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Invariant Domains in the Complexification of a Noncompact Riemannian Symmetric Space

Laura Geatti

Dipartimento di Matematica, Università di Roma 2, Tor Vergata, 00133 Rome, Italy

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Let G/K be a noncompact Riemannian symmetric space and let $G^{\mathbb{C}}/K^{\mathbb{C}}$ be its complexification. Then G acts on $G^{\mathbb{C}}/K^{\mathbb{C}}$ by left translations. We study the invariant CR-structure of the closed G-orbits of maximal dimension and determine which ones can lie in the boundary of an invariant Stein domain. In this way, we obtain information on the G-invariant Stein domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Key Words: semisimple Lie group; symmetric space; homogeneous CR-structure.

INTRODUCTION

Let G/K be an irreducible Riemannian symmetric space. Its complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a Stein manifold and left translations by elements of G are holomorphic automorphisms of $G^{\mathbb{C}}/K^{\mathbb{C}}$. Invariant Stein domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and invariant plurisubharmonic functions are natural objects to investigate. In the case when the symmetric space G/K is compact, such objects are well understood. Every G-invariant domain $\Omega \subseteq G^{\mathbb{C}}/K^{\mathbb{C}}$ intersects a certain analytic set D in a lower dimensional domain Ω_D, biholomorphic to a Reinhardt domain in $(\mathbb{C}^*)^r$ ($r = \operatorname{rank} G/K$). Invariant Stein domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ are precisely the ones for which this intersection is connected and Stein (cf. [FH, La]). Moreover, there is a one-to-one correspondence between invariant plurisubharmonic functions on Ω and logarithmically convex Weyl-invariant functions on Ω_D (see [AL]).

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In this paper, we consider the case when G/K is a noncompact Riemannian symmetric space. Under these assumptions, global results like in the compact case are no longer true. Local slices only exist at closed G-orbits and there are no nonconstant global G-invariant plurisubharmonic functions on $G^{\mathbb{C}}/K^{\mathbb{C}}$ (see [L]).

The G-action determines a finite number of invariant regions, whose union is dense in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and which roughly correspond to the different types of closed orbits of maximal dimension (*generic orbits*). We study the invariant CR-structure that generic orbits inherit from the complex manifold $G^{\mathbb{C}}/K^{\mathbb{C}}$. We compute the corresponding Levi form and Levi cone, which governs the local extension of CR-functions to holomorphic functions. In this way, we determine which generic orbits can be contained in the boundary of an invariant Stein domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ or in a level set of an invariant plurisubharmonic function. It turns out that regions associated to different types of generic orbits have rather different complex analytic properties: for example, only some of them contain invariant Stein subdomains and admit nonconstant invariant plurisubharmonic functions.

One of them is the region $\overline{\mathbf{X}}_0$, which consists of all G-orbits intersecting the compact dual symmetric space $U \cdot \bar{e} \cong U/K$, embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as the U-orbit of the base point $\bar{e}$ (here U denotes the compact real form of $G^{\mathbb{C}}$) (see [AG]). In general, $\overline{\mathbf{X}}_0$ contains several copies of the symmetric space G/K, and each of them comes with a distinguished invariant neighborhood $D \subset \overline{\mathbf{X}}_0$. D is the largest connected invariant domain which admits a retraction to G/K and carries a canonical G-invariant Kaehler structure compatible with the Riemannian structure of G/K (see [GS1, GS2, LS, Sz]). These domains, say $D_0, \dots, D_m$, contain Stein-invariant subdomains and carry nonconstant invariant plurisubharmonic functions. They are conjectured to be Stein [AG] and to be related to the parameter space of linear cycles in flag domains [WZ].

When the group G is of Hermitian type and $G^{\mathbb{C}}/K^{\mathbb{C}}$ contains compactly causal symmetric spaces G/H as minimal orbits, then there are other regions in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing invariant Stein subdomains. Let p be a point on one such orbit G/H and let W (resp. $-W$) be the maximal Ad_H-stable regular elliptic cone in the tangent space $T(G/H)_p$. Then $S_W := G \exp iW$ and $S_{-W} := G \exp i(-W)$ are G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing G/H in their boundary. The domains $S_{\pm W}$ were showed to be Stein in [Ne]. Moreover, their invariant plurisubharmonic functions and Stein subdomains were completely characterized.

In this paper, we show that, with few possible exceptions, all proper G-invariant Stein domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ are contained either in one of the domains $D_0, \dots, D_m$ or in one of the domains $S_{\pm W_1}, \dots, S_{\pm W_s}$. The same holds for domains admitting nonconstant invariant plurisubharmonic functions.

The possible exceptions are among the domains whose boundary entirely consists of nongeneric orbits. The domains $D_0, \dots, D_m$ are of this type. Unfortunately, the techniques of this paper do not apply to such domains and their Steinness remains an open question.

One of the motivations for this study comes from representation theory. A natural G-manifold like $G^{\mathbb{C}}/K^{\mathbb{C}}$ may provide a setting for the geometric realization of unitary representations of G. For example, the invariant Stein domains contained in $S_{\pm W}$ carry Hilbert spaces of holomorphic functions where the group G acts in a unitary fashion. The representations of G which are realized in this way are unitary highest weight representations with spherical lowest K-type [Ne]. On the other hand, the results of this paper show that being Stein is an uncommon property among G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$. One may wonder whether $G^{\mathbb{C}}/K^{\mathbb{C}}$ contains invariant domains which are q-complete or the generic orbits themselves carry some natural Hilbert space where the group G acts by a unitary representation. In other words, these results are just a first step in the investigation of the G-invariant objects in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

The paper is organized as follows. In Section 1, we recall some general facts about CR-structures; in Section 2, we recall Matsuki's results about the double coset decomposition of a complex reductive group $G^{\mathbb{C}}$ under the action by the fixed point sets of a pair of involutions. Matsuki's results yield a parametrization of the generic orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ in terms of Cartan subsets, i.e., cross sections of the form $C = \exp J\mathfrak{c} \cdot p$, where $\mathfrak{c}$ is an r-dimensional abelian subspace of $\mathfrak{g}$ and p is a base point in $G^{\mathbb{C}}/K^{\mathbb{C}}$. We show that each Cartan subset admits a base point p which satisfies some very restrictive algebraic conditions. This fact is crucial for the computation of the Levi form of the generic orbits. For most reduced restricted root systems, such conditions imply that the G-orbit of p is a semisimple symmetric space G/H, embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as a totally real submanifold. In particular, G/H has minimal dimension. In Section 3, we determine the invariant CR-structure of generic orbits. This is done by explicitly computing the vector fields generating the tangent space and the complex tangent space at a reference point in terms of some generalized restricted root systems of $\mathfrak{g}^{\mathbb{C}}$. We also determine all the isotropy types of generic orbits. In Section 4, we set up the general formulas for the calculation of the Levi form. In Section 5, we carry out the computation of the Levi form and the Levi cone of all generic orbits. In Section 6, we apply the results of Section 5 to the study of invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and of their invariant plurisubharmonic functions. As an example, we carry out the rank-one case.

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1. CR-STRUCTURES AND LEVI FORM

Generalities about CR-structures

Let M be a complex manifold with tangent bundle TM and let $J\colon TM \to TM$ denote its complex structure. Let $T^{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$ denote the formal complexification of TM. Then J extends to a $\mathbb{C}$-linear morphism of $T^{\mathbb{C}}M$ and decomposes $T^{\mathbb{C}}M$ as

$$T^{\mathbb{C}}M = HM \oplus AM \tag{1.1}$$

into the holomorphic and antiholomorphic tangent bundles of M. The bundles HM and AM are by definition the $\pm i$-eigenspaces of J on $T^{\mathbb{C}}M$, respectively. The complex conjugation

$$^{-}\colon T^{\mathbb{C}}M \longrightarrow T^{\mathbb{C}}M, \qquad X \otimes z \longmapsto X \otimes \bar{z},$$

defines a $\mathbb{C}$-antilinear bundle isomorphism $^{-}\colon HM \to AM$. The map $X \mapsto X \otimes 1$ defines a canonical embedding $TM \to T^{\mathbb{C}}M$ identifying TM with the real part of $T^{\mathbb{C}}M$ with respect to the complex conjugation, i.e.,

$$TM = \{Z \in T^{\mathbb{C}}M \mid \overline{Z} = Z\}.$$

The bundle maps $\pi^H\colon TM \to HM$ given by $X \mapsto \frac{1}{2}(X - iJX)$ and $\pi^A\colon TM \to AM$ given by $X \mapsto \frac{1}{2}(X + iJX)$ define $\mathbb{R}$-isomorphisms satisfying $\pi^H(JX) = i\pi^H(X)$ and $\pi^A(JX) = -i\pi^A(X)$, respectively.

Let S be a real submanifold of M with tangent bundle TS. Let $x \in S$. Denote by TS_x the tangent space to S at x and by

$$T_{\mathbb{C}}S_x := TS_x \cap JTS_x$$

the maximal J-stable subspace of TS_x. If the complex dimension d of $T_{\mathbb{C}}S_x$ is independent of x, then S is a CR-manifold and d is called the CR-dimension of S. Moreover, $T_{\mathbb{C}}S = TS \cap JTS$ is a well-defined J-stable subbundle of TS. Denote by $\operatorname{codim}_{\mathbb{R}}(S, M)$ the real codimension of S in M.

DEFINITION 1.1. A CR-submanifold $S \subset M$ is called *generic* if

$$\operatorname{codim}_{\mathbb{R}}(S, M) \le \dim_{\mathbb{C}} M - d.$$

The vector bundles $T_{\mathbb{C}}S$ and TS can be formally complexified as well. The decomposition (1.1) induces a decomposition of $T_{\mathbb{C}}^{\mathbb{C}}S$ as

$$T_{\mathbb{C}}^{\mathbb{C}}S = HS \oplus AS,$$

where $HS = T^{\mathbb{C}}S \cap HM$ and $AS = T^{\mathbb{C}}S \cap AM$. One has that a CR-submanifold S of a complex manifold is involutive: if Z, W are local sections of HS, so is $[Z, W]$. As a consequence, if X, Y are local sections of $T_{\mathbb{C}}S$, the same is true for both

$$[X, Y] - [JX, JY] \qquad \text{and} \qquad [JX, Y] + [X, JY]. \tag{1.2}$$

The Levi Form

Let S be a CR-submanifold of a complex manifold M. We recall the definition of the Levi form of S (see [Bo] or [Tu]).

Let $x \in S$. Denote by Z_x a tangent vector in $T_{\mathbb{C}}S_x$ and by $\widehat{Z}$ an arbitrary extension of Z_x to a local section of $T_{\mathbb{C}}S$. Then the vector fields $\pi^H(\widehat{Z}) = \frac{1}{2}(\widehat{Z} - iJ\widehat{Z})$ and $\pi^A(\widehat{Z}) = \frac{1}{2}(\widehat{Z} + iJ\widehat{Z})$ are local sections of the bundles HS and AS.

DEFINITION 1.2. The Levi form $\mathbf{L}$ of S at x is the map $\mathbf{L}$: $T_{\mathbb{C}}S_x \times T_{\mathbb{C}}S_x \to T^{\mathbb{C}}S_x/T^{\mathbb{C}}_{\mathbb{C}}S_x$ given by

$$\mathbf{L}(X_x, Y_x) := \frac{i}{4}[\widehat{X} - iJ\widehat{X}, \widehat{Y} + iJ\widehat{Y}]_x \qquad \text{mod } T^{\mathbb{C}}_{\mathbb{C}}S_x.$$

Remark 1.3. The Levi form $\mathbf{L}$ at x is well defined, as it does not depend on the choice of the extensions $\widehat{X}$ and $\widehat{Y}$. Moreover, $\mathbf{L}$ is an $\mathbb{R}$-bilinear Hermitian form satisfying

$$\mathbf{L}(X_x, Y_x) = \mathbf{L}(JX_x, JY_x) \qquad \text{and} \qquad \mathbf{L}(X_x, Y_x) = \overline{\mathbf{L}(Y_x, X_x)},$$

where the conjugation on $T^{\mathbb{C}}S_x/T^{\mathbb{C}}_{\mathbb{C}}S_x$ is the restriction of the conjugation on $T^{\mathbb{C}}M_x$. It follows that $\mathbf{L}(X_x, X_x)$ is real valued, i.e., $\mathbf{L}(X_x, X_x) \in TS_x/T_{\mathbb{C}}S_x$. By (1.2), for all $X_x, Y_x \in T_{\mathbb{C}}S_x$ the following identities hold modulo $T^{\mathbb{C}}_{\mathbb{C}}S_x$:

$$\mathbf{L}(X_x, Y_x) = \frac{i}{2}[\widehat{X}, \widehat{Y}]_x - \frac{1}{2}[\widehat{X}, J\widehat{Y}]_x \qquad \text{and} \qquad \mathbf{L}(X_x, X_x) = \frac{1}{2}[J\widehat{X}, \widehat{X}]_x.$$

A key geometric object associated with the Levi form is the Levi cone, which is the higher codimensional analogue of the signature of the classical Levi form of a CR-hypersurface.

DEFINITION 1.4. Let S be a CR-manifold in M and let $x \in S$. The Levi cone $\mathscr{C}_x(S)$ of S at x is defined by

$$\mathscr{C}_x(S) := \{\mathbf{L}(X_x, X_x) \mid X_x \in T_{\mathbb{C}}S_x\} \subset TS_x/T_{\mathbb{C}}S_x.$$

Observe that $\mathscr{C}_x(S)$ is a real cone (i.e., satisfies the condition $\mathbb{R}^+ \cdot \mathscr{C}_x(S) \subset \mathscr{C}_x(S)$) and may have an empty interior. The cone $\mathscr{C}_x(S)$ governs the holomorphic extension of CR-functions defined on a neighborhood of x in S. In this regard, we mention a theorem which will be applied in Section 6 (cf. [Bo, p. 202]).

THEOREM 1.5. *Let S be a generic CR-submanifold of a complex manifold M. Let $x \in S$ and assume that the Levi cone at x satisfies the condition*

$$\mathscr{C}_x(S) = TS_x/T_{\mathbb{C}}S_x.$$

Then, for each neighborhood ω of x in S, there exists a neighborhood Ω of x in M satisfying $\Omega \cap S \subset \omega$ and with the property that every CR-function of class C^1 on $\Omega \cap S$ extends to a unique holomorphic function on Ω.

2. GENERIC ORBITS IN $G^{\mathbb{C}}/K^{\mathbb{C}}$

2.1. *Preliminaries*

A semisimple Riemannian symmetric space of the noncompact type is a coset space G/K, where G is a real semisimple Lie group and $K \subset G$ is a maximal compact subgroup. Even when G is a complex group, it is viewed as a real Lie group. Since G/K is simply connected, it decomposes as the Riemannian product of irreducible symmetric spaces

$$G/K = G_1/K_1 \times \cdots \times G_n/K_n.$$

Throughout the paper, we assume for simplicity that G/K is irreducible. (Later on, we show how to extend our results from the irreducible case to the general case (Section 6).) Without loss of generality, G can be assumed to be a connected real simple Lie group and to admit a faithful linear representation. Then G and K have complexifications $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$, and the coset space $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a Stein manifold. The group $G^{\mathbb{C}}$ can be assumed to be simply connected. When G itself is a complex group, it can be assumed simply connected.

Denote by π the canonical projection $\pi: G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$. Throughout the paper, we indicate the image of an object in $G^{\mathbb{C}}$ under π by overlining the corresponding symbol (e.g., $\overline{S} := \pi(S)$ for $S \subset G^{\mathbb{C}}$). Left translations by elements of G on $G^{\mathbb{C}}/K^{\mathbb{C}}$ are defined by

$$L_g \cdot \bar{x} = g \cdot \bar{x} := \overline{gx}, \qquad g \in G,\ \bar{x} \in G^{\mathbb{C}}/K^{\mathbb{C}},$$

and are holomorphic automorphisms of $G^{\mathbb{C}}/K^{\mathbb{C}}$.

The aim of this section is to give a parametrization of the closed G-orbits of maximal dimension in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Such a parametrization is based on a result by Matsuki about the double coset decomposition of a complex reductive group $G^{\mathbb{C}}$ under the action by the fixed point sets of a pair of involutions [Ma1]. In the case we consider, one involution is the conjugation σ of $G^{\mathbb{C}}$ corresponding to the noncompact real form G, and the other one is $\tau = \theta^{\mathbb{C}}$, the complexification of the Cartan involution θ of G. It is easy to check that

$$\sigma\tau = \tau\sigma \qquad \text{and} \qquad \sigma\tau = \Theta,$$

where Θ denotes the Cartan involution of $G^{\mathbb{C}}$. The fixed point sets of the above involutions are given by

$$(G^{\mathbb{C}})^{\sigma} = G, \qquad (G^{\mathbb{C}})^{\tau} = K^{\mathbb{C}}, \qquad (G^{\mathbb{C}})^{\Theta} = U,$$

respectively, where U is the compact real form of $G^{\mathbb{C}}$.

We denote the Lie algebra of a group by the corresponding gothic letter. For example, $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ denote the Lie algebras of G and $G^{\mathbb{C}}$, respectively.

We denote by the same symbol an involution of a group and the derived involution of its Lie algebra. Recall that there is a one-to-one correspondence between involutions of a simply connected group and those of its Lie algebra. Moreover, the fixed point set of a simply connected group is always connected [S].

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ and let J denote the complex structure of $\mathfrak{g}^{\mathbb{C}}$. To σ, τ, and Θ there correspond three decompositions of $\mathfrak{g}^{\mathbb{C}}$ into ± 1-eigenspaces

$$\mathfrak{g}^{\mathbb{C}} = (\mathfrak{g}^{\mathbb{C}})^{\sigma} \oplus (\mathfrak{g}^{\mathbb{C}})^{-\sigma} = \mathfrak{g} \oplus J\mathfrak{g}, \qquad \mathfrak{g}^{\mathbb{C}} = (\mathfrak{g}^{\mathbb{C}})^{\tau} \oplus (\mathfrak{g}^{\mathbb{C}})^{-\tau} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}},$$
$$\mathfrak{g}^{\mathbb{C}} = (\mathfrak{g}^{\mathbb{C}})^{\Theta} \oplus (\mathfrak{g}^{\mathbb{C}})^{-\Theta} = \mathfrak{u} \oplus J\mathfrak{u},$$

where $\mathfrak{u} = \mathfrak{k} \oplus J\mathfrak{p}$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$.

2.2. *Semisimple Elements, Cartan Subsets, and Generic Orbits*

The group $G \times K^{\mathbb{C}}$ acts on $G^{\mathbb{C}}$ by

$$(g, k) \cdot x \longmapsto gxk^{-1},$$

and *two elements* x, y *sit on the same* $G \times K^{\mathbb{C}}$*-orbit in* $G^{\mathbb{C}}$ *if and only if* $\bar{x}, \bar{y}$ *sit on the same* G*-orbit in* $G^{\mathbb{C}}/K^{\mathbb{C}}$. Before we can state Matsuki's result about closed $G \times K^{\mathbb{C}}$-orbits in $G^{\mathbb{C}}$, we need some preliminaries.

For $g \in G^{\mathbb{C}}$, consider the involution of $\mathfrak{g}^{\mathbb{C}}$ defined by

$$\tau_g = \mathrm{Ad}_g \tau \mathrm{Ad}_{g^{-1}} \tag{2.1}$$

and the (real) automorphism of $\mathfrak{g}^{\mathbb{C}}$ given by

$$f_g := \sigma\tau_g.$$

DEFINITION 2.1. An element $g \in G^{\mathbb{C}}$ is called *semisimple with respect to* σ, τ if the corresponding automorphism f_g is semisimple. The set of semisimple elements with respect to σ, τ in $G^{\mathbb{C}}$ is denoted by $G^{\mathbb{C}}_{\mathrm{ss},\sigma,\tau}$.

DEFINITION 2.2. An element $g \in G^{\mathbb{C}}_{\mathrm{ss},\sigma,\tau}$ is called *regular semisimple with respect to* σ, τ if the intersection $(\mathfrak{g}^{\mathbb{C}})^{-\sigma} \cap (\mathfrak{g}^{\mathbb{C}})^{-\tau_g}$ is commutative. The set of *regular semisimple elements with respect to* σ, τ in $G^{\mathbb{C}}$ is denoted by $G^{\mathbb{C}}_{\mathrm{rs},\sigma,\tau}$.

Both sets of semisimple and regular semisimple elements with respect to σ, τ are $G \times K^{\mathbb{C}}$-stable and there are inclusions $G^{\mathbb{C}}_{\mathrm{rs},\sigma,\tau} \subset G^{\mathbb{C}}_{\mathrm{ss},\sigma,\tau} \subset G^{\mathbb{C}}$. The set $G^{\mathbb{C}}_{\mathrm{ss},\sigma,\tau}$ is open and dense in $G^{\mathbb{C}}$ and consists of the elements in $G^{\mathbb{C}}$ sitting on closed $G \times K^{\mathbb{C}}$-orbits (cf. [Ma1, Proposition 4, p. 67]). The set $G^{\mathbb{C}}_{\mathrm{rs},\sigma,\tau}$ is open and dense in $G^{\mathbb{C}}_{\mathrm{ss},\sigma,\tau}$ and consists of the elements in $G^{\mathbb{C}}$ sitting on closed $G \times K^{\mathbb{C}}$-orbits of maximal dimension (cf. Section 3).

Following Matsuki [Ma1, Sect. 4.4], we now introduce Cartan subsets in $G^{\mathbb{C}}$ in our special case, together with the appropriate notion of conjugacy and of Weyl group. Cartan subsets are cross sections for the closed $G \times K^{\mathbb{C}}$-orbits in $G^{\mathbb{C}}$: every such orbit intersects some Cartan subset and every closed orbit of maximal dimension intersects precisely one Cartan subset (up to conjugacy) along the orbit of the corresponding Weyl group.

DEFINITION 2.3. Let $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{g}$ be a maximal abelian subspace. $\mathfrak{a}$ is called a *fundamental Cartan subspace*. The torus group $A := \exp J\mathfrak{a} \subset G^{\mathbb{C}}$ is called a *fundamental Cartan subset*.

DEFINITION 2.4. A *standard Cartan subset* is a set of the form

$$C := \exp J\mathfrak{c} \cdot p \subset G^{\mathbb{C}},$$

where p is a point in $A = \exp J\mathfrak{a}$ and

$$\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \oplus \mathfrak{c}_{\mathfrak{p}} \qquad \text{for } \mathfrak{c}_{\mathfrak{k}} = \mathfrak{c} \cap \mathfrak{k},\ \mathfrak{c}_{\mathfrak{p}} = \mathfrak{c} \cap \mathfrak{p} \subset \mathfrak{a},$$

is a θ-stable (maximal) abelian subspace in $\mathfrak{g} \cap \mathrm{Ad}_p \mathfrak{p}^{\mathbb{C}}$ with $\dim \mathfrak{c} = \dim \mathfrak{a}$. The space $\mathfrak{c}$ is called a *standard Cartan subspace*.

Remark 2.5. Observe that in general the base point p of a standard Cartan subset $C = \exp J\mathfrak{c} \cdot p$ is not uniquely determined: for all $p' \in \exp J\mathfrak{c}_{\mathfrak{p}} \cdot p$, one has that $\mathfrak{c}$ is a maximal abelian subspace of $\mathfrak{g} \cap \mathrm{Ad}_{p'} \mathfrak{p}^{\mathbb{C}}$ and $\exp J\mathfrak{c} \cdot p'$ defines the same Cartan subset C.

For a fundamental Cartan subset A, define a *Weyl group* as

$$W_{K\times K}(A) = N_{K\times K}(A)/Z_{K\times K}(A),$$

where

$$N_{K\times K}(A) = \{(h,l) \in K \times K \mid hAl^{-1} = A\}$$

and

$$Z_{K\times K}(A) = \{(h,l) \in K \times K \mid hal^{-1} = a, \forall\, a \in A\}.$$

There is a notion of conjugacy between Cartan subsets which goes as follows.

Let $C_1 = \exp J\mathfrak{c}_{\mathfrak{k}}^1 \cdot A_1$ and $C_2 = \exp J\mathfrak{c}_{\mathfrak{k}}^2 \cdot A_2$ be standard Cartan subsets, where $A_1 = \exp J\mathfrak{c}_{\mathfrak{p}}^1 \cdot p_1$ and $A_2 = \exp J\mathfrak{c}_{\mathfrak{p}}^2 \cdot p_2$. Then C_1 and C_2 are said to be $W_{K\times K}(A)$-*conjugate* if

$$A_2 = hA_1k^{-1} \qquad \text{for some } (h,k) \in N_{K\times K}(A).$$

For a standard Cartan subset $C = \exp J\mathfrak{c} \cdot p$, define a *Weyl group* as

$$W_{K\times K}(C) = N_{K\times K}(C)/Z_{K\times K}(C),$$

where

$$N_{K\times K}(C) = \{(h,l) \in K \times K \mid hCl^{-1} = C\}$$

and

$$Z_{K\times K}(C) = \{(h,l) \in K \times K \mid hcl^{-1} = c, \forall\, c \in C\}.$$

Remark 2.6. (i) By [Ma1, Lemma 10(ii)], if two Cartan subsets C_1 and C_2 are conjugate, then $C_2 = mC_1l^{-1}$ for some $(m,l) \in K \times K$. In other words, conjugate Cartan subsets intersect the same $G \times K^{\mathbb{C}}$-orbits in $G^{\mathbb{C}}$.

(ii) The group $W_{K\times K}(A)$ and its action on A have been described in [AG, Proposition 6]:

$$W_{K\times K}(A) = \{(h,l) \in K \times K \mid h \in N_K(\mathfrak{a}),\ hl^{-1} \in \exp J\Gamma\}$$

is isomorphic to the semidirect product of the Weyl group $W_K(\mathfrak{a})$ and Γ, the kernel of the map exp: $\mathfrak{a} \to \exp J\mathfrak{a}/\exp J\mathfrak{a} \cap K$. Precisely, Γ is the lattice in $\mathfrak{a}$ given by

$$\Gamma = \sum_{\alpha\in\Delta_{\mathfrak{a}}} \mathbb{Z}\pi \frac{2h_\alpha}{B(h_\alpha, h_\alpha)},$$

where, for each root α, the vector h_α is defined by $\alpha(H) = B(H, h_\alpha)$, $H \in \mathfrak{a}$.

If $(h,l) \in W_{K\times K}(A)$ and $x = e^{JX} \in A$, then

$$(h,l)\cdot x = e^{J\mathrm{Ad}_h X} hl^{-1} = e^{J(\mathrm{Ad}_h X + \gamma)}, \qquad \gamma \in \Gamma.$$

In particular, there is an embedding $N_K(\mathfrak{a}) \hookrightarrow N_{K\times K}(A)$, $h \mapsto (h,h)$.

(iii) Let $C = \exp J\mathfrak{c}\cdot p$ be a Cartan subset and let H be the isotropy subgroup in G of the base point $\bar{p} \in G^{\mathbb{C}}/K^{\mathbb{C}}$. Then, for each $h \in N_{K\cap H}(\mathfrak{c})$, one has that

$$h \exp J\mathfrak{c}\cdot ph^{-1} = \exp J\mathrm{Ad}_h\mathfrak{c}\cdot hph^{-1} = \exp J\mathfrak{c}\cdot pkh^{-1}, \qquad k, kh^{-1} \in K^{\mathbb{C}}.$$

This means that C and hCh^{-1} are Cartan subsets in $G^{\mathbb{C}}$ with the same image $\overline{C}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$. In particular, they intersect precisely the same G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Standard Cartan subsets can be described in terms of *orthogonal systems* of restricted root vectors in $\mathfrak{g}$. In our situation, this description is equivalent to the one given in [Ma1, p. 83]. Let $\Delta_{\mathfrak{a}}$ denote the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and let

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha\in\Delta_{\mathfrak{a}}} \mathfrak{g}^\alpha, \qquad \mathfrak{g}^0 = Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a}, \qquad \mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$$

be the corresponding root decomposition.

An *orthogonal system* of restricted root vectors is a set $Q = \{X_{\alpha_1}, \dots, X_{\alpha_m}\}$ of restricted root vectors $X_{\alpha_j} \in \mathfrak{g}^{\alpha_j}$ satisfying the conditions $[X_{\alpha_i}, X_{\alpha_j}] = [X_{\alpha_i}, \theta X_{\alpha_j}] = 0$ for $i \neq j$.

Define

$$\mathfrak{t}_Q := \mathbb{R}(X_{\alpha_1} + \theta X_{\alpha_1}) \oplus \cdots \oplus \mathbb{R}(X_{\alpha_m} + \theta X_{\alpha_m}) \subset \mathfrak{k},$$

$$\mathfrak{a}_Q := \bigcap_{j=1}^{m} \mathrm{Ker}\{\alpha_j\} = \{H \in \mathfrak{a} \mid \alpha_j(H) = 0 \text{ for } j = 1, \dots, m\} \subset \mathfrak{a}, \quad (2.2)$$

$$A_Q := \{e^{JH} \in A = \exp J\mathfrak{a} \mid e^{2i\alpha_j(H)} = -1 \text{ for } j = 1, \dots, m\}.$$

Each connected component of the set

$$C_Q = \exp J\mathfrak{t}_Q \cdot A_Q \quad (2.3)$$

is a standard Cartan subset. All standard Cartan subsets arise in this way.

The connected components of C_Q are in one-to-one correspondence with the ones of A_Q. A connected component C of C_Q can be written as

$$C = \exp J\mathfrak{t}_Q \cdot \exp J\mathfrak{a}_Q \cdot p, \quad (2.4)$$

where the $(r - m)$-dimensional torus

$$\exp J\mathfrak{a}_Q = A_Q^0 = \{e^{2JH} \in A \mid e^{2i\alpha_j(H)} = 1 \text{ for } j = 1, \dots, m\}$$

is the connected component of the identity of A_Q, and p is a base point satisfying the conditions

$$p = e^{JA_0} \in \exp J\mathfrak{a} \qquad \text{and} \qquad \alpha_j(A_0) \equiv \pi/2 \qquad \mathrm{mod}\, \pi, \;\; j = 1, \dots, m. \quad (2.5)$$

Remark 2.7. (i) Denote by $R_Q = \{\alpha_1, \dots, \alpha_m\}$ the set of restricted roots corresponding to an orthogonal system of restricted root vectors $Q = \{X_{\alpha_1}, \dots, X_{\alpha_m}\}$. Then the roots in R_Q are orthogonal with respect to the Killing form, i.e., $\langle \alpha_i, \alpha_j \rangle = 0$ for all $i \neq j$ (see [Ma2]). In general, they may not be strongly orthogonal. For example, when $G = SO_0(6, 8)$ there are orthogonal systems of restricted root vectors arising from orthogonal, nonstrongly orthogonal roots. However, a set of orthogonal, nonstrongly orthogonal restricted roots may not admit an orthogonal system of restricted root vectors.

(ii) *Conjugacy* of Cartan subsets can be formulated in terms of orthogonal systems as follows:

- Let $Q_1 = \{X_{\alpha_1}, \dots, X_{\alpha_m}\}$ be an orthogonal system of restricted root vectors of an orthogonal set of roots $R_{Q_1} = \{\alpha_1, \dots, \alpha_m\}$. Let $w \in N_K(\mathfrak{a})$ and let $R_{Q_2} = w \cdot R_{Q_1} = \{w\alpha_1, \dots, w\alpha_m\}$. Then Q_1 and $Q_2 = \{wX_{\alpha_1}, \dots, wX_{\alpha_m}\}$ give rise to families of pairwise conjugate Cartan subsets C_{Q_1} and C_{Q_2}.

• Let $C_1 \cong C_2$ be conjugate standard Cartan subsets. If C_1 is associated to an orthogonal system of root vectors $Q_1 = \{X_{\alpha_1}, \dots, X_{\alpha_m}\}$ of an orthogonal set of roots R_{Q_1}, then C_2 is conjugate to a Cartan subset associated to the orthogonal system $Q_2 = w \cdot Q_1$ of the set of orthogonal roots $R_{Q_2} = w \cdot R_{Q_1}$. Precisely, A_2 is a connected component of A_{Q_2}, for $Q_2 = w \cdot Q_1$.

