1. Let $F = \mathbf{Q}(\sqrt[3]{2})$.
   (a) Determine characteristic polynomial, norm and trace of the element $-1 + \sqrt[3]{2}$.
   (b) Determine the discriminant of $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ (here $\sqrt[3]{4}$ denotes $\sqrt[3]{2}^2$).
   (c) Determine the discriminant of $\{\sqrt[3]{2}, 2\sqrt[3]{2}, 3\sqrt[3]{2}\}$.

2. Let F be the number field $\mathbf{Q}[X]/(X^4 - 2)$.
   (a) Describe the field homomorphisms $F \hookrightarrow \mathbf{C}$.
   (b) The ring $\mathbf{R}[X]/(X^4 - 2)$ is isomorphic to $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$. Determine r_1 and r_2.
   (c) Show that $\sqrt{2}$ is an element of F. In other words, show that F contains an element whose square is equal to 2. Determine characteristic polynomial, norm and trace of the element $\sqrt{2}$.

3. Let $F = \mathbf{Q}(\sqrt{2}, \sqrt{3})$. Determine $\alpha \in F$ so that $F = \mathbf{Q}(\alpha)$. Determine its characteristic polynomial.

4. Let $F = \mathbf{Q}(\sqrt{2}, i)$.
   (a) Show that $\sqrt{2}$ and i are in the ring of integers O_F.
   (b) Determine the minimum polynomial of $i\sqrt{2}$ and of $i + \sqrt{2}$.

5. Give an example of $x = a + bi \in \mathbf{Q}(i)$ (with $a, b \in \mathbf{Q}$)
   (a) such that $N(x) \in \mathbf{Z}$ but $\mathrm{Tr}(x) \notin \mathbf{Z}$;
   (b) such that $N(x) \notin \mathbf{Z}$ but $\mathrm{Tr}(x) \in \mathbf{Z}$.

6. Show that the following numbers are algebraic over $\mathbf{Q}$, determine the minimal polynomial of each and find all of its conjugates over $\mathbf{Q}$:
$$\sqrt{2+\sqrt{2}}, \qquad \frac{\sqrt{3}+\sqrt{2}}{2}, \qquad \frac{\sqrt{-2}}{1+i}$$
   Which among the numbers given above are algebraic integers?

7. Draw pictures of the rings of integers of $\mathbf{Q}(\sqrt{-2})$, $\mathbf{Q}(\sqrt{3})$ and $\mathbf{Q}(\sqrt{-7})$.

8. Let F be a number field of degree n and let $\omega_1, \ldots, \omega_n$ be elements of the ring of integers O_F of F. Show that if $\Delta(\omega_1, \ldots, \omega_n)$ is a squarefree integer, then $\omega_1, \ldots, \omega_n$ is a $\mathbf{Z}$-basis of O_F. In other words, we have $\Delta_F = \Delta(\omega_1, \ldots, \omega_n)$.

9. Let $F = \mathbf{Q}(\alpha)$ be a number field of degree n. Let $f \in \mathbf{Q}[X]$ be the minimum polynomial of α.
   (a) Let $k \in \mathbf{Q}$. Show that $N(k - \alpha) = f(k)$.
   (b) Let $k, m \in \mathbf{Z}$ with $m \neq 0$. Show that $N(k - m\alpha) = m^n f(k/m)$.

10. Let $F = \mathbf{Q}(\sqrt{-5})$.
    (a) Show that $(2 + \sqrt{-5})(2 - \sqrt{-5})$ is a square in $\mathbf{Z}[\sqrt{-5}]$.
    (b) Show that $2 + \sqrt{-5}$ and $2 - \sqrt{-5}$ are coprime in $\mathbf{Z}[\sqrt{-5}]$. (Hint: a common divisor divides both the sum and the product)
    (c) Show that $2 + \sqrt{-5}$ and $2 - \sqrt{-5}$ themselves are not squares.