GEOMETRY OF HERMITIAN SYMMETRIC SPACES UNDER THE ACTION OF A MAXIMAL UNIPOTENT GROUP

LAURA GEATTI AND ANDREA IANNUZZI

Abstract. Let G/K be a non-compact irreducible Hermitian symmetric space of rank r and let NAK be an Iwasawa decomposition of G. The group N acts on G/K by biholomorphisms and the real r-dimensional subset $A \cdot eK$ intersects every N-orbit trasversally in a single point. Moreover $A \cdot eK$ is contained in a complex r-dimensional submanifold of G/K biholomorphic to $\mathbb{H}^r$, the product of r copies of the upper half-plane in $\mathbb{C}$. This fact leads to a ono-to-one correspondence between N-invariant domains in G/K and tube domains in $\mathbb{H}^r$. In this setting we prove a generalization of Bochner's tube theorem. Namely, an N-invariant domain D in G/K is Stein if and only if the base Ω of the associated tube domain is convex and "cone invariant".

We also prove the univalence of N-invariant holomorphically separable Riemann domains over G/K. This yields a precise description of the envelope of holomorphy of an arbitrary N-invariant domain in G/K. Finally, we obtain a characterization of several classes of N-invariant plurisubharmonic funtions on D in terms of the corresponding classes of convex functions on Ω. As an application we present an explicit Lie group theoretical description of all N-invariant potentials of the Killing metric on G/K and of the associated moment maps.

1. Introduction

The classical Bochner's tube theorem states that the envelope of holomorphy of a tube domain $\mathbb{R}^n + i\Omega$ in $\mathbb{C}^n$ is univalent and coincides with its convex envelope $\mathbb{R}^n + i\,\mathrm{conv}(\Omega)$. Moreover, there is a one-to-one correspondence between the class of $\mathbb{R}^n$-invariant plurisubharmonic functions on a Stein tube domain in $\mathbb{C}^n$ and the class of convex functions on its base in $\mathbb{R}^n$ (cf. [Gun90], Thm.13, p.111).

In this paper we obtain generalizations of the above results in the setting of irreducible Hermitian symmetric spaces of the non-compact type, under the action of a maximal unipotent group of holomorphic automorphisms.

Any such space can be realized as a quotient G/K of a non-compact real simple Lie group G over a maximal compact subgroup K. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ be an Iwasawa decomposition of $\mathfrak{g}$, where $\mathfrak{n}$ is a maximal nilpotent subalgebra, $\mathfrak{a}$

Mathematics Subject Classification (2010): 32M15, 31C10, 32T05.

Key words: Hermitian symmetric spaces, Stein domains, plurisubharmonic functions.

The authors acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome "Tor Vergata", CUP E83C18000100006. This research was partially supported by GNSAGA-INDAM.

is a maximal split abelian subalgebra and $\mathfrak{k}$ is the Lie algebra of K. The integer $r := \dim \mathfrak{a}$ is by definition the rank of G/K.

Let NAK be the corresponding Iwasawa decomposition of G, where $A := \exp \mathfrak{a}$ and $N := \exp \mathfrak{n}$. The group N acts on G/K by biholomorphisms and every N-orbit in G/K intersects the smooth, real r-dimensional submanifold $A \cdot eK$ transversally in a single point.

As the space G/K is *Hermitian* symmetric, the Lie algebra $\mathfrak{g}$ of G contains a subalgebra which is the direct sum of r pairwise commuting copies of $\mathfrak{sl}(2, \mathbb{R})$. The orbit of the base point $eK \in G/K$ under the corresponding subgroup of G is a closed complex submanifold of G/K which contains $A \cdot eK$ and is biholomorphic to the product of r copies of the upper half-plane $\mathbb{H}$ in $\mathbb{C}$. This biholomorphism, which restricts to a diffeomorphism between $A \cdot eK$ and the positive imaginary octant in $\mathbb{C}^r$, determines a one-to-one correspondence between N-invariant domains D in G/K and tube domains $\mathbb{R}^r + i\Omega$ in $\mathbb{H}^r$ (cf. Sect. 3). If D is an N-invariant domain in G/K, then the properties of D and of the N-invariant functions on D can be best described in terms of Ω. Define the cone

$$C := \begin{cases} (\mathbb{R}^{>0})^r, & \text{in the non-tube case,} \\ (\mathbb{R}^{>0})^{r-1} \times \{0\}, & \text{in the tube case.} \end{cases}$$

A set $\Omega \subset \mathbb{R}^r$ is C-invariant if $\mathbf{y} \in \Omega$ implies $\mathbf{y} + \mathbf{v} \in \Omega$, for all $\mathbf{v} \in C$. Our generalizion of Bochner's tube thorem is as follows.

Theorem 3.4. *Let G/K be a non-compact irreducible Hermitian symmetric space of rank r. Let D be an N-invariant domain in G/K and let $\mathbb{R}^r + i\Omega$ be the associated r-dimensional tube domain. Then D is Stein if and only if Ω is convex and C-invariant.*

We also show that a holomorphically separable, N-equivariant, Riemann domain over G/K is necessarily univalent (cf. Prop. 3.7). This implies the following corollary.

Corollary 3.8. *The envelope of holomorphy $\widehat{D}$ of an N-invariant domain D in G/K coincides with the N-invariant domain whose associated r-dimensional tube is $\mathbb{R}^r + i\widehat{\Omega}$, where $\widehat{\Omega}$ is the convex, C-invariant hull of Ω.*

A first proof of Theorem 3.4 is obtained by realizing G/K as a Siegel domain and by combining some results from the theory of normal J-algebras with some convexity arguments. An alternative proof relies on the special features of the smooth N-invariant plurisubharmonic functions on G/K. There is a one-to-one correspondence between N-invariant functions on D and functions on Ω, and such correspondence preserves regularity.

Let $\widehat{f} \colon \Omega \to \mathbb{R}$ be a function defined on a C-invariant domain in $(\mathbb{R}^{>0})^r$ and let $\overline{C}$ be the closure of the cone C. Then $\widehat{f}$ is said $\overline{C}$-decreasing if for every $\mathbf{y} \in \Omega$ and $\mathbf{v} \in \overline{C}$ the restriction of $\widehat{f}$ to the half-line $\{\mathbf{y} + t\mathbf{v} \ : \ t \geqslant 0\}$ is decreasing. The following theorem is a generalization to our setting of the well known correspondence between $\mathbb{R}^r$-invariant plurisubharmonic functions on a tube domain and convex functions on its base (see. Thm. 5.11 for a detailed statement).

Theorem. *Let D be a Stein, N-invariant domain in a non-compact, irreducible Hermitian symmetric space G/K of rank r. Let Ω be the base of the associated r-dimensional tube domain.*

An N-invariant function $f\colon D \to \mathbb{R}$ is (strictly) plurisubharmonic if and only if the corresponding function $\widehat{f}\colon \Omega \to \mathbb{R}$ is (stably) convex and $\overline{C}$-decreasing.
In particular, every N-invariant plurisubharmonic function on D is continuous.

In the smooth case, an important ingredient for the proof of the above theorem is the computation of the Levi form of an N-invariant function $f\colon D \to \mathbb{R}$ in terms of the Hessian and the gradient of $\widehat{f}\colon \Omega \to \mathbb{R}$ (cf. Prop. 4.1). This is done in a Lie theoretic way, with no use of explicit models nor of the classification of the symmetric spaces G/K. Instead, we use a simple moment map identity which enables us to maximally exploit the symmetries at hand. In the non-smooth case, the result is obtained by adapting to our context a classical approximation method.

As an application of the above theorem, we determine all the N-invariant potentials of the Killing metric on G/K in a Lie theoretical fashion.

The paper is organized as follows. Section 2 contains the preliminaries. In Section 3 we prove Theorem 3.4. In Section 4 we compute the Levi form of a smooth N-invariant function on G/K. In Section 5 we investigate N-invariant plurisubharmonic functions on Stein N-invariant domains in G/K. In Section 6 we determine all the N-invariant potentials of the Killing metric and their associated moment maps. In Section 7 we give an alternative proof of Theorem 3.4 by using N-invariant plurisubharmonic functions.

We wish to thank the referee for carefully reading the manuscript and for giving us precious advise.

2. Preliminaries

Let G/K be an irreducible Hermitian symmetric space, where G can be assumed to be a real non-compact simple Lie group and K is a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the respective Lie algebras. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$, with Cartan involution θ. Denote by $B(\cdot\,,\,\cdot)$ both the Killing form of $\mathfrak{g}$ and its $\mathbb{C}$-linear extension to $\mathfrak{g}^{\mathbb{C}}$.

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. The dimension of $\mathfrak{a}$ is by definition the *rank* r of G/K. Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^{\alpha}$ be the restricted root decomposition of $\mathfrak{g}$ determined by the adjoint action of $\mathfrak{a}$, where $\mathfrak{m}$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. For a simple Lie algebra of Hermitian type $\mathfrak{g}$, the restricted root system is either of type C_r (if G/K is of tube type) or of type BC_r (if G/K is not of tube type), i.e. there exists a basis $\{e_1, \dots, e_r\}$ of $\mathfrak{a}^*$ for which a positive system Σ^+ is given by

$$\Sigma^+ = \{2e_j,\ 1 \leqslant j \leqslant r,\ e_k \pm e_l,\ 1 \leqslant k < l \leqslant r\}, \quad \text{for type } C_r,$$

$$\Sigma^+ = \{e_j,\ 2e_j,\ 1 \leqslant j \leqslant r,\ e_k \pm e_l,\ 1 \leqslant k < l \leqslant r\}, \quad \text{for type } BC_r\,.$$

The roots $2e_1, \dots, 2e_r$ form a maximal set of long strongly orthogonal positive restricted roots. The root spaces $\mathfrak{g}^{2e_1}, \dots, \mathfrak{g}^{2e_r}$ are one-dimensional and one can choose generators $E^j \in \mathfrak{g}^{2e_j}$ such that the $\mathfrak{sl}(2)$-triples $\{E^j,\ \theta E^j,\ H_j := [\theta E^j,\ E^j]\}$ are normalized as follows

$$[H_j,\, E^l] = \delta_{jl} 2E^l, \quad \text{for} \quad j, l = 1, \dots, r. \tag{1}$$

Denote by I_0 the G-invariant complex structure of G/K. By changing sign of the generators E^j if necessary, we may assume that $I_0(E^j - \theta E^j) = H_j$. By the strong orthogonality of $2e_1, \dots, 2e_r$, the vectors $H_1, \dots, H_r$ form a B-orthogonal basis of $\mathfrak{a}$, dual to $e_1, \dots, e_r$ of $\mathfrak{a}^*$, and the associated $\mathfrak{sl}(2)$-triples pairwise commute.

Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ be the Iwasawa decomposition subordinate to Σ^+, where $\mathfrak{n} = \oplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$, and let $G = NAK$ be the corresponding Iwasawa decomposition of G. Then $S = NA$ is a real split solvable group acting freely and transitively on G/K. In particular, the tangent space to G/K at the base point eK can be identified with the Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$.

The map $\phi\colon \mathfrak{s} \to \mathfrak{p}$, given by $\phi(X) := \frac{1}{2}(X - \theta X)$, is an isomorphism of vector spaces. As a consequence,

$$\langle X\,, Y \rangle := B(\phi(X)\,, \phi(Y)) = -\tfrac{1}{2} B(X\,, \theta Y), \tag{2}$$

for X, $Y \in \mathfrak{s}$, defines a positive definite symmetric bilinear form on $\mathfrak{s}$. Moreover, the map $J\colon \mathfrak{s} \to \mathfrak{s}$, given by

$$JX := \phi^{-1} \circ I_0 \circ \phi(X), \tag{3}$$

defines a complex structure on $\mathfrak{s}$, such that $\phi(JX) = I_0 \phi(X)$. The complex structure J permutes the restricted root spaces of $\mathfrak{s}$ (cf. [RoVe73]), namely

$$J\mathfrak{a} = \bigoplus_{j=1}^{r} \mathfrak{g}^{2e_j}, \quad J\mathfrak{g}^{e_j - e_l} = \mathfrak{g}^{e_j + e_l}, \quad J\mathfrak{g}^{e_j} = \mathfrak{g}^{e_j}\,. \tag{4}$$

In order to obtain a precise description of J on $\mathfrak{s}$, we recall a few more facts. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\mu \in \Delta} \mathfrak{g}^\mu$ be the root decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to a maximally split Cartan subalgebra $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}$ of $\mathfrak{g}$, where $\mathfrak{b}$ is an abelian subalgebra of $\mathfrak{m}$. Let σ be the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. Let θ denote also the $\mathbb{C}$-linear extension of θ to $\mathfrak{g}^{\mathbb{C}}$. One has $\theta\sigma = \sigma\theta$. Write $\overline{Z} := \sigma Z$, for $Z \in \mathfrak{g}^{\mathbb{C}}$. As σ and θ stabilize $\mathfrak{h}$, they induce actions on Δ, defined by $\bar{\mu}(H) := \overline{\mu(H)}$ and $\theta\mu(H) := \mu(\theta(H))$, for $H \in \mathfrak{h}$, respectively. Fix a positive root system Δ^+ compatible with Σ^+, meaning that $\mu|_{\mathfrak{a}} = Re(\mu) \in \Sigma^+$ implies $\mu \in \Delta^+$. Then $\sigma\Delta^+ = \Delta^+$.