As a consequence, $W_K(\mathfrak{a})$-equivalence classes of sets of positive orthogonal restricted roots, admitting an orthogonal system of restricted root vectors, yield a complete set of representatives of $W_{K\times K}(\mathfrak{a})$-conjugacy classes of standard Cartan subsets.

Let $\{C_i = \exp J\mathfrak{c}_i \cdot p_i\}_{i\in I}$ be a complete set of representatives of the $W_{K\times K}(A)$-conjugacy classes of standard Cartan subsets in $G^{\mathbb{C}}$. By the above description, such a set is finite. For $i \in I$, denote by

$$\mathbf{X}_i := G \cdot \exp J\mathfrak{c}_i \cdot p_i \cdot K^{\mathbb{C}}$$

the set of the $G \times K^{\mathbb{C}}$-orbits in $G^{\mathbb{C}}$ intersecting C_i, and by

$$\mathbf{X}_0 = G \cdot \exp J\mathfrak{a} \cdot K^{\mathbb{C}}$$

the set of the $G \times K^{\mathbb{C}}$-orbits in $G^{\mathbb{C}}$ intersecting the fundamental Cartan subset $A := \exp J\mathfrak{a}$. In our situation, Theorem 3 in [Ma1, p. 80], can be summarized as follows.

THEOREM 2.8. (i) *Every closed $G \times K^{\mathbb{C}}$-orbit in $G^{\mathbb{C}}$ intersects a standard Cartan subset C_i for some $i \in I$.*

(ii) *Closed $G \times K^{\mathbb{C}}$-orbits consisting of regular semisimple elements with respect to σ, τ intersect precisely one Cartan subset C_i in the orbit of the Weyl group $W_{K\times K}(C_i)$.*

(iii) *There is identification of orbit spaces $\mathbf{X}_0/G \times K^{\mathbb{C}} \cong A/W_{K\times K}(A)$. In other words, $x, y \in A$ lie on the same $G \times K^{\mathbb{C}}$-orbit if and only if they lie on the same $W_{K\times K}(A)$-orbit.*

Since the canonical projection π is equivariant, continuous, and closed, the sets of elements in $G^{\mathbb{C}}/K^{\mathbb{C}}$ sitting on closed G-orbits and on closed G-orbits of maximal dimension are given by $\overline{G^{\mathbb{C}}_{\mathrm{ss},\sigma,\tau}}$ and $\overline{G^{\mathbb{C}}_{\mathrm{rs},\sigma,\tau}}$, respectively. We call the closed G-orbits of maximal dimension *generic orbits*. By the above theorem,

$$\overline{G^{\mathbb{C}}_{\mathrm{ss},\sigma,\tau}} = \bigcup \overline{\mathbf{X}}_i.$$

Moreover, every generic orbit S admits a reference point on a set $\overline{C_i}$, where C_i, $i \in I$, is a uniquely determined Cartan subset. Sometimes, for simplicity, we call $\overline{C_i}$ a Cartan subset as well.

2.3. *Standard Cartan Subsets and Minimal Orbits*

In this section, we prove that every standard Cartan subset admits a base point p which satisfies some very restrictive algebraic conditions. Under such conditions, the G-orbit of $\bar{p} \in G^{\mathbb{C}}/K^{\mathbb{C}}$ has locally minimal dimension (cf. Sections 3.2 and 3.3). For most reduced restricted root systems, such conditions also imply that the G-orbit of $\bar{p}$ is a semisimple symmetric space, embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as a totally real submanifold. In this case, the G-orbit of $\bar{p}$ has absolute minimal dimension.

LEMMA 2.9. *Let $\mathfrak{g}$ be a simple real Lie algebra. Let $C \subset C_Q$ be a standard Cartan subset associated to an orthogonal system of restricted root vectors $Q = \{X_{\alpha_1}, \dots, X_{\alpha_m}\}$.*

(i) *If the restricted root system $\Delta_{\mathfrak{a}}$ of $\mathfrak{g}$ is reduced, not of type C_r or F_4, there exists a base point $p = e^{JA_0} \in A_Q$ of C satisfying*

$$\alpha(A_0) \equiv 0 \qquad \mathrm{mod}\, \pi/2 \qquad \text{for all } \alpha \in \Delta_{\mathfrak{a}}. \tag{2.6}$$

(ii) *If the restricted root system $\Delta_{\mathfrak{a}}$ of $\mathfrak{g}$ is of type C_r, $(BC)_r$, or F_4, there exists a base point $p = e^{JA_0} \in A_Q$ of C satisfying either conditions* (2.6) *or*

$$2\alpha(A_0) \equiv 0 \qquad \mathrm{mod}\, \pi/2 \qquad \text{for all } \alpha \in \Delta_{\mathfrak{a}}. \tag{2.7}$$

Proof. Let $\mathfrak{g}$ be a simple real Lie algebra of real rank r (i.e., $\dim_{\mathbb{R}} \mathfrak{a} = r$). Then the restricted root system $\Delta_{\mathfrak{a}}$ is irreducible and it is either isomorphic to one of the rank-r reduced root systems, or it is of type $(BC)_r$ (see [He, Kn2]). Let $R_Q = \{\alpha_1, \dots, \alpha_m\}$ be the set of (orthogonal) restricted roots corresponding to Q.

By definition, a base point $p = e^{JA_0}$ of a Cartan subset $C \subset C_Q$ satisfies the system of equations

$$\begin{cases} \alpha_1(A_0) \;=\; (2n_1 + 1)\dfrac{\pi}{2} \\ \quad\vdots \\ \alpha_m(A_0) = (2n_m + 1)\dfrac{\pi}{2} \end{cases} \qquad \text{for some integers } n_1, \dots, n_m \in \mathbb{Z}. \tag{2.8}$$

The proof of the lemma is equivalent to showing that in A_Q there exists a base point $p = e^{JA_0}$ for C, satisfying either

$$\begin{cases} \alpha_j(A_0) = (2n_j + 1)\dfrac{\pi}{2}, & j = 1, \dots, m, \\ \alpha(A_0) \in \mathbb{Z}\dfrac{\pi}{2}, & \forall\, \alpha \in \Delta_{\mathfrak{a}}, \end{cases} \tag{2.9}$$

or

$$\begin{cases} \alpha_j(A_0) = (2n_j + 1)\dfrac{\pi}{2}, & j = 1, \dots, m, \\ 2\alpha(A_0) \in \mathbb{Z}\dfrac{\pi}{2}, & \forall\, \alpha \in \Delta_{\mathfrak{a}}. \end{cases} \tag{2.9$'$}$$

Observe that when the orthogonal roots $\alpha_1, \ldots, \alpha_m$ span the space $\mathfrak{a}^*$, then $\mathfrak{a}_Q = \{0\}$ and the base point p is uniquely determined by conditions (2.8). So it is not a priori obvious that conditions (2.9) or (2.9)′ can be fulfilled by some base point p.

When $\Delta_\mathfrak{a}$ is of type $A_r, D_r, G_2, E_6, E_7, E_8$, orthogonal restricted roots are automatically strongly orthogonal (cf. [Ma2]). In all these cases, the sets R_Q of strongly orthogonal restricted roots, up to Weyl group equivalence, have been determined in [Su]. (They coincide with the sets of strongly orthogonal roots of the split real form of a complex Lie algebra of type $A_r, D_r, G_2, E_6, E_7, E_8$, respectively.) For the sake of completeness we list them hereby.

A_r $\Delta = \{\pm(e_i - e_j),\ 1 \leq i < j \leq r+1\} \subset \mathbb{R}^{r+1} \cap (e_1 + \cdots + e_{r+1})^\perp$.
$\Pi = \{(e_1 - e_2), \ldots, (e_r - e_{r+1})\}$.
Every set of strongly orthogonal roots is conjugate to a set of the form $R_Q = \{(e_{i_1} - e_{i_2}), \ldots, (e_{i_{2k-1}} - e_{i_{2k}}), k \geq 1, i_1 < i_2, \ldots, i_{2k-1} < i_{2k}\}$, where $i_1, i_2, \ldots, i_{2k} \in \{1, 2, \ldots, r+1\}$ are distinct integers.

D_r $\Delta = \{\pm(e_i \pm e_j),\ 1 \leq i < j \leq r\} \subset \mathbb{R}^r$.
$\Pi = \{(e_1 - e_2), \ldots, (e_{r-1} - e_r), (e_{r-1} + e_r)\}$.
Every set of strongly orthogonal roots is conjugate to a set of the form $R_Q = R(l,k) = \{(e_1 \pm e_2), \ldots, (e_{2l-1} \pm e_{2l}), (e_{2l+1} - e_{2l+k+1}), \ldots, (e_{2l+k} - e_{2l+2k})\}$ for $l, k \geq 0$, $2l + 2k \leq r$.
For r even, there is also the possibility $R_Q = R(0, r/2 - 1) \cup \{e_{r-1} + e_r\}$.

G_2 $\Delta = \{\pm(e_2 - e_3), \pm(e_3 - e_1), \pm(e_1 - e_2), \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\} \subset \mathbb{R}^3 \cap (e_1 + e_2 + e_3)^\perp$.
$\Pi = \{(e_1 - e_2), (-2e_1 + e_2 + e_3)\}$.
Every set R_Q of strongly orthogonal roots is conjugate to a subset of the maximal set $R = \{(e_2 - e_3), (2e_1 - e_2 - e_3)\}$.

E_8 $\Delta = \{\pm(e_i \pm e_j),\ 1 \leq i < j \leq 8,\ \frac{1}{2}\sum_{i=1}^{8}(-1)^{\nu(i)}e_i,\ \sum \nu(i) \text{ even}\} \subset \mathbb{R}^8$.
$\Pi = \{\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), (e_2 + e_1), (e_2 - e_1), (e_3 - e_2), (e_4 - e_3), (e_5 - e_4), (e_6 - e_5), (e_7 - e_6)\}$.
Every set R_Q of strongly orthogonal roots is conjugate to a subset of the maximal set
$R = \{\frac{1}{2}(e_1 - e_2 + e_3 - e_4 + e_5 - e_6 - e_7 + e_8), \frac{1}{2}(e_1 + e_2 - e_3 + e_4 - e_5 - e_6 - e_7 + e_8), (e_3 + e_4), (e_2 + e_5), (e_1 + e_6), \frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 - e_7 + e_8), \frac{1}{2}(-e_1 + e_2 + e_3 - e_4 - e_5 + e_6 - e_7 + e_8), (e_8 + e_7)\}$.

E_7 $\Delta = \{\text{roots in } E_8 \text{ orthogonal to the root } (e_8 + e_7)\}$.
$\Pi = \{\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), (e_2 + e_1), (e_2 - e_1), (e_3 - e_2), (e_4 - e_3), (e_5 - e_4), (e_6 - e_5)\}$.

Every set R_Q of strongly orthogonal roots in E_7 is conjugate to a subset of the maximal set given by the first seven vectors of $R \subset E_8$.

E_6 $\quad \Delta = \{$roots in E_8 orthogonal to the roots $(e_8 + e_7), (e_8 + e_6)\}$.
$\Pi = \{\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), (e_2 + e_1), (e_2 - e_1), (e_3 - e_2), (e_4 - e_3), (e_5 - e_4)\}$.
Every set R_Q of strongly orthogonal roots in E_6 is conjugate to a subset of the maximal set given by the first four vectors of $R \subset E_8$.

When $\Delta_\mathfrak{a}$ is of type $B_r, C_r, (BC)_r, F_4$, sets of orthogonal restricted roots are not necessarily strongly orthogonal. For these types of root systems, sets of orthogonal roots can be described as follows.

B_r $\quad \Delta = \{\pm e_i,\ 1 \le i \le r,\ \pm(e_i \pm e_j), 1 \le i < j \le r\} \subset \mathbb{R}^r$.
$\Pi = \{(e_1 - e_2), \dots, (e_{r-1} - e_r), e_r\}$.
Every set of orthogonal roots is conjugate to a set of the form
$R_Q = \{(e_{i_1} \pm e_{j_1}), \dots, (e_{i_k} \pm e_{j_k}), e_{h_1}, \dots, e_{h_n}\}$, where $k, n \ge 0$, $i_1 < j_1, \dots, i_k < j_k$, and $i_1, j_1, \dots, i_k, j_k, h_1, \dots, h_n \in \{1, 2, \dots, r\}$ are distinct integers.

C_r $\quad \Delta = \{\pm 2e_i,\ 1 \le i \le r,\ \pm(e_i \pm e_j),\ 1 \le i < j \le r\} \subset \mathbb{R}^r$.
$\Pi = \{(e_1 - e_2), \dots, (e_{r-1} - e_r), 2e_r\}$.
Every set of orthogonal roots is conjugate to a set of the form
$R_Q = \{(e_{i_1} \pm e_{j_1}), \dots, (e_{i_k} \pm e_{j_k}), 2e_{h_1}, \dots, 2e_{h_n}\}$, where $k, n \ge 0, i_1 < j_1, \dots, i_k < j_k$, and $i_1, j_1, \dots, i_k, j_k, h_1, \dots, h_n \in \{1, 2, \dots, r\}$ are distinct integers.

$(BC)_r$ $\quad \Delta = \{\pm e_i, \pm 2e_i,\ 1 \le i \le r,\ \pm(e_i \pm e_j),\ 1 \le i < j \le r\} \subset \mathbb{R}^r$.
$\Pi = \{(e_1 - e_2), \dots, (e_{r-1} - e_r), e_r\}$.
Every set of orthogonal roots is conjugate to a set of the form
$R_Q = \{(e_{i_1} \pm e_{j_1}), \dots, (e_{i_k} \pm e_{j_k}), e_{m_1}, \dots, e_{m_p}, 2e_{h_1}, \dots, 2e_{h_n}\}$, where $k, p, n \ge 0$, $i_1 < j_1, \dots, i_k < j_k$, and $i_1, j_1, \dots, i_k, j_k, m_1, \dots, m_p, h_1, \dots, h_n \in \{1, 2, \dots, r\}$ are distinct integers.

F_4 $\quad \Delta = \{\pm e_i,\ 1 \le i \le 4,\ \pm(e_i \pm e_j),\ 1 \le i < j \le 4,\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\} \subset \mathbb{R}^4$.
$\Pi = \{\frac{1}{2}(e_1 - e_2 - e_3 - e_4), e_4, (e_3 - e_4), (e_2 - e_3)\}$.
Every orthogonal set R_Q of roots is conjugate to a subset of
$R_1 = \{(e_1 \pm e_2), (e_3 \pm e_4)\}$, $R_2 = \{e_1, \dots, e_4\}$,
$R_3 = \{(e_i \pm e_j), e_h, e_k,\ i < j,\ i, j, h, k, \text{ all distinct}\}$.

In the above lists, we have identified $\mathfrak{a}^*$ with a subspace of some euclidean space $\mathbb{R}^n$. In this way, every root in $\Delta_\mathfrak{a}$ can be written as a linear combination of the functionals $e_1, \dots, e_n \in \mathbb{R}^n$, and conditions (2.8) translate into a linear system of equations in $e_{i_1}(A_0), \dots, e_{i_k}(A_0)$ for some $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$.

For $\Delta_{\mathfrak{a}} = A_r, B_r, D_r$, the systems (2.8) arising in this way have the following properties:

- The equations are either of the form

$$e_k(A_0) \pm e_h(A_0) = (2s+1)\frac{\pi}{2} \qquad \text{or} \qquad e_i(A_0) = (2t+1)\frac{\pi}{2}, \quad t, s \in \mathbb{Z}. \tag{2.10}$$

- Different equations involve e_i's with different indices, except for pairs of equations of the form

$$\begin{cases} e_p(A_0) + e_q(A_0) = (2a+1)\frac{\pi}{2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad a, b \in \mathbb{Z}. \\ e_p(A_0) - e_q(A_0) = (2b+1)\frac{\pi}{2}, \end{cases} \tag{2.11}$$

One can check that it is always possible to find $A_0 \in \mathfrak{a}$ such that $e_i(A_0) \in \mathbb{Z}\frac{\pi}{2}$, for all i and conditions (2.9) are satisfied.

For $\Delta_{\mathfrak{a}} = G_2, E_6, E_7, E_8$ and every set of strongly orthogonal roots R_Q, it is always possible to find $A_0 \in \mathfrak{a}$ such that conditions (2.9) are satisfied. An easy calculation proves it for G_2.

Let R be the maximal set of strongly orthogonal roots in E_8. The corresponding 8×8 system (2.8) uniquely determines the values $\{e_i(A_0)\}_{i=1,\dots,8}$ and the values of the simple roots in Π

$$\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), (e_2 + e_1), (e_2 - e_1), (e_3 - e_2),$$
$$(e_4 - e_3), (e_5 - e_4), (e_6 - e_5), (e_7 - e_6)$$

on A_0, which are respectively given by

$$\begin{aligned} &(n_1 + n_2 - n_3 - n_4)\frac{\pi}{2}, && (1 + n_2 + n_4 + n_5 - n_6)\frac{\pi}{2}, \\ &-(n_1 - n_4 + n_5 - n_7)\frac{\pi}{2}, && (n_1 - n_2 + n_3 - n_4)\frac{\pi}{2}, \\ &(-n_1 + n_2 + n_6 - n_7)\frac{\pi}{2}, && (n_1 - n_2 - n_3 + n_4)\frac{\pi}{2}, \\ &(-n_1 - n_4 + n_5 + n_7)\frac{\pi}{2}, && -(1 + n_5 + n_6 + n_7 - n_8)\frac{\pi}{2}. \end{aligned}$$

It follows that A_0 satisfies conditions (2.9) for the maximal set R and therefore for every subset $R_Q \subset R$. The discussion of E_6 and E_7 is included in that of E_8.

For $\Delta_{\mathfrak{a}} = C_r$, systems (2.8) arising from a set R_Q of orthogonal roots contain a number of systems like (2.11) and equations of the form

$$2e_j(A_0) = (2u+1)\frac{\pi}{2}, \qquad u \in \mathbb{Z},$$

which have solutions in $\frac{1}{2}\mathbb{Z}\frac{\pi}{2}$. If the roots in R_Q are actually strongly orthogonal, it is always possible to find $A_0 \in \mathfrak{a}$ satisfying conditions (2.9). However, there are sets of orthogonal, nonstrongly orthogonal roots for which it is only possible to find $A_0 \in \mathfrak{a}$ satisfying conditions (2.9)$'$. (In some cases, such roots also admit an orthogonal systems of restricted root vectors: for example when $\mathfrak{g} = EVII$.)

For $\Delta_\mathfrak{a} = F_4$, conditions (2.9) can be satisfied for all orthogonal sets of roots contained in R_1 and R_2 and all strongly orthogonal sets contained in R_3. For orthogonal sets contained in R_3, conditions (2.9)$'$ can be satisfied. (However, only when $\mathfrak{g} = EVI$ and $\mathfrak{g} = EIX$ do such sets of roots actually admit an orthogonal system of restricted root vectors.)

In the *nonreduced case* $(BC)_r$, because of double roots, conditions (2.9) generally need to be replaced by conditions (2.9)$'$, even for sets of strongly orthogonal roots. Then the arguments used for B_r apply to this case as well.

We conclude by observing that when $\mathfrak{g}$ admits a complex structure, the restricted root system $\Delta_\mathfrak{a}$ is isomorphic to the ordinary root system Δ and orthogonal systems of restricted root vectors only occur in connection with strongly orthogonal roots. In particular, every Cartan subset admits a base point satisfying conditions (2.6).

Conditions (2.6) and (2.7) put severe restrictions on a point $p = e^{JA_0} \in \exp J\mathfrak{a}$ on the associated involution τ_p of $\mathfrak{g}^\mathbb{C}$ (see (2.1)) and on the G-orbit of $\bar{p} \in G^\mathbb{C}/K^\mathbb{C}$.

LEMMA 2.10. *Let* $p = e^{JA_0} \in \exp J\mathfrak{a}$. *Then*

(i) $\tau_p = \mathrm{Ad}_{p^2}\tau$ *and* $\tau_p\tau_{p^{-1}} = \mathrm{Ad}_{p^4}$.

(ii) $\sigma\tau_p = \tau_{p^{-1}}\sigma$, $\tau\tau_p = \tau_{p^{-1}}\tau$, *and* $\tau_p\Theta = \Theta\tau_p$.

(iii) *Let* $C = \exp J\mathfrak{c}_Q \cdot p$ *be a Cartan subset associated to the orthogonal system of restricted root vectors* $Q = \{X_{\alpha_1}, \dots, X_{\alpha_m}\}$. *Then the base point* p *satisfies*

$$\tau_p|\mathfrak{c}_Q^\mathbb{C} = \tau_{p^{-1}}|\mathfrak{c}_Q^\mathbb{C} = -\mathrm{Id} \qquad \textit{and} \qquad \tau_p\tau_{p^{-1}}|\mathfrak{c}_Q^\mathbb{C} = \mathrm{Ad}_{p^4}|\mathfrak{c}_Q^\mathbb{C} = \mathrm{Id}.$$

Proof. If $p \in \exp J\mathfrak{a}$, then $\tau(p) = \sigma(p) = p^{-1}$. It follows that

$$\sigma\mathrm{Ad}_p = \mathrm{Ad}_{p^{-1}}\sigma, \qquad \tau\mathrm{Ad}_p = \mathrm{Ad}_{p^{-1}}\tau.$$

The above relations together with the definitions of τ_p and $\tau_{p^{-1}}$ imply parts (i) and (ii). Observe that $\tau_p|\mathfrak{a}_Q = \tau_{p^{-1}}|\mathfrak{a}_Q = -\mathrm{Id}$, since $p \in \exp J\mathfrak{a}$. Moreover, by (2.2), one has that

$$\tau_p(X_{\alpha_i} + \theta X_{\alpha_i}) = \tau_{p^{-1}}(X_{\alpha_i} + \theta X_{\alpha_i}) = -(X_{\alpha_i} + \theta X_{\alpha_i}) \qquad \text{for all } X_{\alpha_i} \in Q.$$

All the statements in part (iii) are now immediate.

LEMMA 2.11. *Let* $p = e^{JA_0} \in \exp J\mathfrak{a}$ *be a point satisfying conditions* (2.6). *Then*

(i) $\mathrm{Ad}_{p^4} = \mathrm{Id}$, $\tau_p = \tau_{p^{-1}}$, and $\tau_p \sigma = \sigma \tau_p$.

(ii) τ_p *preserves the real form* G *of* $G^{\mathbb{C}}$.

(iii) *The restriction of* τ_p *to* $\mathfrak{g}$ *is an involution of* $\mathfrak{g}$ *commuting with the Cartan involution* θ.

(iv) $\sigma\tau_p$ *is a conjugation of* $\mathfrak{g}^{\mathbb{C}}$ *with real form* $\mathfrak{g}^c = \mathfrak{h} \oplus J\mathfrak{q}$, *where* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *is the* τ_p*-decomposition of* $\mathfrak{g}$.

Proof. By conditions (2.6), one has that $\mathrm{Ad}_{p^4} = \mathrm{Id}$ on $\mathfrak{g}^{\mathbb{C}}$ and $G^{\mathbb{C}}$. As a consequence, $\tau_p\sigma = \sigma\tau_p$ and $\tau_p(G) = G$. Statement (iii) is immediate. (iv) Since $\sigma\tau_p = \tau_p\sigma$, one has that $\sigma\tau_p$ is a conjugation of $\mathfrak{g}^{\mathbb{C}}$ commuting with τ_p. In this way, the τ_p-decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{q}^{\mathbb{C}}$ is both σ- and $\sigma\tau_p$-stable. In particular, $\mathrm{Fix}(\sigma\tau_p, \mathfrak{g}^{\mathbb{C}}) = \mathrm{Fix}(\sigma\tau_p, \mathfrak{h}^{\mathbb{C}}) \oplus \mathrm{Fix}(\sigma\tau_p, \mathfrak{q}^{\mathbb{C}}) = \mathfrak{h} \oplus J\mathfrak{q}$.

COROLLARY 2.12. *By Lemma* 2.11, *one has that*

$$(G^{\mathbb{C}})^{\tau_p} = (G^{\tau_p})^{\mathbb{C}} = \mathrm{Ad}_p K^{\mathbb{C}}.$$

Remark 2.13. Let $p = e^{JA_0} \in \exp J\mathfrak{a}$ be a base point for a Cartan subset $C = \exp J\mathfrak{c} \cdot p$. Assume p satisfies conditions (2.6). Denote by $G_{\bar{p}}$ and $(G^{\mathbb{C}})_{\bar{p}}$ the isotropy subgroups of the point $\bar{p} \in G^{\mathbb{C}}/K^{\mathbb{C}}$ in G and $G^{\mathbb{C}}$, respectively. Then

$$(G^{\mathbb{C}})_{\bar{p}} = (G^{\mathbb{C}})^{\tau_p} = \mathrm{Ad}_p K^{\mathbb{C}} \qquad \text{and} \qquad G_{\bar{p}} = G \cap (G^{\mathbb{C}})^{\tau_p} = G^{\tau_p}.$$

In other words, $G_{\bar{p}}$ is a (possibly disconnected) real form of $(G^{\mathbb{C}})_{\bar{p}}$. The G-orbit of $\bar{p}$ is a semisimple (generally non-Riemannian) symmetric space G/G^{τ_p} of the same dimension and rank as G/K. The map $G^{\mathbb{C}} \to G^{\mathbb{C}}$, $g \mapsto g \cdot p$ induces an identification

$$G^{\mathbb{C}}/K^{\mathbb{C}} \cong G^{\mathbb{C}}/(G^{\tau_p})^{\mathbb{C}},$$

and the space G/G^{τ_p} embeds in $G^{\mathbb{C}}/(G^{\tau_p})^{\mathbb{C}}$ as a totally real submanifold of maximal dimension. If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the τ_p-decomposition of $\mathfrak{g}$, there is a canonical identification $\mathfrak{q} \cong T(G/G^{\tau_p})_{\bar{p}}$ and the Cartan subspace $\mathfrak{c}$ is a maximal abelian subspace of $\mathfrak{q}$.

LEMMA 2.14. *Let* $p = e^{JA_0} \in \exp J\mathfrak{a}$ *be a point satisfying conditions* (2.7). *Then*

(i) $\mathrm{Ad}_{p^8} = \mathrm{Id}$ *and* Ad_{p^4} *is a complex involution of* $\mathfrak{g}^{\mathbb{C}}$.

(ii) Ad_{p^4} *commutes with* τ, τ_p, *and* σ.

(iii) Ad_{p^4} *measures the noncommutativity of* σ *and* τ_p:

$$\sigma\tau_p = \tau_p\sigma\,\mathrm{Ad}_{p^4}, \qquad (\sigma\tau_p)^4 = \mathrm{Id}.$$

Proof. The proof of the lemma is a straightforward application of the definitions and of the results of Lemma 2.10.

Remark 2.15. Let $p = e^{JA_0} \in \exp J\mathfrak{a}$ be a point satisfying conditions (2.7). In general, under these assumptions, $\sigma\tau_p \neq \tau_p\sigma$ and the τ_p-decomposition of $\mathfrak{g}^{\mathbb{C}}$

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{q}^{\mathbb{C}}$$

is not σ-stable. It follows that

$$\dim_{\mathbb{R}} \mathfrak{g} \cap \mathfrak{h}^{\mathbb{C}} < \dim_{\mathbb{C}} \mathfrak{h}^{\mathbb{C}} \qquad \text{and} \qquad \dim_{\mathbb{R}} G/G_p > \dim_{\mathbb{R}} G/K.$$

In this case, the G-orbit of the point $\bar{p} \in G^{\mathbb{C}}/K^{\mathbb{C}}$ is not a totally real submanifold of $G^{\mathbb{C}}/K^{\mathbb{C}}$.

3. THE CR-STRUCTURE AND THE ORBIT TYPE OF A GENERIC ORBIT

3.1. *Vector Fields Induced by the G-action*

Fix the trivialization of the tangent bundle $TG^{\mathbb{C}} = G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$ given by the right-invariant vector fields on $G^{\mathbb{C}}$. A vector in $\mathfrak{g}^{\mathbb{C}}$ and the corresponding right-invariant vector field on $G^{\mathbb{C}}$ are denoted by the same symbol. Let $G^{\mathbb{C}} \times_{K^{\mathbb{C}}} \mathfrak{p}^{\mathbb{C}}$ denote the G-equivariant bundle defined as the quotient of $G^{\mathbb{C}} \times \mathfrak{p}^{\mathbb{C}}$ by the equivalence relation $[x, v] \sim [xk^{-1}, \mathrm{Ad}_k v]$, $k \in K^{\mathbb{C}}$. The map

$$(\bar{x}, v) \longmapsto [x, (L_x)_*^{-1} v] = [x, \mathrm{Ad}_{x^{-1}} v], \qquad \bar{x} \in G^{\mathbb{C}}/K^{\mathbb{C}},\ v \in T(G^{\mathbb{C}}/K^{\mathbb{C}})_{\bar{x}},$$

provides a G-equivariant identification between the tangent bundle $T(G^{\mathbb{C}}/K^{\mathbb{C}})$ and $G^{\mathbb{C}} \times_{K^{\mathbb{C}}} \mathfrak{p}^{\mathbb{C}}$. Here $(L_x)_* := (dL_x)_e$. Under this identification, the tangent space to $G^{\mathbb{C}}/K^{\mathbb{C}}$ at $\bar{x}$ is given by

$$T(G^{\mathbb{C}}/K^{\mathbb{C}})_{\bar{x}} = \mathrm{Ad}_x \mathfrak{p}^{\mathbb{C}}, \qquad x \in G^{\mathbb{C}}, \pi(x) = \bar{x}.$$

Left translations by elements of G on $G^{\mathbb{C}}/K^{\mathbb{C}}$ induce a Lie algebra homomorphism associating to $X \in \mathfrak{g}$ the vector field X^* on $G^{\mathbb{C}}/K^{\mathbb{C}}$ generated by the action of the one-parameter subgroup

$$L_{\exp tX}\colon \bar{x} \longmapsto \exp tX \cdot \bar{x}. \tag{3.1}$$

By definition, the value $X^*(\bar{x})$ of X^* at $\bar{x}$ is the tangent vector at $\bar{x}$ to the curve $\exp tX \cdot \bar{x}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

LEMMA 3.1. *One has that*

$$X^*(\bar{x}) = \Pi_x(X),$$

where

$$\Pi_x \colon \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathrm{Ad}_x \mathfrak{p}^{\mathbb{C}}$$

is the projection induced by the Lie algebra decomposition $\mathfrak{g}^{\mathbb{C}} = \mathrm{Ad}_x \mathfrak{k}^{\mathbb{C}} \oplus \mathrm{Ad}_x \mathfrak{p}^{\mathbb{C}}$.

Proof. For $\bar{x} \in G^{\mathbb{C}}/K^{\mathbb{C}}$, the isotropy subgroup of $\bar{x}$ in $G^{\mathbb{C}}$ is given by $G^{\mathbb{C}}_{\bar{x}} = \mathrm{Ad}_x K^{\mathbb{C}}$, where $x \in G^{\mathbb{C}}$ and $\bar{x} = \pi(x)$. Therefore the isotropy algebra is given by $\mathfrak{g}^{\mathbb{C}}_{\bar{x}} = \mathrm{Ad}_x \mathfrak{k}^{\mathbb{C}}$ and $X^*(\bar{x})$ coincides with the projection $\Pi_x(X)$.

