HYPERBOLIC MANIFOLDS WHOSE ENVELOPES OF HOLOMORPHY ARE NOT HYPERBOLIC

L. GEATTI, A. IANNUZZI, AND J.-J. LOEB

Abstract. We present a family of examples of two dimensional, hyperbolic complex manifolds whose envelopes of holomorphy are not hyperbolic.

In this note we present a family of hyperbolic complex manifolds whose envelopes of holomorphy are not hyperbolic. Here we consider hyperbolicity in the sense of Kobayashi. In Isaev's classification [2] of two-dimensional hyperbolic manifolds with three-dimensional automorphism group, these manifolds appear as subdomains of $\Delta \times \mathbb{P}^1$, with Δ the unit disk in $\mathbb{C}$ and $\mathbb{P}^1$ the one-dimensional complex projective space. Here we consider their realizations as $SU(1,1)$-invariant domains in the Lie group complexification $SL(2,\mathbb{C})/U(1)^{\mathbb{C}}$ of the symmetric space $SU(1,1)/U(1)$. Then, by applying the univalence results for Stein, equivariant Riemann domains over Lie group complexifications of rank-one, Riemannian symmetric spaces obtained in [1], we can explicitely determine their envelopes of holomorphy. Such envelopes turn out to be all biholomorphic to $\Delta \times \mathbb{C}$. In particular, they are not hyperbolic.

Let G be the Lie group $SU(1,1)$. Consider the holomorphic action of its universal complexification $G^{\mathbb{C}} = SL(2,\mathbb{C})$ on $\mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$g \cdot ([z_1 : z_2], [w_1 : w_2]) := (\, g \cdot [z_1 : z_2],\ \overline{\sigma(g)} \cdot [w_1 : w_2]\,)\,, \tag{1}$$

where $\sigma(g) = I_{1,1}\, {}^t\bar{g}^{-1} I_{1,1}$ denotes the conjugation of $G^{\mathbb{C}}$ relative to G (here $I_{1,1}$ the diagonal matrix representing the standard hermitian form of signature $(1,1)$). Denote by K a maximal compact subgroup of G and by $K^{\mathbb{C}}$ its complexification. The quotient $G^{\mathbb{C}}/K^{\mathbb{C}}$ can be identified with the unique open $G^{\mathbb{C}}$-orbit in $\mathbb{P}^1 \times \mathbb{P}^1$ given by

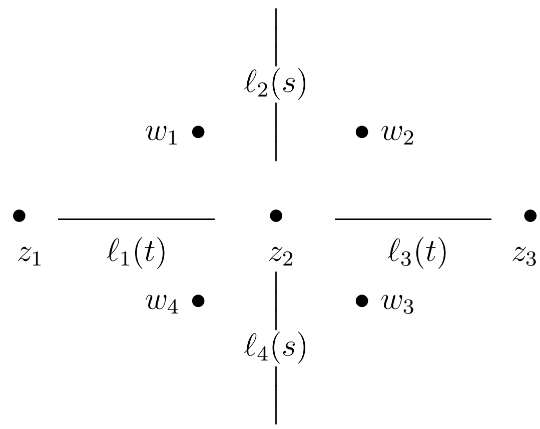
$$G^{\mathbb{C}} \cdot ([1:0],[1:0]) = \{\, ([z_1 : z_2], [w_1 : w_2]) \in \mathbb{P}^1 \times \mathbb{P}^1 \ : \ z_1 w_1 - z_2 w_2 \neq 0 \,\}\,.$$

Mathematics Subject Classification (2000): 32Q45, 32D10, 32M05.

Key words: hyperbolic manifold, envelope of holomorphy, symmetric space.

A global slice for the G-action on $G^{\mathbb{C}}/K^{\mathbb{C}}$ by left translations is represented by the following diagram (cf. [1], Sect. 4.1)

(2)



All elements in the diagram, except for z_1, z_2 and z_3, lie on hypersurface G-orbits. The points $z_1 := ([1:0],[1:0])$ and $z_3 := ([0:1],[0:1])$ lie on G-orbits diffeomorphic to the symmetric space G/K. The point $z_2 := ([1:i],[1:i])$ lies on a G-orbit diffeomorphic to a pseudo-Riemmanian symmetric space of the same dimension as G/K. The slices $\ell_1, \ldots, \ell_4$ are defined by

$$\ell_1(t) = ([\cos\frac{\pi}{4}(1-t) : i\sin\frac{\pi}{4}(1-t)], [\cos\frac{\pi}{4}(1-t) : i\sin\frac{\pi}{4}(1-t)])\,,$$
$$\ell_3(t) = ([\cos\frac{\pi}{4}(1+t) : i\sin\frac{\pi}{4}(1+t)], [\cos\frac{\pi}{4}(1+t) : i\sin\frac{\pi}{4}(1+t)])\,,$$

for $0 < t < 1$, and by

$$\ell_2(s) = ([e^s : ie^{-s}], [e^{-s} : ie^s])\,,$$
$$\ell_4(s) = ([e^{-s} : ie^s], [e^s : ie^{-s}])\,,$$

for $s > 0$. Note that z_2 is the limit point of ℓ_1, ℓ_2, ℓ_3 and ℓ_4 for values of the parameters approaching zero. Similarly one has $\ell_1(1) = z_1$ and $\ell_3(1) = z_3$. The points

$$w_1 := ([1:0],[1:i]), \quad w_2 := ([1:i],[0:1])\,,$$
$$w_3 := ([1:i],[1:0]) \quad w_4 := ([0:1],[1:i])\,.$$

represent the four non-closed hypersurface G-orbits containing $G \cdot z_2$ in their closure.