Given a restricted root $\alpha \in \Sigma$, the corresponding restricted root space $\mathfrak{g}^\alpha$ decomposes into the direct sum of ordinary root spaces with respect to the Cartan subalgebra $\mathfrak{h}$ as follows

$$\mathfrak{g}^\alpha = \Big(\bigoplus_{\substack{\mu\in\Delta,\, \mu\neq\bar\mu \\ Re(\mu)=\alpha}} \mathfrak{g}^\mu \oplus \mathfrak{g}^{\bar\mu} \quad \oplus \mathfrak{g}^\lambda \Big) \cap \mathfrak{g},$$

where $\lambda \in \Delta$ is possibly a root satisfying $\lambda = \bar\lambda = \alpha$. The next lemma is obtained by combining Lemma 2.2 in [GeIa22] with (3).

Lemma 2.1. (the complex structure J on $\mathfrak{s}$).
(a) For $j = 1, \dots, r$, let $H_j \in \mathfrak{a}$ and $E^j \in \mathfrak{g}^{2e_j}$ be elements normalized as in (1). Then $JE^j = \frac{1}{2}H_j$ and $JH_j = -2E^j$.

(b) Let $X = Z^\mu + \overline{Z^\mu} \in \mathfrak{g}^{e_j - e_l}$, where $\mu \in \Delta^+$ is a root satisfying $Re(\mu) = e_j - e_l$ and $Z^\mu \in \mathfrak{g}^\mu$ (if $\bar\mu = \mu$, we may assume $Z^\mu = \overline{Z^\mu}$ and set $X = Z^\mu$). Then $JX = [E^l, X] \in \mathfrak{g}^{e_j+e_l}$.
Let $X = Z^\mu + \overline{Z^\mu} \in \mathfrak{g}^{e_j + e_l}$, where $\mu \in \Delta^+$ is a root satisfying $Re(\mu) = e_j + e_l$ and $Z^\mu \in \mathfrak{g}^\mu$ (if $\bar\mu = \mu$, we may assume $Z^\mu = \overline{Z^\mu}$ and set $X = Z^\mu$). Then $JX = [\theta E^l, X] \in \mathfrak{g}^{e_j-e_l}$.

(c) Let $X = Z^\mu + \overline{Z^\mu} \in \mathfrak{g}^{e_j}$, where μ is a root in Δ^+ satisfying $Re(\mu) = e_j$ and $Z^\mu \in \mathfrak{g}^\mu$ (as $\dim \mathfrak{g}^{e_j}$ is even, one necessarily has $\bar\mu \neq \mu$). Then $JX = iZ^\mu + \overline{iZ^\mu} \in \mathfrak{g}^{e_j}$.

Remark 2.2. (a J-stable basis of $\mathfrak{s}$) *In view of Lemma 2.1, one can choose a J-stable basis of $\mathfrak{s}$, compatible with the restricted root decomposition.*
(a) As a basis of $\mathfrak{a} \oplus J\mathfrak{a}$, take pairs of elements H_j, $JH_j = -2E^j$, for $j = 1, \dots, r$, normalized as in (1).
(b) As a basis of $\mathfrak{g}^{e_j-e_l} \oplus \mathfrak{g}^{e_j+e_l}$, take 4-tuples of elements

$$X = Z^\mu + \overline{Z^\mu}, \quad X' = iZ^\mu + \overline{iZ^\mu}, \quad JX = [E^l, X], \quad JX' = [E^l, X'],$$

parametrized by the pairs of roots $\mu \neq \bar\mu \in \Delta^+$ satisfying $Re(\mu) = e_j - e_l$ (with no repetition), with Z^μ a root vector in $\mathfrak{g}^\mu$. For $\mu = \bar\mu$, one may assume $Z^\mu = \overline{Z^\mu}$ and take the pair $X = Z^\mu$, $JX = [E^l, X]$.
(c) As a basis of $\mathfrak{g}^{e_j}$ (non-tube case), take pairs of elements

$$X = Z^\mu + \overline{Z^\mu}, \quad JX = iZ^\mu + \overline{iZ^\mu},$$

parametrized by the pairs of roots $\mu \neq \bar\mu \in \Delta^+$ satisfying $Re(\mu) = e_j$ (with no repetition), with $Z^\mu \in \mathfrak{g}^\mu$.

The next lemma contains some identities which are needed in Section 3. Its proof is essentially contained in [GeIa22], Lemma 2.4.

Lemma 2.3. *Let $\mu \in \Delta^+$ be a root satisfying $Re(\mu) = e_j - e_l$ and let Z^μ a root vector in $\mathfrak{g}^\mu$. Let $X = Z^\mu + \overline{Z}^\mu \in \mathfrak{g}^{e_j - e_l}$ and $JX = [E^l, X] \in \mathfrak{g}^{e_j+e_l}$. If $\overline{\mu} \neq \mu$, let $X' = iZ^\mu + \overline{iZ^\mu}$ and $JX' = [E^l, X']$. Then* *(a)* $[JX, X] = [JX', X'] = sE^j$, *for some* $s \in \mathbb{R}$, $s \neq 0$*;*

(b) $[JX', X] = 0$.

Let μ *be a root in* Δ^+, *with* $Re(\mu) = e_j$ *(non-tube case) and let* Z^μ *be a root vector in* $\mathfrak{g}^\mu$. *Let* $X = Z^\mu + \overline{Z}^\mu$ *and* $JX = iZ^\mu + \overline{iZ^\mu}$. *Then*

(c) $[JX, X] = tE^j$, *for some* $t \in \mathbb{R}$, $t \neq 0$.

Hermitian symmetric spaces and Siegel domains. Let $S = NA$ be the real split solvable group arising from the Iwasawa decomposition of G subordinate to Σ^+. The group S acts simply transitively on G/K. With the complex structure J described in (3) and the linear form $f_0 \in \mathfrak{s}^*$ defined by $f_0(X) := B(X, Z_0)$, where $Z_0 \in Z(\mathfrak{k})$ is the element inducing the complex structure on $\mathfrak{p}$, the Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ of S has the structure of a *normal* J*-algebra* (see [GPSV68], p. 49, [PS69], Sect. 3, p. 51, and [RoVe73], Sect. 5, A).

This means in particular that $\omega(X, Y) := -f_0([X, Y])$ is a non-degenerate skew-symmetric J-invariant bilinear form on $\mathfrak{s}$ and that the symmetric bilinear form $\langle X, Y\rangle := -f_0([JX, Y])$ is the J-invariant positive definite inner product on $\mathfrak{s}$ defined in (2).

The adjoint action of $\mathfrak{a}$ on $\mathfrak{s}$ is symmetric with respect to $\langle\cdot,\cdot\rangle$ and decomposes $\mathfrak{s}$ into the orthogonal direct sum of the restricted root spaces. Moreover, the adjoint action of the element $H_0 := \frac{1}{2}\sum_j H_j \in \mathfrak{a}$ decomposes $\mathfrak{s}$ and $\mathfrak{n}$ as

$$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_{1/2} \oplus \mathfrak{s}_1, \qquad \mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_{1/2} \oplus \mathfrak{n}_1\,,$$

where $\mathfrak{n}_j = \mathfrak{n} \cap \mathfrak{s}_j$ and

$$\mathfrak{s}_0 = \mathfrak{a} \oplus \bigoplus_{1 \leq j < l \leq r} \mathfrak{g}^{e_j - e_l}, \quad \mathfrak{s}_{1/2} = \oplus_{1 \leq j \leq r}\, \mathfrak{g}^{e_j}, \quad \mathfrak{s}_1 = \oplus_{1 \leq j \leq r}\, \mathfrak{g}^{2e_j} \oplus \bigoplus_{1 \leq j < l \leq r} \mathfrak{g}^{e_j + e_l}.$$

Set $E_0 := \sum E^j$. The orbit

$$V := Ad_{\exp \mathfrak{s}_0} E_0$$

is a sharp convex homogeneous selfadjoint cone in $\mathfrak{s}_1$ and

$$F \colon \mathfrak{s}_{1/2} \times \mathfrak{s}_{1/2} \to \mathfrak{s}_1^{\mathbb{C}} := \mathfrak{s}_1 + i\mathfrak{s}_1, \qquad F(W, W') = \frac{1}{4}([JW', W] - i[W', W]),$$

is a V-valued Hermitian form, i.e. it is sesquilinear and $F(W, W) \in \overline{V}$ (the closure of V), for all $W \in \mathfrak{s}_{1/2}$. The group S acts on $\mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$ by affine transformations, given by

$$g \cdot (Z, W) = (Ad_s Z + \xi + 2iF(Ad_s W, \zeta) + iF(\zeta, \zeta), Ad_s W + \zeta), \tag{5}$$

where $g = \exp\zeta \exp\xi\, \exp\sigma$, where $\zeta \in \mathfrak{s}_{1/2}$, $\xi \in \mathfrak{s}_1$, $\sigma \in \mathfrak{s}_0$, and $s = \exp\sigma$. Then the Hermitian symmetric space $G/K \cong S \cdot eK$ is biholomorphic to the Siegel domain given by

$$D(V, F) = \{(Z, W) \in \mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2} \mid Im(Z) - F(W, W) \in V\},$$

via the map

$$\mathcal{L}\colon\ S\cdot eK \mapsto D(V,F), \qquad s\cdot eK \mapsto s\cdot(iE_0,0). \tag{6}$$

If $\mathfrak{s}_{1/2} = \{0\}$ (if and only if the restricted roots system of $\mathfrak{g}$ is of type C_r), then G/K is of *tube type*, otherwise it is of *non-tube type*. If $\dim_{\mathbb{R}}\mathfrak{s}_1 = n$ and $\dim_{\mathbb{R}}\mathfrak{s}_{1/2} = 2m$, then $\mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$ can be identified with $\mathbb{C}^{n+m}$ and $D(V,F)$ with an open convex affinely homogeneous domain therein.

3. N-invariant Stein domains in G/K

The goal of this section is to obtain a characterization of N-invariant Stein domains in an irreducible non-compact Hermitian symmetric space G/K of rank r. To an invariant domain D we associate an r-dimensional tube domain in $\mathbb{H}^r$. Then we prove that D is Stein if and only if the base of the associated tube is convex and satisfies an additional geometric condition.

For this we exploit the realization of G/K as a Siegel domain (cf. Sect. 2). We also prove the univalence of holomorphically separable, N-equivariant, Riemann domains over G/K. This result yields a precise description of the envelope of holomorphy of an N-invariant domain in G/K.

Consider the unipotent abelian subgroup of G, isomorphic to $\mathbb{R}^r$, defined by $R := \exp J\mathfrak{a}$ (cf. (4)). The R-invariant set

$$R\exp(\mathfrak{a})\cdot eK$$

is an r-dimensional closed complex submanifold of G/K, intersecting all N-orbits in G/K. Consider the positive octant $J\mathfrak{a}^+ := \{\sum y_k E^k \mid y_k > 0,\ k = 1,\dots,r\}$ in $J\mathfrak{a}$. One easily verifies that if $E \in J\mathfrak{a}^+$, then

$$Ad_{\exp\mathfrak{a}}E = J\mathfrak{a}^+ \qquad \text{and} \qquad iJ\mathfrak{a}^+ = \exp\mathfrak{a}\cdot(iE_0,0),$$

where $(iE_0,0)$ is the base point of $D(V,F)$ (see (5)). The map $\mathcal{L}$ restricts to a biholomorphism

$$R\exp(\mathfrak{a})\cdot eK \to J\mathfrak{a}\oplus iJ\mathfrak{a}^+,$$

given by

$$\exp(\textstyle\sum_j e_j E^j)\exp(\sum_k h_k H_k)K \mapsto \sum_j e_j E^j + iAd_{\exp(\sum_k h_k H_k)}E_0. \tag{7}$$

In particular $\mathcal{L}_{|\exp(\mathfrak{a})K}$ determines a diffeomorphism

$$L\colon \mathfrak{a} \to J\mathfrak{a}^+, \qquad \sum_k h_k H_k \mapsto Ad_{\exp(\sum_k h_k H_k)}E_0 = \sum_j e^{2h_j}E^j. \tag{8}$$

N-invariant domains in G/K and tube domains in $\mathbb{H}^r$. In view of the above discussion, the following facts hold true.

(i) An N-invariant domain D in G/K is uniquely determined by a domain $\mathcal{D}$ in $\mathfrak{a}$ via

$$D := N \exp \mathcal{D} \cdot eK.$$

(ii) The intersection of D with the closed complex submanifold $R\exp(\mathfrak{a}) \cdot eK$ is given by $R\exp(\mathcal{D}) \cdot eK$.

(iii) A tube domain $J\mathfrak{a} + i\Omega$ in $J\mathfrak{a} + iJ\mathfrak{a}^+$ determines a unique N-invariant domain in G/K via

$$D = N\exp(\mathcal{D}) \cdot eK, \qquad \text{where } \mathcal{D} = L^{-1}(\Omega).$$

Definition 3.1. *The r-dimensional tube domain associated to D is the image of the set $R\exp(\mathcal{D}) \cdot eK$ under $\mathcal{L}$, namely*

$$J\mathfrak{a} + i\Omega, \qquad \textit{where } \Omega := L(\mathcal{D})\,.$$

Remark 3.2. (i) If we identify $J\mathfrak{a} + iJ\mathfrak{a}$ with $\mathbb{R}^r + i\mathbb{R}^r$ (by sending E^j into the j^{th} vector of the canonical basis of $\mathbb{R}^r$, for $j = 1, \dots, r$), then the set $J\mathfrak{a} + iJ\mathfrak{a}^+$ is identified with $\mathbb{H}^r$, where $\mathbb{H}$ is the upper half-plane in $\mathbb{C}$. In particular, the tube domain associated to D is just a tube domain in $\mathbb{H}^r$.