3.2. *Tangent Space, Complex Tangent Space, and Isotropy Subgroup of a Generic Orbit Lying in* $\overline{\mathbf{X}}_0$

In this section, we calculate the tangent space and the complex tangent space to a generic orbit in the invariant subset $\overline{\mathbf{X}}_0$ associated to the fundamental Cartan subset $A = \exp J\mathfrak{a}$. Fix $\bar{x}_0 \in \overline{A}$ for some $x_0 = e^{JH_0} \in A$. Denote by S the G-orbit of $\bar{x}_0$. The tangent space $TS_{\bar{x}_0}$ to S at $\bar{x}_0$ is generated by the vectors $X^*(\bar{x}_0)$ for $X \in \mathfrak{g}$. By Lemma 3.1, the vectors $X^*(\bar{x}_0)$ are obtained by computing the projection

$$\Pi_{x_0} \colon \mathfrak{g} \longrightarrow \mathrm{Ad}_{x_0} \mathfrak{p}^{\mathbb{C}}. \tag{3.2}$$

Let $\Delta_{\mathfrak{a}}$ be the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and let

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{a}}} \mathfrak{g}^{\alpha}, \qquad \mathfrak{g}^0 = Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a}, \qquad \mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$$

be the corresponding root decomposition. Since $\theta\mathfrak{g}^{\alpha} = \mathfrak{g}^{-\alpha}$, consider the θ-stable subspace of $\mathfrak{g}$

$$\mathfrak{g}[\alpha] := \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha}, \qquad \alpha \in \Delta^+_{\mathfrak{a}}.$$

For $Z_\alpha \in \mathfrak{g}^{\alpha}$, consider the two-dimensional subspace of $\mathfrak{g}[\alpha]$

$$Z[\alpha] := \mathrm{span}_{\mathbb{R}}\{Z_\alpha, \theta Z_\alpha\}.$$

If Z_α runs through a basis of $\mathfrak{g}^{\alpha}$, one has

$$\mathfrak{g}[\alpha] = \bigoplus_{Z_\alpha} Z[\alpha].$$

Each $Z[\alpha]$ is θ-stable, with Cartan decomposition $Z[\alpha] = Z[\alpha]_{\mathfrak{k}} \oplus Z[\alpha]_{\mathfrak{p}}$, and a θ-invariant basis is given by

$$\{K_\alpha := Z_\alpha + \theta Z_\alpha,\ P_\alpha := Z_\alpha - \theta Z_\alpha\}.$$

Denote by $Z[\alpha]^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ the complexification of $Z[\alpha]$. Define

$$\begin{aligned} E_\alpha &:= \mathrm{Ad}_{x_0} K_\alpha = \cos\alpha(H_0) K_\alpha + \sin\alpha(H_0) J P_\alpha, \\ F_\alpha &:= \mathrm{Ad}_{x_0} P_\alpha = \cos\alpha(H_0) P_\alpha + \sin\alpha(H_0) J K_\alpha. \end{aligned} \tag{3.3}$$

Then

$$\{E_\alpha, JE_\alpha, F_\alpha, JF_\alpha\}$$

is a real basis of $\mathrm{Ad}_{x_0} Z[\alpha]^{\mathbb{C}}$ compatible with the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathrm{Ad}_{x_0} \mathfrak{k}^{\mathbb{C}} \oplus \mathrm{Ad}_{x_0} \mathfrak{p}^{\mathbb{C}}$.

In the next proposition, we explicitly compute the map (3.2). For simplicity, we write X^* for $X^*(\bar{x}_0)$.

Proposition 3.2. (i) *If* $X \in \mathfrak{m}$, *then* $X^* = 0$.

(ii) *If* $X \in \mathfrak{a}$, *then* $X^* = X$.

(iii) *If* $K_\alpha, P_\alpha \in Z[\alpha]$, *then*

$$K_\alpha^* = -\sin\alpha(H_0) J F_\alpha, \qquad P_\alpha^* = \cos\alpha(H_0) F_\alpha. \tag{3.4}$$

Proof. Statements (i) and (ii) follow directly from the fact that $\mathfrak{m} \subset \mathrm{Ad}_{x_0} \mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{a} \subset \mathrm{Ad}_{x_0} \mathfrak{p}^{\mathbb{C}}$. To prove statement (iii), observe that relations (3.3) imply that

$$\begin{aligned} K_\alpha &= -\sin\alpha(H_0) J F_\alpha + \cos\alpha(H_0) E_\alpha \qquad \text{and} \\ P_\alpha &= \cos\alpha(H_0) F_\alpha - \sin\alpha(H_0) J E_\alpha. \end{aligned}$$

Therefore $K_\alpha^* = -\sin\alpha(H_0) J F_\alpha$ and $P_\alpha^* = \cos\alpha(H_0) F_\alpha$, as requested.

Define

$$\mathfrak{s} := \bigoplus_{\alpha \in \Delta_{\mathfrak{a}}^+} \mathfrak{g}[\alpha]$$

and consider the map

$$*\colon \mathfrak{a} \oplus \mathfrak{s} \longrightarrow \mathfrak{a} \oplus \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{p}}^{\mathbb{C}}, \qquad X \longmapsto X^*(\bar{x}_0). \tag{3.5}$$

Proposition 3.3. *The map* (3.5) *is an isomorphism if and only if* x_0 *satisfies the condition*

$$\alpha(H_0) \not\equiv 0 \qquad \mathrm{mod}\ \pi/2, \qquad \forall\, \alpha \in \Delta_{\mathfrak{a}}. \tag{3.6}$$

Proof. By Proposition 3.2, the map (3.5) is the identity on $\mathfrak{a}$. Moreover, on each subspace $Z[\alpha] \subset \mathfrak{g}[\alpha]$ the restriction of the map $*$

$$*|Z[\alpha]\colon\ Z[\alpha] \longrightarrow \mathrm{Ad}_{x_0} Z[\alpha]^{\mathbb{C}}_{\mathfrak{p}},$$

with respect to the real bases $\{K_\alpha, P_\alpha\}$ and $\{F_\alpha, JF_\alpha\}$, is given by the matrix

$$M_\alpha = \begin{pmatrix} 0 & -\sin\alpha(H_0) \\ \cos\alpha(H_0) & 0 \end{pmatrix}.$$

Since, for each root α, one has that $\det M_\alpha = \sin\alpha(H_0)\cdot\cos\alpha(H_0) \neq 0$ if and only if $\alpha(H_0) \not\equiv 0 \mod \pi/2$, the statement follows.

Denote by $G_{\bar{x}_0}$ the isotropy subgroup of $\bar{x}_0$ in G, by $\mathfrak{g}_{\bar{x}_0}$ and $\mathfrak{g}^{\mathbb{C}}_{\bar{x}_0}$ the isotropy subalgebras of $\bar{x}_0$ in $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$, respectively. In the next proposition, we determine the orbit type of the generic orbits in $\overline{\mathbf{X}}_0$.

PROPOSITION 3.4. *Let $x_0 = e^{JH_0} \in \exp J\mathfrak{a}$. The point $\bar{x}_0 \in G^{\mathbb{C}}/K^{\mathbb{C}}$ lies on a generic G-orbit if and only if the map* (3.5) *is an isomorphism. In this case, the isotropy subgroup of $\bar{x}_0$ is given by the centralizer of $\mathfrak{a}$ in K*

$$G_{\bar{x}_0} = Z_K(\mathfrak{a}).$$

Proof. At Lie algebra level, we have

$$\mathfrak{g}_{\bar{x}_0} = \mathfrak{g} \cap \mathrm{Ad}_{x_0}\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \cap \mathrm{Ad}_{x_0}\mathfrak{k} \oplus \mathfrak{p} \cap J\mathrm{Ad}_{x_0}\mathfrak{k}. \tag{3.7}$$

Since

$$\mathrm{Ad}_{x_0}\mathfrak{k} = \mathfrak{m} \oplus \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha] \qquad \text{and}$$

$$\mathrm{Ad}_{x_0} K_\alpha = \cos\alpha(H_0) K_\alpha + \sin\alpha(H_0) J P_\alpha,$$

the intersection (3.7) has minimal dimension when

$$\mathfrak{k} \cap \mathrm{Ad}_{x_0}\mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \qquad \text{and} \qquad \mathfrak{p} \cap J\mathrm{Ad}_{x_0}\mathfrak{k} = \{0\}.$$

This happens when (3.6) holds and the map $*$ in (3.5) is an isomorphism. In this case, $\mathfrak{g}_{\bar{x}_0} = \mathfrak{g} \cap \mathrm{Ad}_{x_0}\mathfrak{k}^{\mathbb{C}} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and $G_{\bar{x}_0} = Z_K(\mathfrak{a})$.

Remark 3.5. (i) The argument of Proposition 3.4 also shows the following facts:

- The isotropy subgroup $G_{\bar{x}_0}$ is compact whenever

$$\alpha(H_0) \not\equiv \pi/2 \qquad \mathrm{mod}\ \pi, \qquad \forall\, \alpha \in \Delta_{\mathfrak{a}}. \tag{3.8}$$

It was shown in [AG] that conditions (3.8) define precisely the subset of $\overline{\mathbf{X}}_0$ where the G-action is proper. Indeed, if $\alpha(H_0) \equiv \pi/2 \mod \pi$ for some root $\alpha \in \Delta_{\mathfrak{a}}$, the isotropy subgroup $G_{\bar{x}_0}$ is noncompact.

- The isotropy subgroup $G_{\bar{x}_0}$ is maximal if

$$\alpha(H_0) \equiv 0 \mod \pi/2, \qquad \forall\, \alpha \in \Delta_{\mathfrak{a}}.$$

In this case, the G-orbit of $\bar{x}_0$ is minimal and by Remark 2.13 is a semisimple symmetric space, embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as a totally real submanifold. Two points $\bar{x}_0$ and $\bar{x}_1$ sit on a minimal orbit of the same type if and only if $\alpha(H_0 - H_1) \equiv 0 \mod \pi$, $\forall\, \alpha \in \Delta_{\mathfrak{a}}$. In particular, the G-orbit of $\bar{x}_0$ is of type G/K whenever

$$\alpha(H_0) \equiv 0 \mod \pi, \qquad \forall\, \alpha \in \Delta_{\mathfrak{a}}.$$

(ii) The set of generic orbits in $\overline{\mathbf{X}}_0$ is parametrized by the complement in $\mathfrak{a}$ of the set of hyperplanes

$$\bigcup_{\alpha \in \Delta_{\mathfrak{a}}} \{H \in \mathfrak{a} \mid \alpha(H) \equiv 0 \bmod \pi/2\},$$

modulo the action of the Weyl group $W_{K\times K}(A)$. Generic orbits in $\overline{\mathbf{X}}_0$ form an open dense subset and they are all of the same type.

The next remark relates points on generic orbits in $\overline{A}$ with regular semisimple elements $G^{\mathbb{C}}_{\mathrm{rs},\,\sigma,\,\tau}$ in $G^{\mathbb{C}}$ (cf. Definition 2.2 and [Ma1]).

Remark 3.6. Let $x_0 = e^{JH_0}$ be a point in $\exp J\mathfrak{a}$. Then $\bar{x}_0$ sits on a generic orbit if and only if $x_0 \in G^{\mathbb{C}}_{\mathrm{rs},\,\sigma,\,\tau}$.

Proof. By Definition 2.2, a point x_0 belongs to $G^{\mathbb{C}}_{\mathrm{rs},\,\sigma,\,\tau}$ if and only if the intersection $(\mathfrak{g}^{\mathbb{C}})^{-\sigma} \cap (g^{\mathbb{C}})^{-\tau_{x_0}} = J\mathfrak{g} \cap \mathrm{Ad}_{x_0}\mathfrak{p}^{\mathbb{C}}$ is abelian. Since

$$\mathrm{Ad}_{x_0}\mathfrak{p}^{\mathbb{C}} = \mathfrak{a}^{\mathbb{C}} \oplus \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]^{\mathbb{C}}_{\mathfrak{p}}$$

and

$$\mathrm{Ad}_{x_0} P_\alpha = \cos\alpha(H_0) P_\alpha + \sin\alpha(H_0) JK_\alpha, \qquad P_\alpha \in Z[\alpha]_{\mathfrak{p}},$$

the intersection $J\mathfrak{g} \cap \mathrm{Ad}_{x_0}\mathfrak{p}^{\mathbb{C}}$ is abelian if and only if it coincides with $J\mathfrak{a}$. This happens if and only if condition (3.6) holds, and the statement follows by Proposition 3.4.

COROLLARY 3.7. *The tangent space and the complex tangent space to a generic orbit S at $\bar{x}_0$ are given by*

$$TS_{\bar{x}_0} = \mathfrak{a} \oplus \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]^{\mathbb{C}}_{\mathfrak{p}} \qquad \text{and} \qquad T_{\mathbb{C}}S_{\bar{x}_0} = \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]^{\mathbb{C}}_{\mathfrak{p}},$$

where Z_α runs through a basis of $\mathfrak{g}^\alpha$ for $\alpha \in \Delta^+_{\mathfrak{a}}$. The orbit S is also generic as a CR-manifold (*cf. Definition* 1.1).

Remark 3.8. Later on, in the computation of the Levi form of generic orbits (cf. Lemma 4.3 and Proposition 5.4), we need the inverses of relations (3.4), namely

$$(*)^{-1}F_\alpha = \frac{1}{\cos\alpha(H_0)} P_\alpha \qquad \text{and} \qquad (*)^{-1}JF_\alpha = \frac{-1}{\sin\alpha(H_0)} K_\alpha. \tag{3.9}$$

3.3. *Tangent Space, Complex Tangent Space, and Isotropy Subgroup of a Generic Orbit beyond* $\overline{\mathbf{X}}_0$

In this section, we consider the generic orbits outside $\overline{\mathbf{X}}_0$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$, namely the orbits intersecting a standard Cartan subset $\overline{C}$ other than the fundamental one, where $C = \exp J\mathfrak{c} \cdot p$ is a standard Cartan subset with $\mathfrak{c} \neq \mathfrak{a}$.

Let $p = e^{JA_0} \in \exp J\mathfrak{a}$ be a base point for C. Let τ_p be the associated involution of $\mathfrak{g}^{\mathbb{C}}$ and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{q}^{\mathbb{C}}$$

the corresponding decomposition, where $\mathfrak{h}^{\mathbb{C}} = \mathrm{Ad}_p \mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{q}^{\mathbb{C}} = \mathrm{Ad}_p \mathfrak{p}^{\mathbb{C}}$. Denote by $H^{\mathbb{C}} = \mathrm{Ad}_p K^{\mathbb{C}}$ the isotropy subgroup of $\bar{p} \in G^{\mathbb{C}}/K^{\mathbb{C}}$ in $G^{\mathbb{C}}$. Then

$$G^{\mathbb{C}}/K^{\mathbb{C}} \cong G^{\mathbb{C}}/H^{\mathbb{C}}$$

and the tangent bundle $T(G^{\mathbb{C}}/K^{\mathbb{C}})$ can also be identified with $G^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathfrak{q}^{\mathbb{C}}$, where $\mathfrak{q}^{\mathbb{C}} \cong T(G^{\mathbb{C}}/K^{\mathbb{C}})_{\bar{p}}$.

Fix $\overline{x_0 \cdot p} \in \overline{C}$, where $x_0 = e^{JX_0} \in \exp J\mathfrak{c}$. By Lemma 3.1, the tangent space $TS_{\overline{x_0 \cdot p}}$ and the complex tangent space $T_{\mathbb{C}}S_{\overline{x_0 \cdot p}}$ can be determined by computing the projection

$$\Pi_{x_0 \cdot p} \colon \mathfrak{g} \longrightarrow \mathrm{Ad}_{x_0} \mathfrak{q}^{\mathbb{C}}$$

subordinated to the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathrm{Ad}_{x_0} \mathfrak{h}^{\mathbb{C}} \oplus \mathrm{Ad}_{x_0} \mathfrak{q}^{\mathbb{C}}. \tag{3.10}$$

In dealing with generic orbits outside $\overline{\mathbf{X}}_0$, the restricted root system $\Delta_{\mathfrak{a}}$ is replaced by the restricted root system $\Delta_{\mathfrak{c}}$ determined by the adjoint action of $\mathfrak{c}^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$.

Observe that, by Lemma 2.10(iii), the standard Cartan subspace $\mathfrak{c}$ is contained in $\mathfrak{g} \cap \mathfrak{q}^{\mathbb{C}}$. Since $\mathfrak{c}$ is θ-stable, it decomposes as $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \oplus \mathfrak{c}_{\mathfrak{p}}$, where $\mathfrak{c}_{\mathfrak{k}} \subset \mathfrak{k}$ and $\mathfrak{c}_{\mathfrak{p}} \subset \mathfrak{p}$. Since $\mathfrak{c}_{\mathfrak{k}}$ and $\mathfrak{c}_{\mathfrak{p}}$ are commuting abelian subspaces consisting of semisimple elements, $\mathfrak{c}$ is an abelian semisimple subspace of $\mathfrak{g}$. By the same argument, its complexification $\mathfrak{c}^{\mathbb{C}}$ is a maximal abelian semisimple subspace of $\mathfrak{q}^{\mathbb{C}}$. Denote by $\Delta_{\mathfrak{c}}$ the set of nonzero roots of $\mathfrak{g}^{\mathbb{C}}$ under the adjoint action of $\mathfrak{c}^{\mathbb{C}}$, and let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{c}}} \mathfrak{g}^{\alpha}, \qquad \mathfrak{g}^0 = \mathfrak{m}^{\mathbb{C}} \oplus \mathfrak{c}^{\mathbb{C}}, \qquad \mathfrak{m}^{\mathbb{C}} = \mathfrak{z}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}}) \tag{3.11}$$

be the corresponding root decomposition of $\mathfrak{g}^{\mathbb{C}}$. One can verify that $\Delta_{\mathfrak{c}}$ is a root system, possibly nonreduced. One has that $\dim \mathfrak{g}^{\alpha} \geq 1$ and $\tau_p \mathfrak{g}^{\alpha} = \mathfrak{g}^{-\alpha}$.

Moreover, since $\mathfrak{c}^{\mathbb{C}}$ is σ-stable, there is an induced action of σ on $\Delta_{\mathfrak{c}}$ given by

$$\bar{\alpha}(H) := \overline{\alpha(\sigma(H))} \qquad \text{and} \qquad \sigma(\mathfrak{g}^{\alpha}) = \mathfrak{g}^{\bar{\alpha}}.$$

Recall that, by Lemma 2.9, the base point p can be assumed to satisfy either conditions (2.6) or (2.7). As we already saw in Lemmas 2.10, 2.11, and 2.14, these conditions have important implications for the behavior of the involution τ_p with respect to the conjugation σ.

LEMMA 3.9. *Let $C = \exp J\mathfrak{c} \cdot p$ be a standard Cartan subset. Assume the base point p satisfies either conditions* (2.6) *or* (2.7). *If $\alpha \in \Delta_{\mathfrak{c}}$, then*

$$\mathrm{Ad}_{p^4}\colon \mathfrak{g}^{\alpha} \longrightarrow \mathfrak{g}^{\alpha}$$

is either the identity (*under conditions* (2.6)) *or a semisimple involution of* $\mathfrak{g}^{\alpha}$ (*under conditions* (2.7)). *Let $\mathfrak{g}_{\alpha} = \mathfrak{g}^{\alpha}_{+} \oplus \mathfrak{g}^{\alpha}_{-}$ be the decomposition of $\mathfrak{g}_{\alpha}$ into the corresponding ± 1-eigenspaces and consider the map $\sigma\tau_p\colon \mathfrak{g}^{\alpha} \to \mathfrak{g}^{-\bar{\alpha}}$. Then $\sigma\tau_p = \tau_p\sigma$ on $\mathfrak{g}^{\alpha}_{+}$, while $\sigma\tau_p = -\tau_p\sigma$ on $\mathfrak{g}^{\alpha}_{-}$.*

Proof. By Lemma 2.10(iii), one has that $\mathrm{Ad}_{p^4}|\mathfrak{c} = \mathrm{Id}_{\mathfrak{c}}$. Hence $\mathrm{Ad}_{p^4}\mathfrak{g}_{\alpha} = \mathfrak{g}_{\alpha}$. Moreover, by Lemmas 2.11 and 2.14, one has that $(\mathrm{Ad}_{p^4})^2 = \mathrm{Id}$. If $Z_{\alpha} \in \mathfrak{g}^{\alpha}_{+}$, then $\mathrm{Ad}_{p^4} Z_{\alpha} = \tau_{p^{-1}}\tau_p Z_{\alpha} = Z_{\alpha}$. Hence $\sigma\tau_p Z_{\alpha} = \sigma\tau_{p^{-1}} Z_{\alpha} = \tau_p\sigma Z_{\alpha}$.
If $Z_{\alpha} \in \mathfrak{g}^{\alpha}_{-}$, then $\mathrm{Ad}_{p^4} Z_{\alpha} = \tau_{p^{-1}}\tau_p Z_{\alpha} = -Z_{\alpha}$. Hence $\sigma\tau_p Z_{\alpha} = -\sigma\tau_{p^{-1}} Z_{\alpha} = -\tau_p\sigma Z_{\alpha}$.

For $\alpha \in \Delta_{\mathfrak{c}}^{+}$, define

$$\mathfrak{g}[\alpha] := \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{\bar{\alpha}} \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{-\bar{\alpha}}.$$

Since the involution Ad_{p^4} is semisimple, there exists a basis of $\mathfrak{g}^{\alpha}$ consisting of ± 1-eigenvectors. For each such $Z_{\alpha} \in \mathfrak{g}_{\alpha}$, define

$$Z[\alpha] := \mathrm{span}_{\mathbb{C}}\{Z_{\alpha}, \sigma Z_{\alpha}, \tau_p Z_{\alpha}, \tau_p\sigma Z_{\alpha}\}$$

and write $Z[\alpha]_{+}$ or $Z[\alpha]_{-}$ to emphasize that the generator Z_{α} belongs to $\mathfrak{g}^{\alpha}_{+}$ or to $\mathfrak{g}^{\alpha}_{-}$, respectively. Both $\mathfrak{g}[\alpha]$ and $Z[\alpha]$ are Ad_{p^4}, τ_p, and σ-stable. If Z_{α} runs through a suitable basis of $\mathfrak{g}^{\alpha}$ consisting of Ad_{p^4}-eigenvectors, then

$$\mathfrak{g}[\alpha] = \bigoplus_{Z_{\alpha}} Z[\alpha]_{+} \oplus \bigoplus_{Z_{\alpha}} Z[\alpha]_{-}.$$

Let

$$Z[\alpha] = Z[\alpha]_{\mathfrak{h}^{\mathbb{C}}} \oplus Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}} = \mathrm{Re}\, Z[\alpha] \oplus J\,\mathrm{Im}\, Z[\alpha] \tag{3.12}$$

be the decompositions of $Z[\alpha]$ with respect to τ_p and σ. Observe that on $Z[\alpha]_{+}$ the involutions τ_p and σ commute. Hence there exists a σ-invariant basis of $Z[\alpha]_{+}$ which is also τ_p-stable. On $Z[\alpha]_{-}$, the involutions τ_p and σ anticommute. So there exists a σ-invariant basis of $Z[\alpha]_{-}$, but it cannot be τ_p-stable.

Remark 3.10. *Our next goal is to explicitly exhibit a σ-invariant basis of each $Z[\alpha]_+$ and $Z[\alpha]_-$.* In order to compute the projection (3.10), we also want such a basis to be as close as possible to a τ_p-stable one. Subdivide the roots in $\Delta_\mathfrak{c}$ into real, imaginary, and complex roots, depending on their behavior on $\mathfrak{c}$:

$$\Delta_\mathfrak{c}^r = \{\alpha \in \Delta_\mathfrak{c} \mid \alpha(H) \in \mathbb{R}, H \in \mathfrak{c}\},$$
$$\Delta_\mathfrak{c}^i = \{\alpha \in \Delta_\mathfrak{c} \mid \alpha(H) \in i\mathbb{R}, H \in \mathfrak{c}\},$$
$$\Delta_\mathfrak{c}^c = \{\alpha \in \Delta_\mathfrak{c} \mid \alpha(H) \notin \mathbb{R}, i\mathbb{R}, H \in \mathfrak{c}\}.$$

We need to distinguish several cases.

(1) If $\alpha \in \Delta_\mathfrak{c}^c$ is a *complex* root, then each $Z[\alpha]$ is a four-dimensional space and a τ_p-stable basis of $Z[\alpha]$ is given by

$$H_\alpha = Z_\alpha + \tau_p Z_\alpha, \qquad H_{\bar\alpha} = \sigma Z_\alpha + \tau_p \sigma Z_\alpha,$$
$$Q_\alpha = Z_\alpha - \tau_p Z_\alpha, \qquad Q_{\bar\alpha} = \sigma Z_\alpha - \tau_p \sigma Z_\alpha.$$

(i) Assume $Z_\alpha \in \mathfrak{g}_+^\alpha$. Then $\sigma H_\alpha = H_{\bar\alpha}$ and $\sigma Q_\alpha = Q_{\bar\alpha}$. A σ-invariant basis of $Z[\alpha]_+$, which is also τ_p-stable, is given by

$$\begin{aligned}
\operatorname{Re} H_\alpha &= \frac{1}{2}(H_\alpha + \sigma H_\alpha) = \frac{1}{2}(H_\alpha + H_{\bar\alpha}),\\
\operatorname{Im} H_\alpha &= \frac{-1}{2}J(H_\alpha - \sigma H_\alpha) = \frac{-1}{2}J(H_\alpha - H_{\bar\alpha}),\\
\operatorname{Re} Q_\alpha &= \frac{1}{2}(Q_\alpha + \sigma Q_\alpha) = \frac{1}{2}(Q_\alpha + Q_{\bar\alpha}),\\
\operatorname{Im} Q_\alpha &= \frac{-1}{2}J(Q_\alpha - \sigma Q_\alpha) = \frac{-1}{2}J(Q_\alpha - Q_{\bar\alpha}).
\end{aligned} \tag{3.13}$$

(ii) Assume $Z_\alpha \in \mathfrak{g}_-^\alpha$. Then $\sigma H_\alpha = Q_{\bar\alpha}$ and $\sigma Q_\alpha = H_{\bar\alpha}$. A σ-invariant basis of $Z[\alpha]_-$ is given by

$$\begin{aligned}
\operatorname{Re} H_\alpha &= \frac{1}{2}(H_\alpha + \sigma H_\alpha) = \frac{1}{2}(H_\alpha + Q_{\bar\alpha}),\\
\operatorname{Im} H_\alpha &= \frac{-1}{2}J(H_\alpha - \sigma H_\alpha) = \frac{-1}{2}J(H_\alpha - Q_{\bar\alpha}),\\
\operatorname{Re} Q_\alpha &= \frac{1}{2}(Q_\alpha + \sigma Q_\alpha) = \frac{1}{2}(Q_\alpha + H_{\bar\alpha}),\\
\operatorname{Im} Q_\alpha &= \frac{-1}{2}J(Q_\alpha - \sigma Q_\alpha) = \frac{-1}{2}J(Q_\alpha - H_{\bar\alpha}).
\end{aligned} \tag{3.14}$$

(2) If $\alpha \in \Delta_{\mathfrak{c}}^{r}$ is a *real* root, then $\mathfrak{g}[\alpha] = \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha}$ and both $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{-\alpha}$ are σ-stable. Since Ad_{p^4} and σ commute (Lemma 2.14(ii)), there exists a σ-invariant basis $\{Z_\alpha\}$ of $\mathfrak{g}^{\alpha}$ of Ad_{p^4}-eigenvectors. For each such Z_α,

$$Z[\alpha] = \mathrm{span}\{Z_\alpha, \tau_p Z_\alpha\}.$$

(i) If $Z_\alpha \in \mathfrak{g}_{+}^{\alpha}$, then $\sigma Q_\alpha = Q_{\bar{\alpha}} = Q_\alpha$ and $\sigma H_\alpha = H_{\bar{\alpha}} = H_\alpha$. A σ-invariant basis of $Z[\alpha]_{+}$, which is also τ_p-stable, is given by

$$\{H_\alpha, Q_\alpha\}.$$

(ii) If $Z_\alpha \in \mathfrak{g}_{-}^{\alpha}$, then $\sigma Q_\alpha = H_{\bar{\alpha}} = H_\alpha$ and $\sigma H_\alpha = Q_{\bar{\alpha}} = Q_\alpha$. A σ-invariant basis of $Z[\alpha]_{-}$ is given by

$$\left\{\mathrm{Re}\, H_\alpha = \frac{1}{2}(H_\alpha + Q_\alpha),\ \mathrm{Im}\, H_\alpha = \frac{-1}{2} J(H_\alpha - Q_\alpha)\right\}.$$

(3) If $\alpha \in \Delta_{\mathfrak{c}}^{i}$ is an *imaginary* root, then $\mathfrak{g}[\alpha] = \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha}$ and $\sigma\tau_p\colon \mathfrak{g}^{\alpha} \to \mathfrak{g}^{\alpha}$ is a complex antilinear map preserving both $\mathfrak{g}_{+}^{\alpha}$ and $\mathfrak{g}_{-}^{\alpha}$ (see Lemma 2.14(ii)).

(i) Observe that $(\sigma\tau_p)^2 = \mathrm{Id}$ on $\mathfrak{g}_{+}^{\alpha}$; in other words, $\sigma\tau_p|\mathfrak{g}_{+}^{\alpha}$ is a conjugation of $\mathfrak{g}_{+}^{\alpha}$. In particular, $\mathfrak{g}_{+}^{\alpha}$ admits a $\sigma\tau_p$-invariant basis, whose elements satisfy the condition $\sigma\tau_p Z_\alpha = Z_\alpha$. For each such $Z_\alpha \in \mathfrak{g}_{+}^{\alpha}$,

$$Z[\alpha]_{+} = \mathrm{span}\{Z_\alpha, \tau_p Z_\alpha\}.$$

Since $\sigma H_\alpha = H_{\bar{\alpha}} = H_\alpha$ and $\sigma Q_\alpha = Q_{\bar{\alpha}} = -Q_\alpha$, a σ-invariant basis of $Z[\alpha]_{+}$, which is also τ_p-stable, is given by

$$\{\mathrm{Re}\, H_\alpha = H_\alpha,\ \mathrm{Im}\, Q_\alpha = -JQ_\alpha\}.$$

(ii) On $\mathfrak{g}_{-}^{\alpha}$, one has that $(\sigma\tau_p)^2 = -\mathrm{Id}$. It follows that $\mathfrak{g}_{-}^{\alpha}$ is even dimensional and admits a basis consisting of pairs $\{Z_\alpha, \sigma\tau_p Z_\alpha\}$. For each such $Z_\alpha \in \mathfrak{g}_{-}^{\alpha}$, the space $Z[\alpha]_{-}$ is four dimensional and formulas (3.14) hold.

For the zero root space, one has

$$\mathfrak{g}^{0} = \mathfrak{g}[0] = \mathfrak{m}^{\mathbb{C}} \oplus \mathfrak{c}^{\mathbb{C}} = \mathfrak{g}_{+}^{0} \oplus \mathfrak{g}_{-}^{0}, \qquad \mathfrak{m}^{\mathbb{C}} = \mathfrak{z}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}}).$$

By Lemma 2.10(iii), one has that $\mathfrak{c}^{\mathbb{C}} \subset \mathfrak{g}_{+}^{0}$ and therefore

$$\mathfrak{g}_{+}^{0} = \mathfrak{c}^{\mathbb{C}} \oplus \mathfrak{m}_{+}^{\mathbb{C}} \qquad \text{and} \qquad \mathfrak{g}_{-}^{0} = \mathfrak{m}_{-}^{\mathbb{C}}.$$

LEMMA 3.11. *One has that* $\mathfrak{m}_{-}^{\mathbb{C}} = \{0\}$. *In particular,* $\mathfrak{m}^{\mathbb{C}}$ *is* σ*-stable.*

Proof. Let $M \in \mathfrak{m}_{-}^{\mathbb{C}}$. Since $\tau_p(\sigma(M)) = -\sigma(\tau_p(M)) = -\sigma(M)$, one has that $\sigma(M) \in \mathfrak{c}^{\mathbb{C}}$. On the other hand, $\mathfrak{c}^{\mathbb{C}}$ is σ-stable and consequently $M = \sigma(\sigma(M)) \in \mathfrak{c}^{\mathbb{C}} \cap \mathfrak{m}^{\mathbb{C}} = \{0\}$. It follows that $\mathfrak{m}_{-}^{\mathbb{C}} = \{0\}$ and $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}_{+}^{\mathbb{C}}$. This means that τ_p and σ commute on $\mathfrak{m}^{\mathbb{C}}$. In particular, $\mathfrak{m}^{\mathbb{C}}$ is σ-stable.

Consider the decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}) \oplus \bigoplus_{\alpha, Z_\alpha} Z[\alpha] \cap \mathfrak{g}, \qquad \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{g} \cap \mathfrak{m}^{\mathbb{C}} \oplus \mathfrak{c}.$$

Then a basis of $\mathfrak{g}$ compatible with the above decomposition can be obtained from a σ-invariant basis of $\mathfrak{g}^0$, compatible with the τ_p-decomposition $\mathfrak{g}^0 = \mathfrak{m}^{\mathbb{C}} \oplus \mathfrak{c}^{\mathbb{C}}$, and a σ-invariant basis of each subspace $Z[\alpha]$.

