathfrak{sl}_2$, and $\mathbb{C}_{\epsilon_1} \oplus \mathbb{C}_{-\epsilon_1}$ equals V_2. By Proposition 9, a Cartan subspace of this representation is of the form $c = c_1 \oplus c_2$, where $c_1 \subset \mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{-\lambda_1} \oplus U$ and c_2 is the diagonal subspace of $\mathbb{C}_{\epsilon_1} \oplus \mathbb{C}_{-\epsilon_1}$; by Proposition 13, c is a Cartan subspace of V, too. However V_2 does not contain non-zero G-semisimple elements of V, and this is a contradiction. □

Lemma 19. *The combination of $\langle S.13\rangle$ with itself or any other representation in Table S′ different from $\langle S.9\rangle$ is not polar.*

Proof. Write $V = V_1 \oplus V_2$ where V_1 is $\langle S.13\rangle$. Knop reduction with respect to the highest weight λ_2 of V_2 contains as a summand $T(\mathfrak{sp}_{2m})\underline{\oplus}\mathfrak{sp}_{2m}$ by the last sentence before Lemma 18. If $m \geq 2$, this is a connected saturated representation without $\mathfrak{sl}_2$-links and has already been shown not to be polar. If $m = 1$, we have $T(\mathfrak{sl}_2)\underline{\oplus}\mathfrak{sl}_2$ which is not polar by Lemma 18. Hence in neither case is V polar. □

Let now V be an arbitrary connected saturated decomposable coisotropic representation of $\mathfrak{g}$ with $\mathfrak{sl}_2$-links. We will show that V is not polar. In view of Proposition 9(a), we may assume that V has two components and both are polar. Now $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{s} + \mathfrak{g}_2$, $V = V_1 \oplus V_2$, and V_i is an indecomposable symplectic representation of $\mathfrak{g}_i + \mathfrak{s}$ given by Table S′, $i = 1, 2$, where $\mathfrak{s}$ is an $\mathfrak{sl}_2$-link. Owing to Lemmata 18 and 19, we may assume V_1 and V_2 are both different from $\langle S.13\rangle$ and $\langle S.9\rangle$. Let $v \in V_1$ be a semisimple point as in (4·1). Then $\mathfrak{g}_{v_1} = (\mathfrak{g}_1 + \mathfrak{s})_{v_1} + \mathfrak{g}_2 = (\mathfrak{g}_1)_v + \mathfrak{t}^1 + \mathfrak{g}_2$. If V were G-polar then, due to Proposition 9(c), the set of closed orbits of $T^1 \cdot G_2$ in V_2 would have to coincide with the set of closed orbits of $SL_2 \cdot G_2$, but it follows from (4·2) that this is not the case. This finishes the proof of Theorem 2.

5. Symplectic symmetric spaces

A *symplectic symmetric space* is a symmetric space which is endowed with a symplectic structure invariant by the symmetries. We refer to [Bie95, Bie98] for the basic theory of such spaces. Our interest in them is that the (complexified) isotropy representations of symplectic symmetric spaces provide examples of symplectic *θ-groups* (namely, adjoint groups of graded Lie algebras) thus, polar symplectic representations [PV94, §8.5, 8.6]. Indeed simply-connected symplectic symmetric spaces are parametrized by symplectic involutive Lie algebras. A *symplectic involutive Lie algebra* is a triple $(\mathfrak{g}, \sigma, \omega)$ where $\mathfrak{g}$ is a real Lie algebra, σ is an involution of $\mathfrak{g}$, with respect to which there is an eigenspace decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, and ω is an $\mathrm{ad}_{\mathfrak{h}}$-invariant non-degenerate 2-form on $\mathfrak{q}$.