(ii) If the domain D is Stein, then $R\exp\mathcal{D} \cdot eK$ is Stein and so is the tube domain associated to D. In particular, by Bochner's theorem, its base Ω is convex.

We are going to give a precise characterization of the open convex sets Ω arising from Stein N-invariant domains in G/K.

Assume that the symmetric space G/K is realized as a Siegel domain $D(V, F)$, and let D be an N-invariant domain therein. Then

$$D = \{(Z, W) \in D(V, F) \mid Im(Z) - F(W, W) \in \mathbf{\Omega}\},$$

where $\mathbf{\Omega}$ is the $Ad_{\exp \mathfrak{n}_0}$-invariant open subset in V, determined by

$$i\mathbf{\Omega} := D \cap iV,$$

and, by (6), (7) and (8), the base of the associated tube is

$$\Omega = \mathbf{\Omega} \cap J\mathfrak{a}^+.$$

Define a cone in $J\mathfrak{a}$ as follows

$$C := \begin{cases} \mathcal{C}_r, & \text{in the non-tube case,} \\ \mathcal{C}_{r-1}, & \text{in the tube case,} \end{cases} \tag{9}$$

where

$$\mathcal{C}_r := cone(E^1, \dots, E^r) = J\mathfrak{a}^+ \quad \text{and} \quad \mathcal{C}_{r-1} := cone(E^1, \dots, E^{r-1}) \times \{0\}.$$

Definition 3.3. *A set $\Omega \subset J\mathfrak{a}$ is C-invariant if $E \in \Omega$ implies $E + C \subset \Omega$ Equivalently, if $E \in \Omega$ implies $E + \overline{C} \subset \Omega$, where $\overline{C}$ denotes the closure of C.*

The characterization of N-invariant Stein domains in G/K can be stated as follows.

Theorem 3.4. *Let G/K be a non-compact irreducible Hermitian symmetric space of rank r. Let D be an N-invariant domain in G/K and let $\mathbb{R}^r + i\Omega$ be the associated tube domain. Then D is Stein if and only if Ω is convex and C-invariant.*

In order to prove the above theorem, we need some preliminary results. For this we separate the tube and the non-tube case.

The tube case. Denote by $conv(\boldsymbol{\Omega})$ the convex hull of $\boldsymbol{\Omega}$ in $\mathfrak{s}_1$. Since $\boldsymbol{\Omega}$ is $Ad_{\exp \mathfrak{n}_0}$-invariant and the action is linear, then also $conv(\boldsymbol{\Omega})$ is $Ad_{\exp \mathfrak{n}_0}$-invariant. Denote by $p\colon \mathfrak{s}_1 \to J\mathfrak{a}$ the projection onto $J\mathfrak{a}$, parallel to $\oplus \mathfrak{g}^{e_j+e_l}$. Denote by

$$(E^1)^*, \ldots, (E^r)^* \tag{10}$$

the elements in the dual $\mathfrak{n}^*$ of $\mathfrak{n}$, with the property that $(E^j)^*(E^l) = \delta_{jl}$ and $(E^j)^*(X^\alpha) = 0$, for all $X^\alpha \in \mathfrak{g}^\alpha$, with $\alpha \in \Sigma^+ \backslash \{2e_1, \ldots, 2e_r\}$.

Lemma 3.5. *The following statements hold true*

(i) *Let $E = \sum x_k E^k \in J\mathfrak{a}^+$, where $x_k \in \mathbb{R}^{>0}$, for $k = 1, \ldots, r$. Then*

$$p(Ad_{\exp \mathfrak{n}_0} E) = E + \mathcal{C}_{r-1}.$$

In particular, the coordinate x_r of E remains unchanged under the $Ad_{\exp \mathfrak{n}_0}$-action.

(ii) *Let $X \in \mathfrak{g}^{e_j - e_l}$ be a non-zero element. Then $[[E^l, X], X] = sE^j$, for some $s \in \mathbb{R}^{>0}$.*

(iii) *One has $p(conv(\boldsymbol{\Omega})) = conv(p(\boldsymbol{\Omega}))$.*

Proof. (i) Let $E \in J\mathfrak{a}^+$ and let $h_0 \in \exp \mathfrak{n}_0$, where $\mathfrak{n}_0 = \bigoplus_{1 \leqslant i < j \leqslant r} \mathfrak{g}^{e_i - e_j}$. By Theorem 4.10 in [RoVe73], for every $1 \leqslant i < j \leqslant r$ there exists a basis $\{E_{ij}^p\}$ of $\mathfrak{g}^{e_i - e_j}$, with coordinates $\{x_{ij}^p\}_p$, such that

$$(E^i)^*(Ad_{h_0} E) = x_i(1 + \textstyle\sum_{p,\, j>i} (x_{ij}^p)^2)$$

(formula (4.13) in [RoVe73]). Since $i < r$, one has $p(Ad_{\exp X} E) = E + \mathcal{C}_{r-1}$, as claimed. In particular the r^{th} coordinate of E does not vary under the $Ad_{\exp \mathfrak{n}_0}$-action.

(ii) Let $X \in \mathfrak{g}^{e_j - e_l}$ be a non-zero element. Then $\exp tX \in \exp \mathfrak{n}_0$ and $ad_X^3(E) \in \mathfrak{g}^{3e_j - e_l} = \{0\}$. It follows that the curve

$$Ad_{\exp tX} E_0 = \exp ad_{tX}(E_0) = E_0 + t[X, E^l] + \tfrac{t^2}{2}[X, [X, E^l]], \ t \in \mathbb{R},$$

is contained in V. By Lemma 2.3 (a), its projection onto $J\mathfrak{a}$ is given by

$$p(Ad_{\exp tX} E_0) = (E^j)^*(Ad_{\exp tX} E_0) E^j = (1 + \tfrac{t^2}{2} s) E^j,$$

for some $s \in \mathbb{R}$, $s \neq 0$. By (i) the quantity $1 + \frac{t^2}{2}s > 0$, for all $t \in \mathbb{R}$. Therefore $s > 0$, as claimed.

(iii) We prove the two inclusions. By the linearity of p, the set $p(conv(\mathbf{\Omega}))$ is convex and contains $p(\mathbf{\Omega})$. Hence, $p(conv(\mathbf{\Omega})) \supset conv(p(\mathbf{\Omega}))$. Conversely, let $Z \in conv(\mathbf{\Omega})$. Then there exist $t_0 \in (0,1)$ and $X, Y \in \mathbf{\Omega}$ such that $Z = t_0 X + (1-t_0)Y$. Since $p(Z) = t_0 p(X) + (1-t_0)p(Y)$, one has $p(conv(\mathbf{\Omega})) \subset conv(p(\mathbf{\Omega}))$. □

The non-tube case. Denote by $\widetilde{p}\colon \mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2} \to iJ\mathfrak{a}$ the projection onto $iJ\mathfrak{a}$ parallel to $\mathfrak{s}_1 \oplus i(\oplus \mathfrak{g}^{e_j+e_l}) \oplus \mathfrak{s}_{1/2}$.

Lemma 3.6. *Let $E \in J\mathfrak{a}^+$. Then $\widetilde{p}(N \cdot (iE, 0)) = i(E + \overline{\mathcal{C}}_r)$.*

Proof. The N-orbit of the point $(iE, 0) \in \mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$ is given by

$$\begin{aligned} N \cdot (iE, 0) &= \exp \mathfrak{s}_{1/2} \exp \mathfrak{s}_1 \exp \mathfrak{n}_0 \cdot (iE, 0) \\ &= \{(\xi + i(Ad_{\exp \mathfrak{n}_0} E + F(\zeta, \zeta)), \zeta) \mid \xi \in \mathfrak{s}_1,\ \zeta \in \mathfrak{s}_{1/2}\} \end{aligned} \tag{11}$$

By (11) and Lemma 3.5 (i), one has $\widetilde{p}(N \cdot (iE, 0)) = i(E + \mathcal{C}_{r-1} + \tilde{p}(F(\mathfrak{s}_{1/2}, \mathfrak{s}_{1/2})))$. In the symmetric case, $\mathfrak{g}^{e_j} \neq \{0\}$, for all $j = 1, \ldots, r$. Moreover, by Lemma 2.3(c), for every $X \neq 0$ in $\mathfrak{g}^{e_j}$, the element $[JX, X] = F(X, X) \in \overline{V} \cap J\mathfrak{a}$ is a positive multiple of E^j. It follows that $\{[J\zeta, \zeta],\ \zeta \in \mathfrak{s}_{1/2}\} = \overline{J\mathfrak{a}^+}$, and $\tilde{p}(N \cdot (iE, 0)) = i(E + \overline{\mathcal{C}}_r)$, as claimed. □

Proof of Theorem 3.4. **The tube case**. An N-invariant domain D in a symmetric tube domain $D(V)$ is itself a tube domain with base the $Ad_{\exp \mathfrak{n}_0}$-invariant set $\mathbf{\Omega}$. Hence all we have to prove is that $\mathbf{\Omega}$ is convex if and only if Ω is convex and $\Omega + \mathcal{C}_{r-1} \subset \Omega$.

Assume that $\mathbf{\Omega}$ is convex. Then Ω is convex, being the intersection of $\mathbf{\Omega}$ with the positive octant $J\mathfrak{a}^+$. To prove that Ω is C-invariant, let $E = \sum_j x_j E^j \in \Omega$, where $x_j > 0$, for $j = 1, \ldots, r$, and let $X \in \mathfrak{g}^{e_j - e_l}$ be a non-zero element. One has $ad_X^3(E) \in \mathfrak{g}^{3e_j - e_l} = \{0\}$. Hence, for every $t \in \mathbb{R}$,

$$Ad_{\exp tX} E = E + t x_l [X, E^l] + \tfrac{1}{2} t^2 x_l [X, [X, E^l]]$$

lies in $\mathbf{\Omega}$. As $\mathbf{\Omega}$ is convex, by replacing t with $-t$ and adding terms, one has that also $E + \frac{1}{2} t^2 x_l [X, [X, E^l]] = E + t^2 s x_l E^j$ lies in $\mathbf{\Omega}$, for some $s > 0$ (cf. Lemma 3.5 (ii)). This argument applied to all $j = 1, \ldots, r-1$ and the convexity of Ω imply that $\Omega + \mathcal{C}_{r-1} \subset \Omega$, as desired.

Conversely, assume that Ω convex and C-invariant. We prove the convexity of $\mathbf{\Omega}$ by showing that $conv(\mathbf{\Omega}) \subset \mathbf{\Omega}$. From Lemma 3.5 (ii) and the C-invariance of Ω, one has

$$p(\mathbf{\Omega}) = p(Ad_{\exp \mathfrak{n}_0} \Omega) = \Omega + \mathcal{C}_{r-1} \subset \Omega.$$

Moreover, from Lemma 3.5 (iii), the above inclusion and the convexity of Ω, one has

$$conv(\mathbf{\Omega}) \cap J\mathfrak{a} \subset p(conv(\mathbf{\Omega})) = conv(p(\mathbf{\Omega})) \subset \Omega.$$

Finally, from the $Ad_{\exp \mathfrak{n}_0}$-invariance of $conv(\boldsymbol{\Omega})$ it follows that

$$conv(\boldsymbol{\Omega}) = Ad_{\exp \mathfrak{n}_0}(conv(\boldsymbol{\Omega}) \cap J\mathfrak{a}) \subset Ad_{\exp \mathfrak{n}_0}\Omega = \boldsymbol{\Omega}.$$

This completes the proof of the theorem in the tube case.

The non-tube case. Let D be an N-invariant domain in a symmetric Siegel domain $D(V,F)$. Denote by $conv(D)$ the convex hull of D in $\mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$. As N acts on D by affine transformations, also $conv(D)$ is N-invariant.

If D is Stein, then $D \cap \{W = 0\}$ is a Stein tube domain in $\mathfrak{s}_1^{\mathbb{C}}$ with base $\boldsymbol{\Omega}$. By Theorem 3.4 for the tube case and Lemma 3.6, the set Ω is convex and $\Omega + \overline{\mathcal{C}}_r \subset \Omega$.

Conversely, assume that Ω is convex and C-invariant, i.e. $\Omega + \overline{\mathcal{C}}_r \subset \Omega$ (see Def. 3.3). We are going to prove that D is convex. By Lemma 3.6, one has

$$\widetilde{p}(D) = \widetilde{p}(N \cdot \Omega) = i(\Omega + \overline{\mathcal{C}}_r) \subset i\Omega.$$

Moreover,

$$conv(D) \cap iJ\mathfrak{a} \subset \widetilde{p}(conv(D)) = conv(\widetilde{p}(D)) \subset i\Omega.$$

By the N-invariance of $conv(D)$, one obtains

$$conv(D) = N \cdot (conv(D) \cap iJ\mathfrak{a}) \subset N \cdot i\Omega = D.$$

Hence D is convex and therefore Stein (cf. [Gun90], Vol.1, Thm.10, p. 67). This concludes the proof of the theorem. □

Remark. *If D is an N-invariant Stein domain in G/K, then the associated tube domain $\mathbb{R}^r + i\Omega$ is Stein, being biholomorphic to the intersection of D with a closed submanifold in G/K. In particular its base Ω is an open convex set. Theorem 3.4 shows that $\mathbb{R}^r + i\Omega$ is not an arbitrary Stein tube domain, as Ω must also be C-invariant.*

We conclude this section with a univalence result for holomorphically separable, N-equivariant, Riemann domains over G/K.