For $\alpha \in \Delta_{\mathfrak{c}}^+$ and $Z_\alpha \in \mathfrak{g}^\alpha$, define

$$\begin{aligned} E_\alpha &:= \mathrm{Ad}_{x_0} H_\alpha = \cos\alpha(X_0) H_\alpha + \sin\alpha(X_0) J Q_\alpha, \\ F_\alpha &:= \mathrm{Ad}_{x_0} Q_\alpha = \cos\alpha(X_0) Q_\alpha + \sin\alpha(X_0) J H_\alpha. \end{aligned} \tag{3.15}$$

Then

$$\{E_\alpha, JE_\alpha, E_{\bar\alpha}, JE_{\bar\alpha}, F_\alpha, JF_\alpha, F_{\bar\alpha}, JF_{\bar\alpha}\}$$

is a real basis of $\mathrm{Ad}_{x_0} Z[\alpha]$ compatible with the decomposition (3.10). For real or imaginary roots, some of the above vectors may coincide (cf. Remark 3.10).

For simplicity, we write

$$\cos\alpha(X_0) = A + iB \qquad \text{and} \qquad \sin\alpha(X_0) = C + iD \tag{3.16}$$

for $A = \mathrm{Re}\cos\alpha(X_0)$, $B = \mathrm{Im}\cos\alpha(X_0)$, $C = \mathrm{Re}\sin\alpha(X_0)$, $D = \mathrm{Im}\sin\alpha(X_0)$. The next proposition is an analogue of Proposition 3.2 and computes the vector fields induced by the G-action on S at a reference point. It is obtained by analyzing the restrictions of the map * to the different components of $\mathfrak{g}$. We write X^* for $X(\overline{x_0 \cdot p})$.

PROPOSITION 3.12. (i) *If* $X \in \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{g}$, *then* $X^* = 0$.

(ii) *If* $X \in \mathfrak{c}$, *then* $X^* = X$.

(iii) *If* $Z_\alpha \in \mathfrak{g}_+^\alpha$ *and* $\{\mathrm{Re}\, H_\alpha, \mathrm{Im}\, H_\alpha, \mathrm{Re}\, Q_\alpha, \mathrm{Im}\, Q_\alpha\}$ *is the basis of* $Z[\alpha]_+ \cap \mathfrak{g}$ *given in* (3.13), *then*

$$\begin{aligned} \mathrm{Re}\, H_\alpha^* &= \frac{1}{2}(DF_\alpha - CJF_\alpha - DF_{\bar\alpha} - CJF_{\bar\alpha}), \\ \mathrm{Im}\, H_\alpha^* &= \frac{1}{2}(-CF_\alpha - DJF_\alpha + CF_{\bar\alpha} - DJF_{\bar\alpha}), \\ \mathrm{Re}\, Q_\alpha^* &= \frac{1}{2}(AF_\alpha + BJF_\alpha + AF_{\bar\alpha} - BJF_{\bar\alpha}), \\ \mathrm{Im}\, Q_\alpha^* &= \frac{1}{2}(BF_\alpha - AJF_\alpha + BF_{\bar\alpha} + AJF_{\bar\alpha}). \end{aligned} \tag{3.17}$$

(iv) *If* $Z_\alpha \in \mathfrak{g}_-^\alpha$ *and* $\{\operatorname{Re} H_\alpha, \operatorname{Im} H_\alpha, \operatorname{Re} Q_\alpha, \operatorname{Im} Q_\alpha\}$ *is the basis of* $Z[\alpha]_- \cap \mathfrak{g}$ *given in* (3.14), *then*

$$\begin{aligned} \operatorname{Re} H_\alpha^* &= \frac{1}{2}(DF_\alpha - CJF_\alpha + AF_{\bar\alpha} - BJF_{\bar\alpha}), \\ \operatorname{Im} H_\alpha^* &= \frac{1}{2}(-CF_\alpha - DJF_\alpha + BF_{\bar\alpha} + AJF_{\bar\alpha}), \\ \operatorname{Re} Q_\alpha^* &= \frac{1}{2}(AF_\alpha + BJF_\alpha - DF_{\bar\alpha} - CJF_{\bar\alpha}), \\ \operatorname{Im} Q_\alpha^* &= \frac{1}{2}(BF_\alpha - AJF_\alpha + CF_{\bar\alpha} - DJF_{\bar\alpha}). \end{aligned} \tag{3.18}$$

Proof. The proof consists of long but straightforward computations and it is omitted.

Remark 3.13. For simplicity, we have stated the result for four-dimensional spaces $Z[\alpha]$. Formulas (3.17) and (3.18) simplify in the cases of real and imaginary roots discussed in Remark 3.10 (2)(i), (2)(ii), (3)(i) and respectively become

$$H_\alpha^* = \operatorname{Re} H_\alpha^* = -\sin\alpha(X_0)JF_\alpha, \qquad Q_\alpha^* = \operatorname{Re} Q_\alpha^* = \cos\alpha(X_0)F_\alpha,$$

$$\operatorname{Re} H_\alpha^* = \frac{\cos\alpha(X_0)}{2}F_\alpha - \frac{\sin\alpha(X_0)}{2}JF_\alpha,$$

$$\operatorname{Im} H_\alpha^* = \frac{-\sin\alpha(X_0)}{2}F_\alpha + \frac{\cos\alpha(X_0)}{2}JF_\alpha,$$

$$\operatorname{Re} H_\alpha^* = -\sin\alpha(X_0)JF_\alpha, \qquad \operatorname{Im} Q_\alpha^* = -\cos\alpha(X_0)JF_\alpha.$$

Define

$$\mathfrak{s} := \bigoplus_{\alpha,\, Z_\alpha} Z[\alpha] \cap \mathfrak{g}$$

and consider the map

$$*\colon \mathfrak{c} \oplus \mathfrak{s} \longrightarrow \mathfrak{c} \oplus \bigoplus_{\alpha,\, Z_\alpha} \operatorname{Ad}_{x_0} Z[\alpha]_{\mathfrak{g}^{\mathbb{C}}}, \qquad X \longmapsto X^*(\overline{x_0 \cdot p}). \tag{3.19}$$

PROPOSITION 3.14. (i) *If the base point* p *satisfies conditions* (2.6), *the map* (3.19) *is an isomorphism if and only if* x_0 *satisfies the conditions*

$$\sin 2\alpha(X_0) \neq 0, \qquad \forall\, \alpha \in \Delta_{\mathfrak{c}}.$$

(ii) *If the base point* p *satisfies conditions* (2.7), *the map* (3.19) *is an isomorphism if and only if* x_0 *satisfies the conditions*

$$\begin{aligned} \sin 2\alpha(X_0) &\neq 0, \qquad \forall\, \alpha \in \Delta_{\mathfrak{c}} \text{ such that } \mathfrak{g}_-^\alpha = \{0\}, \\ \cos 2\alpha(X_0) &\neq 0, \qquad \forall\, \alpha \in \Delta_{\mathfrak{c}} \text{ such that } \mathfrak{g}_+^\alpha = \{0\}, \\ \cos 2\alpha(X_0)\sin 2\alpha(X_0) &\neq 0, \qquad \forall\, \alpha \in \Delta_{\mathfrak{c}} \text{ such that } \mathfrak{g}_+^\alpha, \mathfrak{g}_-^\alpha \neq \{0\}. \end{aligned}$$

Proof. By Proposition 3.12, the map $*$ is the identity on $\mathfrak{c}$. On each four-dimensional subspace subspace $Z[\alpha] \cap \mathfrak{g}$ the restriction

$$*|Z[\alpha] \cap \mathfrak{g}\colon Z[\alpha] \cap \mathfrak{g} \longrightarrow \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}},$$

with respect to the real bases

$$\{\mathrm{Re}\, H_\alpha, \mathrm{Im}\, H_\alpha, \mathrm{Re}\, Q_\alpha, \mathrm{Im}\, Q_\alpha\} \qquad \text{and} \qquad \{F_\alpha, JF_\alpha, F_{\bar{\alpha}}, JF_{\bar{\alpha}}\},$$

is given by one of the following matrices

$$M^+ = \begin{pmatrix} D & -C & -D & -C \\ -C & -D & C & -D \\ A & B & A & -B \\ B & -A & B & A \end{pmatrix}, \qquad Z_\alpha \in \mathfrak{g}_+^\alpha,$$

$$M^- = \begin{pmatrix} D & -C & A & -B \\ -C & -D & B & A \\ A & B & -D & -C \\ B & -A & C & -D \end{pmatrix}, \qquad Z_\alpha \in \mathfrak{g}_-^\alpha.$$

Computing the determinants of these matrices and substituting relations (3.16), one gets

$$\det M^+ = \frac{1}{4} |\sin 2\alpha(X_0)|^2 \qquad \text{and} \qquad \det M^- = |\cos 2\alpha(X_0)|^2.$$

The same formulas hold as well in the special cases of real and imaginary roots discussed in Remarks 3.10 and 3.13. Hence the statement follows.

PROPOSITION 3.15. *Let $x_0 \cdot p = e^{JX_0} \cdot p \in \exp J\mathfrak{c} \cdot p$. The point $\overline{x_0 \cdot p} \in G^{\mathbb{C}}/K^{\mathbb{C}}$ lies on a generic G-orbit if and only if the map* (3.19) *is an isomorphism. Then the isotropy subgroup is given by*

$$G_{\overline{x_0 \cdot p}} = G \cap Z_{H^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}}), \qquad \text{where } H^{\mathbb{C}} = \mathrm{Ad}_p K^{\mathbb{C}}.$$

Proof. The isotropy subalgebra of $\overline{x_0 \cdot p}$ in $\mathfrak{g}$ is given by

$$\mathfrak{g}_{\overline{x_0 \cdot p}} = \mathfrak{g} \cap \mathrm{Ad}_{x_0} \mathfrak{h}^{\mathbb{C}} = \mathrm{Ker}(*\colon \mathfrak{g} \longrightarrow \mathrm{Ad}_{x_0} \mathfrak{q}^{\mathbb{C}}),$$

where

$$\mathrm{Ad}_{x_0} \mathfrak{h}^{\mathbb{C}} = \mathfrak{z}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}}) \oplus \bigoplus_{\alpha, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{h}^{\mathbb{C}}}.$$

By the results of Proposition 3.12 and 3.14, the kernel of the map $*$ has minimal dimension when the map (3.19) is an isomorphism. In this case, the isotropy subalgebra is given by $\mathfrak{g}_{\overline{x_0 \cdot p}} = \mathfrak{g} \cap \mathfrak{z}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}})$ and the isotropy subgroup is given by $G_{\overline{x_0 \cdot p}} = G \cap Z_{H^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}})$.

Remark 3.16. (i) By the above proposition, closed generic orbits intersecting $\overline{C}$ are all of the same type. By Lemma 3.11, the group $Z_{H^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}})$ is σ-stable and $\dim G \cap Z_{H^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}}) = \dim Z_K(\mathfrak{a})$, as expected. Closed generic orbits in $G \cdot \overline{C}$ form an open sense subset of $G \cdot \overline{C}$. By proposition 3.14(i)–(ii), they are parametrized by the complement in $\mathfrak{c}$ of the set

$$\bigcup_{\substack{\alpha \in \Delta_{\mathfrak{c}}^{\mathfrak{r}} \\ \mathfrak{g}_-^{\alpha} = \{0\}}} \{\alpha(X) \equiv 0 \bmod \pi/2\} \bigcup_{\substack{\alpha \in \Delta_{\mathfrak{c}}^{\mathfrak{r}} \\ \mathfrak{g}_+^{\alpha} = \{0\}}} \{\alpha(X) \equiv \pi/4 \bmod \pi/2\}$$

$$\bigcup_{\substack{\alpha \in \Delta_{\mathfrak{c}}^{\mathfrak{r}} \\ \mathfrak{g}_-^{\alpha},\, \mathfrak{g}_+^{\alpha} \neq \{0\}}} \{\alpha(X) \equiv 0 \bmod \pi/4\} \bigcup_{\substack{\alpha \in \Delta_{\mathfrak{c}}^{\mathfrak{i}} \\ \mathfrak{g}_+^{\alpha} \neq \{0\}}} \{\alpha(X) = 0\}$$

$$\bigcup_{\substack{\alpha \in \Delta_{\mathfrak{c}}^{\mathfrak{c}} \\ \mathfrak{g}_-^{\alpha} = \{0\}}} \left\{ \begin{cases} \operatorname{Im} \alpha(X) = 0 \\ \operatorname{Re} \alpha(X) \equiv 0 \bmod \pi/2 \end{cases} \right\}$$

$$\bigcup_{\substack{\alpha \in \Delta_{\mathfrak{c}}^{\mathfrak{c}} \\ \mathfrak{g}_+^{\alpha} = \{0\}}} \left\{ \begin{cases} \operatorname{Im} \alpha(X) = 0 \\ \operatorname{Re} \alpha(X) \equiv \pi/4 \bmod \pi/2 \end{cases} \right\}$$

$$\bigcup_{\substack{\alpha \in \Delta_{\mathfrak{c}}^{\mathfrak{c}} \\ \mathfrak{g}_-^{\alpha},\, \mathfrak{g}_+^{\alpha} \neq \{0\}}} \left\{ \begin{cases} \operatorname{Im} \alpha(X) = 0 \\ \operatorname{Re} \alpha(X) \equiv 0 \bmod \pi/4 \end{cases} \right\},$$

modulo the action of the Weyl group. Observe that only the hyperplanes in $\mathfrak{c}$ defined corresponding to real or imaginary roots in $\Delta_{\mathfrak{c}}$, disconnect the above set.

(ii) A point $\overline{x_0 \cdot p} \in \overline{C}$ sits on a generic orbit if and only if $x_0 \cdot p \in G^{\mathbb{C}}_{\mathrm{rs},\,\sigma,\,\tau}$ (cf. Definition 2.2). In this case,

$$(\mathfrak{g}^{\mathbb{C}})^{-\sigma} \cap (\mathfrak{g}^{\mathbb{C}})^{-\tau_{x_0 p}} = J\mathfrak{g} \cap \operatorname{Ad}_{x_0} \mathfrak{q}^{\mathbb{C}} = J\mathfrak{c}.$$

The proof is similar to that of Remark 3.6 and follows from Proposition 3.14 and the decomposition

$$\operatorname{Ad}_{x_0} \mathfrak{q}^{\mathbb{C}} = \mathfrak{c}^{\mathbb{C}} \oplus \bigoplus_{\alpha,\, Z_\alpha} \operatorname{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}}.$$

(iii) *Properness of action.* There are cases when the G-action is proper also on some subset of $G^{\mathbb{C}}/K^{\mathbb{C}}$ outside $\overline{\mathbf{X}}_0$ (cf. Remark 3.5). This can be checked directly in Example 6.8. In general, this happens when G is a Hermitian Lie group and $G^{\mathbb{C}}/K^{\mathbb{C}}$ is also the complexification of a compactly causal symmetric space of G (cf. Section 6).

COROLLARY 3.17. *The tangent space and the complex tangent space to a generic orbit S at* $\overline{x_0 \cdot p}$ *are given by*

$$TS_{\overline{x_0\cdot p}} = \mathfrak{c} \oplus \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}} \qquad \textit{and} \qquad T_{\mathbb{C}}S_{\overline{x_0\cdot p}} = \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}}.$$

The orbit S is also generic as a CR-manifold (*cf. Definition* 1.1).

Remark 3.18. Later on, in the computation of the Levi form of a generic orbit S (cf. Lemma 4.3 and Propositions 5.14–5.16), we need the inverses of relations (3.17) and (3.18), namely for $Z_\alpha \in \mathfrak{g}_+^\alpha$,

$$\begin{aligned}
(*)^{-1}F_\alpha &= \frac{1}{2}\left(\frac{1}{\cos\alpha}Q_\alpha + \frac{1}{\cos\bar\alpha}Q_{\bar\alpha} + \frac{1}{\sin\alpha}JH_\alpha - \frac{1}{\sin\bar\alpha}JH_{\bar\alpha}\right),\\
(*)^{-1}F_{\bar\alpha} &= \frac{1}{2}\left(\frac{1}{\cos\alpha}Q_\alpha + \frac{1}{\cos\bar\alpha}Q_{\bar\alpha} - \frac{1}{\sin\alpha}JH_\alpha + \frac{1}{\sin\bar\alpha}JH_{\bar\alpha}\right),\\
(*)^{-1}JF_\alpha &= \frac{1}{2}\left(\frac{1}{\cos\alpha}JQ_\alpha - \frac{1}{\cos\bar\alpha}JQ_{\bar\alpha} - \frac{1}{\sin\alpha}H_\alpha - \frac{1}{\sin\bar\alpha}H_{\bar\alpha}\right),\\
(*)^{-1}JF_{\bar\alpha} &= \frac{1}{2}\left(-\frac{1}{\cos\alpha}JQ_\alpha + \frac{1}{\cos\bar\alpha}JQ_{\bar\alpha} - \frac{1}{\sin\alpha}H_\alpha - \frac{1}{\sin\bar\alpha}H_{\bar\alpha}\right),
\end{aligned} \tag{3.20}$$

and for $Z_\alpha \in \mathfrak{g}_-^\alpha$,

$$\begin{aligned}
(*)^{-1}F_\alpha &= \frac{\cos\alpha}{\cos 2\alpha}Q_\alpha + \frac{\sin\bar\alpha}{\cos 2\bar\alpha}JQ_{\bar\alpha} - \frac{\sin\alpha}{\cos 2\alpha}JH_\alpha + \frac{\cos\bar\alpha}{\cos 2\bar\alpha}H_{\bar\alpha},\\
(*)^{-1}F_{\bar\alpha} &= \frac{\sin\alpha}{\cos 2\alpha}JQ_\alpha + \frac{\cos\bar\alpha}{\cos 2\bar\alpha}Q_{\bar\alpha} + \frac{\cos\alpha}{\cos 2\alpha}H_\alpha - \frac{\sin\bar\alpha}{\cos 2\bar\alpha}JH_{\bar\alpha},\\
(*)^{-1}JF_\alpha &= \frac{\cos\alpha}{\cos 2\alpha}JQ_\alpha + \frac{\sin\bar\alpha}{\cos 2\bar\alpha}Q_{\bar\alpha} + \frac{\sin\alpha}{\cos 2\alpha}H_\alpha - \frac{\cos\bar\alpha}{\cos 2\bar\alpha}JH_{\bar\alpha},\\
(*)^{-1}JF_{\bar\alpha} &= \frac{\sin\alpha}{\cos 2\alpha}Q_\alpha + \frac{\cos\bar\alpha}{\cos 2\bar\alpha}JQ_{\bar\alpha} - \frac{\cos\alpha}{\cos 2\alpha}JH_\alpha + \frac{\sin\bar\alpha}{\cos 2\bar\alpha}H_{\bar\alpha}.
\end{aligned} \tag{3.21}$$

Let $\alpha \in \Delta_{\mathfrak{c}}^r$. For $Z_\alpha \in \mathfrak{g}_+^\alpha$, satisfying $\sigma Z_\alpha = Z_\alpha$, one has

$$(*)^{-1}F_\alpha = \frac{1}{\cos\alpha}Q_\alpha, \qquad (*)^{-1}JF_\alpha = -\frac{1}{\sin\alpha}H_\alpha.$$

For $Z_\alpha \in \mathfrak{g}_-^\alpha$, satisfying $\sigma Z_\alpha = Z_\alpha$, one has

$$\begin{aligned}
(*)^{-1}F_\alpha &= \frac{\cos\alpha Q_\alpha + \sin\alpha\, JQ_\alpha}{\cos 2\alpha} + \frac{\cos\alpha H_\alpha - \sin\alpha\, JH_\alpha}{\cos 2\alpha},\\
(*)^{-1}JF_\alpha &= \frac{\sin\alpha Q_\alpha + \cos\alpha\, JQ_\alpha}{\cos 2\alpha}Q_\alpha + \frac{\sin\alpha H_\alpha - \cos\alpha\, JH_\alpha}{\cos 2\alpha}.
\end{aligned}$$

Let $\alpha \in \Delta_{\mathfrak{c}}^i$. If $Z_\alpha \in \mathfrak{g}_+^\alpha$ satisfies $\sigma\tau_p Z_\alpha = Z_\alpha$, then

$$(*)^{-1}F_\alpha = \frac{1}{\sin\alpha}JH_\alpha, \qquad (*)^{-1}JF_\alpha = \frac{1}{\cos\alpha}JQ_\alpha.$$

If $Z_\alpha \in \mathfrak{g}_-^\alpha$, then $Z[\alpha]$ is four dimensional and formulas (3.21) become

$$(*)^{-1}F_\alpha = \frac{\cos\alpha}{\cos 2\alpha}(Q_\alpha + H_{\bar\alpha}) - \frac{\sin\alpha}{\cos 2\alpha}J(H_\alpha + Q_{\bar\alpha}),$$

$$(*)^{-1}F_{\bar\alpha} = \frac{\sin\alpha}{\cos 2\alpha}J(Q_\alpha + H_{\bar\alpha}) + \frac{\cos\alpha}{\cos 2\alpha}(H_\alpha + Q_{\bar\alpha}),$$

$$(*)^{-1}JF_\alpha = \frac{\cos\alpha}{\cos 2\alpha}J(Q_\alpha - H_{\bar\alpha}) + \frac{\sin\alpha}{\cos 2\alpha}(H_\alpha - Q_{\bar\alpha}),$$

$$(*)^{-1}JF_{\bar\alpha} = \frac{\sin\alpha}{\cos 2\alpha}(Q_\alpha - H_{\bar\alpha}) + \frac{\cos\bar\alpha}{\cos 2\bar\alpha}J(Q_{\bar\alpha} - H_\alpha).$$

4. THE LEVI FORM OF A GENERIC ORBIT: GENERAL FORMULAS

In this section, we establish the general formulas for the Levi form of a generic orbit S at a base point $\bar x_0$. Given arbitrary tangent vectors $Z, W \in T_{\mathbb{C}}S_{\bar x_0}$, it is necessary to extend them to local sections of the subbundle $T_{\mathbb{C}}S$ of the tangent bundle TS and to compute their brackets at $\bar x_0$ (see Definition 1.2).

4.1. *Extending Vector Fields*

Let $C = \exp J\mathfrak{c} \cdot p$ be a Cartan subset. In particular, if $p = e$ and $\mathfrak{c} = \mathfrak{a}$, then C is the fundamental Cartan subset. Let $x_0 \cdot p = \exp JX_0 \cdot p$ be a point in C. Let $\overline{x_0 \cdot p}$ be the corresponding point in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and S the G-orbit of $\overline{x_0 \cdot p}$. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{q}^{\mathbb{C}}$ be the decomposition of $\mathfrak{g}^{\mathbb{C}}$ induced by the involution τ_p. The complex tangent space to S at $\overline{x_0 \cdot p}$ is given by

$$T_{\mathbb{C}}(S)_{\overline{x_0\cdot p}} = \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}} \subset \mathrm{Ad}_{x_0}\mathfrak{q}^{\mathbb{C}}.$$

Let $Z \in T_{\mathbb{C}}S_{\overline{x_0\cdot p}}$. We need to extend Z to a local vector field in a neighborhood of $\overline{x_0 \cdot p}$ in S. If the G-orbit of $\overline{x_0 \cdot p}$ is generic, a neighborhood $U_{\overline{x_0\cdot p}}$ of $\overline{x_0 \cdot p}$ in S can be parametrized by a suitable neighborhood V_0 of zero in

$$\mathfrak{c} \oplus \hat{\mathfrak{s}} \subset \mathfrak{g}, \quad \hat{\mathfrak{s}} := \bigoplus_{\alpha, Z_\alpha} Z[\alpha] \cap \mathfrak{g}, \quad Z[\alpha] = \mathrm{span}\{\mathrm{Re}\, H_\alpha, \mathrm{Im}\, H_\alpha, \mathrm{Re}\, Q_\alpha, \mathrm{Im}\, Q_\alpha\}$$

(cf. Propositions 3.3 and 3.14), via the map

$$V_0 \longrightarrow U_{\overline{x_0\cdot p}} = \exp V_0 \cdot \overline{x_0 \cdot p}, \qquad X \longmapsto \bar x = \exp X \cdot \overline{x_0 \cdot p}.$$

Here the vector $X = X(\bar x) \in V_0$ and the group element $\gamma = \gamma(\bar x) = \exp X(\bar x) \in G$, satisfying

$$\bar x = \exp X \cdot \overline{x_0 \cdot p} = \gamma \cdot \overline{x_0 \cdot p},$$

are uniquely determined by the point $\bar x \in U_{\overline{x_0\cdot p}}$.

LEMMA 4.1. *A vector field on $U_{\overline{x_0 \cdot p}}$ extending Z can be defined by*

$$\widehat{Z}(\bar{x}) := (dL_\gamma)_{\overline{x_0 \cdot p}} Z = \mathrm{Ad}_\gamma Z.$$

Proof. The above extension is well defined. If $\bar{x} \in U_{\overline{x_0 \cdot p}}$, then $\widehat{Z}(\bar{x})$ is a local section of $T_{\mathbb{C}} S_{\bar{x}}$. In fact,

$$(dL_\gamma)_{\overline{x_0 \cdot p}} \colon TS_{\overline{x_0 \cdot p}} \longrightarrow TS_{\bar{x}} \qquad \text{and} \qquad (dL_\gamma)_{\overline{x_0 \cdot p}} \colon T_{\mathbb{C}} S_{\overline{x_0 \cdot p}} \longrightarrow T_{\mathbb{C}} S_{\bar{x}}.$$

It follows that $\widehat{Z}(\bar{x}) \in T_{\mathbb{C}} S_{\bar{x}}$.

4.2. *The Calculation of the Brackets*

Let $Z, W \in T_{\mathbb{C}} S_{\overline{x_0 \cdot p}}$ and let $\widehat{Z}, \widehat{W}$ be the extensions defined in Lemma 4.1. In order to calculate the brackets $[\widehat{Z}, \widehat{W}]$, choose a complex basis of $\mathfrak{g}^{\mathbb{C}}$ compatible with the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{q}^{\mathbb{C}},$$

namely

$$\{\{H_j\}, \{M_l\}, \{E_\alpha^i\}, \{F_\alpha^i\}\},$$

where $\{H_j\}_{j=1,\dots,\dim \mathfrak{c}^{\mathbb{C}}}$ is an orthonormal basis of $\mathfrak{c}_{\mathbb{R}} = \{H \in \mathfrak{c}^{\mathbb{C}} \mid \alpha(H) \in \mathbb{R},\ \forall \alpha \in \Delta_{\mathfrak{c}}\}$, $\{M_l\}_{l=1,\dots,\dim \mathfrak{m}^{\mathbb{C}}}$ is a basis of $\mathfrak{m}^{\mathbb{C}}$, and, for $\alpha \in \Delta_{\mathfrak{c}}^+$, $\{F_\alpha^i\}_{i=1,\dots,\dim \mathfrak{g}^\alpha}$ and $\{E_\alpha^i\}_{i=1,\dots,\dim \mathfrak{g}^\alpha}$ are bases of $\mathrm{Ad}_{x_0} \mathfrak{g}[\alpha]_{\mathfrak{q}^{\mathbb{C}}}$ and $\mathrm{Ad}_{x_0} \mathfrak{g}[\alpha]_{\mathfrak{h}^{\mathbb{C}}}$, respectively (cf. Sections 3.2 and 3.3). Write

$$\widehat{Z}(\bar{x}) = m_l(\bar{x}) M_l + h_j(\bar{x}) H_j + e_\alpha^i(\bar{x}) E_\alpha^i + f_\alpha^i(\bar{x}) F_\alpha^i$$

and

$$\widehat{W}(\bar{x}) = m_l'(\bar{x}) M_l + h_j'(\bar{x}) H_j + (e_\alpha^i)'(\bar{x}) E_\alpha^i + (f_\alpha^i)'(\bar{x}) F_\alpha^i,$$

with the summation convention. When $\bar{x}$ varies in a neighborhood of $\overline{x_0 \cdot p}$ in S, the vector fields $\widehat{Z}, \widehat{W}$ are vectors in $\mathfrak{g}^{\mathbb{C}}$ whose coefficients are complex-valued functions of $\bar{x}$. Since $\widehat{Z}(\overline{x_0 \cdot p}), \widehat{W}(\overline{x_0 \cdot p}) \in \mathrm{span}\{F_\alpha^i\}_{i,\alpha}$, the coefficients satisfy the relations

$$m_i(\overline{x_0 \cdot p}) = m_i'(\overline{x_0 \cdot p}) = h_j(\overline{x_0 \cdot p}) = h_j'(\overline{x_0 \cdot p}) = e_\alpha(\overline{x_0 \cdot p}) = e_\alpha'(\overline{x_0 \cdot p}) = 0.$$

Calculating the brackets by the formula $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$ and observing that brackets of tangent vector fields are tangent vector fields, one obtains

$$[\widehat{Z}, \widehat{W}]_{\overline{x_0 \cdot p}} = \sum_j \big(Z(h_j') - W(h_j)\big) H_j \qquad \mathrm{mod}\ T_{\mathbb{C}} S_{\overline{x_0 \cdot p}}.$$

It remains to calculate the first derivatives $Z(h_j')$ and $W(h_j)$ of the coefficient functions h_j, h_j' with respect to the tangent vectors Z and W at $\overline{x_0 \cdot p}$. Let B denote the Killing form of $\mathfrak{g}^{\mathbb{C}}$.

LEMMA 4.2. *One has*

$$Z(h'_j) = B([(*)^{-1}Z, W], H_j), \qquad W(h_j) = B([(*)^{-1}W, Z], H_j),$$

where $(*)^{-1}\colon TS_{\overline{x_0\cdot p}} \to \mathfrak{c} \oplus \mathfrak{s}$ *is the inverse of the map defined in* (3.5) *and* (3.19).

Proof. Consider the curve $c(t) := \exp t(*)^{-1}Z \cdot \overline{x_0 \cdot p}$. Since $(*)^{-1}Z$ belongs to $\mathfrak{c} \oplus \mathfrak{s} \subset \mathfrak{g}$, the curve $c(t)$ is all contained in the G-orbit of $\overline{x_0 \cdot p}$, has initial point $c(0) = \overline{x_0 \cdot p}$, and has initial tangent vector $c'(0) = Z$. In this way,

$$Z(h'_j) = \frac{d}{dt}\bigg|_{t=0} h'_j\big(\exp\big(t(*)^{-1}Z\big) \cdot \overline{x_0 \cdot p}\big). \tag{4.1}$$

The functions $h'_j(\bar{x})$ can be expressed as

$$h'_j(\bar{x}) = B(\widehat{W}(\bar{x}), H_j) = B(\mathrm{Ad}_{\gamma(\bar{x})}W, H_j).$$

If $\bar{x} = \exp\big((*)^{-1}Z\big) \cdot \overline{x_0 \cdot p}$ for some $Z \in T_{\mathbb{C}}S_{\overline{x_0\cdot p}}$, then

$$\gamma\big(\exp\big((*)^{-1}Z\big) \cdot \overline{x_0 \cdot p}\big) := \exp\big((*)^{-1}Z\big).$$

If follows that

$$\begin{aligned} Z(h'_j) &= \frac{d}{dt}\bigg|_{t=0} h'_j\big(\exp\big(t(*)^{-1}Z\big) \cdot \overline{x_0 \cdot p}\big) = B\left(\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp((*)^{-1}tZ)}W, H_j\right) \\ &= B([(*)^{-1}Z, W], H_j). \end{aligned}$$

The second identity is obtained in a similar way.

