The Prime Number Theorem

The Prime Number Theorem says that the probability that an integer $n \gg 0$ is prime is approximately

$$\frac{1}{\ln(n)}.$$

The logarithm $\ln(n)$ measures the size of n: it is roughly the number of digits of n.

The prime numbers get sparser as their size increases.

On the other hand the function $1/\ln(x)$ decreases rather slowly.

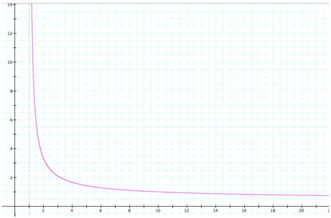

Figure: The function $1/\ln(x)$

Consequence:

Prime numbers of all sizes are plentiful.

If $n \sim 10^{20}$, then

$$\frac{1}{\ln(n)} \sim 0,022$$

In the interval

$$[10^{20} - 500, 10^{20} + 500]$$

by the PNT one expects about 22 prime numbers.

In reality: there are 26 of them.

Below is the list.

99999999999999999533
99999999999999999607
99999999999999999617
99999999999999999631
99999999999999999689
99999999999999999701
99999999999999999773
99999999999999999799
99999999999999999803
99999999999999999857
99999999999999999931
99999999999999999941
99999999999999999973
99999999999999999989
100000000000000000039
100000000000000000129
100000000000000000151
100000000000000000193
100000000000000000207
100000000000000000301
100000000000000000349
100000000000000000361
100000000000000000391
100000000000000000393
100000000000000000441
100000000000000000477

If $n \sim 10^{40}$, then

$$\frac{1}{\ln(n)} \sim 0,010$$

In the interval

$$[10^{40} - 500, 10^{40} + 500]$$

by the PNT one expects about 11 prime numbers.

In reality: there are 16 of them.

Below is the list.

9999999999999999999999999999999999999547
9999999999999999999999999999999999999563
9999999999999999999999999999999999999631
9999999999999999999999999999999999999727
9999999999999999999999999999999999999769
9999999999999999999999999999999999999787
9999999999999999999999999999999999999799
9999999999999999999999999999999999999827
9999999999999999999999999999999999999883
9999999999999999999999999999999999999973
9999999999999999999999999999999999999983
10000000000000000000000000000000000000121
10000000000000000000000000000000000000139
10000000000000000000000000000000000000301
10000000000000000000000000000000000000391
10000000000000000000000000000000000000457

If $n \sim 10^{60}$, then

$$\frac{1}{\ln(n)} \sim 0,007$$

In the interval

$$[10^{60} - 500, 10^{60} + 500]$$

by the PNT one expects about 7 prime numbers.

In reality: there are 12 of them.

Below is the list

999999999999999999999999999999999999999999999999999999999587
999999999999999999999999999999999999999999999999999999999619
999999999999999999999999999999999999999999999999999999999721
999999999999999999999999999999999999999999999999999999999829
999999999999999999999999999999999999999999999999999999999839
1000000000000000000000000000000000000000000000000000000000007
1000000000000000000000000000000000000000000000000000000000067
1000000000000000000000000000000000000000000000000000000000079
1000000000000000000000000000000000000000000000000000000000099
1000000000000000000000000000000000000000000000000000000000151
1000000000000000000000000000000000000000000000000000000000423
1000000000000000000000000000000000000000000000000000000000451

Try yourself to list the primes in the interval

$$[n - M, n + M]$$

by choosing n and M and gluing the function

```
P(n,M)=for(k=-M,M, if(isprime(n+k),print(n+k ),))
```

into PARI/GP.

Example

```
? P(53,20)
37
41
43
47
53
59
61
67
71
73
```

Figure: The primes in the interval $[53 - 20, 53 + 20]$