An indecomposable (i.e. non-isomorphic to a product of symplectic involutive Lie algebras) non-flat (i.e. satisfying $[\mathfrak{q}, \mathfrak{q}] \neq 0$) reductive symplectic involutive Lie algebra $(\mathfrak{g}, \sigma, \omega)$ is simple [Bie95, Prop. 3.5.4]. The symplectic structures ω on a simple involutive Lie algebra $(\mathfrak{g}, \sigma)$ are parametrized by the non-zero elements in the center $Z(\mathfrak{h})$ of $\mathfrak{h}$ [Bie98, Th. 2.1]. Moreover, if $\mathfrak{g}$ is a complex Lie algebra viewed as real, then σ is a complex automorphism, ω is complex bilinear, $\dim_{\mathbb{C}} Z(\mathfrak{h}) = 1$ and $(\mathfrak{h}, \mathfrak{q})$ is a θ-group; otherwise $\mathfrak{g}$ is absolutely simple, $\dim_{\mathbb{R}} Z(\mathfrak{h}) = 1$ and the complexification $(\mathfrak{h}^{\mathbb{C}}, \mathfrak{q}^{\mathbb{C}})$ is a θ-group [Bie98, Prop. 2.2 and Thm. 2.2]. In any case, the (indecomposable) polar symplectic representations thus obtained are exacly the complexified isotropy representations of irreducible Hermitian Riemannian symmetric spaces [Bie98, § 10], hence of type 2 and listed in Table B. On the other hand, every representation in Table B is closed orbit equivalent to the complexification of the isotropy representation of an irreducible Hermitian Riemannian symmetric space [Dad85, EH99].

A complex semisimple symplectic involutive Lie algebra is the product of complex simple symplectic involutive Lie algebras [BCG95, Prop. 3], each of which with an indecomposable (polar) symplectic representation (of type 2) as isotropy representation. Now an arbitrary polar symplectic representation can be assumed saturated, up to closed orbit equivalence (Proposition 10(b)), and then it is the product of indecomposable polar symplectic representations (Theorem 2); finally, it is closed orbit equivalent to the isotropy representation of a complex semisimple symplectic symmetric space if and only if each of its factors is. This completes the proof of Theorem 3.

6. The moment map

Recall the idea of a moment map. In our context, an action of an algebraic group G on a symplectic variety (X, ω) is called *Hamiltonian* if there exists a *moment map*, that is, an equivariant map $\mu : X \to \mathfrak{g}^*$ (where $\mathfrak{g}^*$ is regarded with the coadjoint representation) such that $\omega(\xi x, v) = \langle d\mu_x(v) | \xi \rangle$ for all $\xi \in \mathfrak{g}$, $v \in T_x X$, $x \in X$ (compare [Kno07, section 2]). In our particular case of interest $X = V$ is a symplectic representation of G, there is a canonical moment map given by

$$\mu : V \to \mathfrak{g}^*, \qquad \langle \mu(v) | \xi \rangle = \frac{1}{2} \omega(\xi v, v).$$

Assume now V is polar symplectic. Apply Knop reduction to get a terminal representation with set of weights $\{\lambda_1, \ldots, \lambda_r\}$. Let v_{λ_j} be an h-unit λ_j-weight vector, and $v_{-\lambda_j} = \epsilon(v_{\lambda_j})$ so that $\omega(v_{\lambda_j}, v_{-\lambda_j}) = 1$. Owing to Proposition 13 and Remark 15, $c = \langle v_{\lambda_1} + v_{-\lambda_1}, \ldots, v_{\lambda_r} + v_{-\lambda_r} \rangle$ is a Cartan subspace of (G, V).

We say that a set of weights of a representation is *strongly orthogonal* if neither the sum nor the difference of weights in the set is a root.

Proposition 20. (a) *If the set* $\{\lambda_1, \dots, \lambda_r\}$ *is strongly orthogonal, then*

$$\mu\left(\sum_{j=1}^{r} a_j(v_{\lambda_j} + v_{-\lambda_j})\right) = \sum_{j=1}^{r} a_j^2 \lambda_j,$$

where $a_j \in \mathbb{C}$*. In particular,*

$$\mu(c) = \langle \lambda_1, \dots, \lambda_r \rangle =: \mathfrak{a}^* \subset \mathfrak{h}^*.$$

(b) *Every saturated polar symplectic representation admits a set* $\{\lambda_1, \dots, \lambda_r\}$ *satisfying the assumption in (a).*