The following bracket formulas are now immediate:

$$\left.\begin{aligned} [\widehat{Z}, \widehat{W}]_{\overline{x_0\cdot p}} &= [(*)^{-1}Z, W] - [(*)^{-1}W, Z] \\ [\widehat{Z}, \widehat{JW}]_{\overline{x_0\cdot p}} &= [(*)^{-1}Z, JW] - [(*)^{-1}JW, Z] \end{aligned}\right\} \mod T_{\mathbb{C}}S_{\overline{x_0\cdot p}}. \tag{4.2}$$

4.3. *The Levi Form*: *General Formulas*

Recall that

$$T_{\mathbb{C}}S_{\overline{x_0\cdot p}} = \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}}$$

(cf. Sections 3.2 and 3.3) and observe that the map $X \otimes 1 + Y \otimes i \mapsto X + JY$ provides the identification

$$(TS_{\overline{x_0\cdot p}})^{\mathbb{C}}/(T_{\mathbb{C}}S_{\overline{x_0\cdot p}})^{\mathbb{C}} \cong \mathfrak{c}^{\mathbb{C}} = \mathfrak{c} \oplus J\mathfrak{c}.$$

In particular, the quadratic Levi form $\mathbf{L}(Z, Z)$ is valued in $\mathfrak{c}$ and is given by

$$\mathbf{L}(Z, Z) = -\frac{1}{2}[\widehat{Z}, \widehat{JZ}]_{x_0} \qquad \mod T_{\mathbb{C}}S_{\overline{x_0\cdot p}}.$$

Denote by $(*)^{-1}$ the inverse of the map $*$ defined in (3.5) and (3.19).

LEMMA 4.3. *Let* $Z, W \in T_{\mathbb{C}} S_{\overline{x_0 \cdot p}}$. *Then*

$$\mathbf{L}(Z, W) = \frac{1}{2}[(*)^{-1}JW, Z] - \frac{i}{2}[(*)^{-1}W, Z] \qquad \text{mod } T_{\mathbb{C}}S_{\overline{x_0 \cdot p}},$$

$$\mathbf{L}(Z, Z) = \frac{1}{2}[(*)^{-1}JZ, Z] - \frac{i}{2}[(*)^{-1}Z, Z] \qquad \text{mod } T_{\mathbb{C}}S_{\overline{x_0 \cdot p}}.$$

Proof. The above formulas follow directly from the bracket formulas (4.2).

5. CALCULATION OF THE LEVI FORM AND THE LEVI CONE OF A GENERIC ORBIT

5.1. *The Calculation in* $\overline{X}_0$

In this section, we compute the Levi form and the Levi cone of the generic orbits intersecting $\overline{A}$, where $A := \exp J\mathfrak{a}$ is the fundamental Cartan subset. We resume the notation introduced in Section 3.2

Recall that by Corollary 3.7 the complex tangent space to a generic orbit S at a base point $\bar{x}_0 \in \overline{A}$, where $x_0 = e^{JH_0}$, $H_0 \in \mathfrak{a}$, is given by

$$T_{\mathbb{C}}S_{\bar{x}_0} = \bigoplus_{\alpha \in \Delta_{\mathfrak{a}}^+} \mathrm{Ad}_{x_0}\mathfrak{g}[\alpha]_{\mathfrak{p}}^{\mathbb{C}}, \qquad \text{where } \mathrm{Ad}_{x_0}\mathfrak{g}[\alpha]_{\mathfrak{p}}^{\mathbb{C}} = \bigoplus_{Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{p}}^{\mathbb{C}} \tag{5.1}$$

and Z_α runs through a basis of $\mathfrak{g}^\alpha$ for every $\alpha \in \Delta_{\mathfrak{a}}^+$.

In order to compute the Levi form by the formulas developed in Section 4, we explicitly construct a basis of $T_{\mathbb{C}}S_{\bar{x}_0}$ where the calculations turn out particularly simple. Such a basis depends on the choice of a convenient basis $\mathcal{B}_\alpha = \{Z_\alpha\}$ for each restricted root space $\mathfrak{g}^\alpha \subset \mathfrak{g}$.

Extend $\mathfrak{a}$ to a θ-invariant Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ of $\mathfrak{g}$, with $\mathfrak{t} \subset \mathfrak{k}$. Let Δ be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ and let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}^\lambda$$

be the corresponding root decomposition. Since $\mathfrak{h}$ is both θ- and σ-invariant, there are induced actions of θ and σ on Δ, defined by

$$\theta\lambda(H) := \lambda(\theta(H)), \qquad \bar{\lambda}(H) := \overline{\lambda(\sigma(H))}, \qquad H \in \mathfrak{h}^{\mathbb{C}}.$$

One has that

$$\theta\bar{\lambda} = \overline{\theta\lambda} \qquad \text{and} \qquad \theta\bar{\lambda} = -\lambda.$$

Observe that, if λ is a root in Δ, then $\lambda + \theta\lambda$ is not a root (cf. [He, p. 530]). For $\alpha \in \Delta_{\mathfrak{a}}$, define

$$\Delta_\alpha := \{\lambda \in \Delta | \lambda_{|\mathfrak{a}} = \alpha\}.$$

Then

$$\lambda \in \Delta_\alpha \Longrightarrow \bar{\lambda} \in \Delta_\alpha, \qquad \theta\lambda, \theta\bar{\lambda} \in \Delta_{-\alpha}.$$

If $0 \neq \alpha \in \Delta_{\mathfrak{a}}$, the complexified restricted root space $(\mathfrak{g}^\alpha)^{\mathbb{C}}$ decomposes as

$$(\mathfrak{g}^\alpha)^{\mathbb{C}} = \bigoplus_{\substack{\lambda \in \Delta_\alpha \\ \lambda \neq \bar{\lambda}}} \mathfrak{g}^\lambda \oplus \mathfrak{g}^{\bar{\lambda}} \qquad \text{or} \qquad (\mathfrak{g}^\alpha)^{\mathbb{C}} = \bigoplus_{\substack{\lambda \in \Delta_\alpha \\ \lambda \neq \bar{\lambda}}} \mathfrak{g}^\lambda \oplus \mathfrak{g}^{\bar{\lambda}} \oplus \mathfrak{g}^\mu, \quad \mu \in \Delta_\alpha, \ \mu = \bar{\mu},$$

where the root spaces on the right-hand side are ordinary root spaces in $\mathfrak{g}^{\mathbb{C}}$. (The roots $\lambda \in \Delta_\alpha$ are chosen so that each summand appears precisely once in the above decomposition.) For the complexified restricted zero-root space, one has $(\mathfrak{g}^0)^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$, where $\mathfrak{m}^{\mathbb{C}} = \mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$. Observe that $\mathfrak{m}^{\mathbb{C}}$ is a reductive subalgebra containing the ordinary root spaces $\mathfrak{g}^\lambda$, with $\lambda|\mathfrak{a}^{\mathbb{C}} \equiv 0$.

Starting from the above decomposition, for each $\alpha \in \Delta_{\mathfrak{a}}$, we construct a basis $\mathscr{B}_\alpha$ of the restricted root space $\mathfrak{g}^\alpha \subset \mathfrak{g}$. In order to do this, fix a set of root vectors $\{W_\lambda\}_{\lambda \in \Delta}$, with $W_\lambda \in \mathfrak{g}^\lambda$, normalized as follows

$$B(W_\lambda, W_{-\lambda}) = 1, \qquad B(W_\lambda, W_\mu) = 0, \qquad \lambda + \mu \neq 0.$$

(Here B denotes the Killing form of $\mathfrak{g}^{\mathbb{C}}$.) In addition, if $\lambda = \bar{\lambda}$ is a real root, we also assume $\sigma W_\lambda = W_\lambda$. Then

$$\mathscr{B}_\alpha = \left\{ \{W_\lambda + \sigma W_\lambda, -J(W_\lambda - \sigma W_\lambda)\}_{\substack{\lambda \in \Delta_\alpha \\ \lambda \neq \bar{\lambda}}}, \{W_\lambda\}_{\substack{\lambda \in \Delta_\alpha \\ \lambda = \bar{\lambda}}} \right\} \tag{5.2}$$

is a σ-invariant basis of $(\mathfrak{g}^\alpha)^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ and hence a basis of $\mathfrak{g}^\alpha \subset \mathfrak{g}$.

For $\lambda \in \Delta$, let $h_\lambda \in \mathfrak{h}_{\mathbb{R}} = \{H \in \mathfrak{h}^{\mathbb{C}} \mid \lambda(H) \in \mathbb{R}, \forall\, \lambda \in \Delta\}$ be the vector defined by

$$\lambda(H) = B(H, h_\lambda), \qquad H \in \mathfrak{h}^{\mathbb{C}},$$

and for $\alpha \in \Delta_{\mathfrak{a}}$, let $h_\alpha \in \mathfrak{a}$ be the vector defined by

$$\alpha(H) = B(H, h_\alpha), \qquad H \in \mathfrak{a}^{\mathbb{C}}.$$

Then $h_\alpha = \frac{1}{2}(h_\lambda - \theta h_\lambda)$.

LEMMA 5.1. *Let* $\alpha \in \Delta_{\mathfrak{a}}$ *and let* $\mathscr{B}_\alpha$ *be the basis of* $\mathfrak{g}^\alpha$ *defined in* (5.2). *The following relations hold*:

(i) *Let* $Z_\alpha \in \mathfrak{g}^\alpha$. *Then* $[Z_\alpha, \theta Z_\alpha] = B(Z_\alpha, \theta Z_\alpha) h_\alpha \in \mathfrak{a}$, *where* $B(Z_\alpha, \theta Z_\alpha)$ *is a negative real constant*.

(ii) *For* $Z_\alpha \neq Z'_\alpha \in \mathscr{B}_\alpha$, *one has that* $[Z_\alpha, \theta Z'_\alpha] \in \mathfrak{k}^{\mathbb{C}}$.

Proof. (i) If $Z_\alpha \in \mathfrak{g}^\alpha$, then $\theta Z_\alpha \in \mathfrak{g}^{-\alpha}$ and $[Z_\alpha, \theta Z_\alpha] \in \mathfrak{g}^0 = \mathfrak{z}_\mathfrak{g}(\mathfrak{a})$. On the other hand, $\theta[Z_\alpha, \theta Z_\alpha] = -[Z_\alpha, \theta Z_\alpha]$. Hence $[Z_\alpha, \theta Z_\alpha] \in \mathfrak{p} \cap \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) = \mathfrak{a}$. For all $H \in \mathfrak{a}$, one has

$$B(H, [Z_\alpha, \theta Z_\alpha]) = B(H, B(Z_\alpha, \theta Z_\alpha)h_\alpha),$$

and by the nondegeneracy of $B|\mathfrak{a} \times \mathfrak{a}$, it follows that $[Z_\alpha, \theta Z_\alpha] = B(Z_\alpha, \theta Z_\alpha)h_\alpha$. Since $B(X, \theta Y)$ is negative definite on $\mathfrak{g}$, the statement follows.

(ii) Let $\lambda, \mu \in \Delta_\alpha$, with $\lambda \neq \mu, \bar{\mu}$. Calculating

$$[W_\lambda \pm \sigma W_\lambda, \theta(W_\mu \pm \sigma W_\mu)] \\ = [W_\lambda, \theta W_\mu] \pm [W_\lambda, \theta\sigma W_\mu] \pm [\sigma W_\lambda, \theta W_\mu] \pm [\sigma W_\lambda, \theta\sigma W_\mu],$$

one has that all the brackets on the right-hand side belong to root spaces $\mathfrak{g}^\gamma$, where $\gamma \in \Delta$ is a nonzero root which vanishes identically on $\mathfrak{a}^\mathbb{C}$. Such root spaces are contained in $\mathfrak{k}^\mathbb{C}$ and the statement follows.

LEMMA 5.2. (i) *Let $\alpha \neq \beta \in \Delta_\mathfrak{a}^+$. Let $Z_\alpha \in \mathfrak{g}^\alpha$ and $Z_\beta \in \mathfrak{g}^\beta$. Then*

$$B([Z_\alpha \pm \theta Z_\alpha, Z_\beta \pm \theta Z_\beta], \mathfrak{a}^\mathbb{C}) = 0.$$

(ii) *Let Z_α, Z'_α be distinct elements of the basis $\mathscr{B}_\alpha$ of $\mathfrak{g}^\alpha$ defined in* (5.2). *Then*

$$[Z_\alpha - \theta Z_\alpha, Z_\alpha + \theta Z_\alpha] = 2[Z_\alpha, \theta Z_\alpha], \qquad B([Z_\alpha \pm \theta Z_\alpha, Z'_\alpha \pm \theta Z'_\alpha], \mathfrak{a}^\mathbb{C}) = 0.$$

Proof. The lemma follows by direct computations and Lemma 5.1(ii).

COROLLARY 5.3. *Let $\alpha \in \Delta_\mathfrak{a}$ and let $\mathscr{B}_\alpha$ be the basis of $\mathfrak{g}^\alpha$ defined in* (5.2). *Only the brackets of the form $[Z_\alpha - \theta Z_\alpha, Z_\alpha + \theta Z_\alpha]$ give some contribution in the $\mathfrak{a}$ direction.*

For $\alpha \in \Delta_\mathfrak{a}^+$, fix the basis $\mathscr{B}_\alpha$ of $\mathfrak{g}^\alpha$ defined in (5.2). Recall from Section 3.2 that to each $Z_\alpha \in \mathscr{B}_\alpha$ there are associated $K_\alpha \in Z[\alpha]_\mathfrak{k}$, $P_\alpha \in Z[\alpha]_\mathfrak{p}$, and $F_\alpha = \mathrm{Ad}_{x_0} P_\alpha$ in $\mathrm{Ad}_{x_0} Z[\alpha]_\mathfrak{p}^\mathbb{C}$ and recall the decomposition (5.1) of the complex tangent space $T_\mathbb{C} S_{\bar{x}_0}$. We calculate the Levi form with respect to the basis $\{\{F_\alpha\}_{Z\alpha \in \mathscr{B}\alpha}\}_{\alpha \in \Delta_\mathfrak{a}^+}$ of $T_\mathbb{C} S_{\bar{x}_0}$.

PROPOSITION 5.4. *For the Levi form one has*:

(a) $$\mathbf{L}(F_\alpha, F_\alpha) = \frac{2}{\sin 2\alpha(H_0)}[Z_\alpha, \theta Z_\alpha] = \frac{2B(Z_\alpha, \theta Z_\alpha)}{\sin 2\alpha(H_0)} h_\alpha$$

for $\alpha \in \Delta_\mathfrak{a}^+$.

(b) $$\mathbf{L}(F_\alpha, F'_\alpha) = 0 \text{ for all } F_\alpha, \neq F'_\alpha, \text{ when } \dim \mathfrak{g}^\alpha > 1.$$

(c) $$\mathbf{L}(F_\alpha, F_\beta) = 0 \text{ for all } \alpha \neq \beta \in \Delta_\mathfrak{a}^+.$$

Proof. We compute the Levi form by the formulas of Lemma 4.3 together with the identities of Remark 3.8, Lemmas 5.1 and 5.2, and Corollary 5.3. For example, writing α for $\alpha(H_0)$ and computing modulo $T_{\mathbb{C}}S_{\bar{x}_0}$, we have

$$\begin{aligned}
\mathbf{L}(F_\alpha, F_\alpha) &= \frac{1}{2}[\widehat{JF_\alpha}, \widehat{F}_\alpha]_{x_0} \\
&= \frac{1}{2}([(*)^{-1}JF_\alpha, F_\alpha] - [(*)^{-1}F_\alpha, JF_\alpha]) \\
&= \frac{1}{2}\left(\left[\frac{1}{\sin\alpha}K_\alpha, \cos\alpha P_\alpha + \sin\alpha JK_\alpha\right]\right. \\
&\qquad \left. -\left[\frac{1}{\cos\alpha}P_\alpha, \cos\alpha JP_\alpha - \sin\alpha K_\alpha\right]\right) \\
&= \frac{1}{2}\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}\right)[P_\alpha, K_\alpha] \\
&= \frac{1}{\sin 2\alpha}[P_\alpha, K_\alpha] = \frac{2}{\sin 2\alpha}[Z_\alpha, \theta Z_\alpha] = \frac{2B(Z_\alpha, \theta Z_\alpha)}{\sin 2\alpha}h_\alpha.
\end{aligned}$$

The other identities follow in a similar way.

We conclude this section by computing the Levi cone of the generic orbits in $\overline{\mathbf{X}}_0$.

Remark 5.5. Denote by $\Pi_{\mathfrak{a}} = \{\alpha_1, \dots, \alpha_r\}$ the set of simple roots in $\Delta_{\mathfrak{a}}$. Consider the intersection points of the hyperplanes

$$\bigcap_{\alpha_i \in \Pi_{\mathfrak{a}}} \{H \in \mathfrak{a} \mid \alpha_i(H) = m_i\pi\}, \qquad m_i \in \mathbb{Z},$$

and call them vertices. To each such vertex V, we can associate a cell

$$\omega_V = \{H \in \mathfrak{a} |\, |\alpha_i(H) - \alpha_i(V)| < \pi/2\}.$$

When $V = O \in \mathfrak{a}$, we denote

$$\omega_0 = \{H \in \mathfrak{a} |\, |\alpha_i(H)| < \pi/2, \forall\, \alpha_i \in \Pi_{\mathfrak{a}}\}.$$

By Remark 3.5(ii), the union of such cells contains the parameter space of generic orbits in $\overline{\mathbf{X}}_0$, and the union of the closures of such cells exhausts the whole $\mathfrak{a}$. By Remark 3.5(i), the point $\bar{v} \in G^{\mathbb{C}}/K^{\mathbb{C}}$ corresponding to a vertex V sits on a minimal orbit of type G/K. By Remark 3.5(ii), there are finitely many G-orbits of type G/K in $\overline{\mathbf{X}}_0$.

DEFINITION. Given a set of vectors $\{X_1, \ldots, X_k\}$ in a real vector space W, the cone generated by $\{X_1, \ldots, X_k\}$ is by definition the set of linear combinations with nonnegative coefficients of $\{X_1, \ldots, X_k\}$ and it is denoted by

$$\operatorname{cone}\{X_1, \ldots, X_k\}.$$

PROPOSITION 5.6. *Let S be a generic G-orbit in $\overline{\mathbf{X}}_0$. Let $\bar{x}_0$ be a base point of S, where $x_0 = \exp JH_0 \in \exp J\mathfrak{a}$. Assume $H_0 \in \omega_{V_0}$ for some vertex $V_0 \in \mathfrak{a}$ defined by the conditions $\alpha_i(V_0) \equiv 0 \bmod \pi$ for all $\alpha_i \in \Pi_{\mathfrak{a}}$. Then there are the following possibilities.*

(i) *If $|\alpha(H_0) - \alpha(V_0)| < \pi/2$ for all $\alpha \in \Delta_{\mathfrak{a}}$, then the Levi cone $\mathscr{C}_{\bar{x}_0}(S)$ has nonempty interior and it is sharp.*

(ii) *If $|\alpha(H_0) - \alpha(V_0)| > \pi/2$ for some $\alpha \in \Delta_{\mathfrak{a}} \backslash \Pi_{\mathfrak{a}}$, then the Levi cone is given by $\mathscr{C}_{\bar{x}_0}(S) = \mathfrak{a}$.*

Proof. The proposition follows from the formulas of Proposition 5.4. Since $B(Z_\alpha, \theta Z_\alpha) < 0$ for all $Z_\alpha \in \mathfrak{g}^\alpha$, the Levi cone is given by

$$\mathscr{C}_{\bar{x}_0}(S) = \operatorname{cone}\left\{ \frac{-1}{\sin 2\alpha(H_0)} h_\alpha \right\}_{\alpha \in \Delta_{\mathfrak{a}}^+} \subset \mathfrak{a}. \tag{5.3}$$

Hence it has nonempty interior. By Remark 5.5, expression (5.3), and the periodicity of the sin function, it is sufficient to consider $H_0 \in \omega_0$. Moreover, by Remark 2.6(ii), there is no loss in generality in assuming $0 < \alpha_i(H_0) < \pi/2$ for all $\alpha_i \in \Pi_{\mathfrak{a}}$.

Then

(i) is obvious.

(ii) Assume $\alpha(H_0) > \pi/2$ for some $\alpha \in \Delta_{\mathfrak{a}} \backslash \Pi_{\mathfrak{a}}$. We claim that $\mathbb{R} \cdot h_{\alpha_j} \subset \mathscr{C}_{\bar{x}_0}(S)$ for every simple root $\alpha_j \in \Pi_{\mathfrak{a}}$.

Let $\Lambda = \sum_{j=1}^r m_j \alpha_j$ be the highest root in $\Delta_{\mathfrak{a}}$. Observe that Λ has all the coefficients $m_j \in \mathbb{Z}_{>0}$ and can be obtained from an arbitrary simple root α_i by adding simple roots. Since $\alpha(H_0) > \pi/2$, also $\Lambda(H_0) > \pi/2$. Then, for every simple root $\alpha_i \in \Pi_{\mathfrak{a}}$, there exists a root $\lambda = \sum_{j=1}^r n_j \alpha_j$, with coefficient $n_i \in \mathbb{Z} > 0$, such that $\lambda(H_0) \in]\pi/2, \pi[$ and $\lambda - \alpha_j(H_0) \in]0, \pi/2[$, whenever $\lambda - \alpha_j \in \Delta_{\mathfrak{a}}$. Now fix $\alpha_i \in \Pi_{\mathfrak{a}}$ and λ as above and let $\alpha_j \in \Pi_{\mathfrak{a}}$ be a simple root such that $\lambda - \alpha_j \in \Delta_{\mathfrak{a}}$. Then consider the triplet of roots $\lambda - \alpha_j, \alpha_j, \lambda$. Since $\alpha_j(H_0)$, $\lambda - \alpha_j(H_0) \in]\pi/2[$, and $\lambda(H_0) \in]\pi/2, \pi[$, by (5.3)

$$-h_{\lambda-\alpha_j}, -h_{\alpha_j}, h_\lambda \in \mathscr{C}_{\bar{x}_0}(S) \qquad \text{and} \qquad \operatorname{span}_{\mathbb{R}}\{h_{\alpha_j}, h_\lambda\} \subset \mathscr{C}_{\bar{x}_0}(S).$$

Similarly, if $\alpha_k \in \Pi_{\mathfrak{a}}$ is a simple root such that $\lambda - \alpha_j - \alpha_k \in \Delta_{\mathfrak{a}}$, consider the triple of roots $\lambda - \alpha_j - \alpha_k, \alpha_k, \lambda - \alpha_j$. By the result of the previous step and by (5.3),

$$-h_{\lambda-\alpha_j-\alpha_k}, -h_{\alpha_k}, \pm h_{\lambda-\alpha_j} \in \mathscr{C}_{\bar{x}_0}(S) \quad \text{and} \quad \operatorname{span}_{\mathbb{R}}\{h_{\lambda-\alpha_j}, h_{\alpha_k}\} \subset \mathscr{C}_{\bar{x}_0}(S).$$

Iterating this argument, one obtains that $\pm h_{\alpha_i} \in \mathscr{C}_{\bar{x}_0}(S)$ for every simple root α_k appearing in λ with nonzero coefficient. In particular, this holds for α_i. Since α_i was arbitrary, the claim follows. Since $\Delta_{\mathfrak{a}}$ is irreducible, the proof of the proposition is complete.

5.2. *The Calculation beyond* $\overline{\mathbf{X}}_0$

In this section, we compute the Levi form and the Levi cone of the generic orbits intersecting $\overline{C}$, where $C = \exp J\mathfrak{c} \cdot p$ is a standard Cartan subset different from the fundamental Cartan subset A. We resume the notations introduced in Section 3.3. By Corollary 3.17, the complex tangent space $T_{\mathbb{C}}S_{\overline{x_0 \cdot p}}$ to a generic orbit S at a base point $\overline{x_0 \cdot p} \in \overline{C}$, where $x_0 = e^{JX_0}$, $X_0 \in \mathfrak{c}$, is given by

$$T_{\mathbb{C}}S_{\overline{x_0 \cdot p}} = \bigoplus_{\alpha \in \Delta_{\mathfrak{c}}^+} \mathrm{Ad}_{x_0}\mathfrak{g}[\alpha]_{\mathfrak{q}^{\mathbb{C}}}, \quad \text{where } Ad_{x_0}\mathfrak{g}[\alpha]_{\mathfrak{q}^{\mathbb{C}}} = \bigoplus_{Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}} \tag{5.4}$$

and Z_α runs through a basis of $\mathfrak{g}^\alpha \subset \mathfrak{g}^{\mathbb{C}}$ for each $\alpha \in \Delta_{\mathfrak{c}}^+$.

As in the previous case (cf. Section 5.1), we explicitly construct a basis of $T_{\mathbb{C}}S_{\overline{x_0 \cdot p}}$ where the calculation of the Levi form turns out particularly simple. This depends on the choice of a convenient basis $\mathscr{B}_\alpha$ of each restricted root space $\mathfrak{g}^\alpha \subset \mathfrak{g}^{\mathbb{C}}$.

Recall that, by Lemma 2.9, the base point p of the Cartan subset $C = \exp J_{\mathfrak{c}} \cdot p$ can be assumed to satisfy either conditions (2.6) or (2.7). As a consequence, Ad_{p^4} is either the identity or an involution of $\mathfrak{g}^{\mathbb{C}}$ commuting with τ_p and σ. Also recall that Ad_{p^4} preserves each restricted root space $\mathfrak{g}^\alpha$ and acts as the identity on $\mathfrak{g}^0 = \mathfrak{c}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ (cf. Lemmas 2.14, 3.9, and 3.11).

LEMMA 5.7. *There exists a Cartan subalgebra $\mathfrak{l}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$, extending $\mathfrak{c}^{\mathbb{C}}$, of the form*

$$\mathfrak{l}^{\mathbb{C}} = \mathfrak{b}^{\mathbb{C}} \oplus \mathfrak{c}^{\mathbb{C}}, \qquad \mathfrak{b}^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}},$$

which is stable under Ad_{p^4}, τ_p, Θ, and σ. In particular, if $\mathfrak{b} = \mathfrak{b}^{\mathbb{C}} \cap \mathfrak{g}$, then $\mathfrak{l} = \mathfrak{b} \oplus \mathfrak{c}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$.

Proof. Since τ_p and Θ commute (cf. Lemma 2.10(ii)), the subspace $\mathfrak{h}^{\mathbb{C}}$ is Θ-stable. Since $\mathfrak{c}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ are Θ-stable, the complex reductive subalgebra $\mathfrak{m}^{\mathbb{C}} = \mathfrak{z}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}})$ is Θ-stable as well. By Lemma 3.11, $\mathfrak{m}^{\mathbb{C}}$ is also σ-stable. Let $\mathfrak{b}^{\mathbb{C}}$ be a σ and Θ-stable Cartan subalgebra of $\mathfrak{m}^{\mathbb{C}}$. Since Ad_{p^4} and τ_p act on $\mathfrak{m}^{\mathbb{C}}$ as the identity, $\mathfrak{l}^{\mathbb{C}} = \mathfrak{b}^{\mathbb{C}} \oplus \mathfrak{c}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ with the required properties and the corresponding real form $\mathfrak{l} = \mathfrak{b} \oplus \mathfrak{c}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$.

Every θ-stable Cartan subalgebra in $\mathfrak{m} = \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{g}$ provides an extension of $\mathfrak{c}$ with the required properties. In general, not all θ-stable Cartan subalgebras $\mathfrak{l} = \mathfrak{b} \oplus \mathfrak{c}$ obtained as above are conjugate in $\mathfrak{g}$, unless $\mathfrak{m}$ is compact. In that case, $\mathfrak{b}$ is compact as well.

Let Δ be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{l}^{\mathbb{C}}$ and let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}^{\lambda}$$

be the corresponding root decomposition. The roots in $\Delta_{\mathfrak{c}}$ are restrictions to $\mathfrak{c}^{\mathbb{C}}$ of the roots in Δ. Observe that $\mathfrak{m}^{\mathbb{C}}$ consists of the root spaces $\mathfrak{g}^{\lambda}$, where $\lambda \in \Delta$ is a nonzero root such that $\lambda|\mathfrak{c}^{\mathbb{C}} \equiv 0$. Since $\mathfrak{l}^{\mathbb{C}}$ is both σ and τ_p-stable, there are induced actions of σ and τ_p on Δ defined by

$$\bar{\lambda}(H) := \overline{\lambda(\sigma(H))}, \qquad \tau_p\lambda(H) := \lambda(\tau_p(H)), \qquad H \in \mathfrak{l}^{\mathbb{C}}.$$

Since σ and τ_p commute on $\mathfrak{l}^{\mathbb{C}}$, one has that

$$\overline{\tau_p\lambda} = \tau_p\bar{\lambda}.$$

Observe that, for every root $\lambda \in \Delta$,

$$\lambda + \tau_p\lambda \notin \Delta.$$

Since Ad_{p^4} acts trivially on $\mathfrak{l}^{\mathbb{C}}$, it acts as plus or minus the identity on each root space $\mathfrak{g}^{\lambda}$, $\lambda \in \Delta$. Moreover, since Ad_{p^4} preserves the Killing form and commutes with both σ and τ_p, it acts in the same way on all the root spaces $\mathfrak{g}^{\pm\lambda}$, $\mathfrak{g}^{\pm\bar{\lambda}}$, $\mathfrak{g}^{\pm\tau_p\lambda}$, and $\mathfrak{g}^{\pm\tau_p\bar{\lambda}}$.

For $\alpha \in \Delta_{\mathfrak{c}}$, define

$$\Delta_\alpha := \{\lambda \in \Delta \mid \lambda|\mathfrak{c}^{\mathbb{C}} = \alpha\}.$$

Then

$$\lambda \in \Delta_\alpha \Longrightarrow -\tau_p\lambda \in \Delta_\alpha$$

and each restricted root space $\mathfrak{g}^{\alpha}$ decomposes as

$$\mathfrak{g}^{\alpha} = \bigoplus_{\substack{\lambda \in \Delta_\alpha \\ \lambda \neq -\tau_p\lambda}} \mathfrak{g}^{\lambda} \oplus \mathfrak{g}^{-\tau_p\lambda} \qquad \text{or} \qquad \mathfrak{g}^{\alpha} = \bigoplus_{\substack{\lambda \in \Delta_\alpha \\ \lambda \neq -\tau_p\lambda}} \mathfrak{g}^{\lambda} \oplus \mathfrak{g}^{-\tau_p\lambda} \oplus \mathfrak{g}^{\mu},$$

$$\mu \in \Delta_\alpha, \ \ \mu = -\tau_p\mu.$$

The root spaces appearing on the right-hand side are ordinary root spaces in $\mathfrak{g}^{\mathbb{C}}$. (The roots $\lambda \in \Delta_\alpha$ are chosen so that each summand appears precisely once in the above decomposition.)

For each restricted root space $\mathfrak{g}^{\alpha}$, we use the above decomposition to construct a basis $\mathscr{B}_\alpha$ compatible with the decomposition $\mathfrak{g}^{\alpha} = \mathfrak{g}^{\alpha}_{+} \oplus \mathfrak{g}^{\alpha}_{-}$. If the root α is either real or imaginary, we also require the elements $Z_\alpha \in \mathcal{B}_\alpha$ to satisfy the properties stated in parts (2) and (3) of Remark 3.10: if α is real, $\mathcal{B}_\alpha$ consists of σ-invariant elements; if α is imaginary, $\mathcal{B}_\alpha$ consists of $\sigma\tau_p$-invariant elements generating $\mathfrak{g}^\alpha_+$ and of pairs of elements $\{Z_\alpha, \sigma\tau_p Z_\alpha\}$ generating $\mathfrak{g}^\alpha_-$.

Fix a set of root vectors $\{W_\lambda\}_{\lambda\in\Delta}$, with $W_\lambda \in \mathfrak{g}^\lambda$, satisfying

$$B(W_\lambda, W_{-\lambda}) = 1, \qquad B(W_\lambda, W_\mu) = 0, \qquad \lambda + \mu \neq 0.$$

Then $[W_\lambda, W_{-\lambda}] = h_\lambda$, where $h_\lambda \in \mathfrak{l}_\mathbb{R} = \{h \in \mathfrak{l}^\mathbb{C} \mid \lambda(H) \in \mathbb{R}\ \forall\, \lambda \in \Delta\}$ is the element defined by $\lambda(H) = B(H, h_\lambda)$, $H \in \mathfrak{l}^\mathbb{C}$.

In addition, for every real root $\lambda \in \Delta$, the vectors W_λ are assumed to be σ-invariant, i.e., $\sigma(W_\lambda) = W_\lambda$, while for every imaginary root $\lambda \in \Delta$, the vectors W_λ are assumed to satisfy

$$\sigma(W_\lambda) = \pm W_{-\lambda},$$

depending on whether the real form of the complex three-dimensional σ-stable subalgebra generated by $\{h_\lambda, W_\lambda, W_{-\lambda}\}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ or to $\mathfrak{su}(2)$. Imaginary roots with such properties are called "noncompact" and "compact," respectively.

