1. Let $A = \mathbf{Z}[X]$, let I be the ideal (X) and let $J = (2, X)$. Then we have $I \subset J \subset A$. Show that there is no ideal $K \subset \mathbf{Z}[X]$ for which $I = JK$.

2. Let $L = \{(x, y, z) \in \mathbf{Z}^3 : 2x + 3y + 5z \equiv 0 \pmod 7\}$. Show that L is a lattice in $\mathbf{R}^3$ and determine its covolume.

3. Let F be a number field and let $\mu_F = \{\zeta \in O_F : \zeta^d = 1 \text{ for some exponent } d \geq 1\}$ denote the group of roots of unity. Prove that μ_F is a cyclic group.

4. Let F be a number field with $r_1 \geq 1$. Prove that $\mu_F = \{\pm 1\}$.

5. Let F be a number field of degree n and let $\Phi : F \longrightarrow F_{\mathbf{C}} = \prod_{k=1}^{n} \mathbf{C}$ the homomorphism given by $\Phi(x) = (\phi_1(x), \ldots, \phi_n(x))$. Here $\phi_1, \ldots, \phi_n$ are the field homomorphisms $F \to \mathbf{C}$. In class we have seen that the image of F is contained in $F_{\mathbf{R}}$ and O_F is a lattice in $F_{\mathbf{R}}$ of covolume $\sqrt{|\Delta_F|}$.
   (a) Let $I \subset O_F$ be a non-zero ideal. Show that its image in $F_{\mathbf{R}}$ is also a lattice.
   (b) Show that the lattice of part (a) has covolume $N(I)\sqrt{|\Delta_F|}$.

6. The Minkowski constant of a number field F of degree n and discriminant Δ_F is defined by $M_F = \frac{n!}{n^n} \left(\frac{\pi}{4}\right)^{r_2} \sqrt{|\Delta_F|}$. Here r_2 denotes the number of pairs of complex embeddings $F \to \mathbf{C}$. Compute the Minkowski constants of the fields $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt[3]{2})$ and $\mathbf{Q}(\sqrt[4]{2})$.

7. Let $F = \mathbf{Q}(\sqrt{-6})$.
   (a) Compute r_1, r_2 and the discriminant of F. Show that the Minkowski constant M_F of F is $3.11\ldots$
   (b) Show that $\mathfrak{p} = (2, \sqrt{-6})$ and $\mathfrak{q} = (3, \sqrt{-6})$ are the only prime ideals of O_F of norm $< M_F$.
   (c) Show that $\mathfrak{p}\mathfrak{q} = (\sqrt{-6})$ and deduce that $\mathfrak{p}$ generates the class group $Cl(O_F)$.
   (d) Show that $\mathfrak{p}^2 = (2)$ and that $\mathfrak{p}$ is not principal. Deduce that $Cl(O_F) \cong \mathbf{Z}/2\mathbf{Z}$.

8. Let $F = \mathbf{Q}(\sqrt{10})$.
   (a) Compute r_1, r_2 and the discriminant of F. Show that the Minkowski constant M_F of F is $\sqrt{10} = 3.16\ldots$
   (b) Show that $\mathfrak{p} = (2, \sqrt{10})$, $\mathfrak{q} = (3, \sqrt{10}+1)$ and $\mathfrak{q}' = (3, \sqrt{10}-1)$ are the only prime ideals of O_F of norm $< M_F$.
   (c) Show that the principal ideal $(2+\sqrt{10})$ is equal to $\mathfrak{p}\mathfrak{q}'$ and that $(2-\sqrt{10})$ is equal to $\mathfrak{p}\mathfrak{q}$. Deduce that $\mathfrak{p}$ generates the class group $Cl(O_F)$.
   (d) Show that $\mathfrak{p}^2 = (2)$ and that $\mathfrak{p}$ is not principal. Deduce that $Cl(O_F) \cong \mathbf{Z}/2\mathbf{Z}$.
   (e) Show that $\varepsilon = 3 + \sqrt{10}$ and -1 generate the unit group O_F^*.

9.* Let f be the polynomial $X^3 + 3X^2 - 3X + 5$, let α denote a zero and put $F = \mathbf{Q}(\alpha)$. You may use the fact that $O_F = \mathbf{Z}[\alpha]$.
   (a) Show that f is irreducible and has only one real zero.
   (b) Show that the Minkowski constant of F is < 12.13.
   (c) Determine the class group and the unit group of O_F.