Proof. (a) We first note that $\omega(\mathfrak{g}_\alpha(v_{\lambda_j} + v_{-\lambda_j}), v_{\lambda_k} + v_{-\lambda_k}) = 0$ for all $\alpha \in \Delta$, by strong orthogonality of λ_j, λ_k in case $j \neq k$, and by non-singularity of v_{λ_j} in case $j = k$. This already shows $\mu(c) \subset \mathfrak{h}^*$. To finish, let $\xi \in \mathfrak{h}$ and compute

$$\begin{aligned} \omega(\xi(v_{\lambda_j} + v_{-\lambda_j}), v_{\lambda_k} + v_{-\lambda_k}) &= \langle \lambda_j | \xi \rangle\, \omega(v_{\lambda_j} - v_{-\lambda_j}, v_{\lambda_k} + v_{-\lambda_k}) \\ &= \begin{cases} 0, \text{if } j \neq k, \\ 2\langle \lambda_j | \xi \rangle, \text{if } j = k. \end{cases} \end{aligned}$$

The desired formula follows.

(b) In view of Theorem 2, it suffices to consider the indecomposable case. We run Knop's algorithm for the representations listed in Tables A and B, where we can omit those of rank at most one, and find explicitly a set $\{\lambda_1, \dots, \lambda_r\}$. We obtain the following table. (We use Bourbaki's notation for weights [Bou68, Planches I-IX], and denote by η the weight associated to $\mathfrak{t}^1$.)

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_p$	$\epsilon_1 + \epsilon_1', \dots, \epsilon_r + \epsilon_r' \quad (r = \min\{[\frac{p}{2}], m\})$
$\mathfrak{sl}_2 \otimes \mathfrak{spin}_9$	$\epsilon_1 + \varpi_4', \epsilon_1 + \varpi_1' - \varpi_4'$
$\mathfrak{spin}_{13}$	$\varpi_6, \varpi_3 - \varpi_6$
$T(\mathfrak{sl}_m \otimes \mathfrak{sl}_n)$	$\epsilon_1 + \epsilon_1' + \eta, \dots, \epsilon_r + \epsilon_r' + \eta \quad (r = \min\{m, n\})$
$T(\Lambda^2 \mathfrak{sl}_n)$	$\epsilon_1 + \epsilon_2 + \eta, \dots, \epsilon_{2r-1} + \epsilon_{2r} + \eta \quad (r = [\frac{n}{2}])$
$T(S^2 \mathfrak{sl}_n)$	$2\epsilon_1 + \eta, \dots, 2\epsilon_n + \eta$
$T(\mathfrak{so}_m)$	$\varpi_1 + \eta, \varpi_1 - \eta$
$T(\mathfrak{spin}_7)$	$\varpi_3 + \eta, \varpi_3 - \eta$
$T(\mathfrak{spin}_{10})$	$\varpi_4 - \eta, \varpi_5 + \eta,$
$T(\mathfrak{g}_2)$	$\varpi_1 + \eta, \varpi_1 - \eta$
$T(\mathfrak{e}_6)$	$\varpi_1 + \eta, \varpi_6 - \eta, -\varpi_1 + \varpi_6 + \eta$

TABLE C: KNOP ALGORITHM FOR SOME POLAR REPRESENTATIONS.

It is readily checked that $\{\lambda_1, \dots, \lambda_r\}$ is always strongly orthogonal. $\square$

Corollary 21. *The moment map μ of a saturated polar representation maps closed orbits to closed orbits.*

Proof. This is clear from the proposition since the closed orbits of G in V (resp. $\mathfrak{g}^*$) are exactly those that meet c (resp. $\mathfrak{h}^*$). $\square$

Let (G, V) be a symplectic representation. Since the moment map is equivariant, there is an induced *invariant moment map*:

$$\begin{array}{ccc} V & \xrightarrow{\mu} & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ V/\!\!/G & \xrightarrow[\psi := \mu/\!\!/G]{} & \mathfrak{g}^*/\!\!/G \end{array}$$