We write $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_+$, or $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_-$, depending on whether Ad_{p^4} acts as plus or minus the identity on $\mathfrak{g}^\lambda$.

Remark 5.8. For each root $\alpha \in \Delta_\mathfrak{c}$, the basis $\mathcal{B}_\alpha$ is given as follows.

(1) Let $\alpha \in \Delta_\mathfrak{c}^+$ be a real root.

(i) If $\lambda \in \Delta_\alpha$ is a root satisfying $\lambda \neq \bar\lambda$ and $\tau_p\lambda \neq -\lambda, \bar\lambda$, then $\lambda, -\tau_p\lambda, \bar\lambda, -\tau_p\bar\lambda$ are distinct roots in Δ_α and the restricted root space $\mathfrak{g}^\alpha$ contains the four-dimensional subspace

$$\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\tau_p\lambda} \oplus \mathfrak{g}^{\bar\lambda} \oplus \mathfrak{g}^{-\tau_p\bar\lambda}. \tag{5.5}$$

If $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_+$, then a σ-invariant basis of subspace (5.5) is given by the vectors

$$\begin{aligned} \mathrm{Re}(W_\lambda + \tau_p W_{-\lambda}) &= \frac{1}{2}((W_\lambda + \sigma W_\lambda) + \tau_p(W_{-\lambda} + \sigma W_{-\lambda})),\\ \mathrm{Im}(W_\lambda + \tau_p W_{-\lambda}) &= \frac{-1}{2}J((W_\lambda - \sigma W_\lambda) + \tau_p(W_{-\lambda} - \sigma W_{-\lambda})),\\ \mathrm{Re}(W_\lambda - \tau_p W_{-\lambda}) &= \frac{1}{2}((W_\lambda + \sigma W_\lambda) - \tau_p(W_{-\lambda} + \sigma W_{-\lambda})),\\ \mathrm{Im}(W_\lambda - \tau_p W_{-\lambda}) &= \frac{-1}{2}J((W_\lambda - \sigma W_\lambda) - \tau_p(W_{-\lambda} - \sigma W_{-\lambda})). \end{aligned} \tag{5.6}$$

If $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_-$, then a σ-invariant basis of subspace (5.5) is given by the vectors

$$\begin{aligned}
\mathrm{Re}(W_\lambda + J\tau_p W_{-\lambda}) &= \frac{1}{2}((W_\lambda + \sigma W_\lambda) + J\tau_p(W_{-\lambda} + \sigma W_{-\lambda})),\\
\mathrm{Im}(W_\lambda + J\tau_p W_{-\lambda}) &= \frac{-1}{2}J((W_\lambda - \sigma W_\lambda) + J\tau_p(W_{-\lambda} - \sigma W_{-\lambda})),\\
\mathrm{Re}(W_\lambda - J\tau_p W_{-\lambda}) &= \frac{1}{2}((W_\lambda + \sigma W_\lambda) - J\tau_p(W_{-\lambda} + \sigma W_{-\lambda})),\\
\mathrm{Im}(W_\lambda - J\tau_p W_{-\lambda}) &= \frac{-1}{2}J((W_\lambda - \sigma W_\lambda) - J\tau_p(W_{-\lambda} - \sigma W_{-\lambda})).
\end{aligned} \tag{5.7}$$

(ii) If $\lambda \in \Delta_\alpha$ is a root satisfying $\lambda = \bar{\lambda}$ and $\tau_p\lambda = -\lambda$, subspace (5.5) reduces to $\mathfrak{g}^\lambda$ and a σ-invariant basis of $\mathfrak{g}^\lambda$ is given by

$$W_\lambda. \tag{5.8}$$

(iii) If $\lambda \in \Delta_\alpha$ is a root satisfying $\lambda = \bar{\lambda}$ and $\tau_p\lambda \neq -\lambda$, subspace (5.5) reduces to $\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\tau_p\lambda}$ and a σ-invariant basis of is given by

$$\{W_\lambda + \tau_p W_{-\lambda}, W_\lambda - \tau_p W_{-\lambda}\} \qquad \text{if } \mathfrak{g}^\lambda = \mathfrak{g}^\lambda_+ \tag{5.9}$$

or by

$$\{W_\lambda + J\tau_p W_{-\lambda}, W_\lambda - J\tau_p W_{-\lambda}\} \qquad \text{if } \mathfrak{g}^\lambda = \mathfrak{g}^\lambda_-. \tag{5.10}$$

(iv) If $\lambda \in \Delta_\alpha$ is a root satisfying $\lambda \neq \bar{\lambda}, -\tau_p\lambda$, and $-\tau_p\lambda = \bar{\lambda}$, subspace (5.5) reduces to $\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\tau_p\lambda} = \mathfrak{g}^\lambda \oplus \mathfrak{g}^{\bar{\lambda}}$ and a σ-invariant basis is given by

$$\{W_\lambda + \sigma W_\lambda, -J(W_\lambda - \sigma W_\lambda)\}. \tag{5.11}$$

(2) Let $\alpha \in \Delta_{\mathfrak{c}}^+$ be an *imaginary* root.

(i) If $\lambda \in \Delta_\alpha$ satisfies $\lambda \neq -\bar{\lambda}$ and $\tau_p\lambda \neq -\lambda, \bar{\lambda}$, then $\lambda, -\tau_p\lambda, -\bar{\lambda}, \tau_p\bar{\lambda}$ are all distinct roots in Δ_α, the restricted root space $\mathfrak{g}^\alpha$ contains the four-dimensional subspace

$$\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\tau_p\lambda} \oplus \mathfrak{g}^{-\bar{\lambda}} \oplus \mathfrak{g}^{\tau_p\bar{\lambda}}, \tag{5.12}$$

and the vectors

$$\begin{aligned}
Z_\alpha^1 &= \frac{1}{2}((W_\lambda + \tau_p\sigma W_\lambda) + \tau_p(W_{-\lambda} + \tau_p\sigma W_{-\lambda})),\\
Z_\alpha^2 &= \frac{1}{2}((W_\lambda + \tau_p\sigma W_\lambda) - \tau_p(W_{-\lambda} + \tau_p\sigma W_{-\lambda})),\\
Z_\alpha^3 &= \frac{1}{2}((W_\lambda - \tau_p\sigma W_\lambda) + \tau_p(W_{-\lambda} - \tau_p\sigma W_{-\lambda})),\\
Z_\alpha^4 &= \frac{1}{2}((W_\lambda - \tau_p\sigma W_\lambda) - \tau_p(W_{-\lambda} - \tau_p\sigma W_{-\lambda}))
\end{aligned} \tag{5.13}$$

are linearly independent. If $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_+$, then $\sigma\tau_p$ is a conjugation of subspace (5.12) and the elements $\{Z^1_\alpha, Z^2_\alpha, JZ^3_\alpha, JZ^4_\alpha\}$ form a $\sigma\tau_p$-invariant basis. If $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_-$, the elements $\{Z^1_\alpha, Z^2_\alpha, Z^3_\alpha, Z^4_\alpha\}$ are a basis of subspace (5.12) satisfying the conditions

$$\sigma\tau_p Z^1_\alpha = Z^4_\alpha, \qquad \sigma\tau_p Z^2_\alpha = Z^3_\alpha. \tag{5.13$'$}$$

(ii) If $\lambda \in \Delta_\alpha$ satisfies $\lambda \neq -\bar{\lambda}$, $\tau_p\lambda \neq -\lambda$, $\tau_p\lambda = \bar{\lambda}$, subspace (5.12) reduces to $\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\tau_p\lambda}$. Since $\sigma\tau_p$ is a complex antilinear endomorphism of $\mathfrak{g}^\lambda$, it follows that $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_+$. A $\sigma\tau_p$-invariant basis of $\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\tau_p\lambda}$ is given by the following vectors from (5.13)

$$\begin{aligned} &\{Z^1_\alpha, Z^2_\alpha\} && \text{if } \tau_p\sigma W_\lambda + W_\lambda \neq 0,\\ &\{JZ^3_\alpha, JZ^4_\alpha\} && \text{if } \tau_p\sigma W_\lambda + W_\lambda = 0. \end{aligned} \tag{5.14}$$

(iii) If $\lambda \in \Delta_\alpha$ satisfies $\bar{\lambda} = \tau_P\lambda = -\lambda$, subspace (5.12) reduces to $\mathfrak{g}^\lambda$. In particular, $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_+$. If λ is noncompact, a $\sigma\tau_p$-invariant basis of $\mathfrak{g}^\lambda$ is given by

$$\begin{aligned} &Z^1_\alpha, = \frac{1}{2}(W_\lambda + \tau_p W_{-\lambda}) && \text{if } W_\lambda + \tau_p W_{-\lambda} \neq 0,\\ &JZ^4_\alpha = JW_\lambda && \text{if } W_\lambda + \tau_p W_{-\lambda} = 0. \end{aligned} \tag{5.15}$$

If λ is compact, a $\sigma\tau_p$-invariant basis of $\mathfrak{g}^\lambda$ is given by

$$\begin{aligned} &JZ^3_\alpha = \frac{1}{2}J(W_\lambda + \tau_p W_{-\lambda}) && \text{if } W_\lambda + \tau_p W_{-\lambda} \neq 0,\\ &Z^2_\alpha \ = W_\lambda && \text{if } W_\lambda + \tau_p W_{-\lambda} = 0. \end{aligned} \tag{5.15$'$}$$

(iv) If $\lambda \in \Delta_\alpha$ satisfies $\lambda = -\bar{\lambda}$ and $\tau_p\lambda \neq -\lambda, \bar{\lambda}$, subspace (5.12) reduces to $\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\tau_p\lambda}$. The vectors in (5.13) become

$$\begin{aligned} &Z^1_\alpha = W_\lambda + \tau_p W_{-\lambda}, && Z^4_\alpha = W_\lambda - \tau_p W_{-\lambda} && \text{for } \lambda \text{ noncompact},\\ &Z^3_\alpha = W_\lambda + \tau_p W_{-\lambda}, && Z^2_\alpha = W_\lambda - \tau_p W_{-\lambda} && \text{for } \lambda \text{ compact}. \end{aligned}$$

Assume $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_+$. Then a $\sigma\tau_p$-invariant basis of $\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\tau_p\lambda}$ is given by

$$\begin{aligned} &\{Z^1_\alpha, JZ^4_\alpha\} && \text{for } \lambda \text{ noncompact},\\ &\{Z^2_\alpha, JZ^3_\alpha\} && \text{for } \lambda \text{ compact}. \end{aligned} \tag{5.16}$$

Assume now $\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_-$. Then a basis of $\mathfrak{g}^\lambda \oplus \mathfrak{g}^{-\tau_p\lambda}$ is given by

$$\begin{aligned} &\{Z^1_\alpha, Z^4_\alpha\}, && \sigma\tau_p Z^1_\alpha = Z^4_\alpha && \text{for } \lambda \text{ noncompact},\\ &\{Z^2_\alpha, Z^3_\alpha\}, && \sigma\tau_p Z^2_\alpha = Z^3_\alpha && \text{for } \lambda \text{ compact}. \end{aligned} \tag{5.17}$$

(3) Let $\alpha \in \Delta_{\mathfrak{c}}$ be a complex root. Then a basis of $\mathfrak{g}^{\alpha}$, compatible with the decomposition $\mathfrak{g}^{\alpha} = \mathfrak{g}^{\alpha}_{+} \oplus \mathfrak{g}^{\alpha}_{-}$, is given by

$$\{W_{\lambda} + \tau_p W_{-\lambda}, W_{\lambda} - \tau_p W_{-\lambda}\}_{\substack{\lambda\in\Delta_{\alpha} \\ \lambda\neq-\tau_p\lambda}}, \{W_{\lambda}\}_{\substack{\lambda\in\Delta_{\alpha} \\ \lambda\neq-\tau_p\lambda}}. \tag{5.18}$$

Remark 5.9. For $\alpha \in \Delta_{\mathfrak{c}}$, let $h_{\alpha} \in \mathfrak{c}^{\mathbb{C}}$ be the vector defined by $\alpha(H) = B(H, h_{\alpha})$ for $H \in \mathfrak{c}^{\mathbb{C}}$. Then

$$h_{\alpha} = \frac{1}{2}(h_{\lambda} - \tau_p h_{\lambda}) \in \mathfrak{c}_{\mathbb{R}}, \qquad \lambda \in \Delta_{\alpha}.$$

Observe that $\mathfrak{l}_{\mathbb{R}} = \mathfrak{b}_{\mathbb{R}} \oplus \mathfrak{c}_{\mathbb{R}}$, where $\mathfrak{b}_{\mathbb{R}} = J\mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{b}_{\mathfrak{p}}$ and $\mathfrak{c}_{\mathbb{R}} = J\mathfrak{c}_{\mathfrak{k}} \oplus \mathfrak{c}_{\mathfrak{p}}$.

LEMMA 5.10. *Let $\alpha \in \Delta_{\mathfrak{c}}$. Let $\mathcal{B}_{\alpha}$ be the basis of $\mathfrak{g}^{\alpha}$ defined in Remark* 5.8. *The following relations hold*:

(i) *If $Z_{\alpha} \in \mathfrak{g}^{\alpha}$, then $[Z_{\alpha}, \tau_p Z_{\alpha}] = B(Z_{\alpha}, \tau_p Z_{\alpha})h_{\alpha} \in \mathfrak{c}^{\mathbb{C}}$.*

(ii) *Let Z_{α}, Z'_{α} be distinct elements in $\mathcal{B}_{\alpha}$. Then*

$$[Z_{\alpha}, \tau_p Z'_{\alpha}] \in \mathfrak{h}^{\mathbb{C}}.$$

Proof. The proof of (i) is similar to the proof of the corresponding statement in Lemma 5.1, and it is based on the nondegeneracy of the Killing form restricted to $\mathfrak{c}^{\mathbb{C}} \times \mathfrak{c}^{\mathbb{C}}$.

The proof of (ii) consists of direct computations.

LEMMA 5.11. (i) *Let $\alpha \neq \beta \in \Delta_{\mathfrak{c}}^{+}$. Let $Z_{\alpha} \in \mathfrak{g}^{\alpha}$ and $Z_{\beta} \in \mathfrak{g}^{\beta}$. Then*

$$B([Z_{\alpha} \pm \tau_p Z_{\alpha}, Z_{\beta} \pm \tau_p Z_{\beta}], \mathfrak{c}^{\mathbb{C}}) = 0.$$

(ii) *Let $Z_{\alpha}, Z'_{\alpha} \in \mathfrak{g}^{\alpha}$ be distinct elements of the basis $\mathcal{B}_{\alpha}$ defined in Remark* 5.8. *Then*

$$[Z_{\alpha} - \tau_p Z_{\alpha}, Z_{\alpha} + \tau_p Z_{\alpha}] = 2[Z_{\alpha}, \tau_p Z_{\alpha}],$$
$$B([Z_{\alpha} \pm \tau_p Z_{\alpha}, Z'_{\alpha} \pm \tau_p Z'_{\alpha}], \mathfrak{c}^{\mathbb{C}}) = 0.$$

Proof. The lemma follows from direct computations and Lemma 5.10(ii).

COROLLARY 5.12. *Let $\alpha \in \Delta_{\mathfrak{c}}$ and let $\mathcal{B}_{\alpha}$ be the basis of $\mathfrak{g}^{\alpha}$ defined in Remark* 5.8. *Only the brackets of the form $[Z_{\alpha} - \tau_p Z_{\alpha}, Z_{\alpha} + \tau_p Z_{\alpha}]$ give some contribution in the $\mathfrak{c}^{\mathbb{C}}$ direction.*

For $\alpha \in \Delta_{\mathfrak{c}}^{+}$ fix the basis $\mathcal{B}_{\alpha}$ of $\mathfrak{g}^{\alpha}$ constructed in Remark 5.8. Recall from Section 3.3 that to each $Z_{\alpha} \in \mathcal{B}_{\alpha}$ there are associated $H_{\alpha} \in Z[\alpha]_{\mathfrak{h}^{\mathbb{C}}}$, $Q_{\alpha} \in Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}}$, and $F_{\alpha} = \mathrm{Ad}_{x_0} Q_{\alpha}$ in $\mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}}$ and also recall the decomposition (5.4) of the complex tangent space $T_{\mathbb{C}} S_{\overline{x_0 \cdot p}}$. We calculate the quadratic Levi form with respect to this basis of $T_{\mathbb{C}}S_{\overline{x_0 \cdot p}}$. By Remark 3.10, one has that

$\mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}} = \mathrm{span}_{\mathbb{C}}\{F_\alpha\}$, for α real, or for α imaginary and $Z_\alpha \in \mathfrak{g}^\alpha_+$.

$\mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}} = \mathrm{span}_{\mathbb{C}}\{F_\alpha, F_{\bar\alpha}\}$, for α complex, or for α imaginary and $Z_\alpha \in \mathfrak{g}^\alpha_-$.

The proofs of Propositions 5.13–5.16 follow from direct computations using the formulas of Lemma 4.3 and Remark 3.18 together with the results of Lemmas 5.10 and 5.11 and Corollary 5.12.

PROPOSITION 5.13. *Let $\alpha \neq \beta, \bar\beta \in \Delta^+_{\mathfrak{c}}$. Then*

$$\mathbf{L}(F_\alpha, F_\beta) = \mathbf{L}(F_\alpha, F_{\bar\beta}) = \mathbf{L}(F_{\bar\alpha}, F_\beta) = \mathbf{L}(F_{\bar\alpha}, F_{\bar\beta}) = 0.$$

Let $\alpha \in \Delta^+_{\mathfrak{c}}$ with $\dim \mathfrak{g}^\alpha > 1$. Let Z_α, Z'_α be distinct elements of the basis $\mathcal{B}_\alpha$ such that $Z[\alpha] \cap Z'[\alpha] = \{0\}$. Let F_α, F'_α be the corresponding elements in $\mathrm{Ad}_{x_0}\mathfrak{g}[\alpha]_{\mathfrak{q}^{\mathbb{C}}}$. Then

$$\mathbf{L}(F_\alpha, F'_\alpha) = \mathbf{L}(F_{\bar\alpha}, F'_\alpha) = \mathbf{L}(F_\alpha, F'_{\bar\alpha}) = \mathbf{L}(F_{\bar\alpha}, F'_{\bar\alpha}) = 0.$$

PROPOSITION 5.14. *Let $\alpha \in (\Delta^+_{\mathfrak{c}})^r$ be a real root. Let $Z_\alpha \in \mathfrak{g}^\alpha$ be a root vector satisfying $\sigma(Z_\alpha) = Z_\alpha$.*

(i) *If $Z_\alpha \in \mathfrak{g}^\alpha_+$, then*

$$\mathbf{L}(F_\alpha, F_\alpha) = \frac{2}{\sin 2\alpha(X_0)}[Z_\alpha, \tau_p Z_\alpha].$$

(ii) *If $Z_\alpha \in \mathfrak{g}^\alpha_-$, then*

$$\mathbf{L}(F_\alpha, F_\alpha) = \frac{2}{\cos 2\alpha(X_0)} J[Z_\alpha, \tau_p Z_\alpha].$$

PROPOSITION 5.15. *Let $\alpha \in (\Delta^+_{\mathfrak{c}})^i$ be an imaginary root.*

(i) *If $Z_\alpha \in \mathfrak{g}^\alpha_+$ and $\sigma\tau_p Z_\alpha = Z_\alpha$, then*

$$\mathbf{L}(F_\alpha, F_\alpha) = -\frac{2}{\sinh 2\,\mathrm{Im}\,\alpha(X_0)} J[Z_\alpha, \tau_p Z_\alpha].$$

(ii) *Let $Z_\alpha \in \mathfrak{g}^\alpha_-$. Then $\dim_{\mathbb{C}} Z[\alpha] = 4$ and*

$$\mathbf{L}(F_\alpha, F_\alpha) = \mathbf{L}(F_{\bar\alpha}, F_{\bar\alpha}) = 0, \qquad \mathbf{L}(F_\alpha, F_{\bar\alpha}) = \frac{2}{\cosh 2\,\mathrm{Im}\,\alpha(X_0)} J[Z_\alpha, \tau_p Z_\alpha].$$

In particular, for $z_\alpha, z_{\bar\alpha} \in \mathbb{C}$, one has

$$\mathbf{L}(z_\alpha F_\alpha + z_{\bar\alpha} F_{\bar\alpha}, z_\alpha F_\alpha + z_{\bar\alpha} F_{\bar\alpha}) = 2\,\mathrm{Re}(z_\alpha \overline{z_{\bar\alpha}} \mathbf{L}(F_\alpha, F_{\bar\alpha})).$$

PROPOSITION 5.16. *Let $\alpha \in (\Delta^+_{\mathfrak{c}})^c$ be a complex root.*

(i) *If* $Z_\alpha \in \mathfrak{g}_+^\alpha$, *then*

$$\mathbf{L}(F_\alpha, F_\alpha) = \mathbf{L}(F_{\bar\alpha}, F_{\bar\alpha}) = 0, \qquad \mathbf{L}(F_\alpha, F_{\bar\alpha}) = \frac{2}{\sin 2\alpha(X_0)}[Z_\alpha, \tau_p Z_\alpha].$$

(ii) *If* $Z_\alpha \in \mathfrak{g}_-^\alpha$, *then*

$$\mathbf{L}(F_\alpha, F_\alpha) = \mathbf{L}(F_{\bar\alpha}, F_{\bar\alpha}) = 0, \qquad \mathbf{L}(F_\alpha, F_{\bar\alpha}) = \frac{2}{\cos 2\alpha(X_0)}J[Z_\alpha, \tau_p Z_\alpha].$$

In particular, for $z_\alpha, z_{\bar\alpha} \in \mathbb{C}$, *one has*

$$\mathbf{L}(z_\alpha F_\alpha + z_{\bar\alpha} F_{\bar\alpha}, z_\alpha F_{\bar\alpha} + z_{\bar\alpha} F_{\bar\alpha}) = 2\,\mathrm{Re}(z_\alpha \overline{z_{\bar\alpha}} \mathbf{L}(F_\alpha, F_{\bar\alpha})).$$

Now we are ready to compute the Levi cone of generic orbits.

DEFINITION 5.17. For $\alpha \in \Delta_{\mathfrak{c}}$ denote by m_α the cardinality of the set Δ_α or, equivalently, the complex dimension of the restricted root space $\mathfrak{g}^\alpha$.

LEMMA 5.18. *Let* S *be a generic orbit with base point* $\overline{x_0 \cdot p}$, *where* $x_0 \cdot p = e^{JX_0} \cdot p \in C = \exp J\mathfrak{c} \cdot p$. *Let* $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \oplus \mathfrak{c}_{\mathfrak{p}}$ *be the Cartan decomposition of* $\mathfrak{c}$. *The Levi cone* $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ *at* $\overline{x_0 \cdot p}$ *is the cone in* $\mathfrak{c}$ *generated by the following vectors*:

(1) $\pm\mathrm{Re}\, h_\alpha$, $\pm\mathrm{Im}\, h_\alpha \in \mathfrak{c}$ *for all* $\alpha \in (\Delta_{\mathfrak{c}}^+)^c$.

(2) $$-\frac{1}{\sinh 2\,\mathrm{Im}\,\alpha(X_0)} J h_\alpha \in \mathfrak{c}_{\mathfrak{k}}$$

for all $\alpha \in (\Delta_{\mathfrak{c}}^+)^i$ *for which* $\mathfrak{g}^\alpha = \mathfrak{g}_+^\alpha$, $m_\alpha \geq 1$, *and all roots* $\lambda \in \Delta_\alpha$ *are noncompact imaginary roots*.

(3) $$\frac{1}{\sinh 2\,\mathrm{Im}\,\alpha(X_0)} J h_\alpha \in \mathfrak{c}_{\mathfrak{k}}$$

for all $\alpha \in (\Delta_{\mathfrak{c}}^+)^i$ *for which* $\mathfrak{g}^\alpha = \mathfrak{g}_+^\alpha$, $m_\alpha \geq 1$, *and all roots* $\lambda \in \Delta_\alpha$ *are compact imaginary roots*.

(4) $\pm J h_\alpha \in \mathfrak{c}_{\mathfrak{k}}$ *for all other* $\alpha \in (\Delta_{\mathfrak{c}}^+)^i$, *with* $m_\alpha > 1$.

(5) *Either* h_α *or* $-h_\alpha \in \mathfrak{c}_{\mathfrak{p}}$ *for all roots* $\alpha \in (\Delta_{\mathfrak{c}}^+)^r$.

In particular, $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ *has nonempty interior.*

Proof. Statements (1)–(5) of the proposition follow by applying Propositions 5.13–5.16 to the basis $\mathscr{B}_\alpha$ defined in Remark 5.8 for $\alpha \in \Delta_{\mathfrak{c}}$. We examine the various cases.

Let $\alpha \in \Delta_{\mathfrak{c}}^c$ be a complex root. One can easily check that, for all the vectors Z_α in (5.18), one has

$$[Z_\alpha, \tau_p Z_\alpha] = C h_\alpha,$$

where C is a nonzero complex number. Together with Proposition 5.16, this proves (1).

Let $\alpha \in \Delta_{\mathfrak{c}}^{i}$ be an imaginary root. If $\dim \mathfrak{g}^{\alpha} = 1$, then $\mathfrak{g}^{\alpha} = \mathfrak{g}_{+}^{\alpha}$ (cf. Remark 3.8(2) (iii)). Let $Z_{\alpha} \in \mathfrak{g}^{\alpha}$ be one of the vectors in (5.15) or in (5.15)′. Then

$$[Z_{\alpha}, \tau_{p} Z_{\alpha}] = \pm h_{\alpha},$$

depending on whether α is a noncompact or a compact imaginary root. Assume now $\dim \mathfrak{g}^{\alpha} > 1$. The vectors in (5.16) in Remark 3.8 (2)(iv) satisfy

$$[Z_{\alpha}^{1}, \tau_{p} Z_{\alpha}^{1}] = [JZ_{\alpha}^{4}, \tau_{p} JZ_{\alpha}^{4}] = h_{\alpha}$$

and

$$[Z_{\alpha}^{2}, \tau_{p} Z_{\alpha}^{2}] = [JZ_{\alpha}^{3}, \tau_{p} JZ_{\alpha}^{3}] = -h_{\alpha},$$

respectively. This proves (2) and (3). It remains to show that $\pm h_{\alpha} \in \mathfrak{c}$ in all other cases, when $\dim \mathfrak{g}^{\alpha} > 1$. If $\mathfrak{g}_{+}^{\alpha}$ contains the subspace (5.12), the vectors $\{Z_{\alpha}^{1}, Z_{\alpha}^{2}\}$ in (5.13) are a pair of vectors $Z_{\alpha} \neq Z_{\alpha}' \in \mathscr{B}_{\alpha}$ such that

$$[Z_{\alpha}, \tau_{p} Z_{\alpha}] = -[Z_{\alpha}', \tau_{p} Z_{\alpha}'] = Ch_{\alpha},$$

where C is a nonzero real constant. The same is true for the vectors $\{Z_{\alpha}^{1}, Z_{\alpha}^{2}\}$ or $\{JZ_{\alpha}^{3}, JZ_{\alpha}^{4}\}$ in (5.14). If $\mathfrak{g}_{-}^{\alpha}$ contains the subspace (5.12), the vectors $\{Z_{\alpha}^{1}, Z_{\alpha}^{4}\}$ and $\{Z_{\alpha}^{2}, Z_{\alpha}^{3}\}$ in (5.13) satisfy conditions (5.13)′ and the spaces $Z^{1}[\alpha]$ and $Z^{2}[\alpha]$ are four dimensional. By Proposition 5.15(iii), one has that $\pm Jh_{\alpha} \in \mathfrak{c}$. The same is true for the vectors given in (5.17). The spaces $Z^{1}[\alpha]$ and $Z^{2}[\alpha]$ are four dimensional and, by Proposition 5.15(iii), one has that $\pm Jh_{\alpha} \in \mathfrak{c}$. This concludes the proof of (4).

Let $\alpha \in \Delta_{\mathfrak{c}}^{r}$ be a real root. If $Z_{\alpha} \in \mathfrak{g}^{\alpha}$ is a σ-invariant element, then $B(Z_{\alpha}, \tau_{p} Z_{\alpha})$ is either a real or a purely imaginary number, depending on whether $Z_{\alpha} \in \mathfrak{g}_{+}^{\alpha}$ or $Z_{\alpha} \in \mathfrak{g}_{-}^{\alpha}$. One can check that for each $\alpha \in (\Delta_{\mathfrak{c}}^{+})^{r}$ there exists an element Z_{α} for which $B(Z_{\alpha}, \tau_{p} Z_{\alpha}) \neq 0$ (cf. Remark 5.8(1)(i)–(1)(iv)). By Lemma 5.10(ii), this proves (5).

Since $\Delta_{\mathfrak{c}}$ is a root system, it follows from (1)–(5) that the cone $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ has nonempty interior.

Before proving the next proposition, we need to quickly review the definition of a compactly causal symmetric space and the properties of the associated symmetric algebra. We take as references [FO, HO, KN].

DEFINITION 5.19. Compactly causal symmetric spaces.

Let G/H be a semisimple pseudo-Riemannian symmetric space, i.e., a triple (G, H, τ), where G is a real semisimple Lie group, τ is an involution of G (commuting with the Cartan involution), and H is an open subgroup of the fixed point subgroup of τ in G. Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau)$ be the corresponding symmetric algebra. The space G/H is called "compactly causal" if $\mathfrak{q}$ contains Ad_H-stable regular convex cones V consisting of elliptic elements (regular means $V \cap -V = \{0\}$ and $\langle V \rangle = \mathfrak{q}$).

Remark 5.20.

- If G/H is irreducible, it is compactly causal if and only if G/K is a bounded symmetric domain and the induced involution τ: $G/K \to G/K$ is antiholomorphic.
- The symmetric Lie algebras arising from irreducible compactly causal symmetric spaces are precisely the ones where $\mathfrak{g}$ is simple hermitian and $\mathfrak{z}(\mathfrak{k})$ is contained in $\mathfrak{q}$. In particular, $\mathfrak{q}$ admits an elliptic maximal abelian subpace $\mathfrak{c} \subset \mathfrak{q}$ and $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c})$ is compactly embedded in $\mathfrak{g}$.

In the proof of the next proposition, we need a characterization of compactly causal symmetric Lie algebras in terms of the restricted root system $\Delta_{\mathfrak{c}}$. This was essentially done in [KN].

- Let $\mathfrak{g} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{h}_{\mathfrak{p}} \oplus \mathfrak{q}_{\mathfrak{k}} \oplus \mathfrak{q}_{\mathfrak{p}}$ be the combined decomposition of $\mathfrak{g}$ with respect to both τ and the Cartan involution θ (here $\mathfrak{h}_{\mathfrak{k}} := \mathfrak{h} \cap \mathfrak{k}$ etc.). Consider the Lie subalgebra $\mathfrak{r} = \mathfrak{q}_{\mathfrak{k}} \oplus [\mathfrak{q}_{\mathfrak{k}}, \mathfrak{q}_{\mathfrak{k}}] \subset \mathfrak{k}$. Let $\Delta_{\mathfrak{c}}$ be the restricted root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{c}^{\mathbb{C}}$. A root $\alpha \in \Delta_{\mathfrak{c}}$ is called compact if $\mathfrak{g}^{\alpha} \cap \mathfrak{r}^{\mathbb{C}} \neq \{0\}$, and noncompact otherwise. If α is a noncompact root, then $\mathfrak{g}^{\alpha}$ is contained in $\mathfrak{p}^{\mathbb{C}}$. Denote by $(\Delta_{\mathfrak{c}})_k$ and $(\Delta_{\mathfrak{c}})_n$ the compact and noncompact roots in $\Delta_{\mathfrak{c}}$, respectively. The root system $\Delta_{\mathfrak{c}}$ is called split if $\mathfrak{g}^{\alpha} \subset \mathfrak{k}^{\mathbb{C}}$ for all compact roots $\alpha \in (\Delta_{\mathfrak{c}})_k$. The Weyl group $W_H(\mathfrak{c}) = N_H(\mathfrak{c})/Z_H(\mathfrak{c})$ is isomorphic to the group W_k generated by the reflections in the compact roots ([KN, Definition III.9 and Proposition V.2.i]). If the positive noncompact roots $(\Delta_{\mathfrak{c}}^+)_n$ are stable under the Weyl group, the system $\Delta_{\mathfrak{c}}^+$ is called $\mathfrak{r}$-adapted.