Proposition 3.7. *A holomorphically separable, N-equivariant, Riemann domain $\pi\colon Z \to G/K$ is univalent, i.e. the holomorphic map π is globally injective.*

Proof. Let $\pi\colon Z \to G/K$ be a holomorphically separable, N-equivariant, Riemann domain over G/K. By [Ros63], Thm. 4.6, Z admits a holomorphic, N-equivariant open embedding into its envelope of holomorphy, which is a Stein N-equivariant, Riemann domain over G/K. Hence, without loss of generality, we may assume that Z is Stein.

Let $\pi(Z) = N\exp(\mathcal{D}) \cdot eK$ be the image of Z under π. Define $\Sigma := \exp(\mathcal{D}) \cdot eK$ and $\widetilde{\Sigma} := \pi^{-1}(\Sigma)$. Note that $\widetilde{\Sigma}$ is a closed submanifold of Z.

Claim. *The map $\widetilde{\phi} : N \times \widetilde{\Sigma} \to Z$, given by $(n,x) \to n \cdot x$, is a diffeomorphism.*

Proof of the claim. Since $\Sigma = \pi(Z) \cap \exp(\mathfrak{a}) \cdot eK$ is a closed real submanifold of $\pi(Z)$ and π is a local biholomorphism, the restriction $\pi|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \Sigma$ is a local diffeomorphism. Moreover there is the commutative diagram

$$\begin{array}{ccc} N \times \widetilde{\Sigma} & \xrightarrow{\ \widetilde{\phi}\ } & Z \\ {\scriptstyle Id\times(\pi|_{\widetilde{\Sigma}})}\downarrow & & \downarrow{\scriptstyle \pi} \\ N \times \Sigma & \xrightarrow{\ \phi\ } & N \exp \mathcal{D} \cdot eK \end{array}$$

where the maps $Id \times (\pi|_{\widetilde{\Sigma}})$ and π are local diffeomorphisms, and ϕ is a global diffeomorphism. Hence the map $\widetilde{\phi}$ is a local diffeomorphism.

To prove that $\widetilde{\phi}$ is surjective, let z be an arbitrary element in Z. Note that $\pi(z) = n \exp(H)K$, for some $n \in N$ and $H \in \mathcal{D}$. Then the element $w := n^{-1} \cdot z \in \widetilde{\Sigma}$ satisfies $n \cdot w = z$, implying the surjectivity of $\widetilde{\phi}$.

To prove that $\widetilde{\phi}$ is injective, assume that $n \cdot w = n' \cdot w'$, for some $n, n' \in N$ and $w, w' \in \widetilde{\Sigma}$. From the equivariance of π it follows that $n \cdot \pi(w) = n' \cdot \pi(w')$. As ϕ is bijective, it follows that $n = n'$ and $\pi(w) = \pi(w')$. Thus $w = (n^{-1}n') \cdot w' = w'$, implying the injectivity of $\widetilde{\phi}$ and concluding the proof of the claim.

Now, to prove the univalence of π, it is sufficient to show that the restriction $\pi|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \Sigma$ of π to $\widetilde{\Sigma}$ is injective. For this, consider the closed complex submanifold $R \cdot \widetilde{\Sigma} = \pi^{-1}(R \cdot \Sigma)$ of Z, where $R = \exp \mathfrak{a}$. As Z is Stein, so is $R \cdot \widetilde{\Sigma}$. Hence the restriction $\pi|_{R \cdot \widetilde{\Sigma}} : R \cdot \widetilde{\Sigma} \to R \cdot \Sigma$ defines an R-equivariant, Stein, Riemann domain over the Stein tube $R \cdot \Sigma$. As R is isomorphic to $\mathbb{R}^r$, from [CoLo86], p. 60, it follows that $\pi|_{R \cdot \widetilde{\Sigma}}$ is injective. Hence the same is true for $\pi|_{\widetilde{\Sigma}}$ and π, as wished. ☐

Corollary 3.8. *The envelope of holomorphy $\widehat{D}$ of an N-invariant domain D in G/K is the smallest Stein domain in G/K containing D. Namely, $\widehat{D}$ is the N-invariant domain such that the base $\widehat{\Omega}$ of the associated tube is the convex C-invariant hull of Ω.*

4. The Levi form of an N-invariant function on G/K

Let G/K be a non-compact, irreducible Hermitian symmetric space of rank r. Let $f : D \to \mathbb{R}$ be an N-invariant function. Then f is uniquely determined by the functions

$$\widetilde{f}(H) := f(\exp H \cdot eK), \tag{12}$$

and

$$\widehat{f}(\mathbf{y}) := f(\exp(L^{-1}(\mathbf{y})) \cdot eK) = \widetilde{f}(L^{-1}(\mathbf{y})) \tag{13}$$

where $H \in \mathcal{D}$ and $\mathbf{y} = L(H) \in \Omega \subset (\mathbb{R}^{>0})^r$, according to the following commutative diagram

$$\begin{array}{ccc} \Omega & & \\ {\scriptstyle L^{-1}}\downarrow & \searrow^{\widehat{f}} & \\ \mathcal{D} & \xrightarrow{\widetilde{f}} & \mathbb{R}. \\ {\scriptstyle \exp}\downarrow & \nearrow_{f} & \\ D & & \end{array}$$

Since the N-action on D is proper and every N-orbit intersects transversally the smooth slice $\exp(\mathcal{D}) \cdot eK$ in a single point, it is easy to check that the map $f \to \widetilde{f}$ is a bijection between the class $C^0(D)^N$ of continuous N-invariant functions on D and the class $C^0(\mathcal{D})$ of continuous functions on $\mathcal{D}$. By Theorem 4.1 in [Fle78], such a map is also a bijection between $C^\infty(D)^N$ and $C^\infty(\mathcal{D})$. Analogous statements hold true for the map $f \to \widehat{f}$.

The goal of this section is to express the real symmetric J-invariant bilinear form

$$\mathbf{h}_f(\,\cdot\,,\,\cdot\,) := -dd^c f(\,\cdot\,, J\,\cdot\,),$$

of a smooth N-invariant function f on D, in terms of the first and second derivatives of the corresponding function $\widetilde{f}$ on $\mathcal{D}$ (Prop. 4.1). Recall that a function f on D is plurisubharmonic (resp. strictly plurisubharmonic) if and only if the Levi form

$$\mathbf{L}_f^{\mathbb{C}}(Z, \overline{W}) = 2(\mathbf{h}_f(X,Y) + i\mathbf{h}_f(X, JY))$$

is positive semidefinite (resp. positive definite), where $Z = X - iJX$ and $W = Y - iJY$ are vectors of type $(1,0)$.

Since $\mathbf{L}_f^{\mathbb{C}}$ is positive semidefinite (resp. positive definite) if and only if $\mathbf{h}_f$ is positive semidefinite (resp. positive definite), the calculation of $\mathbf{h}_f$ will enable us to characterize smooth N-invariant plurisubharmonic functions on a Stein N-invariant domain D in G/K by suitable conditions on the corresponding functions $\widetilde{f}$ on $\mathcal{D}$ and $\widehat{f}$ on Ω (cf. Thm. 5.1).

If f is N-invariant, then so is $\mathbf{h}_f$. Therefore it will be sufficient to determine $\mathbf{h}_f$ along the slice $\exp(\mathcal{D}) \cdot eK$.

For $X \in \mathfrak{g}$, denote by $\widetilde{X}$ the vector field on G/K induced by the left G-action. Its value at $z \in G/K$ is given by

$$\widetilde{X}_z := \tfrac{d}{ds}\big|_{s=0} \exp sX \cdot z.$$

Let $X \in \mathfrak{g}^\alpha$, for $\alpha \in \Sigma^+ \cup \{0\}$ (here $X \in \mathfrak{a}$, when $\alpha = 0$). If $z = aK$, with $a = \exp H$ and $H \in \mathfrak{a}$, then the vector field $\widetilde{X}$ can also be expressed as

$$\widetilde{X}_z = e^{-\alpha(H)} a_* X. \tag{14}$$

Set

$$\mathbf{b} := B(H_1, H_1) = \ldots = B(H_r, H_r), \tag{15}$$

which is a real positive constant only depending on the Lie algebra $\mathfrak{g}$.

Proposition 4.1. *Let D be an N-invariant domain in G/K and let $f : D \to \mathbb{R}$ be a smooth N-invariant function. Fix $a = \exp H$, with $H = \sum_j h_j H_j \in \mathcal{D}$. Then, in the basis of $\mathfrak{s}$ defined in Remark 2.2, the form $\mathbf{h}_f$ at $z = aK \in D$ is given as follows.*

(i) *The spaces $a_*\mathfrak{a}$, $a_*J\mathfrak{a}$, $a_*\mathfrak{g}^{e_j-e_l}$, $a_*\mathfrak{g}^{e_j+e_l}$ and $a_*\mathfrak{g}^{e_j}$ are pairwise $\mathbf{h}_f$-orthogonal.*

(ii) *For $H_j, H_l \in \mathfrak{a}$ one has* $\mathbf{h}_f(a_*H_j, a_*H_l) = -2\delta_{jl}\frac{\partial \widetilde{f}}{\partial h_l}(H) + \frac{\partial^2 \widetilde{f}}{\partial h_j \partial h_l}(H)$.

On the blocks $a_*\mathfrak{g}^{e_j-e_l}$ and $a_*\mathfrak{g}^{e_j}$ the restriction of $\mathbf{h}_f$ is diagonal and the only non-zero values are given as follows.

(iii) *For X, $X' \in \mathfrak{g}^{e_j-e_l}$ as in Remark 2.2(b), one has*

$$\mathbf{h}_f(a_*X, a_*X) = -2\tfrac{\|X\|^2}{\mathbf{b}}\tfrac{\partial \widetilde{f}}{\partial h_j}(H), \qquad \mathbf{h}_f(a_*X', a_*X') = -2\tfrac{\|X'\|^2}{\mathbf{b}}\tfrac{\partial \widetilde{f}}{\partial h_j}(H).$$

(iv) (*non-tube case*) *For $X \in \mathfrak{g}^{e_j}$ as in Remark 2.2(c), one has*

$$\mathbf{h}_f(a_*X, a_*X) = -2\tfrac{\|X\|^2}{\mathbf{b}}\tfrac{\partial \widetilde{f}}{\partial h_j}(H).$$

On the remaining blocks the form $\mathbf{h}_f$ is determined by (4), its J-invariance, (i) and (iii) above.

Proof. Let $f\colon G/K \to \mathbb{R}$ be a smooth N-invariant function. The computation of $\mathbf{h}_f$ uses the fact that, for $X \in \mathfrak{n}$, the function $\mu_f^X\colon G/K \to \mathbb{R}$, given by $\mu_f^X(z) := d^cf(\widetilde{X}_z)$, is N-equivariant and satisfies the identity

$$d\mu_f^X = -\iota_{\widetilde{X}} dd^c f, \tag{16}$$

where $d^cf := df \circ J$ (see [HeSc07], Lemma 7.1). We begin by determining $d^cf(\widetilde{X}_z)$, for $X \in \mathfrak{n}$ and $z \in G/K$. By the N-invariance of f and of J one has

$$d^cf(\widetilde{X}_{n\cdot z}) = d^cf(\widetilde{\mathrm{Ad}_{n^{-1}}X}_z)\,, \tag{17}$$

for every $z \in G/K$ and $n \in N$. Thus it is sufficient to take $z = aK \in \exp(\mathcal{D})\cdot eK$. Let $H = \sum h_j H_j \in \mathcal{D}$ and $a = \exp H$. Then

$$d^cf(\widetilde{X}_z) = \begin{cases} \frac{1}{2}e^{-2h_j}\frac{\partial \widetilde{f}}{\partial h_j}(H)\,, & \text{for } X = E^j \in \mathfrak{g}^{2e_j} \\ \quad 0\,, & \text{for } X \in \mathfrak{g}^{\alpha}, \text{ with } \alpha \in \Sigma^+ \backslash \{2e_1, \dots, 2e_r\}. \end{cases} \tag{18}$$

The first part of equation (18) follows from (14) and Lemma 2.1 (a):

$$d^cf((\widetilde{E^j})_z) = e^{-2e_j(H)}df(a_*JE^j) = \tfrac{1}{2}e^{-2h_j}\tfrac{d}{ds}\big|_{s=0}\widetilde{f}(H + sH_j) = \tfrac{1}{2}e^{-2h_j}\tfrac{\partial \widetilde{f}}{\partial h_j}(H).$$