Recall that we denote the Cartan subalgebra of $\mathfrak{g}$ by $\mathfrak{h}$. Let $\pi : \mathfrak{h}^* \to \mathfrak{h}^*/W_G$ denote the projection, where W_G denotes the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Knop proved in [Kno07] that there is an essentially unique subspace $\mathfrak{a}^*$ of $\mathfrak{h}^*$ and a subgroup W_V of $\Gamma := N_{W_G}(\mathfrak{a}^*)/Z_{W_G}(\mathfrak{a}^*)$ acting on $\mathfrak{a}^*$ as a group generated by reflections such that the morphism $\mathfrak{a}^*/W_V \to \pi[\mathfrak{a}^*]$ is finite, the image of the (invariant) moment map is $\pi[\mathfrak{a}^*]$, and ψ factors through a map $V/\!\!/G \to \mathfrak{a}^*/W_V$; moreover, (G, V) is coisotropic if and only if the morphism $V/\!\!/G \to \mathfrak{a}^*/W_V$ is an isomorphism. Namely, $\mathfrak{a}^*$ is the span of $\{\lambda_1, \dots, \lambda_r\}$.

Proof of Theorem 4. It remains only to prove the second assertion. Recall that, by Chevalley's theorem, $\mathfrak{g}^*/\!\!/G \cong \mathfrak{h}^*/W_G$. Similarly, $V/\!\!/G \cong c/W(c)$, where $W(c) = N_G(c)/Z_G(c)$ is the Weyl group of (G, V) with respect to c; in addition, since G is connected, $\mathbb{C}[c]^{W(c)}$ is a polynomial algebra [DK85, Th. 2.9 and 2.10]. Hence $V/\!\!/G \cong \mathbb{C}^{\dim c}$. It follows that $\psi^* : \mathbb{C}[\mathfrak{g}^*]^G \to \mathbb{C}[V]^G$ factors as the composition

$$\mathbb{C}[\mathfrak{h}^*]^{W_G} \xrightarrow{\alpha} \mathbb{C}[\mathfrak{a}^*]^{\Gamma} \xrightarrow{\beta} \mathbb{C}[\mathfrak{a}^*]^{W_V}$$

We may assume that V is indecomposable. According to the last column of Tables 1 and 2 in [Kno06], $W_V = \Gamma$ so that β is the identity map, and α is surjective in all cases but $T(\mathfrak{e}_6)$. We finish the proof by proving directly in this case that ψ is injective.

We need to show that $W_G \cdot \xi \cap \mathfrak{a}^* = \Gamma \cdot \xi$ for all $\xi \in \mathfrak{a}^*$ in case $V = T(\mathfrak{e}_6)$. We have $\mathfrak{a}^* = \langle \varpi_1 + \eta, -\varpi_6 + \eta, -\varpi_1 + \varpi_6 + \eta \rangle$. Let $\xi_1, \xi_2 \in \mathfrak{a}^*$ such that $w\xi_1 = \xi_2$ for some $w \in W_G$. The action of W_G fixes η and preserves the complex span $\tilde{\mathfrak{a}}^*$ of ϖ_1, ϖ_6, so $w\tilde{\xi}_1 = \tilde{\xi}_2$, where $\tilde{\xi}_i$ is the $\tilde{\mathfrak{a}}^*$-component of ξ_i with respect to $\mathfrak{a}^* = \tilde{\mathfrak{a}}^* \oplus \mathbb{C}\eta$. Moreover the action of W_G preserves the real span $\tilde{\mathfrak{a}}^*_{\mathbb{R}}$ of ϖ_1, ϖ_6, so, by taking real and imaginary components, we may assume that $\tilde{\xi}_1, \tilde{\xi}_2 \in \tilde{\mathfrak{a}}^*_{\mathbb{R}}$. Since the action of Γ on $\tilde{\mathfrak{a}}^*_{\mathbb{R}}$ is generated by the reflections on the real lines through ϖ_1, ϖ_6, we may replace $\tilde{\xi}_1$, $\tilde{\xi}_2$ by suitable Γ-conjugates and further assume that they are linear combinations of ϖ_1, ϖ_6 with non-negative coefficients. The fact that ϖ_1, ϖ_6 are fundamental weights of E_6 now implies that $\tilde{\xi}_1$, $\tilde{\xi}_2$ belong to the closed positive Weyl chamber and hence $w = 1$. This finishes the proof of of the theorem.