Define the following cones in $\mathfrak{c}_{\mathbb{R}}$:

$$C_{\min} := \operatorname{cone}\big(\{h_\alpha\}_{\alpha \in (\Delta_{\mathfrak{c}}^+)_n}\big),$$
$$C_{\max} = (C_{\min})^* := \big\{X \in \mathfrak{c}_{\mathbb{R}} \mid B(X, h_\alpha) \geq 0,\ \ \alpha \in (\Delta_{\mathfrak{c}}^+)_n\big\}.$$

The symmetric algebra $(\mathfrak{g}, \tau)$ is compactly causal if and only if $\mathfrak{g}$ is hermitian, there exists an elliptic maximal abelian subspace $\mathfrak{c} \subset \mathfrak{q}$, and the restricted root system $\Delta_{\mathfrak{c}}$ (with respect to $\mathfrak{c}$) is split and admits an $\mathfrak{r}$-adapted positive system. In particular, $C_{\max}$ is W_k-stable (cf. [KN, Proposition V.10]).

• In the special case when the abelian subspace $\mathfrak{c} \subset \mathfrak{q}$ is also a compact Cartan subalgebra of $\mathfrak{g}$, one has that the symmetric algebra $(\mathfrak{g}, \tau)$ is compactly causal if and only if $\mathfrak{g}$ is hermitian. The root system $\Delta_{\mathfrak{c}}$ coincides the ordinary root system Δ; compact roots Δ_k (resp. noncompact roots Δ_n) are the ones for which the corresponding root space is contained in $\mathfrak{k}^{\mathbb{C}}$ (resp. $\mathfrak{p}^{\mathbb{C}}$). An $\mathfrak{r}$-adapted positive system is the usual $\mathfrak{k}$-adapted positive system for which the positive noncompact roots Δ_n^+ are stable under W_k, i.e., the reflections in the compact roots Δ_k (equivalently, every positive noncompact root is larger than an arbitrary compact root).

PROPOSITION 5.21. *Let S be a generic G-orbit in $G^{\mathbb{C}}/K^{\mathbb{C}}$, intersecting a standard Cartan subset*

$$\overline{C} = \overline{\exp J\mathfrak{c} \cdot p}$$

different from the fundamental Cartan subset (*i.e.*, $\mathfrak{c} \neq \mathfrak{a}$). *Let $\overline{x_0 \cdot p} \in S$ be a base point*, *where $x_0 \cdot p = \exp JX_0 \cdot p \in C$. Then*

$$\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$$

in all cases with only one possible exception: *assume that the Cartan subspace $\mathfrak{c}$ is compact*, *the base point p satisfies conditions* (2.6), *and the G-orbit of p is a compactly causal pseudo-Riemannian symmetric space*. *Then*, *if $X_0 \in \pm JC_{\max} \subset \mathfrak{c}$*, *the Levi cone $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ is sharp*.

Proof. Identify $\mathfrak{c}$ with $\mathfrak{c}_{\mathbb{R}} = J\mathfrak{c}_k \oplus \mathfrak{c}_p$ via the map $(T, A) \mapsto (-JT, A)$. Consider the image of the cone $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ in $\mathfrak{c}_{\mathbb{R}}$ under this map and denote it by $\mathscr{C}$. Denote by $\langle -, - \rangle$ the restriction of the Killing form to $\mathfrak{c}_{\mathbb{R}} \times \mathfrak{c}_{\mathbb{R}}$. Observe that the restricted root system $\Delta_{\mathfrak{c}}$ is always connected. When $\mathfrak{g}$ admits a complex structure, $\Delta_{\mathfrak{c}}$ is isomorphic to the ordinary root system Δ. We need to distinguish several cases.

• Assume that $\mathfrak{c}$ is a noncompact Cartan subspace, $\mathfrak{c} \neq \mathfrak{a}$. By Lemma 5.18, the cone $\mathscr{C}$ contains $\mathbb{R} \cdot h_\alpha$ for every complex root α. We want to show that $\mathscr{C}$ contains $\mathbb{R} \cdot h_\alpha$, for every simple root α. Since real and imaginary simple roots are strongly orthogonal, there exists a simple complex root $\alpha \in \Delta_{\mathfrak{c}}$. Let β be a real (resp. imaginary) simple root such that $\alpha + \beta \in \Delta_{\mathfrak{c}}$ for some complex simple root α. Then $\langle \alpha, \beta \rangle < 0$. If $\alpha + \beta$ is complex, then from $\mathbb{R} \cdot h_\alpha, \mathbb{R} \cdot h_{\alpha+\beta} \subset \mathscr{C}$, one obtains $\mathbb{R} \cdot h_\beta \subset \mathscr{C}$. If $\alpha + \beta$ is imaginary (resp. real), then $\langle \alpha + \beta, \beta \rangle = 0$ and the Cartan integer $a_{\alpha\beta}$ is equal to -2. At this point, there are two possibilities: either $|\alpha| > |\beta|$ or $|\alpha| < |\beta|$. In the first case, $\alpha + 2\beta$ is a root, while, in the second case, $2\alpha + \beta$ is a root. Both $\alpha + 2\beta$ and $2\alpha + \beta$ are complex roots, so either $\mathbb{R} \cdot h_{\alpha+2\beta}$ or $\mathbb{R} \cdot h_{2\alpha+\beta}$ is contained in $\mathscr{C}$. It follows that $\mathbb{R} \cdot h_\beta \subset \mathscr{C}$. If γ is a simple real (resp. imaginary) root such that $\beta + \gamma$ is a root, then $\alpha + \beta + \gamma$ is also a root and it is complex. It follows that $\mathbb{R} \cdot h_\gamma \subset \mathscr{C}$. Iterating these arguments we exhaust all simple roots and obtain $\mathscr{C} = \mathfrak{c}_{\mathbb{R}}$. It follows that $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$.

• Assume that $\mathfrak{c}$ is a compact Cartan subspace and that there exists a *noncompact* Cartan subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ extending $\mathfrak{c}$ (see Lemma 5.7). All roots in $\Delta_{\mathfrak{c}}$ are imaginary.

We claim that *there exists a simple root* $\alpha_0 \in \Delta_{\mathfrak{c}}$ *which is the restriction of a complex root* $\lambda \in \Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{l}^{\mathbb{C}})$. Let $\lambda_1, \ldots, \lambda_n$ be the simple roots in Δ. Recall that the restrictions of the simple roots in Δ form a basis of $\Delta_{\mathfrak{c}}$ and that the root system Δ contains a complex simple root. If none of the simple roots in Δ vanishes on $\mathfrak{c}$, the claim is immediate. Assume now that some simple roots in Δ vanish on $\mathfrak{c}$ and that all the simple roots which do not vanish on $\mathfrak{c}$ are purely imaginary. Then in Δ there exist simple roots $\lambda_{k_1}|\mathfrak{c} \equiv 0, \ldots, \lambda_{k_s}|\mathfrak{c} \equiv 0$ and $\lambda_m|\mathfrak{c} \not\equiv 0$ such that $\lambda_0 = \lambda_{k_1} + \cdots + \lambda_{k_s} + \lambda_m \in \Delta$ is a complex root restricting to a simple root $\alpha_0 \in \Delta_{\mathfrak{c}}$. This concludes the proof of the claim.

By Lemma 5.18(4), one has that $\mathbb{R} \cdot h_{\alpha_0} \subset \mathscr{C}$. Consider now the root $\lambda_1 = \lambda_0 + \lambda_{h_1} + \lambda_{h_t} + \lambda_p$, where $\{\lambda_{h_1}, \ldots, \lambda_{h_t}\}$ is a (possibly empty) set of simple roots in Δ vanishing on $\mathfrak{c}$ and λ_p is a simple root in Δ such that $\alpha_1 = \lambda_p|\mathfrak{c} \not\equiv 0$ is a simple root in $\Delta_{\mathfrak{c}}$. If $\alpha_1 \neq \alpha_0$, then $\alpha_0 + \alpha_1 \in \Delta_{\mathfrak{c}}$ and either α_1 or $\alpha_0 + \alpha_1$ is the restriction of a complex root. In both cases, $\mathbb{R} \cdot h_{\alpha_1} \subset \mathscr{C}$. If $\alpha_1 = \alpha_0$, then $2\alpha_0 \in \Delta_{\mathfrak{c}}$. In this case, consider the root $\lambda_2 = \lambda_1 + \lambda_{u_1} + \lambda_{u_n} + \lambda_q$, where $\{\lambda_{u_1}, \ldots, \lambda_{u_n}\}$ is a (possibly empty) set of simple roots in Δ vanishing on $\mathfrak{c}$ and λ_q is a simple root in Δ such that $\alpha_2 = \lambda_q|\mathfrak{c} \not\equiv 0$ is a simple root in $\Delta_{\mathfrak{c}}$. Then $\alpha_2 \neq \alpha_1 = \alpha_0$ and either α_2 or $\alpha_0 + \alpha_2$ is the restriction of a complex root. In both cases $\mathbb{R} \cdot h_{\alpha_2} \subset \mathscr{C}$. Iterating this argument, we obtain that $\mathbb{R} \cdot h_{\alpha_i} \subset \mathscr{C}$, for all simple roots in $\Delta_{\mathfrak{c}}$. Hence, $\mathscr{C} = \mathfrak{c}_{\mathbb{R}}$ and $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$.

• Assume that $\mathfrak{c}$ is a compact Cartan subspace and that every Cartan subalgebra of $\mathfrak{g}$ extending $\mathfrak{c}$ is compact. It follows that the Lie algebra $\mathfrak{g}$ is *equal-rank* and that the base point p satisfies conditions (2.6) (see Remark 3.10(3)(ii)). In particular, the G-orbit of p is a semisimple symmetric space G/H and $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{c})$ is compact. We claim that:

if G/H is compactly causal, then

- $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$ *for all* $X_0 \notin \pm JC_{\max}$,
- $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ *is sharp for all* $X_0 \in \pm JC_{\max}$;

if G/H is not compactly causal, then $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$.

We first deal with the case when $\mathfrak{c}$ itself is a compact Cartan subalgebra of $\mathfrak{g}$. Later we reduce the general case to this case.

If $\mathfrak{c}$ is a compact Cartan subalgebra of $\mathfrak{g}$, then by Remark 5.20 we have to show that:

if $\mathfrak{g}$ is hermitian, then

(a) $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$ *for all* $X_0 \notin \pm JC_{\max}$,

(b) $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ *is sharp for all* $X_0 \in \pm JC_{\max}$;

if $\mathfrak{g}$ *is not hermitian, then* $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$.

Simple "equal-rank" real Lie algebras admit a set of simple roots $\Pi = \Pi_k \cup \Pi_n$ with a unique noncompact root (cf. [W], Lemma 4, [Kn2], Appendix C).

Let $\{Z_\alpha\}_{\alpha\in\Delta}$, $Z_\alpha \in \mathfrak{g}^\alpha$, be a σ-stable set of root vectors. After a normalization we can assume

$$[Z_\alpha, \sigma Z_\alpha] = h_\alpha,\ \alpha \in \Delta_n^+ \qquad \text{and} \qquad [Z_\alpha, \sigma Z_\alpha] = -h_\alpha,\ \alpha \in \Delta_k^+.$$

Let W_k be the Weyl group generated by the reflections in the compact roots. Since Δ_k and Δ_n are W_k-stable, by Remark 2.6 the base point $\overline{x_0 \cdot p}$ can be assumed to satisfy:

$$\operatorname{Im} \alpha(X_0) > 0 \qquad \text{for all } \alpha \in \Delta_k^+.$$

Then, by Lemma 5.18, the cone $\mathscr{C}$ is the cone in $\mathfrak{c}_{\mathbb{R}}$ generated by the vectors

$$\left\{ -\frac{1}{\sinh 2 \operatorname{Im} \alpha(X_0)} h_\alpha \right\}_{\alpha\in\Delta_n^+}, \qquad \{h_\alpha\}_{\alpha\in\Delta_k^+}. \tag{5.19}$$

Hermitian case. Fix a $\mathfrak{k}$-adapted positive system Δ^+ (cf. Remark 5.20). Denote by Λ the corresponding highest root. Without loss of generality, we may assume $\operatorname{Im}\Lambda(X_0) > 0$ and only consider the cone $-C_{\max}$.

(a) If $X_0 \in -JC_{\max}$, then $\operatorname{Im}\alpha(X_0) > 0$ for all roots $\alpha \in \Delta^+$. In this case, the cone $\mathscr{C}$ is sharp. It is in fact the image of the dual of the positive Weyl chamber, under the reflections with respect to the highest roots of the simple factors of $\mathfrak{k}$.

(b) If $X_0 \notin -JC_{\max}$, then $\operatorname{Im}\alpha(X_0) < 0$, where α is the simple noncompact root. Since Δ^+ is $\mathfrak{k}$-adapted, all positive noncompact roots are obtained from α by adding simple compact roots. Since Λ is noncompact and $\operatorname{Im}\Lambda(X_0) > 0$, there exists a noncompact root μ such that $\operatorname{Im}\mu(X_0) > 0$. Assume μ is a root of minimal order with this property. Write

$$\mu = \alpha + \sum_{s=1}^{p} n_s \beta_s, \qquad n_s > 0, \tag{5.20}$$

with $\beta_s \in \Pi_k$. Let β be a root in Π_k such that $\mu - \beta$ is a noncompact root, with negative imaginary part on X_0. Consider the triplet of roots $\beta, \mu-\beta, \mu$. By (5.19),

$$h_\mu, -h_\beta, -h_{\mu-\beta} \in \mathscr{C} \qquad \text{and} \qquad \operatorname{span}_{\mathbb{R}}\{h_\mu, h_\beta\} \subset \mathscr{C}.$$

Next take $\gamma \in \Pi_k$ such that $\mu - \beta - \gamma$ is a noncompact root, with negative imaginary part on X_0. By the result of the previous step and the same argument, one has that

$$\mathbb{R} \cdot h_\gamma, \mathbb{R} \cdot h_{\mu-\beta}, \mathbb{R} \cdot h_{\mu-\beta-\gamma} \subset \mathscr{C}.$$

Subtracting simple roots from μ in this way, we finally obtain that $\{\mathbb{R} \cdot h_{\beta_s}, \mathbb{R} \cdot h_\alpha\}$ are contained in $\mathscr{C}$ for all simple roots which appear in (5.20). To obtain the same result for the remaining simple roots, we add them one by one to μ, until we obtain the highest root. Observe that the noncompact roots obtained in this way all have positive imaginary part on X_0. If γ is a root in Π_k, such that $\mu + \gamma \in \Delta$, consider the triplet of roots $\gamma, \mu, \mu + \gamma$. By the results of the previous steps, we have that

$$\mathbb{R} \cdot h_\gamma, \mathbb{R} \cdot h_\mu, \mathbb{R} \cdot h_{\mu+\gamma} \subset \mathscr{C}.$$

Iterating this argument until all simple roots are exhausted, we obtain statement (b).

Non-hermitian case. Fix Δ^+ a positive system with a unique noncompact simple root $\alpha \in \Pi_n$. In this case, the highest root Λ is compact and the coefficient in Λ of the root α is equal to 2 (cf. [Kn2, Appendix C]).

Assume first that all noncompact roots have positive imaginary part on X_0. Since the coefficient of α in Λ is equal to 2, there exists a compact root ν which is a sum of precisely two noncompact roots: $\nu = \lambda + \mu$. Observe that

$$-h_\nu, h_\lambda, h_\mu \in \mathscr{C} \qquad \text{and} \qquad \text{span}\{h_\lambda, h_\mu\} \subset \mathscr{C}.$$

The root λ (resp. μ) contains α with coefficient one and from λ one can construct the highest root by adding simple roots. If $\lambda + \beta \in \Delta$ for some $\beta \in \Pi_k$, then

$$\pm \cdot h_\lambda, -h_\beta, h_{\lambda+\beta} \in \mathscr{C} \qquad \text{and} \qquad \text{span}\{h_\lambda, h_\beta\} \subset \mathscr{C}.$$

When the noncompact root α is added, yielding a compact root, we obtain $\mathbb{R} \cdot h_\alpha \subset \mathscr{C}$.

Claim. If $\mathbb{R} \cdot h_\alpha \in \mathscr{C}$, then $\mathscr{C} = \mathfrak{c}_\mathbb{R}$ and $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$.

Let β be a root in Π_k such that $\alpha + \beta \in \Delta$. For the triplet of roots $\alpha, \beta, \alpha + \beta$, we have that

$$\pm h_\alpha, -h_\beta, h_{\alpha+\beta} \in \mathscr{C} \qquad \text{and} \qquad \text{span}_\mathbb{R}\{h_\alpha, h_\beta\} \subset \mathscr{C}.$$

If γ is a root in Π_k such that $\alpha + \beta + \gamma \in \Delta$, then consider the triplet of roots $\alpha + \beta, \gamma, \alpha + \beta + \gamma$. By the previous step and the same argument, one has that

$$\pm h_{\alpha+\beta}, -h_\gamma, h_{\alpha+\beta+\gamma} \in \mathscr{C} \qquad \text{and} \qquad \text{span}_\mathbb{R}\{h_{\alpha+\beta}, h_\gamma\} \subset \mathscr{C}.$$

By iterating this argument until all the simple roots are exhausted, the claim follows.

Assume now that $\alpha \in \Pi_n$ has negative imaginary part on X_0. Since there exists a compact root which is sum of noncompact roots, there exists a noncompact root with positive imaginary part on X_0. Let λ be a root of minimal order with this property. Then λ is of the form

$$\lambda = \alpha + \sum_{s=1}^{p} n_s \beta_s, \qquad n_s > 0,$$

i.e., it is obtained by adding simple compact roots to α. From now on the proof continues as in case (b).

Finally, assume that $\mathfrak{c}$ is a compact Cartan subspace, but not a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{l} = \mathfrak{b} \oplus \mathfrak{c}$, for some $\mathfrak{b} \subset \mathfrak{h}$, be a compact Cartan subalgebra of $\mathfrak{g}$ extending $\mathfrak{c}$ and let Δ be the corresponding root system. Let

$$\pi_{\mathfrak{q}}\colon \mathfrak{l}_{\mathbb{R}} \longrightarrow \mathfrak{c}_{\mathbb{R}}, \qquad \pi_{\mathfrak{q}}(X) = X - \tau_p X$$

be the projection of $\mathfrak{l}_{\mathbb{R}}$ on to $\mathfrak{c}_{\mathbb{R}}$. Since $h_\alpha = h_\lambda - \tau_p h_\lambda$ and $\alpha(X_0) = \lambda(X_0)$ for all $\lambda \in \Delta_\alpha$, the cone $\mathscr{C}$ is the $\pi_{\mathfrak{q}}$-projection of an appropriate cone $\mathscr{C}_{\mathfrak{l}} \subset \mathfrak{l}_{\mathbb{R}}$.

Assume that the G-orbit of p is a compactly causal symmetric space G/H. Fix an $\mathfrak{r}$-adapted positive system in $\Delta_{\mathfrak{c}}$ and compatible orderings for Δ^+ and $\Delta_{\mathfrak{c}}^+$. Since $(\Delta_{\mathfrak{c}}^+)_n$ is W_k-stable, without loss of generality, X_0 may be assumed to satisfy

$$\operatorname{Im} \alpha(X_0) > 0 \qquad \text{for all } \alpha \in (\Delta_{\mathfrak{c}}^+)_k.$$

By Lemma 5.18, then $\mathscr{C}$ is the cone in $\mathfrak{c}_{\mathbb{R}}$ generated by the vectors

$$\left\{ -\frac{1}{\sinh 2 \operatorname{Im} \alpha(X_0)} h_\alpha \right\}_{\alpha \in (\Delta_{\mathfrak{c}}^+)_n}, \qquad \{h_\alpha\}_{\alpha \in (\Delta_{\mathfrak{c}}^+)_k}. \tag{5.21}$$

Since the root system $\Delta_{\mathfrak{c}}$ is split, given $\alpha \in \Delta_{\mathfrak{c}}$, the corresponding root space $\mathfrak{g}^\alpha$ can be assumed to be contained either in $\mathfrak{k}^{\mathbb{C}}$ or in $\mathfrak{p}^{\mathbb{C}}$ (cf. Remark 5.20) and $\mathscr{C}_{\mathfrak{l}} \subset \mathfrak{l}_{\mathbb{R}}$ is the cone generated by the vectors

$$\left\{ \left\{ -\frac{1}{\sinh 2 \operatorname{Im} \lambda(X_0)} h_\lambda \right\}_{\substack{\lambda \in \Delta_\alpha \\ \alpha \in (\Delta_{\mathfrak{c}}^+)_n}}, \{h_\lambda\}_{\substack{\lambda \in \Delta_\alpha \\ \alpha \in (\Delta_{\mathfrak{c}}^+)_k}} \right\}$$
$$= \left\{ \left\{ -\frac{1}{\sinh 2 \operatorname{Im} \lambda(X_0)} h_\lambda \right\}_{\lambda \in \Delta_n^+}, \{h_\lambda\}_{\lambda \in \Delta_k^+} \right\}.$$

If $X_0 \in -JC_{\max}$, then $\operatorname{Im} \alpha(X_0) > 0$ for all positive noncompact roots in $\Delta_{\mathfrak{c}}$, and

$$\mathscr{C}_{\mathfrak{l}} = \operatorname{cone}\left(\{-h_\lambda\}_{\lambda \in \Delta_n^+}, \{h_\lambda\}_{\lambda \in \Delta_k^+} \right).$$

The cone $\mathscr{C}_{\mathfrak{l}} \subset \mathfrak{l}_{\mathbb{R}}$ is sharp, by (a), and $(-\tau_p)$-stable (in particular has intersection with $\mathfrak{b}_{\mathbb{R}}$ equal to $\{0\}$). It follows that $\mathscr{C}$ and $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ are sharp as well.

If $X_0 \notin -JC_{\max}$, then there exists a noncompact root $\alpha_0 \in (\Delta_{\mathfrak{c}}^+)_n$ such that $\operatorname{Im} \alpha_0(X_0) < 0$. In this case,

$$\mathscr{C}_{\mathfrak{l}} = \operatorname{cone}\Big(\{-c_\lambda h_\lambda\}_{\lambda \in \Delta_n^+}, \{h_\lambda\}_{\lambda \in \Delta_k^+}\Big) \subset \mathfrak{l}_{\mathbb{R}},$$

where at least the coefficients c_λ, $\lambda \in \Delta_{\alpha_0}$, are negative. By (b), one has that $\mathscr{C}_{\mathfrak{l}} = \mathfrak{l}_{\mathbb{R}}$ and therefore $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$.

Assume now that G/H is not compactly causal. If $\Delta_{\mathfrak{c}}$ is spilt, then

$$\mathscr{C}_{\mathfrak{l}} = \operatorname{cone}\left(\{h_\lambda\}_{\lambda \in \Delta_k^+}, \left\{-\frac{1}{\sinh 2 \operatorname{Im} \lambda(X_0)} h_\lambda\right\}_{\lambda \in \Delta_n^+}\right). \tag{5.22}$$

One has that $\mathscr{C}_{\mathfrak{l}} = \mathfrak{l}_{\mathbb{R}}$ and $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$. If $\Delta_{\mathfrak{c}}$ is not split,

$$\mathscr{C}_{\mathfrak{l}} \supseteq \operatorname{cone}\left(\{h_\lambda\}_{\lambda \in \Delta_k^+}, \left\{-\frac{1}{\sinh 2 \operatorname{Im} \lambda(X_0)} h_\lambda\right\}_{\lambda \in \Delta_n^+}\right).$$

Hence $\mathscr{C}_{\mathfrak{l}} = \mathfrak{l}_{\mathbb{R}}$ and $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$.

6. APPLICATIONS TO INVARIANT DOMAINS

In this section, we apply the calculation of the Levi cone of the generic orbits obtained in Section 5 to the study of invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and their invariant plurisubharmonic functions. The results of Propositions 5.6 and 5.21 are summarized in Proposition 6.1. For simplicity, the result is stated in the irreducible case. Remark 6.2 deals with the general case.

PROPOSITION 6.1. *Let G/K be an irreducible Riemannian symmetric space of the noncompact type. Let S be a generic G-orbit in $G^{\mathbb{C}}/K^{\mathbb{C}}$ intersecting $\overline{C}$, where $C = \exp J\mathfrak{c} \cdot p$ is a Cartan subset. Let $\overline{x_0 \cdot p} \in \overline{C}$ be a reference point for S, where $x_0 \cdot p = \exp JX_0 \cdot p$, with $X_0 \in \mathfrak{c}$. Let $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ be the corresponding Levi cone. Then*

$$\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$$

in all cases with the following exceptions:

(i) $\overline{C} = \overline{A}$ *is the fundamental Cartan subset and X_0 satisfies the conditions $|\alpha(X_0) - \alpha(V_0)| < \pi/2$ for all $\alpha \in \Delta_{\mathfrak{a}}$, for some vertex $V_0 \in \mathfrak{a}$ defined by $\alpha(V_0) \equiv 0 \bmod \pi$, $\forall\, \alpha \in \Pi_{\mathfrak{a}}$ (cf. Remark 5.5).*

(ii) $\mathfrak{c}$ *is a compact Cartan subspace*, $\mathfrak{g}$ *is a hermitian simple Lie algebra*, $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau_p)$ *is a compactly causal symmetric pair, and* $X_0 \in \pm JC_{\max} \subset \mathfrak{c}$ (*cf. Remark* 5.20).

In cases (i) *and* (ii) *the cone* $\mathscr{C}(S)_{\overline{x_0 \cdot p}}$ *has nonempty interior and it is sharp.*

Remark 6.2. If G/K is not irreducible, there is a decomposition $G/K = G_1/K_1 \times \cdots \times G_n/K_n$ into irreducible factors, where each G_i is a real or complex simple Lie group. Likewise, there is a decomposition of the complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ and of the G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$. One has that the Levi cone of a generic orbit $S = S_1 \times \cdots \times S_n$ is isomorphic to the direct sum of the Levi cones of the orbits S_i. In particular, it is sharp if and only if all summands are.

Combining Proposition 6.1 with Theorem 1.5, we obtain the main results of the paper.

COROLLARY 6.3. *An invariant domain* $\Omega \subset G^{\mathbb{C}}/K^{\mathbb{C}}$, *which contains in its boundary a generic orbit* S *with Levi cone* $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$, *cannot be Stein. In other words, only the generic orbits satisfying condition* (i) *or condition* (ii) *in Proposition* 6.1 *can be contained in the boundary of an invariant Stein domain.*

Proof. By Theorems 2.8 and 1.5, a generic orbit S satisfying $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$ admits an invariant tubular neighborhood $U(S)$ with the following properties: $U(S)$ consists of generic orbits and every smooth CR-function on S extends to a holomorphic function on $U(S)$. It is important to observe that the size of $U(S)$ does not depend on the function to be extended, but depends continuously on the CR-structure of S. In our case, the CR-structure of S depends in a real analytic way on the reference point $\overline{x_0 \cdot p} \in \overline{C}$ (see Corollaries 3.7 and 3.17). By the above facts, if a domain Ω contains S in its boundary, all holomorphic functions on Ω extend to a larger domain. In particular, Ω is not Stein. For more details, we refer to Corollaries 5.6 and 5.7 in [FG].

COROLLARY 6.4. *A generic orbit* S *with Levi cone* $\mathscr{C}(S)_{\overline{x_0 \cdot p}} = \mathfrak{c}$ *cannot be contained in the level set of a nonconstant invariant plurisubharmonic function.*

The above results show that when G/K is a noncompact Riemannian symmetric space, invariant Stein domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ cannot be arbitrarily large, but have to lie in some distinguished regions. We hereby describe them.

The region $\overline{\mathbf{X}}_0 = \overline{G \cdot A}$ associated to the fundamental Cartan subset

The region $\overline{\mathbf{X}}_0$ associated to the fundamental Cartan subset coincides with the region introduced and studied in [AG]. It is constructed as follows. Denote by $G \times_K \mathfrak{p}$ the G-equivariant bundle over G/K defined as the quotient of $G \times \mathfrak{p}$ by the equivalence relation $(g, v) \sim (gk^{-1}, \mathrm{Ad}_k v)$ for $k \in K$. The bundle $G \times_K \mathfrak{p}$ is equivariantly diffeomorphic to the tangent bundle $T(G/K)$ of G/K. Consider the G-equivariant map

$$\phi\colon G \times_K \mathfrak{p} \longrightarrow G^{\mathbb{C}}/K^{\mathbb{C}}, \qquad [g, v] \longmapsto \overline{g \exp Jv \cdot e}.$$

The map ϕ is singular on the G-invariant subset $\mathcal{S}$ intersecting $\mathfrak{a}$ in the family of hyperplanes $\{H \in \mathfrak{a} \mid \alpha(H) \equiv \pi/2 \bmod \pi\}_{\alpha \in \Delta_{\mathfrak{a}}}$. The restriction of ϕ to each connected component of $G \times_K \mathfrak{p} \setminus \mathcal{S}$ is a diffeomorphism, onto its image. The map ϕ is not surjective. The region $\overline{\mathbf{X}}_0$ is by definition the image of ϕ and coincides with the set of G-orbits intersecting the compact dual symmetric space U/K, embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as the U-orbit of the base point $\bar{e}$. In general, the region $\overline{\mathbf{X}}_0$ contains several copies of the symmetric space G/K. One is the G-orbit of the base point $\bar{e}$; the other ones are the G-orbits of the (finitely many) points $\{\bar{v}_1, \ldots, \bar{v}_m\}$ forming the image under ϕ of the set $\mathcal{A} = \{V \in \mathfrak{a} \mid \alpha(V) \equiv 0 \bmod \pi, \forall\, \alpha \in \Delta_{\mathfrak{a}}\}$ (cf. Remark 3.5).

Consider the bounded $W_K(\mathfrak{a})$-invariant convex set in $\mathfrak{a}$

$$\omega_0 = \{H \in \mathfrak{a} \mid |\alpha(H)| < \pi/2, \forall\, \alpha \in \Delta_{\mathfrak{a}}\}$$

and define $\Omega_0 := G \times_K \mathrm{Ad}_K \omega_0$. Then $D_0 := \phi(\Omega_0) = \overline{G \cdot \exp J\omega_0}$ is an open G-invariant domain in $\overline{\mathbf{X}}_0$ containing G/K and diffeomorphic to a tubular neighborhood of G/K. By construction, D_0 is the largest connected invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ admitting a retraction to G/K. Similarly, each copy of G/K is contained in a G-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ diffeomorphic to D_0, namely $D_j = \overline{G \cdot \exp J\omega_j}$, where $\omega_j = \{H \in \mathfrak{a} \mid |\alpha(H) - \alpha(V_j)| < \pi/2, \forall\, \alpha \in \Delta_{\mathfrak{a}}\}$ for some $V_j \in \mathfrak{a}$.

The boundaries of $D_0, \ldots, D_m$ are contained in $\mathcal{S}$ and entirely consist of nongeneric orbits. In general, the complement of $D_0 \cup \cdots \cup D_m$ in $\overline{\mathbf{X}}_0$ has nonempty interior.

The results of Proposition 5.6 imply that, among the generic orbits in $\overline{\mathbf{X}}_0$, only the ones contained in $D_0 \cup \cdots \cup D_m$ can lie in the boundary of an invariant Stein domain or on a level set of an invariant plurisubharmonic function.

The region $\overline{\mathbf{X}}^{\mathrm{caus}} = \overline{G \cdot C}$ associated to a Cartan subset with a compactly causal minimal orbit

Let $C = \exp J\mathfrak{c} \cdot p$ be a Cartan subset, where $\mathfrak{c}$ is a compact Cartan subspace and the G-orbit G/H of the base point p is a compactly causal symmetric space. This only occurs when G is of hermitian type and, in addition, the involution τ_p is an antiholomorphic automorphism of the hermitian symmetric space G/K (cf. Remark 5.20). When this is the case, let W (resp. $-W$) be the maximal Ad_H-invariant regular elliptic cone in the tangent space $T(G/H)_p \cong \mathfrak{q}$. Consider then the G-equivariant map

$$\psi\colon G \times_H W \longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}}, \qquad [g, v] \longmapsto \overline{g \cdot \exp iv \cdot p}.$$

The image of ψ is an invariant domain S_W in the region $\overline{\mathbf{X}}^{\mathrm{caus}} = \overline{G \cdot C}$, containing the symmetric space G/H in its boundary.