For the second part, let $X \in \mathfrak{g}^{\alpha}$, with $\alpha \in \Sigma^+ \backslash \{2e_1, \ldots, 2e_r\}$. Then $JX \in \mathfrak{g}^{\beta}$, with $\beta \in \Sigma^+$. By (14) and the N-invariance of f, one obtains the desired result

$$d^c f(\widetilde{X}_z) = e^{-\alpha(H)+\beta(H)} df(\widetilde{JX}_z) = 0.$$

(i) Orthogonality of the blocks. Let $X \in \mathfrak{g}^{\alpha}$ and $Y \in \mathfrak{g}^{\gamma}$, where $\alpha \in \Sigma^+$ and $\gamma \in \{0\} \cup (\Sigma^+ \backslash \{2e_1, \ldots, 2e_r\})$ are distinct restricted roots (here $Y \in \mathfrak{a}$, when $\gamma = 0$). Then $JY \in \mathfrak{g}^{\beta}$, for some $\beta \in \Sigma^+$. By (14) and (16), one has

$$\begin{aligned}
\mathbf{h}_f(a_*X, a_*Y) &= -dd^c f(a_*X, a_*JY) = -e^{\alpha(H)+\beta(H)} dd^c f(\widetilde{X}_z, \widetilde{JY}_z) \\
&= e^{\alpha(H)+\beta(H)} d\mu^X(\widetilde{JY}_z) = e^{\alpha(H)+\beta(H)} \tfrac{d}{ds}\big|_{s=0} \mu^X(\exp sJY \cdot z) \\
&= e^{\alpha(H)+\beta(H)} \tfrac{d}{ds}\big|_{s=0} d^c f(\widetilde{X}_{\exp sJY \cdot z}) = e^{\alpha(H)+\beta(H)} \tfrac{d}{ds}\big|_{s=0} d^c f(\widetilde{Ad_{\exp(-sJY)}X}_z) \\
&= e^{\alpha(H)+\beta(H)} \tfrac{d}{ds}\big|_{s=0} d^c f(\widetilde{X}_z - s\widetilde{[JY, X]}_z + o(s^2)) \\
&= -e^{\alpha(H)+\beta(H)} d^c f(\widetilde{[JY, X]}_z).
\end{aligned} \tag{19}$$

The brackets $[JY, X]$ lie in $\mathfrak{g}^{\alpha+\beta}$. Since $\alpha \neq \gamma$, one sees that $\alpha + \beta \neq 2e_1, \ldots, 2e_r$. Then, by (18), the expression (19) vanishes, proving the orthogonality of $a_*\mathfrak{g}^{\alpha}$ and $a_*\mathfrak{g}^{\gamma}$, for all α and γ as above. The J-invariance of $\mathbf{h}_f$ implies that $a_*\mathfrak{a}$ is orthogonal to $a_*\mathfrak{g}^{\beta}$, for all $\beta \in \Sigma^+$, and concludes the proof of (i).

Next we determine the form $\mathbf{h}_f$ on the essential blocks.

(ii) The form h_f on $a_*\mathfrak{a}$.

Let $H_j, H_l \in \mathfrak{a}$. Since $JH_l = -2E^l$, one has

$$\begin{aligned}
\mathbf{h}_f(a_*H_j, a_*H_l) &= -2dd^c f(a_*E^l, a_*H_j) = -2e^{2e_l(H)} dd^c f((\widetilde{E^l})_z, (\widetilde{H_j})_z) \\
&= 2e^{2e_l(H)} d\mu^{E^l}((\widetilde{H_j})_z) = 2e^{2e_l(H)} \tfrac{d}{dt}\big|_{t=0} \mu^{E^l}(\exp tH_j \cdot z) \\
&= 2e^{2e_l(H)} \tfrac{d}{dt}\big|_{t=0} d^c f((\widetilde{E^l})_{\exp tA_j \cdot z}),
\end{aligned}$$

which, by (18), becomes

$$2e^{2e_l(H)} \tfrac{d}{dt}\big|_{t=0} \tfrac{1}{2} e^{-2e_l(H+tH_j)} \tfrac{\partial \tilde{f}}{\partial h_l}(H + tH_j) = -2\tfrac{\partial \tilde{f}}{\partial h_l}(H)\delta_{lj} + \tfrac{\partial^2 \tilde{f}}{\partial h_j \partial h_l}(H).$$

This concludes the proof of (ii).

(iii) The form $\mathbf{h}_f$ on $a_*\mathfrak{g}^{e_j - e_l}$.

Let X, $X' \in \mathfrak{g}^{e_j - e_l}$ be elements of the basis given in Remark 2.2 (b). Then JX, $JX' \in \mathfrak{g}^{e_j + e_l}$. From (19), (18) and Lemma 2.3(a) one has

$$\begin{aligned}
\mathbf{h}_f(a_*X, a_*X) &= -dd^c f(a_*X, a_*JX) \\
&= -e^{(e_j+e_l)(H)} e^{(e_j-e_l)(H)} d^c f(\widetilde{[JX, X]}_z) \\
&= -e^{2e_j(H)} \Big(sd^c f((\widetilde{E^j})_z) \Big) = -\tfrac{s}{2} \tfrac{\partial \tilde{f}}{\partial h_j}(H)\,,
\end{aligned} \tag{20}$$

for some $s \in \mathbb{R}\backslash\{0\}$. By Lemma 3.5 (ii), one has $s > 0$. By the comparison of (20) with the formula obtained in Remark 6.2, one deduces the exact value of s, namely $s = \frac{4\|X\|^2}{\mathbf{b}}$. Therefore, one has

$$\mathbf{h}_f(a_*X, a_*X) = -2\tfrac{\|X\|^2}{\mathbf{b}}\tfrac{\partial \tilde{f}}{\partial h_j}(H), \qquad \mathbf{h}_f(a_*X', a_*X') = -2\tfrac{\|X'\|^2}{\mathbf{b}}\tfrac{\partial \tilde{f}}{\partial h_j}(H),$$

as stated. From (19) and Lemma 2.3(b), one obtains $\mathbf{h}_f(a_*X, a_*X') = 0$. From (19), the skew symmetry of dd^cf and the fact that $2(e_j - e_l) \notin \Sigma^+$, one obtains $\mathbf{h}_f(a_*X, a_*JX) = \mathbf{h}_f(a_*X, a_*JX') = 0$. Finally, let $X = Z^\mu + \overline{Z^\mu}$, and $Y = Z^\nu + \overline{Z^\nu}$ be elements of the basis of $\mathfrak{g}^{e_j - e_l}$ given in Remark 2.2 (b), for $\mu,\ \nu \in \Delta^+$ distinct roots satisfying $\nu \neq \mu,\ \bar{\mu}$. Then, by (19) and Lemma 2.1(b) one has

$$\mathbf{h}_f(a_*X, a_*Y) = -e^{2e_j(H)} d^c f(\widetilde{[JY, X]}_z) = 0,$$

since no non-real roots in Δ have real part equal to $2e_j$. This completes the proof of (iii).

(iv) The form $\boldsymbol{h_f}$ on $\boldsymbol{a_*\mathfrak{g}^{e_j}}$.

Let $X = Z^\mu + \overline{Z^\mu}$ and $JX = iZ^\mu + \overline{iZ^\mu}$ be elements of the basis of $\mathfrak{g}^{e_j}$ given in Remark 2.2 (c). Then, from (19) and Lemma 2.3 (c), one obtains

$$\begin{aligned} \mathbf{h}_f(a_*X, a_*X) &= -e^{2e_j(H)} d^c f(\widetilde{[JX, X]}_z) \\ &= -e^{2e_j(H)} t\, d^c f((\widetilde{E^j})_z) = -\tfrac{t}{2}\tfrac{\partial \tilde{f}}{\partial h_j}(H), \end{aligned} \tag{21}$$

for some $t \in \mathbb{R}\backslash\{0\}$. Since for all $\zeta \in \mathfrak{s}_{1/2}$ the form $F(\zeta, \zeta) = [J\zeta, \zeta]$ takes values in the cone $\overline{J\mathfrak{a}^+}$, then $t > 0$. By the comparison of (21) with the formula obtained in Remark 6.2, one deduces the exact value of t, namely $t = \frac{4\|X\|^2}{\mathbf{b}}$ and

$$\mathbf{h}_f(a_*X, a_*X) = \mathbf{h}_f(a_*JX, a_*JX) = -2\tfrac{\|X\|^2}{\mathbf{b}}\tfrac{\partial \tilde{f}}{\partial h_j}(H).$$

Finally, let $X = Z^\mu + \overline{Z^\mu}$ and $Y = Z^\nu + \overline{Z^\nu}$ be elements of the basis of $\mathfrak{g}^{e_j}$ given in Remark 2.2 (c), for $\mu,\ \nu \in \Delta^+$ distinct roots satisfying $\nu \neq \mu,\ \bar{\mu}$. Then, by (19) and Lemma 2.1(c) one has $\mathbf{h}_f(a_*X, a_*Y) = 0$. This concludes the proof of (iv) and of the proposition. □

Remark 4.2. *Statement (i) in Lemma 3.5 suggests why in Prop.4.1 (iii) no conditions appear on* $\frac{\partial \tilde{f}}{\partial h_r}$.

5. N-INVARIANT PSH FUNCTIONS VS. CONVDEC FUNCTIONS

Let D be a Stein, N-invariant domain in a non-compact, irreducible Hermitian symmetric space G/K of rank r. By Theorem 3.4, the base Ω of the associated r-dimensional tube domain is a open convex, C-invariant set.

In this section, we characterize the N-invariant plurisubharmonic functions on D in terms of the associated functions on Ω. We begin with the smooth case. From Proposition 4.1, we obtain a one-to-one correspondence between the class of smooth N-invariant plurisubharmonic functions on D and the class of smooth convex functions on Ω which satisfy an additional monotonicity condition. Later we obtain an analogous statement for arbitrary N-invariant plurisubharmonic functions. As a result, N-invariant plurisubharmonic functions on D are necessarily continuous. Define

$$C := \begin{cases} (\mathbb{R}^{>0})^r, & \text{in the non-tube case,} \\ (\mathbb{R}^{>0})^{r-1} \times \{0\}, & \text{in the tube case.} \end{cases} \tag{22}$$

The above cone C coincides with the one defined in (9), when $J\mathfrak{a} + iJ\mathfrak{a}^+$ is identified with $\mathbb{H}^r$. Definition 3.3 can be reformulated accordingly.

Denote by " $\cdot$ " the standard inner product on $\mathbb{R}^r$. Let $\widetilde{f}$ and $\widehat{f}$ be the functions associated to an N-invariant function $f\colon D \to \mathbb{R}$ introduced in (12) and (13).

Theorem 5.1. *Let D be an N-invariant Stein domain in G/K and let $f : D \to \mathbb{R}$ be a smooth, N-invariant, plurisubharmonic function. Then the following statements are equivalent:*

(i) *f is plurisubharmonic (resp. strictly plurisubharmonic) at $z = aK$, with $a = \exp(H)$ and $H = \sum_j h_j H_j \in \mathcal{D}$;*

(ii) *the form*
$$\Big(-2\delta_{jl} \tfrac{\partial \widetilde{f}}{\partial h_l}(H) + \tfrac{\partial^2 \widetilde{f}}{\partial h_j \partial h_l}(H) \Big)_{j,l=1,\dots,r}$$
in Proposition 4.1(ii) is positive semidefinite (resp. positive definite) and
$$\operatorname{grad}\widetilde{f}(H) \cdot \mathbf{v} \leqslant 0 \ \textit{(resp. } < 0\textit{)}, \qquad \textit{for all } \mathbf{v} \in \overline{C} \backslash \{\mathbf{0}\};$$

(iii) *the Hessian of $\widehat{f}$ is positive semidefinite (resp. positive definite) at $\mathbf{y} = (y_1, \dots, y_r) = L(H)$ and*
$$\operatorname{grad}\widehat{f}(\mathbf{y}) \cdot \mathbf{v} \leqslant 0 \ \textit{(resp. } < 0\textit{)}, \qquad \textit{for all } \mathbf{v} \in \overline{C} \backslash \{\mathbf{0}\}; \tag{23}$$

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows directly from Proposition 4.1.
$(ii) \Leftrightarrow (iii)$ Since $L^{-1}(y_1, \dots, y_r) = (\frac{1}{2}\ln(y_1), \dots, \frac{1}{2}\ln(y_r))$ (see (8)), one has $\widetilde{f}(h_1, \dots, h_r) = \widehat{f}(e^{2h_1}, \dots, e^{2h_r})$. Therefore

$$\tfrac{\partial \widetilde{f}}{\partial h_j}(h_1, \dots, h_r) = 2 \tfrac{\partial \widehat{f}}{\partial y_j}{}_{(e^{2h_1}, \dots, e^{2h_r})} e^{2h_j} \tag{24}$$

$$\tfrac{\partial^2 \widetilde{f}}{\partial h_j \partial h_l}(H) = 4\tfrac{\partial^2 \widehat{f}}{\partial y_j \partial y_l}(e^{2h_1},\dots,e^{2h_r}) e^{2h_j} e^{2h_l} + 4\tfrac{\partial \widehat{f}}{\partial y_j}(e^{2h_1},\dots,e^{2h_r}) e^{2h_j} \delta_{jl}\,. \tag{25}$$

By combining formulas (24) and (25) one obtains

$$\big(4\tfrac{\partial^2 \widehat{f}}{\partial y_j \partial y_l} e^{2h_j} e^{2h_l}\big)_{j,l} = \big(\tfrac{\partial^2 \widetilde{f}}{\partial h_j \partial h_l} - 2\,\tfrac{\partial \widetilde{f}}{\partial h_j}\delta_{jl}\big)_{j,l}. \tag{26}$$

Also, by (24), the same monotonicity conditions hold both for $\widetilde{f}$ and for $\widehat{f}$. □

Definition 5.2. *A function $\widehat{f}\colon \Omega \to \mathbb{R}$, defined on a convex set, is convex if $\widehat{f}(t\mathbf{x} + (1-t)\mathbf{y}) \leqslant t\widehat{f}(\mathbf{x}) + (1-t)\widehat{f}(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $t \in [0,1]$.*

Remark 5.3. (i) If $\widehat{f}$ is smooth, then it is convex if and only if its Hessian is positive semidefinite.
(ii) A smooth function is *stably convex* if its Hessian is positive definite.