Remark 22. Since $\pi : \mathfrak{h}^* \to \mathfrak{h}^*/W_G$ is a dominant finite morphism between affine varieties, it is a closed map and thus $\pi[\mathfrak{a}^*]$ is an affine variety. We have shown that ψ is a bijective morphism from $V/\!\!/G$ to $\pi[\mathfrak{a}^*]$. Essentially by Zariski's main theorem [Mil12, ch. 8], ψ is an isomorphism onto its image if and only if $\pi[\mathfrak{a}^*]$ is a normal variety. In general, $\psi : V/\!\!/G \cong \mathfrak{a}^*/\Gamma \to \pi[\mathfrak{a}^*]$ is the normalization morphism.

Example 23. We give the details of Knop reduction and Theorem 4 for $T(\mathfrak{e}_6)$. This is polar since it is a θ-group. Now

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le 5\} \cup \left\{ \pm\frac{1}{2}\left(\epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^{5}(-1)^{\nu_i}\epsilon_i\right) \;\middle|\; \sum_{i=1}^{5}\nu_i \text{ even}\right\}$$

and

$$\begin{aligned}\Lambda = &\left\{ \pm\left(\eta + \frac{2}{3}(\epsilon_8 - \epsilon_7 - \epsilon_6)\right)\right\} \\ &\cup \left\{ \pm\left(\eta + \frac{1}{6}(\epsilon_8 - \epsilon_7 - \epsilon_6) - \frac{1}{2}\sum_{i=1}^{5}(-1)^{\nu_i}\epsilon_i\right) \;\middle|\; \sum_{i=1}^{5}\nu_i \text{ even}\right\} \\ &\cup \left\{ \pm\left(\eta - \frac{1}{3}(\epsilon_8 - \epsilon_7 - \epsilon_6) \pm \epsilon_i\right) \;\middle|\; 1 \le i \le 5\right\}\end{aligned}$$

where η corresponds to the center of G. Let $\Omega = \epsilon_8 - \epsilon_7 - \epsilon_6$ and denote the fundamental highest weights of E_6 by $\varpi_1, \ldots, \varpi_6$. We start with the extremal weight $\omega_1 := \eta + \frac{2}{3}\Omega = \eta + \varpi_1$, and we are left with

$$\Delta' = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le 5\} \quad \text{and} \quad \Lambda' = \left\{ \pm\left(\eta - \frac{1}{3}\Omega \pm \epsilon_i\right) \;\middle|\; 1 \le i \le 5\right\}.$$

We take sucessively $\omega_2 := \eta - \frac{1}{3}\Omega + \epsilon_5$, $\omega_3 := \eta - \frac{1}{3}\Omega - \epsilon_5$ and end up with $\Lambda_t^+ = \{\varpi_1 + \eta, \varpi_6 - \eta, -\varpi_1 + \varpi_6 + \eta\}$, which is linearly independent. Now $\mathfrak{a}^* = \langle \varpi_1 + \eta, -\varpi_6 + \eta, \varpi_6 - \varpi_1 + \eta\rangle$. Since the angle between ϖ_1 and ϖ_6 is $\pi/3$, we get $\Gamma = A_2$.

Example 24. In Theorem 4, we cannot drop the assumption that the representation is saturated. In fact, consider the polar symplectic representation of type 2 given by $(SL_n, \mathbb{C}^n \oplus \mathbb{C}^{n*})$. Then $\mu : \mathbb{C}^n \oplus \mathbb{C}^{n*} \to \mathfrak{sl}_n^*$ is given by $\mu(u, \alpha)(\xi) = \alpha(\xi \cdot u)$ and $\psi : \mathbb{C} \to \mathbb{C}^{n-1}$ is given by $\psi(z) = (\sigma_2, \ldots, \sigma_n)$ where $\sigma_j = -(j-1)\binom{n}{j}\left(\frac{z}{n}\right)^j$. In particular, if $n = 2$ then $\psi(z) = -\frac{z^2}{4}$ does not separate closed orbits. Note that if $n = 3$ then $\psi(z) = (-\frac{z^2}{3}, -\frac{2z^3}{27})$ is not an isomorphism onto its image; however, for all n, the enlarged saturated polar symplectic representation $T(\mathfrak{sl}_n)$ has $\mathbb{C} \to \mathbb{C}^n$, $z \mapsto (z, 0, \ldots, 0)$ as moment map, that is, an isomorphism onto its image.

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