Consider the family of G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ defined by

$$D_\beta \;=\; G\cdot(\,z_1 \cup \ell_1((0,1)) \cup w_1 \cup \ell_2((0,\beta))\,) \qquad \text{for} \quad 0 < \beta < \infty\,.$$

By the classification of Stein, G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ (cf. Thm. 6.1 in [1]), none of the domains D_β is Stein. Each of them contains a unique Levi-flat orbit given by

$$G\cdot w_1 = \{\,([1:z],[1:w]) \in \mathbb{P}^1 \times \mathbb{P}^1 \;:\; z \in \Delta, \quad w \in \partial\Delta\,\} \cong \Delta \times \partial\Delta\,,$$

and a unique totally-real orbit given by

$$G \cdot z_1 = \{ ([1:z],[1:\bar{z}]) \in \mathbb{P}^1 \times \mathbb{P}^1 \ : \ z \in \Delta \}.$$

Denote by W the limit domain

$$W = G \cdot (z_1 \cup \ell_1(0,1) \cup w_1 \cup \ell_2(0,\infty)) .$$

By construction W contains every domain D_β. One can check that W coincides with the domain

$$\{ ([1:z],[w_1:w_2]) \in \mathbb{P}^1 \times \mathbb{P}^1 \ : \ z \in \Delta, \ w_1 - z w_2 \neq 0 \} .$$

It follows that W is biholomorphic to $\Delta \times \mathbb{C}$ via the map

$$\Delta \times \mathbb{C} \to W, \quad (u,v) \to ([1:u],[1+uv:v]) .$$

In particular, it is Stein and is not hyperbolic.

Proposition 0.1. *For every $0 < \beta < \infty$ the G-invariant domain D_β is hyperbolic. The envelope of holomorphy of D_β is the domain W, which is not hyperbolic.*

Proof. The fact that D_β is hyperbolic follows from Isaev's classification of two-dimensional hyperbolic manifolds with three dimensional automorphism group. Indeed note that for $g \in G$ one has $\overline{\sigma(g)} = \overline{g}$. Thus the restriction to G of the $G^{\mathbb{C}}$-action defined in (1) agrees with the G-action given in [2], p. 22. As a consequence, D_β coincides with an element of the family denoted there by

$$\widehat{\mathfrak{D}}_t^{(1)} \quad \text{for} \quad 1 < t < \infty$$

and it is hyperbolic.

Denote by $E(D_\beta)$ the envelope of holomorphy of D_β. Since $G^{\mathbb{C}}/K^{\mathbb{C}}$ is Stein, the inclusion of D_β in $G^{\mathbb{C}}/K^{\mathbb{C}}$ extends to a local biholomorphism p from $E(D_\beta)$ to $G^{\mathbb{C}}/K^{\mathbb{C}}$ (cf. [3]). Note that the center $\Gamma = \{\pm Id_2\}$ of G acts ineffectively on $G^{\mathbb{C}}/K^{\mathbb{C}}$ and that G/Γ is isomorphic to $SO_0(2,1)$. Then the $SO_0(2,1)$-action on D_β extends to $E(D_\beta)$ and the map p is equivariant, i.e. $p \colon E(D_\beta) \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is a Stein, $SO_0(2,1)$-equivariant Riemann domain.

By Theorem 7.5 in [1], the map p is necessarily injective. Hence $E(D_\beta)$ coincides with the smallest, Stein, G-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing D_β. Since each D_β contains the Levi-flat orbit $G \cdot w_1$, from the classification of Stein, G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ given in Theorem 6.1 of [1], it follows that the envelope of holomorphy of D_β is the domain W.

$\square$

Remark. Similarly one can show that W is the envelope of holomorphy of every element in the family of hyperbolic, G-invariant subdomains of W denoted in [2], p. 22, by

$$\mathfrak{D}^{(1)}_{s,t} \quad \text{for} \quad -1 \leq s < 1 < t \leq \infty,$$

where $s = -1$ and $t = \infty$ do not hold simultaneously. In terms of diagram (2) the elements of the family $\mathfrak{D}^{(1)}_{s,t}$ correspond to the domains

$$G \cdot (\ell_1((0,\alpha)) \cup w_1 \cup \ell_2((0,\beta))) \quad \text{for} \quad 0 < \alpha \leq 1 \ \text{ and } \ 0 < \beta \leq \infty \, .$$

References

[1] L. Geatti, A. Iannuzzi, *On univalence of equivariant Riemann domains over a complex semisimple Lie group.* arXiv: math.CV/0612169

[2] A. V. Isaev, *Hyperbolic 2-Dimensional Manifolds with 3-Dimensional Automorphism Group.* arXiv: math CV/0509030 v6

[3] H. Rossi, *On envelopes of holomorphy.* Commun. Pure Appl. Math. **16** (1963) 9–17

Laura Geatti and Andrea Iannuzzi: Dip. di Matematica, II Università di Roma "Tor Vergata", Via della Ricerca Scientifica, I-00133 Roma, Italy.
E-mail address: geatti@mat.uniroma2.it, iannuzzi@mat.uniroma2.it

Jean-Jacques Loeb, Département de Mathématiques et Informatiques, Université d'Angers, 2 Boulevard Lavoisier, 49045 Angers cedex 01
E-mail address: Jean-Jacques.Loeb@univ-angers.fr