Definition 5.4. *Let Ω be a convex, C-invariant domain in $(\mathbb{R}^{>0})^r$. A function $\widehat{f} : \Omega \to \mathbb{R}$ is $\overline{C}$-decreasing (resp. strictly decreasing) if for every $\mathbf{y} \in \Omega$ and $\mathbf{v} \in \overline{C}\backslash\{\mathbf{0}\}$ the restriction of $\widehat{f}$ to the half-line $\{\mathbf{y} + t\mathbf{v} \ : \ t \geqslant 0\}$ is decreasing (resp. strictly decreasing).*

Remark 5.5. (i) If $\widehat{f} : \Omega \to \mathbb{R}$ is smooth, then it is $\overline{C}$-decreasing if and only if $\operatorname{grad}\widehat{f}(\mathbf{y}) \cdot \mathbf{v} \leqslant 0$ for every $\mathbf{y} \in \Omega$ and $\mathbf{v} \in \overline{C}$.
(ii) A smooth, stably convex function $\widehat{f} : \Omega \to \mathbb{R}$ is $\overline{C}$-decreasing if and only if $\operatorname{grad}\widehat{f}(\mathbf{y}) \cdot \mathbf{v} < 0$, for every $\mathbf{y} \in \Omega$ and $\mathbf{v} \in \overline{C}\backslash\{\mathbf{0}\}$. This follows from the fact that the directional derivatives $\operatorname{grad}\widehat{f}(\mathbf{y}) \cdot \mathbf{v}$ of a stably convex, $\overline{C}$-decreasing function $\widehat{f}$ never vanish. In particular $\widehat{f}$ is automatically strictly $\overline{C}$-decreasing.

In view of the above definitions, we introduce the following classes of smooth functions:

- $ConvDec^{\infty,+}(\Omega)$: smooth, stably convex, $\overline{C}$-decreasing functions on Ω,
- $ConvDec^{\infty}(\Omega)$: smooth, convex, $\overline{C}$-decreasing functions on Ω,
- $Psh^{\infty,+}(D)^N$: smooth, N-invariant, strictly plurisubharmonic functions on D,
- $Psh^{\infty}(D)^N$: smooth, N-invariant, plurisubharmonic functions on D.

Theorem 5.1 establishes a one-to-one correspondence between $ConvDec^{\infty,+}(\Omega)$ and $Psh^{\infty,+}(D)^N$, and between $ConvDec^{\infty}(\Omega)$ and $Psh^{\infty}(D)^N$. It shows that the function $\widehat{f}$ associated to a smooth N-invariant plurisubharmonic function on a Stein domain $D \subset G/K$ is not an arbitrary smooth convex function on Ω, as it must satisfy the additional monotonicity conditions (23).

The rest of this section is devoted to obtaining analogous results in the non-smooth case. To this aim we adapt to our purposes the notion of a plurisubharmonic function given in [Gun90], Def. 1, p. 118.

Consider the smooth, stably convex, positive, strictly $\overline{C}$-decreasing function $\widehat{h} : (\mathbb{R}^{>0})^r \to \mathbb{R}^{>0}$ defined by

$$\widehat{h}(\mathbf{y}) := \textstyle\sum_j \frac{1}{y_j}, \tag{27}$$

and let h be the N-invariant strictly plurisubharmonic function on G/K associated to $\widehat{h}$.

Definition 5.6. *A function $\widehat{f} : \Omega \to \mathbb{R}$ is stably convex and $\overline{C}$-decreasing non-smooth case by saying that if for every point in Ω there exist a convex $\overline{C}$-invariant neighborhood W and $\varepsilon > 0$ such that $\widehat{f} - \varepsilon\widehat{h}$ is a convex, $\overline{C}$-decreasing function on W.*

Definition 5.7. *An N-invariant function $f \colon D \to \mathbb{R}$ is strictly plurisubharmonic if for every point in D there exist an N-invariant neighborhood U and $\varepsilon > 0$ such that $f - \varepsilon h$ is an N-invariant plurisubharmonic function on U.*

In the smooth case, the above notions coincide with the ones introduced earlier. Now define the following spaces of functions:

- $ConvDec^+(\Omega)$: stably convex and $\overline{C}$-decreasing functions on Ω;
- $ConvDec(\Omega)$: convex, $\overline{C}$-decreasing functions on Ω;
- $Psh^+(D)^N$: strictly plurisubharmonic, N-invariant functions on D;
- $Psh(D)^N$: plurisubharmonic, N-invariant functions on D.

In order to prove our main theorem, we adapt a classical approximation method to the class of convex, $\overline{C}$-decreasing functions on convex, C-invariant domains in $(\mathbb{R}^{>0})^r$.

For a domain Ω in $\mathbb{R}^r$, denote by $d_\Omega \colon \Omega \to \mathbb{R}$ the distance function from the boundary. If $\mathbf{y} \in \Omega$, then $d_\Omega(\mathbf{y})$ is by definition the radius of the largest open ball of center $\mathbf{y}$ contained in Ω.

Lemma 5.8. *Let Ω be a proper, convex, C-invariant subdomain of $\mathbb{R}^r$. Then the function*

$$\widehat{u} := -\ln d_\Omega$$

is convex and $\overline{C}$-decreasing.

Proof. By a well known characterization of convex domains, the function $\widehat{u}$ is convex. If for some $\mathbf{y} \in \Omega$ the open ball $\mathbb{B}_\rho(\mathbf{y})$ of center $\mathbf{y}$ and radius ρ is contained in Ω then, by the C-invariance of Ω, also the ball $\mathbb{B}_\rho(\mathbf{y}+\mathbf{v})$ is contained in Ω, for all $\mathbf{v} \in C$. It follows that $d_\Omega(\mathbf{y}+\mathbf{v}) \geqslant d_\Omega(\mathbf{y})$ and consequently $\widehat{u}(\mathbf{y}+\mathbf{v}) \leqslant \widehat{u}(\mathbf{y})$, for all $\mathbf{v} \in \overline{C}$. Hence $\widehat{u}$ is $\overline{C}$-decreasing, as claimed. □

Fix a smooth, positive, radial function $\sigma : \mathbb{R}^r \to \mathbb{R}$ (only depending on $R^2 = \|\mathbf{w}\|^2$), with support in $\mathbb{B}_1(\mathbf{0})$, such that $\sigma'(R^2) < 0$ and $\int_{\mathbb{R}^r} \sigma(\mathbf{w}) d\mathbf{w} = 1$.

For $\varepsilon > 0$, let $\Omega_\varepsilon := \{\mathbf{y} \in \Omega \ : \ d_\Omega(\mathbf{y}) > \varepsilon\}$. Given a convex, $\overline{C}$-decreasing function $\widehat{f} : \Omega \to \mathbb{R}$, define $\hat{f}_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by

$$\widehat{f}_\varepsilon(\mathbf{y}) := \textstyle\int_{\mathbb{R}^r} \widehat{f}(\mathbf{y} + \varepsilon\mathbf{w})\sigma(\mathbf{w})d\mathbf{w} = \frac{1}{\varepsilon^r}\int_{\mathbb{R}^r} \widehat{f}(\mathbf{z})\sigma(\frac{\mathbf{z}-\mathbf{y}}{\varepsilon})d\mathbf{z}\,.$$

The functions $\widehat{f}_\varepsilon$ are clearly smooth.

Lemma 5.9. *Let Ω be a convex, C-invariant domain in $(\mathbb{R}^{>0})^r$. Then the following facts hold true.*

(i) *For every $\varepsilon > 0$, the domain Ω_ε is convex and C-invariant.*

(ii) *The functions*

$$\widehat{f}^+_\varepsilon := \widehat{f}_\varepsilon(\mathbf{y}) + \varepsilon\widehat{h}(\mathbf{y})\,,$$

are smooth, stably convex, $\overline{C}$-decreasing and, for $\varepsilon \searrow 0$, they decrease to $\widehat{f}$, uniformly on compact subsets of Ω.

Proof. (i) Let $\mathbf{y}$ and $\mathbf{y}+\mathbf{v}$ be elements of Ω_ε. Then $\mathbb{B}_\varepsilon(\mathbf{y})$ and $\mathbb{B}_\varepsilon(\mathbf{y}+\mathbf{v})$ are contained in Ω and, by the convexity of Ω, the same is true for $\mathbb{B}_\varepsilon(\mathbf{y}+\mathbf{tv})$, for every $t \in [0,1]$. This shows that Ω_ε is convex. Moreover, as Ω is C-invariant, if $\mathbb{B}_\varepsilon(\mathbf{y})$ is contained in Ω and $\mathbf{v}$ is an element of the cone C, then also the open ball $\mathbb{B}_\varepsilon(\mathbf{y}+\mathbf{v})$ is contained in Ω. This shows that Ω_ε is C-invariant.

(ii) As $\widehat{f}$ is convex, for $\mathbf{y}$, $\mathbf{y}+\mathbf{v} \in \Omega$ and $t \in [0,1]$, one has

$$\widehat{f}_\varepsilon(\mathbf{y}+t\mathbf{v}) := \int_{\mathbb{R}^r} \widehat{f}(\mathbf{y}+t\mathbf{v}+\varepsilon\mathbf{w})\sigma(\mathbf{w})d\mathbf{w}$$

$$\leqslant \int_{\mathbb{R}^r} \big((1-t)\widehat{f}(\mathbf{y}+\varepsilon\mathbf{w}) + t\widehat{f}(\mathbf{y}+\varepsilon\mathbf{w}+\mathbf{v})\big)\sigma(\mathbf{w})d\mathbf{w} = (1-t)\widehat{f}_\varepsilon(\mathbf{y}) + t\widehat{f}_\varepsilon(\mathbf{y}+\mathbf{v})\,,$$

showing that the smooth function $\widehat{f}_\varepsilon$ is convex. Since $\widehat{h}$ is smooth and stably convex, it follows that $\widehat{f}^+_\varepsilon := \widehat{f}_\varepsilon(\mathbf{y}) + \varepsilon\widehat{h}(\mathbf{y})$ is smooth and stably convex as well.

The inequality

$$\widehat{f}_\varepsilon(\mathbf{y}+\mathbf{v}) = \int_{\mathbb{R}^r} \widehat{f}(\mathbf{y}+\mathbf{v}+\varepsilon\mathbf{w})\sigma(\mathbf{w})d\mathbf{w} \leqslant \int_{\mathbb{R}^r} \widehat{f}(\mathbf{y}+\varepsilon\mathbf{w})\sigma(\mathbf{w})d\mathbf{w} = \widehat{f}_\varepsilon(\mathbf{y})\,,$$

for every $\mathbf{y} \in \Omega_\varepsilon$ and $\mathbf{v} \in \overline{C}\backslash\{0\}$, shows that $\widehat{f}^+_\varepsilon$ is $\overline{C}$-decreasing.

Finally, as convexity implies subharmonicity, the remaining part of statement (ii) follows from [Hör94], Thm 3.2.3(ii), p.143. □

Remark 5.10. *By (ii), the smooth functions $\widehat{f}^+_\varepsilon(\mathbf{y})$ are stably convex. This is not necessarily the case for the functions $\widehat{f}_\varepsilon(\mathbf{y})$.*

The next theorem summarises our results and should be regarded as a generalization of the well known statements for Stein tube domains in $\mathbb{C}^n$.

Theorem 5.11. *Let D be a Stein N-invariant domain in a non-compact, irreducible Hermitian symmetric space G/K of rank r. The map $f \to \widehat{f}$ is a bijection between the following classes of functions*

(i) $Psh^{\infty,+}(D)^N$ *and* $ConvDec^{\infty,+}(\Omega)$,
(ii) $Psh^{\infty}(D)^N$ *and* $ConvDec^{\infty}(\Omega)$,
(iii) $Psh\,(D)^N$ *and* $ConvDec(\Omega)$,
(iv) $Psh^{+}(D)^N$ *and* $ConvDec^{+}(\Omega)$.

In particular, N-invariant plurisubharmonic functions on D are necessarily continuous.

Proof. As we already remarked, (i) and (ii) follow from Theorem 5.1 and Remark 5.5.
(iii) Let f be a function in $Psh\,(D)^N$. Since the r-dimensional submanifold $R\exp(\mathcal{D})\cdot eK \subset D$ is biholomorphic to a Stein tube domain $\mathbb{R}^r \times i\Omega$ and the restriction of f to $R\exp(\mathcal{D})\cdot eK$ is plurisubharmonic and R-invariant, then $\widehat{f}$ is necessarily convex. Assume by contradiction that $\widehat{f}$ is not $\overline{C}$-decreasing. Then there exists $s \in \mathbb{R}$ such that the sublevel set $\{\widehat{f} < s\}$ is not $\overline{C}$-invariant and the corresponding N-invariant domain $\{f < s\}$ is not Stein (cf. Thm. 3.4). This contradicts [Car73], Thm. B, p. 419, asserting that the sublevel sets of a plurisubharmonic function in a Stein domain in $\mathbb{C}^n$ are necessarily Stein. Hence $\widehat{f}$ belongs to $ConvDec(\Omega)$, as claimed.