The results of Proposition 5.21 imply that, among the generic orbits in $\overline{\mathbf{X}}^{\mathrm{caus}}$, only the ones contained in S_W or in S_{-W} can lie in the boundary of an invariant Stein domain or on a level set of an invariant plurisubharmonic function. The Stein domains in S_W (resp. in S_{-W}) have been characterized in [Ne], as well as their invariant plurisubharmonic functions. In particular, the domains S_W and S_{-W} have been proved to be Stein. In general, there may be several Cartan subsets $C_1, \dots, C_s$ with the above properties and likewise domains $S_{\pm W_1}, \dots, S_{\pm W_s}$.

Another way of formulating the results of Corollaries 6.3 and 6.4 is the following.

Corollary 6.5. *Let Ω be an invariant Stein domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$, containing a generic orbit S in its boundary. Then S satisfies either condition* (*i*) *or condition* (*ii*) *in Proposition* 6.1 *and Ω is contained either in $\overline{\mathbf{X}}_0$ or in one of the regions $\overline{\mathbf{X}}^{\mathrm{caus}}$. More precisely, Ω is contained either in one of the domains $D_0, \dots, D_m$ or in one of the domains $S_{\pm W_1}, \dots, S_{\pm W_s}$. The same holds for a domain Ω admitting nonconstant invariant plurisubharmonic functions.*

We illustrate the above results by analyzing the rank-1 case.

Example 6.6 (The rank-1 case). A complete list of rank-1 noncompact Riemannian symmetric spaces G/K is the following:

$$H^n(\mathbb{R}) = SO_0(n,1)/SO(n),\ n \geq 1, \qquad H^n(\mathbb{C}) = SU(n,1)/U(n),\ n \geq 2,$$
$$H^n(\mathbb{H}) = Sp(n,1)/Sp(n) \times Sp(1),\ n \geq 1, \qquad H^2(\mathbb{C}ay) = F_4^*/Spin(9)$$
$$SL(2,\mathbb{C})/SU(2).$$

In this case, the generic G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ are real hypersurfaces.

The space $H^1(\mathbb{R}) \cong SO_0(1,1)$ is one dimensional, so every invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ is automatically Stein.

The space $H^2(\mathbb{R}) \cong H^1(\mathbb{C}) \cong SL(2,\mathbb{R})/SO(2)$ is two dimensional. This case will be studied separately in Example 6.8.

(1) Let G/K be one of the spaces $H^n(\mathbb{R})$, $n \geq 3$, $SL(2\mathbb{C})/SU(2)$.

The restricted root system of $\mathfrak{g}$ is reduced and consists of two roots $\Delta_\mathfrak{a} = \{\pm\alpha\}$. There are two Cartan subspaces in $\mathfrak{g}$: the fundamental Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ and a compact Cartan subspace $\mathfrak{c} = \mathbb{R}(X_\alpha + \theta X_\alpha)$, $X_\alpha \in \mathfrak{g}^\alpha$, corresponding to the orthogonal system $\{\alpha\} \subset \Delta_\mathfrak{a}$.

The generic orbits intersecting the fundamental Cartan subset $\overline{A}$ are parametrized by the set

$$\{T \in \mathfrak{a} \mid \alpha(T) \in \,]0, \pi/2[\cup]\pi, 3\pi/2[\} \subset \mathfrak{a}.$$

By Remark 5.5 and Proposition 5.6, the Levi cone of these orbits is sharp. Define the following invariant domains in $\overline{\mathbf{X}}_0$

$$D_0 = \overline{G \exp JT \cdot e} \qquad \text{and} \qquad D_1 = \overline{G \exp JT \cdot p_1}, \tag{6.1}$$

where $\alpha(T) \in [0, \pi/2[$ and the point $p_1 = e^{JP_1}$ satisfies $\alpha(P_1) = \pi$. In particular, the G-orbit or $\bar{p}_1$ is of type G/K. The domains D_0 and D_1 are connected tubular neighborhoods of the symmetric space G/K, given as the G-orbit of $\bar{e}$ and of $\bar{p}_1$, respectively. Each generic orbit in D_0 or D_1 bounds an invariant Stein subdomain containing G/K. Both D_0 and D_1 are Stein. Moreover, they admit invariant plurisubharmonic functions [LS].

To the compact Cartan subspace $\mathfrak{c}$, there corresponds the Cartan subset $\overline{C} = \overline{\exp J\mathfrak{c} \cdot p_2}$, where $p_2 = e^{JP_2}$ is defined by the condition $\alpha(P_2) = \pi/2$. The G-orbit G/H of $\bar{p}_2$ has minimal dimension and is a non-Riemannian symmetric space. In none of the cases considered here is G/H compactly causal. For example, if $G/K = H^n(\mathbb{R})$, the space G/H is isomorphic to $SO_0(n, 1)/SO(n-1, 1)$. The Levi cone of all the generic orbits intersecting $\overline{C}$ coincides with $\mathfrak{c}$. As a consequence, the invariant region $\overline{\mathbf{X}}_1 = \overline{G \cdot C}$ contains no proper invariant Stein subdomains and admits no nonconstant invariant plurisubharmonic functions.

(2) Let G/K be one of the spaces $H^n(\mathbb{C})$, $n \geq 2$, $H^n(\mathbb{H})$, $H^2(\mathbb{C}ay)$.

The restricted root system of these spaces is nonreduced and consists of four roots $\Delta_\mathfrak{a} = \{\pm\alpha, \pm 2\alpha\}$. There are three Cartan subspaces in $\mathfrak{g}$: the fundamental Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ and two compact standard Cartan subspaces $\mathfrak{c}$ and $\mathfrak{c}'$, corresponding to the orthogonal system $\{\alpha\}$ and $\{2\alpha\}$, respectively.

The generic orbits intersecting the fundamental Cartan subset $\overline{A}$ are parametrized by the subset

$$\{T \in \mathfrak{a} \mid \alpha(T) \in \,]0, \pi/4[\cup]\pi/4, \pi/2[\} \subset \mathfrak{a}.$$

By Proposition 5.6, the Levi cone of these orbits is sharp when $\alpha(T) \in \,]0, \pi/4[$, while it coincides with $\mathfrak{a}$ when $\alpha(T) \in \,]\pi/4, \pi/2[$. Define D_0 as in (6.1) for $\alpha(T) \in [0, \pi/4[$. The domain D_0 is a connected tubular neighborhood of the symmetric space G/K, and each generic orbit in D_0 bounds an invariant Stein subdomain containing G/K. The domain D_0 is itself Stein. Moreover, D_0 admits invariant plurisubharmonic functions [LS]. In contrast with the previous case, the complement of D_0 in $\overline{\mathbf{X}}_0$ has nonempty interior and admits no nonconstant invariant plurisubharmonic functions or Stein subdomains.

To the Cartan subspace $\mathfrak{c}$ there corresponds the Cartan subset $\overline{C} = \exp J\mathfrak{c} \cdot p_2$, where $p_2 = e^{JP_2}$ satisfies $\alpha(P_2) = \pi/2$. The G-orbit G/H of $\bar{p}_2$ has minimal dimension and is a non-Riemannian symmetric space.

To the Cartan subspace $\mathfrak{c}'$ there corresponds a Cartan subset $\overline{C}' = \exp J\mathfrak{c}' \cdot p_3$, with $p_3 = e^{JP_3}$ satisfying $\alpha(P_3) = \pi/4$. The G-orbit of the point $\bar{p}_3$ is not totally real and has only locally minimal dimension. The Levi cone of the generic G-orbits intersecting the Cartan subset $\overline{C}$ and the ones intersecting $\overline{C}'$ coincide with $\mathfrak{c}$ and $\mathfrak{c}'$, respectively. As a consequence, none of the corresponding invariant regions in $G^{\mathbb{C}}/K^{\mathbb{C}}$ admits invariant plurisubharmonic functions nor invariant Stein subdomains.

EXAMPLE 6.7 (The space $\mathbf{X} = SL(2,\mathbb{C})/SO(2,\mathbb{C})$). The manifold $\mathbf{X}$ has complex dimension 2 and can be identified with the space of complex symmetric unimodular matrices

$$\mathbf{X} \cong \left\{ Z = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix} \in M(2,2,\mathbb{C}) \mid Z = Z^t,\ \det Z = 1 \right\},$$

where $SL(2,\mathbb{R})$ acts by $(g, Z) \mapsto gZg^t$. Generic orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ are nondegenerate real hypersurfaces of CR-dimension equal to 1 and their Levi cone is always sharp. As a consequence, they bound invariant Stein domains in $\mathbf{X}$. We use the G-invariant function

$$F\colon \mathbf{X} \longrightarrow \mathbb{R}, \qquad F(Z) := -\det\ \mathrm{Re}(Z) = -\frac{1}{4}((z_1+\bar{z}_1)(z_2+\bar{z}_2) - (z_3+\bar{z}_3)^2)$$

to determine such domains explicitly. The level sets of F are in fact G-invariant hypersurfaces in $\mathbf{X}$ consisting of finitely many G-orbits. Denote by $F_R = \{Z \in \mathbf{X} \mid F(Z) = R\}$ the R-level set of F. For every $R \neq -1, 0$, the set F_R is a regular hypersurface. Observe that the complex Hessian of F is everywhere given by

$$-\frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

If $-1 < R < 0$, the level set F_R is disconnected and consists of two G-orbits, S_0 with base point

$$x_0 = \begin{pmatrix} \cos t + i\sin t & 0 \\ 0 & \cos t - i\sin t \end{pmatrix}, \qquad \text{satisfying } \cos^2 t = -R,\ t \in]0, \pi/2[,$$

and S_1 with base point

$$x_1=\begin{pmatrix}\cos t+i\sin t & 0\\ 0 & \cos t-i\sin t\end{pmatrix},\qquad \text{satisfying } \cos^2 t=-R,\, t\in\,]\pi,3\pi/2[.$$

These are generic orbits intersecting the fundamental Cartan subset $\overline{A}$, where $A=\exp J\left(\begin{smallmatrix}t & 0\\ 0 & -t\end{smallmatrix}\right)$, $t\in\mathbb{R}$. Call $f(Z)=\det(Z)$. The complex tangent space to S_0 at x_0 is given by

$$T_{\mathbb{C}}(S_0)_{x_0}=\{Z=(z_1,z_2,z_3)\mid \partial f_{x_0}\cdot Z=0,\partial F_{x_0}\cdot Z=0\}=\{Z=(0,0,z_3)\},$$

and the Levi form at x_0 is given by $\mathbf{L}(Z,Z)=\frac{1}{2}|z_3|^2>0$. The same result holds for S_1. The domain

$$\Omega_R^-=\{Z\in\mathbf{X}\mid F(Z)-R<0\},\qquad R=F(x_0)=F(x_1)<0,$$

is Stein and consists of two connected components bounded by S_0 and S_1, respectively; each of them is a tubular neighborhood of a minimal orbit of type G/K.

If $R>0$, the level set F_R is disconnected and consists of two G-orbits S_0 and S_1, with base points

$$x_0=\begin{pmatrix}i\cosh t & \sinh t\\ \sinh t & -i\cosh t\end{pmatrix},\qquad \sinh^2 t=R,\, t\in\mathbb{R}^+,$$

$$x_1=\begin{pmatrix}i\cosh t & \sinh t\\ \sinh t & -i\cosh t\end{pmatrix},\qquad \sinh^2 t=R,\, t\in\mathbb{R}^-,$$

respectively. These are generic orbits intersecting the Cartan subset $\overline{C}$, where $C=\exp J\left(\begin{smallmatrix}0 & t\\ -t & 0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}i & 0\\ 0 & -i\end{smallmatrix}\right)$, $t\in\mathbb{R}$. The complex tangent space to the orbit at x_0 is given by

$$T_{\mathbb{C}}(S_0)_{x_0}=\{Z=(z_1,z_2,z_3)\mid \partial f_{x_0}\cdot Z=0,\partial F_{x_0}\cdot Z=0\}=\{Z=(z_1,z_1,0)\},$$

and the Levi form at x_0 is given by $\mathbf{L}(Z,Z)=-|z_1|^2<0$. The same result holds for S_1. The domain

$$\Omega_R^+=\{Z\in\mathbf{X}\mid F(Z)-R>0\},\qquad R=F(x_0)=F(x_1)>0,$$

is Stein and consists of two connected components, bounded by S_0 and S_1, respectively. Observe that the orbit of the base point $p=\left(\begin{smallmatrix}i & 0\\ 0 & -i\end{smallmatrix}\right)$ is the non-Riemannian symmetric space $G/H=SL(2,\mathbb{R})/SO(1,1)$. The space G/H is compactly causal and contains proper Ad_H-invariant regular elliptic convex cones in its tangent space $\mathfrak{q}$. Denote by $\pm W$ the maximal ones among such cones. Then one of the connected components of Ω_R^+ is contained in $S_W=\overline{G\cdot\exp JW}$, and the other one in $S_{-W}=\overline{G\cdot\exp J(-W)}$.

Our techniques do not apply to domains whose boundary entirely consists of nongeneric orbits. The domains $D_0 \ldots, D_m$ are among them and at the moment their Steinness remains an open question. We want to give a bound on the number of invariant domains $D \subset \mathbf{X}$, whose boundary ∂D contains no generic orbits and which are possibly Stein. In order to do this, we need to analyze the complement in $G^{\mathbb{C}}/K^{\mathbb{C}}$ of the union of all generic orbits.

The G-action on $X = G^{\mathbb{C}}/K^{\mathbb{C}}$ fits in the framework of [Lu]: consider the complex affine algebraic variety $X^{\mathbb{C}} := G^{\mathbb{C}}/K^{\mathbb{C}} \times G^{\mathbb{C}}/K^{\mathbb{C}}$ with the real structure given by $\sum(x_1, x_2) := (\sigma(x_2), \sigma(x_1))$. The action of the complex algebraic group $G^{\mathbb{C}}$ on $X^{\mathbb{C}}$, given by $g \cdot (x_1, x_2) := (gx_1, gx_2)$, is defined over $\mathbb{R}$. The restricted action of G on the set of real points $\{(x, \sigma(x)) \in X^{\mathbb{C}}\} \cong G^{\mathbb{C}}/K^{\mathbb{C}}$ corresponds to the left translation action of G on $X = G^{\mathbb{C}}/K^{\mathbb{C}}$.

In this framework, one can consider the map $p\colon X \to X\|G$, which associates to each point $x \in X$ the unique closed orbit in the closure of $G \cdot x$. Here $X\|G$ denotes the set of closed G-orbits in X. Each fiber of p contains a unique closed orbit, which is also the unique orbit of minimum dimension in the fiber.

A subset of X is said to be G-saturated, if it is the counterimage of a subset of $X\|G$. Let $x \in X$ be a point on a closed G-orbit S. Then the isotropy subgroup G_x is reductive and there exists a G-saturated neighborhood of x in X which is real-analytically diffeomorphic to $G \times_{G_x} W$, where W is some open G_x-stable neighborhood of 0 in a complement $W_{\bar{x}}$ of TS_x in TX_x (see [Br, Lu]).

Let H be the G-isotropy subgroup of some point on a closed G-orbit. By definition, the H-stratum $X^{[H]}$ in X consists of the points in X having in their p-fiber a minimal closed orbit of type H.

It turns out that $X^{[H]}$ is a locally closed subset of X, and the smooth points of $X^{[H]}$ form a dense subset. Let $x \in X^{[H]}$ be a nonsingular point on a closed G-orbit. Then there exists a G-saturated neighborhood of x in $X^{[H]}$ which is G-equivariantly diffeomorphic to $G \times_H (W^H \times \mathcal{N})$, where W^H denotes the fixed point set of H on W, and $\mathcal{N}$ denotes the null cone of the H-action on a complement of W^H in W (see [BF, Br]).

In our particular situation, let $\bar{x} \in G^{\mathbb{C}}/K^{\mathbb{C}}$ be a point sitting on a closed G-orbit. To the point $\bar{x}$ there is associated the symmetric pair $((G^{\mathbb{C}})^{\sigma\tau_x}, \sigma)$ (see [Ma1]). Observe that the fixed point subgroup of σ in $(G^{\mathbb{C}})^{\sigma\tau_x}$ is precisely the isotropy subgroup of $\bar{x}$ in G, namely $G_{\bar{x}} = G \cap \mathrm{Ad}_x K^{\mathbb{C}}$. Let

$$\begin{aligned} ((\mathfrak{g}^{\mathbb{C}})^{\sigma\tau_x}, \sigma) &= ((\mathfrak{g}^{\mathbb{C}})^{\sigma} \cap (\mathfrak{g}^{\mathbb{C}})^{\tau_x} \oplus (\mathfrak{g}^{\mathbb{C}})^{-\sigma} \cap (\mathfrak{g}^{\mathbb{C}})^{-\tau_x}, \sigma) \\ &= (\mathfrak{g} \cap \mathrm{Ad}_x \mathfrak{k}^{\mathbb{C}} \oplus J\mathfrak{g} \cap \mathrm{Ad}_x \mathfrak{p}^{\mathbb{C}}, \sigma) \end{aligned} \tag{6.2}$$

be the corresponding symmetric algebra. Then the subgroup $G_{\bar{x}}$ acts on the vector space $(\mathfrak{g}^{\mathbb{C}})^{-\sigma} \cap (\mathfrak{g}^{\mathbb{C}})^{-\tau_x}$ by the adjoint representation.

LEMMA 6.8. *Let $\bar{x} \in G^{\mathbb{C}}/K^{\mathbb{C}}$ be a point on a closed G-orbit S. Then $W_{\bar{x}} = (\mathfrak{g}^{\mathbb{C}})^{-\sigma} \cap (\mathfrak{g}^{\mathbb{C}})^{-\tau_x} = J\mathfrak{g} \cap Ad_x\mathfrak{p}^{\mathbb{C}}$ is a $G_{\bar{x}}$-stable complement of $TS_{\bar{x}}$ in $T(G^{\mathbb{C}}/K^{\mathbb{C}})_{\bar{x}}$, where $G_{\bar{x}}$-action on $W_{\bar{x}}$ coincides with the adjoint representation.*

Proof. We need to show that

$$T(G^{\mathbb{C}}/K^{\mathbb{C}})_{\bar{x}} = TS_{\bar{x}} \oplus W_{\bar{x}}. \tag{6.3}$$

As we just observed, both terms of the above decomposition are stable under the adjoint representation of $G_{\bar{x}}$.

Consider a point on the fundamental Cartan subset $\bar{x}_0 \in \overline{A}$, where $x_0 = e^{JH_0} \in A = \exp J\mathfrak{a}$. If the G-orbit of $\bar{x}_0$ is generic, then by Remark 3.6 and Corollary 3.7,

$$J\mathfrak{g} \cap \mathrm{Ad}_{x_0}\mathfrak{p}^{\mathbb{C}} = J\mathfrak{a} \qquad \text{and} \qquad T(G^{\mathbb{C}}/K^{\mathbb{C}})_{\bar{x}_0} = \mathrm{Ad}_{x_0}\mathfrak{p}^{\mathbb{C}} = TS_{\bar{x}_0} \oplus J\mathfrak{a}.$$

Hence (6.3) holds. Assume that the G-orbit of $\bar{x}_0$ is nongeneric. Resume the notation of Section 3.2 and let $\alpha \in \Delta_{\mathfrak{a}}$ be a root such that $\alpha(H_0) \equiv 0 \bmod \pi/2$ (cf. Remark 3.5(ii)). By formulas (3.3) and (3.4), one has that

$$\begin{cases} \cos\alpha(H_0) = 0 \\ \sin\alpha(H_0) = \pm 1 \end{cases} \Longrightarrow \begin{cases} JF_\alpha = \pm K_\alpha = \pm K_\alpha^* \in TS_{\bar{x}_0} \\ F_\alpha = \pm JK_\alpha \in J\mathfrak{g} \cap \ \mathrm{Ad}_{x_0}\mathfrak{p}^{\mathbb{C}} = W_{x_0} \end{cases}.$$

In the same way, one has that

$$\begin{cases} \sin\alpha(H_0) = 0 \\ \cos\alpha(H_0) = \pm 1 \end{cases} \Longrightarrow \begin{cases} F_\alpha = \pm P_\alpha = \pm P_\alpha^* \in TS_{\bar{x}_0} \\ JF_\alpha = \pm JP_\alpha \in J\mathfrak{g} \cap \ \mathrm{Ad}_{x_0}\mathfrak{p}^{\mathbb{C}} = W_{x_0} \end{cases}.$$

Since $\mathrm{Ad}_{x_0}\mathfrak{p}^{\mathbb{C}} = \mathfrak{a}^{\mathbb{C}} \oplus \bigoplus_{\alpha,\, Z_\alpha} \mathrm{Ad}_{x_0} Z[\alpha]_{\mathfrak{p}}^{\mathbb{C}}$, the above statements prove (6.3).

Consider now a point on a standard Cartan subset $\overline{x_0 \cdot p} \in \overline{C}$ for $x_0 \cdot p = e^{JX_0} \cdot p \in C = \exp J\mathfrak{c} \cdot p$. If the G-orbit of $\overline{x_0 \cdot p}$ is generic, then by Remark 3.16(ii) and Corollary 3.17,

$$J\mathfrak{g} \cap \ \mathrm{Ad}_{x_0}\mathfrak{q}^{\mathbb{C}} = J\mathfrak{c} \qquad \text{and} \qquad T(G^{\mathbb{C}}/K^{\mathbb{C}})_{\overline{x_0 \cdot p}} = \mathrm{Ad}_{x_0}\mathfrak{q}^{\mathbb{C}} = TS_{\overline{x_0 \cdot p}} \oplus J\mathfrak{c}.$$

Hence (6.3) holds. Assume that the G-orbit of $\overline{x_0 \cdot p}$ is nongeneric. Resume the notation in Section 3.3 and let $\alpha \in \Delta_{\mathfrak{c}}^c$ be a complex root such that

$$\begin{cases} \mathrm{Im}\ \alpha(X_0) = 0, \\ \mathrm{Re}\ \alpha(x_0) \equiv 0, \quad \bmod \pi/2 \end{cases}$$

(cf. Remark 3.16(i)). If

$$\begin{cases} \mathrm{Im}\ \alpha(X_0) = 0, \\ \cos \mathrm{Re}\ \alpha(X_0) = 0, \quad \sin \mathrm{Re}\ \alpha(X_0) = \pm 1, \\ Z_\alpha \in \mathfrak{g}_+^\alpha, \end{cases}$$

combining (3.15) and (3.17) one has

$$JF_\alpha + JF_{\bar\alpha}, F_\alpha - F_{\bar\alpha} \in TS_{\overline{x_0\cdot p}}, \qquad F_\alpha + F_{\bar\alpha},\ JF_\alpha - JF_{\bar\alpha} \in W_{\overline{x_0\cdot p}} = J\mathfrak{g} \cap \operatorname{Ad}_{x_0}\mathfrak{q}^{\mathbb{C}}.$$

If

$$\begin{cases} \operatorname{Im}\alpha(X_0) = 0, \\ \cos\operatorname{Re}\alpha(X_0) = \pm 1, \quad \sin\operatorname{Re}\alpha(X_0) = 0, \\ Z_\alpha \in \mathfrak{g}_+^\alpha, \end{cases}$$

combining (3.15) and (3.17) one has

$$JF_\alpha - JF_{\bar\alpha}, F_\alpha + F_{\bar\alpha} \in TS_{\overline{x_0\cdot p}}, \qquad JF_\alpha + JF_{\bar\alpha}, F_\alpha - F_{\bar\alpha} \in W_{\overline{x_0\cdot p}} = J\mathfrak{g} \cap \operatorname{Ad}_{x_0}\mathfrak{q}^{\mathbb{C}}.$$

Similarly, for all roots α satisfying the conditions of Remark 3.16(i) and $Z_\alpha \in \mathfrak{g}^\alpha$, by formulas (3.15), Proposition 3.12, and Remark 3.13, one has that $\operatorname{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}}$ admits a basis consisting of pairs $\{S_\alpha, JS_\alpha\}$, where $S_\alpha \in TS_{\overline{x_0\cdot p}}$ and $JS_\alpha \in W_{\overline{x_0\cdot p}}$. Since $\operatorname{Ad}_{x_0}\mathfrak{q}^{\mathbb{C}} = \mathfrak{c}^{\mathbb{C}} \oplus \bigoplus_{\alpha,\, Z_\alpha} \operatorname{Ad}_{x_0} Z[\alpha]_{\mathfrak{q}^{\mathbb{C}}}$, decomposition (6.3) holds and the lemma follows.

COROLLARY 6.9. *Let $\bar x \in G^{\mathbb{C}}/K^{\mathbb{C}}$ be a point sitting on a closed G-orbit S, with isotropy subgroup L. Let $(G^{\mathbb{C}}/K^{\mathbb{C}})^{[L]}$ be the L-stratum in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Then, at all smooth points $z \in (G^{\mathbb{C}}/K^{\mathbb{C}})^{[L]}$,*

$$\dim_{\mathbb{R}} T((G^{\mathbb{C}}/K^{\mathbb{C}})^{[L]})_z \neq 2\dim_{\mathbb{C}} T_{\mathbb{C}}((G^{\mathbb{C}}/K^{\mathbb{C}})^{[L]})_z.$$

LEMMA 6.10. *Let $D \subset G^{\mathbb{C}}/K^{\mathbb{C}}$ be a G-invariant domain with boundary ∂D consisting of nongeneric orbits. If D is Stein, then it coincides with the interior of the closure of a connected component of the set of generic orbits. The domains $D_1, \dots, D_m \subset \overline{\mathbf{X}}_0$ are of this kind.*

Proof. Observe that every relatively closed subset of real codimension greater than or equal to 3 in a complex manifold M is removable [Sh]. The same is true for a locally closed subset $\Omega \subset M$, of real codimension 2, such that $\dim_{\mathbb{R}} T\Omega_z \neq 2\dim_{\mathbb{C}} T_{\mathbb{C}}\Omega_z$ for all $z \in \Omega$ (cf. [St]). These facts together with Corollary 6.9 and Remark 3.5 prove the lemma.

LEMMA 6.11. *Let $C = \exp J\mathfrak{c}\cdot p$ be a Cartan subset. The set of regular semisimple elements with respect to σ, τ in C consists of finitely many connected components.*

Proof. By Remark 3.16(i), we need to show that the exponential in $G^{\mathbb{C}}$ of the set

$$\bigcup_{\alpha\in\Delta_{\mathfrak{c}}^r} \{H \in \mathfrak{c} \mid \alpha(H) \equiv 0 \bmod \pi/4\}$$

consists of the union of a finite number of subsets. Assume first that the base point p satisfies conditions (2.6). The G-orbit of p is a semisimple symmetric space G/H and the symmetric algebra associated to p is given by ($\mathfrak{g}_c = \mathfrak{h} \oplus J\mathfrak{q}, \sigma$) (cf. (6.2)). The Cartan subspace $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \oplus \mathfrak{c}_{\mathfrak{p}}$ is a τ_p-stable and θ-stable maximal abelian subspace in $\mathfrak{q}$.

Let

$$\mathfrak{b} \oplus \mathfrak{c} = \mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{b}_{\mathfrak{p}} \oplus \mathfrak{c}_{\mathfrak{k}} \oplus \mathfrak{c}_{\mathfrak{p}}$$

be a τ_p- and θ-stable Cartan subalgebra of $\mathfrak{g}$ extending $\mathfrak{c}$ (cf. Lemma 5.7). Then $\mathfrak{t} := \mathfrak{b}_{\mathfrak{k}} \oplus J\mathfrak{b}_{\mathfrak{p}} \oplus \mathfrak{c}_{\mathfrak{k}} \oplus J\mathfrak{c}_{\mathfrak{p}}$ is a Cartan subalgebra of the compact real form $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$. Let $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ be the corresponding root system. Since $G^{\mathbb{C}}$ and hence U are simply connected, the kernel of the exponential map $\exp\colon \mathfrak{u} \to U$ is given by the fundamental lattice

$$\Gamma = \sum_{\lambda \in \Delta} 2\pi i \mathbb{Z} H_\lambda, \qquad H_\lambda = \frac{2h_\lambda}{B(h_\lambda, h_\lambda)} \in i\mathfrak{t}, \qquad \lambda(H_\lambda) = 2.$$

For $\alpha \in \Delta_{\mathfrak{c}}^{\mathfrak{r}}$, consider the hyperplane in $\mathfrak{c}$

$$\pi_k = \left\{ H \in \mathfrak{c} \mid \alpha(H) = k\frac{\pi}{2}, k \in \mathbb{Z} \right\},$$

perpendicular to $H_\alpha \in \mathfrak{c}_{\mathfrak{p}}$ (normalized so that $\alpha(H_\alpha) = 2$) and passing through the point $P = k\frac{\pi}{4}H_\alpha \in \mathfrak{c}_{\mathfrak{p}}$.

Given two such hyperplanes π_{k_1}, π_{k_2}, passing through $P_1 = k_1\frac{\pi}{4}H_\alpha$ and $P_2 = k_2\frac{\pi}{4}H_\alpha$, respectively, then $J\pi_{k_1}, J\pi_{k_2}$ have the same image under the exponential map $\exp\colon J\mathfrak{c} \to G^{\mathbb{C}}$ if and only if JP_1 and JP_2 have the same image in U, if and only if $P_1 - P_2 = (k_1 - k_2)\frac{\pi}{4}H_\alpha$ belongs to the lattice Γ.

This happens for sufficiently large $(k_1 - k_2) \in \mathbb{Z}$ if and only if $H_\alpha \in \mathfrak{c}_{\mathfrak{p}} \subset J\mathfrak{t}$ can be written as a linear combination with integer coefficients of the vectors $\{H_\lambda\}_{\lambda \in \Delta}$.

The map $\lambda \mapsto \frac{1}{2}(\lambda - \tau\lambda)$ is a surjective map of Δ onto the restricted root system $\Delta_{\mathfrak{c}}$. Observe that $H_{\tau\lambda} = \tau H_\lambda$ and that $\lambda + \tau\lambda \notin \Delta$ implies that the Cartan integers $c_{\lambda,\tau\lambda} = c_{\tau\lambda,\lambda}$ are either 0 or 1. Then

$$H_\alpha = \begin{cases} H_\lambda, & \tau\lambda = -\lambda, \\ H_\lambda - H_{\tau\lambda}, & c_{\lambda,\tau\lambda} = c_{\tau\lambda,\lambda} = 0, \\ 2(H_\lambda - H_{\tau\lambda}), & c_{\lambda,\tau\lambda} = c_{\tau\lambda,\lambda} = 1, \end{cases}$$

as requested. If the base point p satisfies conditions (2.7), we can apply the above arguments to the symmetric pair $((\mathfrak{g}^{\mathbb{C}})^{\sigma\tau_p}, \sigma)$, where $(\mathfrak{g}^{\mathbb{C}})^{\sigma\tau_p}$ is a real form of the Θ-stable complex subalgebra $(\mathfrak{g}^{\mathbb{C}})^{(\sigma\tau_p)^2}$ of $\mathfrak{g}^{\mathbb{C}}$. This concludes the proof of the lemma.

Recall that there are finitely many $W_{K\times K}(A)$-equivalence classes of Cartan subsets in $G^{\mathbb{C}}$ and every connected component of closed generic orbits meets precisely one Cartan subset C in some connected components of the regular semisimple elements with respect to σ, τ in C (cf. [Ma1, Theorem 3]). By Lemma 6.11, in each Cartan subset C the set of regular semisimple elements with respect to σ, τ consists of finitely many connected components. From all these facts it follows that there are finitely many connected components of closed generic G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$. By Lemma 6.10 and Lemma 6.11, we conclude that

PROPOSITION 6.12. *There are finitely many G-invariant domains* $D \subset G^{\mathbb{C}}/K^{\mathbb{C}}$, *with boundary* ∂D *consisting of nongeneric orbits*, *which are possibly Stein*.

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