For the converse, let $\widehat{f}$ in $ConvDec(\Omega)$. By Lemma 5.9(ii), the functions $\widehat{f}_\varepsilon^+$ are in $ConvDec^{\infty,+}(\Omega_\varepsilon)$ and, for $\varepsilon \searrow 0$, they decrease to $\widehat{f}$ uniformly on compact subsets of Ω. It follows that the corresponding N-invariant functions f_ε^+ decrease, uniformly on the compact subsets of D, to the N-invariant function f corresponding to $\widehat{f}$. By (i) each f_ε^+ belongs to $Psh^{\infty,+}(D)^N$. Hence $f \in Psh\,(D)^N$, as wished.
(iv) follows directly from the definition of $Psh^+(D)^N$ and of $ConvDec^+(\Omega)$.

Finally, from the inclusions

$$\begin{array}{ccccc} ConvDec^{+}(\Omega) & \subset & ConvDec(\Omega) & \subset & C^0(\Omega) \\ \cup & & \cup & & \\ ConvDec^{\infty,+}(\Omega) & \subset & ConvDec^{\infty}(\Omega) & & \end{array}$$

it follows that all the above functions on Ω are continuous, and so are the corresponding N-invariant plurisubharmonic functions on D. □

6. Applications: the N-invariant potentials of the Killing metric.

Let G/K be a non-compact, irreducible Hermitian symmetric space of rank r. The Killing form B of $\mathfrak{g}$, restricted to $\mathfrak{p}$, induces a G-invariant Kähler metric on G/K, which we refer to as the Killing metric. This metric coincides, up to a positive multiplicative constant, with the Bergman metric of G/K. In this section we exhibit an N-invariant potential of the Killing metric and the associated moment map in a Lie theoretical fashion. Later, we determine all the N-invariant potentials of such metric.

Let $f\colon G/K \to \mathbb{R}$ be a smooth N-invariant function. The map $\mu_f\colon G/K \to \mathfrak{n}^*$, defined by

$$\mu_f(z)(X) := d^c f(\widetilde{X}_z), \tag{28}$$

where $X \in \mathfrak{n}$, is N-equivariant and satisfies (16). If f is strictly plurisubharmonic, then it is referred to as the moment map associated with f.

Proposition 6.1. *Let $z = naK \in G/K$, where $n \in N$, $a = \exp H \in A$ and $H = \sum_j h_j H_j \in \mathfrak{a}$. Let $\mathbf{b}$ be the constant defined in (15).*
(i) The N-invariant function $\rho : G/K \to \mathbb{R}$ defined by

$$\rho(naK) := -\tfrac{1}{2}\textstyle\sum_{j=1}^r B(H,\, H_j) = -\tfrac{\mathbf{b}}{2}(h_1 + \cdots + h_r)\,,$$

is a potential of the Killing metric.
(ii) The moment map $\mu_\rho : G/K \to \mathfrak{n}^*$ associated with ρ is given by

$$\mu_\rho(naK)(X) = -\tfrac{\mathbf{b}}{4}\textstyle\sum_{j=1}^r e^{-2h_j}(E^j)^*(\mathrm{Ad}_{n^{-1}}X) = B(Ad_{n^{-1}}X, Ad_a Z_0)\,, \tag{29}$$

where $X \in \mathfrak{n}$, and the $(E^j)^*$ are defined in (10).

Proof. (i) Let $naK \in G/K$, where $a = \exp H$ and $H = \sum_j h_j H_j$. The function $\widetilde{\rho}\colon \mathfrak{a} \to \mathbb{R}$ associated to ρ is given by $\widetilde{\rho}(H) = -\frac{1}{2}\sum_{j=1}^r h_j B(H_j,\, H_j)$ (cf. (12)). In order to obtain (i), we first prove the identities (29). By (28) and (18), one has

$$\mu_\rho(aK)(X) = d^c\rho(\widetilde{X}_{aK}) = -\tfrac{\mathbf{b}}{4}\textstyle\sum_{j=1}^r e^{-2h_j}(E^j)^*(X).$$

By (2), one has

$$(E^j)^*(X) = B(X, \theta E^j)/B(E^j, \theta E^j) = 2B(X, \tfrac{1}{2}(E^j + \theta E^j))/B(E^j, \theta E^j).$$

Since

$$\mathbf{b} := B(H_j, H_j) = B(I_0 H_j, I_0 H_j) = B(E^j - \theta E^j, E^j - \theta E^j) = -2B(E^j, \theta E^j)$$

and $Z_0 = S_0 + \frac{1}{2}\sum_j E^j + \theta E^j$, for some $S_0 \in \mathfrak{m}$ (cf.[GeIa22], Sect. 2), one obtains

$$-\tfrac{\mathbf{b}}{4}\textstyle\sum_{j=1}^r e^{-2h_j}(E^j)^*(X) = -\tfrac{\mathbf{b}}{2}\sum_{j=1}^r e^{-2h_j}\; B(X, \tfrac{1}{2}(E^j + \theta E^j)/B(E^j, \theta E^j)$$
$$= \textstyle\sum_{j=1}^r\; B(X,\; Ad_a\; \tfrac{1}{2}(E^j + \theta E^j)) \;=\; B(X,\; Ad_a\; Z_0)\,,$$

and (29) follows from the N-equivariance of μ_ρ.

Next we are going to show that on $\mathfrak{p} \times \mathfrak{p}$ one has

$$\mathbf{h}_\rho(\, a_*\cdot\, ,\, a_*\cdot\,) = B(\,\cdot\,,\,\cdot\,).$$

Every $X \in \mathfrak{s}$ decomposes as $X = (X - \phi(X)) + \phi(X) \in \mathfrak{k} \oplus \mathfrak{p}$ (see Sect. 2). Since the projection $\phi : \mathfrak{s} \to \mathfrak{p}$ is a linear isomorphism, the above identity is equivalent to

$$\mathbf{h}_\rho(a_*X, a_*Y) = \mathbf{h}_\rho(a_*\phi(X), a_*\phi(Y)) = B\big(\phi(X),\, \phi(Y)\big) = -\tfrac{1}{2}B\big(X,\, \theta Y\big)\,, \qquad (30)$$

for all X, Y in $\mathfrak{s}$. By Proposition 4.1(i), it is sufficient to consider X, Y both in the same block $a_*\mathfrak{a}$, $a_*\mathfrak{g}^{e_j - e_l}$, and $a_*\mathfrak{g}^{2e_j}$.

Let $H_j, H_l \in \mathfrak{a}$, be as in (1). Then, by (ii) of Proposition 4.1, one has

$$\mathbf{h}_\rho(a_*H_j, a_*H_l) = \delta_{jl}B(H_l,\, H_l) = B(H_j, H_l)\,.$$

Let X, $Y \in \mathfrak{g}^\alpha$, with $\alpha = e_j - e_l$ or $\alpha = e_j$. Then $JY \in \mathfrak{g}^\beta$, for $\beta = e_j + e_l$ or $\beta = e_j$, respectively. From (19) and (i) one obtains

$$\begin{aligned}\mathbf{h}_\rho(a_*X, a_*Y) &= -e^{\alpha(H)+\beta(H)} d^c\rho(\widetilde{[JY, X]_z})\\ &= -e^{\alpha(H)+\beta(H)} B([JY, X], Ad_a Z_0).\end{aligned}$$

From the invariance properties of the Killing form B, the decomposition of X and JY in $\mathfrak{k} \oplus \mathfrak{p}$ and the identity $\phi(J\cdot) = I_0\phi(\cdot)$ (cf. (3)), one has

$$\begin{aligned} B([JY, X], Ad_a Z_0) &= B(Ad_{a^{-1}}[JY, X], Z_0) = e^{-(\alpha(H)+\beta(H))}B([JY, X], Z_0)\\ &= e^{-(\alpha(H)+\beta(H))}\left(B([JY - \phi(JY), X - \phi(X)], Z_0) + B([\phi(JY), \phi(X)], Z_0)\right)\\ &= e^{-(\alpha(H)+\beta(H))}B([[Z_0, \phi(Y)], \phi(X)], Z_0) = e^{-(\alpha(H)+\beta(H))}B(\phi(X), [Z_0, [Z_0, \phi(Y)]])\\ &= -e^{-(\alpha(H)+\beta(H))}B(\phi(X), \phi(Y)) = \tfrac{1}{2}e^{-(\alpha(H)+\beta(H))}B(X, \theta Y).\end{aligned}$$

It follows that

$$\mathbf{h}_\rho(a_*X, a_*Y) = -\tfrac{1}{2}B\big(X,\, \theta Y\big), \qquad (31)$$

as desired. This concludes the proof of (i).

(ii) The identity (31) implies that the N-invariant function ρ is strictly plurisubharmonic. Hence μ_ρ is the moment map associated to ρ. Note that the plurisubharmonicity of ρ also follows by applying Proposition 5.1(iii) to the function $\widehat{\rho}(y_1, \ldots, y_r) = -\frac{\mathbf{b}}{2}\sum_j \sqrt{y_j}$. □

Remark 6.2. *Combining (20) and (21) in Proposition 4.1 with (30), we obtain the exact value of the positive quantities s and t*

$$s = \tfrac{4\|X\|^2}{\mathbf{b}}, \quad \textit{for } X \in \mathfrak{g}^{e_j - e_l}, \quad \textit{and} \quad t = \tfrac{4\|X\|^2}{\mathbf{b}}, \quad \textit{for } X \in \mathfrak{g}^{2e_j}.$$

Remark 6.3. *The map $\mu_G : G/K \to \mathfrak{g}^*$ given by $\mu_G(gK)(\cdot) := B(\mathrm{Ad}_{g^{-1}}\cdot, Z_0)$ is a moment map for the G-action on G/K. The moment map μ_ρ in (ii) of Proposition 6.1 coincides with the restriction of $\mu_G(naK)$ to $\mathfrak{n}$. Namely, for $X \in \mathfrak{n}$ and $naK \in G/K$ one has*

$$\mu_\rho(naK)(X) = \mu_G(naK)(X) = B(\mathrm{Ad}_{(na)^{-1}} X\,, Z_0).$$

In the next remark, all N-invariant potentials of the Killing metric are determined.

Proposition 6.4. *Let $\rho\colon G/K \to \mathbb{R}$ be the potential of the Killing metric given in Proposition 6.1 and let σ be another N-invariant potential. Let $\widehat{\rho}$ and $\widehat{\sigma}$ be the corresponding functions on $(\mathbb{R}^{>0})^r$ defined in (13).*
(a) In the non-tube case, one has $\widehat{\sigma} = \widehat{\rho} + d$, and therefore $\sigma = \rho + d$, for some $d \in \mathbb{R}$;
(b) In the tube case, one has $\widehat{\sigma}(\mathbf{y}) = \widehat{\rho}(\mathbf{y}) + cy_r + d$, for $c, d \in \mathbb{R}$. In particular

$$\sigma(naK) = \rho(naK) + ce^{2h_r} + d,$$

where $\mathbf{y} = (y_1, \ldots, y_r) \in (\mathbb{R}^{>0})^r$, $a = \exp H$, with $H = L^{-1}(\mathbf{y}) = \sum_j h_j H_j$, and $c, d \in \mathbb{R}$.

Proof. Let $f := \sigma - \rho$ be the difference of the two potentials. Then f is a smooth N-invariant function on G/K such that $dd^c f(\cdot, J\cdot) \equiv 0$. Let $\widehat{f}\colon \Omega \to \mathbb{R}$ be the associated function.
(a) In the non-tube case, by Proposition 4.1 (iv) and (24), the function $\widehat{f}$ satisfies $\frac{\partial \widehat{f}}{\partial y_j} \equiv 0$, for all $j = 1, \ldots r$. Hence $\widehat{f}$ is constant on $(\mathbb{R}^{>0})^r$ and f is constant on G/K.
(b) In the tube case, from Proposition 4.1, (26) and (24), it follows that $\frac{\partial \widehat{f}}{\partial y_j} \equiv 0$, for all $j = 1, \ldots r-1$, and $\frac{\partial^2 \widehat{f}}{\partial y_r^2} \equiv 0$. Hence $\widehat{f}$ is an affine function of the variable y_r. Equivalently, $\widehat{\sigma}(\mathbf{y}) = \widehat{\rho}(\mathbf{y}) + cy_r + d$, for $c, d \in \mathbb{R}$, as claimed. □

Remark 6.5. *Let $D(V, F)$ be a symmetric Siegel domain. Then the Bergman kernel function $K(z, z)$, where $z \in G/K$, is N-invariant and $\ln K(z, z)$ is a potential of the Bergman metric. As both the Killing metric and the Bergman metric are G-invariant, they differ by a positive multiplicative constant. It follows that $\ln K(z, z)$ is a positive multiple of one of the N-invariant potentials of the Killing metric described in the above remark.*

Example 6.6. As an application of Proposition 6.4, we exhibit all the N-invariant potentials of the Killing metric for the upper half-plane in $\mathbb{C}$ and for the Siegel upper half-plane of rank 2.
(a) Let $G = SL(2, \mathbb{R})$ and let G/K be the corresponding Hermitian symmetric space. Since $\mathbf{b} = 8$ and $r = 1$, then the potential of the Killing metric given in Proposition 6.1 is

$$\rho(naK) = -4h_1.$$

The subgroup $S = NA$, where

$$N = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \,:\, m \in \mathbb{R} \right\} \quad \text{and} \quad A = \left\{ \begin{pmatrix} e^{h_1} & 0 \\ 0 & e^{-h_1} \end{pmatrix} \,:\, h_1 \in \mathbb{R} \right\},$$

acts on $\mathbb{C}$ by linear fractional transformations. The Siegel realization of G/K is the S-orbit of i, namely the upper half-plane

$$\mathbb{H} = \{x_1 + iy_1 \in \mathbb{C} \mid y_1 > 0\}.$$

By (8), one has

$$\widehat{\rho}(y_1) = \rho(\exp L^{-1}(y_1)K) = \ln \tfrac{1}{y_1^2}.$$

Since $\mathbb{H}$ coincides with its associated tube and $\rho(x_1 + iy_1) = \ln \frac{1}{y_1^2}$, all the N-invariant potentials of the Killing metric are given by

$$\sigma(x_1 + iy_1) = \ln \tfrac{1}{y_1^2} + cy_1 + d, \qquad c, d \in \mathbb{R}.$$

(b) Let $G = Sp(2, \mathbb{R})$ be the real symplectic group and let G/K be the corresponding Hermitian symmetric space. As $\mathbf{b} = 12$, the potential of the Killing metric defined in Proposition 6.1 is given by

$$\rho(naK) = -6(h_1 + h_2).$$

Fix the Iwasawa decomposition for which

$$N = \left\{ \begin{pmatrix} \mathbf{n} & \mathbf{m} \\ \mathbf{0} & {}^t\mathbf{n}^{-1} \end{pmatrix} \right\}, \qquad A = \left\{ \begin{pmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}^{-1} \end{pmatrix} \right\},$$

where $\mathbf{n}$ is unipotent, $\mathbf{n}\,{}^t\mathbf{m}$ is symmetric and $\mathbf{a} = \begin{pmatrix} e^{h_1} & 0 \\ 0 & e^{h_2} \end{pmatrix}$, with h_1, h_1 coordinates in $\mathfrak{a}$ with respect to the basis defined in Lemma 2.2. The Siegel realization of G/K is the Siegel upper half-plane of rank 2

$$\mathcal{P} = \{W = S + iT \in M(2, 2, \mathbb{C}) \mid {}^tW = W,\ T \gg 0\},$$

of 2×2 complex symmetric matrices with positive definite imaginary part. It is the orbit of iI_2 under the action of $S = NA$ by linear fractional transformations. The associated tube is $\mathbb{H} \times \mathbb{H}$ and coincides with the diagonal matrices in $\mathcal{P}$. By (8), one has

$$\widehat{\rho}(y_1, y_2) = \rho(\exp L^{-1}(y_1, y_2)K) = \ln \tfrac{1}{(y_1 y_2)^3}.$$

A matrix $S + iT \in \mathcal{P}$ can be expressed in a unique way as

$$na \cdot iI_2 = n \cdot \begin{pmatrix} ie^{2h_1} & 0 \\ 0 & ie^{2h_2} \end{pmatrix}.$$

If $T = \begin{pmatrix} t_1 & t_3 \\ t_3 & t_2 \end{pmatrix}$, a simple computation shows that $e^{2h_1} = t_1 - t_3^2/t_2$ and $e^{2h_2} = t_2$. Hence $y_1 = t_1 - t_3^2/t_2$, $y_2 = t_2$ and $\rho(S + iT) = \ln \frac{1}{(t_1 t_2 - t_3^2)^3}$. If σ is an arbitrary N-invariant potential of the Killing metric, then by Proposition 6.4,

$$\sigma(S + iT) = \ln \tfrac{1}{(t_1 t_2 - t_3^2)^3} + ct_2 + d, \qquad \text{for some } c, d \in \mathbb{R}.$$

7. N-invariant Stein domains in G/K via N-invariant psh functions.

In this section, we present an alternative proof of Theorem 3.4, which relies on the special features of the N-invariant plurisubharmonic functions. As an example of the role played by N-invariant plurisubharmonic functions in this proof, consider the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, for $n > 1$. The N-orbits in $\mathbb{B}^n$ are real hypersurfaces, and coincide with the horospheres internally tangent to the boundary. Since the N-invariant plurisubharmonic functions on $\mathbb{B}^n$ decrease on the subset $\exp \mathfrak{a} \cdot eK$ (see Thm. 5.11), a horoball containing the set $\exp tH_1 \cdot eK$, for $t \in (c, \infty)$, is an N-invariant Stein domain in $\mathbb{B}^n$. The converse holds true as well. This shows that for an N-invariant Stein domain in the ball, the base of the associated tube domain is a half-line.

This proof of Theorem 3.4 is divided into two parts. If D has smooth boundary, then the argument relies on the computation of the Levi form of smooth, N-invariant functions on D (see Sect. 4).

In the non-smooth case, the result is obtained by realizing D as an increasing union of Stein, N-invariant domains with smooth boundary. This construction is based on Lemma 7.2, where an arbitrary open convex C-invariant set is exhausted by an increasing union of open convex C-invariant sets with smooth boundary.

Proof of Theorem 3.4: the smooth case. The rank-1 tube case is trivial, since every $\mathbb{R}$-invariant domain in the upper half-plane $\mathbb{H}$ is Stein. So we deal with the remaining cases: the rank-one non-tube case and the higher rank cases.

We resume the notation $\mathbf{y} = (y_1, \ldots, y_r)$, for elements in $\mathbb{R}^r$. Let $D \subset G/K$ be a Stein, N-invariant domain with smooth boundary and let $\mathbb{R}^r + i\Omega \subset \mathbb{C}^r$ be its associated tube domain. By Rem. 3.2 (ii), its base Ω is a convex set with smooth boundary.

Assume by contradiction that Ω is not C-invariant, i.e. there exist $\mathbf{y} \in \Omega$ and $\mathbf{z} \in (\mathbf{y} + C) \cap \partial\Omega$. By the convexity of Ω, the open segment from $\mathbf{y}$ to $\mathbf{z}$ is contained in Ω. In addition, the vector $\mathbf{v} = \mathbf{z} - \mathbf{y} \in C$ is transversal to the tangent hyperplane $T_{\mathbf{z}}\partial\Omega$ and points outwards. Therefore, given a smooth local defining function $\widehat{f}$ of $\partial\Omega$ near $\mathbf{z}$, one has

$$\tfrac{\partial \widehat{f}}{\partial \mathbf{v}}(\mathbf{z}) = \mathrm{grad}\widehat{f}(\mathbf{z}) \cdot \mathbf{v} > 0.$$

In the tube case, the above inequality and (24) imply that $\frac{\partial \widetilde{f}}{\partial h_j}(H) > 0$, for some $j \in \{1, \ldots, r-1\}$. Then, by Proposition 4.1 (iii), the Levi form of the corresponding N-invariant function f is negative definite on the J-invariant subspace $a_*\mathfrak{g}^{e_j - e_l} \oplus a_*\mathfrak{g}^{e_j + e_l}$ of $T_{aK}(\partial D)$, the tangent space to ∂D in aK. In the non-tube case, one has $\frac{\partial \widetilde{f}}{\partial h_j}(H) > 0$, for some $j \in \{1, \ldots, r\}$. Then, by Proposition 4.1 (iv), the Levi form of the corresponding N-invariant function f is negative definite on the J-invariant subspace $a_*\mathfrak{g}^{e_j}$ of $T_{aK}(\partial D)$. This contradicts the fact that f is a local defining function of the Stein N-invariant domain D and proves that Ω is C-invariant.

Conversely, assume that Ω is convex and C-invariant. We prove that D is Stein by showing that it is Levi-pseudoconvex, i.e. for all points $aK \in \partial D$ and local defining functions f of D near aK, one has $\mathbf{h}_f(X, X) \geqslant 0$, for every tangent vector $X \in T_{aK}\partial D \cap JT_{aK}\partial D$, the complex tangent space to ∂D at aK.

Let $\mathbf{z} \in \partial\Omega$ and let $aK = \mathcal{L}^{-1}(\mathbf{z})$. Denote by $W := T_{\mathbf{z}}\partial\Omega$ the tangent space to $\partial\Omega$ in $\mathbf{z}$. Then the complex tangent space to ∂D at aK is given by

$$a_*(\bigoplus \mathfrak{g}^{e_j \pm e_l} \oplus \bigoplus \mathfrak{g}^{e_j}) \oplus (\mathcal{L}_*^{-1})_{\mathbf{z}} W \oplus J(\mathcal{L}_*^{-1})_{\mathbf{z}} W.$$

Let $\mathbf{v} = (v_1, \ldots, v_r)$ be an outer normal vector to W in $\mathbb{R}^r$. The convexity and the C-invariance of Ω imply that $v_j \leqslant 0$, for $j = 1, \ldots, r$ in the non-tube case, and $v_j \leqslant 0$, for $j = 1, \ldots, r-1$ in the tube case. Otherwise the space W would intersect $\mathbf{y} + C$, for every $\mathbf{y} \in \Omega$, yielding a contradiction.

Let $\widehat{f}$ be a smooth local defining function of Ω near $\mathbf{z}$. By the convexity of Ω, the Hessian $Hess(\widehat{f})(\mathbf{z})$ is positive definite on W. Moreover, as the gradient $\mathrm{grad}\widehat{f}(\mathbf{z})$ is a positive multiple of $\mathbf{v}$, one has $\frac{\partial \widehat{f}}{\partial y_j}(\mathbf{z}) \leqslant 0$, for all $j = 1, \ldots, r$, in the non-tube case, and $\frac{\partial \widehat{f}}{\partial y_j}(\mathbf{z}) \leqslant 0$, for all $j = 1, \ldots, r-1$, in the tube case.

Let f be the corresponding N-invariant local defining function of D near $aK = \exp L^{-1}(\mathbf{z})K$. By Theorem 5.1, the form $\mathbf{h}_f$ is positive definite on $(\mathcal{L}_*^{-1})_{\mathbf{z}} W \oplus J(\mathcal{L}_*^{-1})_{\mathbf{z}} W \subset a_*\mathfrak{a} \oplus a_* J\mathfrak{a}$.

In addition, by (24) and Proposition 4.1, the form $\mathbf{h}_f$ is positive definite on $a_*(\bigoplus \mathfrak{g}^{e_j \pm e_l} \oplus \bigoplus \mathfrak{g}^{e_j})$. As a result, D is Levi pseudoconvex in $aK = \exp L^{-1}(\mathbf{z})K$. Since aK is an arbitrary point in $\partial D \cap \exp \mathfrak{a} \cdot eK$ and both D and f are N-invariant, the domain D is Levi-pseudoconvex and therefore Stein, as desired.

For the proof of Theorem 3.4 in the non-smooth case we also need the following results.

Lemma 7.1. *Let D be a domain in a Stein manifold, let $D' \subset D$ be a subdomain with smooth boundary and let $z \in \partial D \cap \partial D'$. If D' is not Levi pseudoconvex in z, then D is not Stein.*

Proof. Under our assumption, there exists a one dimensional complex submanifold M through z in D with $M \backslash \{z\} \subset D'$ ([Ran86], proof of Thm. 2.11, p. 56). This implies that D is not Hartogs pseudoconvex ([Ran86], Thm. 2.9, p. 54) and in particular it is not Stein. □

Lemma 7.2. *Let Ω be a convex, C-invariant domain in $(\mathbb{R}^{>0})^r$. For $\epsilon > 0$ let $\Omega_\varepsilon := \{\mathbf{y} \in \Omega \ : \ d_\Omega(\mathbf{y} > \varepsilon\}$, as in Lemma 5.9. Then the following facts hold true.*

(i) *Let $\delta_\varepsilon := -\ln 3\varepsilon$ and $\widehat{u} := -\ln d_\Omega$. The sublevel set $\widetilde{\Omega}_\varepsilon := \{\mathbf{y} \in \Omega_\varepsilon \ : \ \widehat{u}_\varepsilon^+(\mathbf{y}) < \delta_\varepsilon\}$ is convex and C-invariant.*

(ii) *The boundary of* $\widetilde{\Omega}_\varepsilon$ *in* $(\mathbb{R}^{>0})^r$ *coincides with* $\{\mathbf{y} \in \Omega_\varepsilon \ : \ \widehat{u}_\varepsilon^+(\mathbf{y}) = \delta_\varepsilon\}$ *and it is smooth.*

(iii) *As* $n \in \mathbb{N}$ *increases, the sequence of convex, C-invariant subdomains with smooth boundary* $\widetilde{\Omega}_{1/n}$ *exhausts* Ω.

Proof. (i) follows by applying (ii) of Lemma 5.9 to $\widehat{u}$. Since the function $\widehat{u}_\varepsilon^+$ is convex, then the domain $\widetilde{\Omega}_\varepsilon$ is convex. Moreover, as $\widehat{u}$ is $\overline{C}$-decreasing, it follows that $\widehat{u}_\varepsilon^+$ is $\overline{C}$-decreasing. Hence $\widetilde{\Omega}_\varepsilon$ is C-invariant, as desired.

(ii) For $\mathbf{y}$ close to $\partial\Omega_\varepsilon = \{\mathbf{z} \in \Omega : \ d_\Omega(\mathbf{z}) = \varepsilon\}$, a rough extimate shows that $d_\Omega(\mathbf{y} + \varepsilon\mathbf{w}) < 3\varepsilon$, for every $\mathbf{w} \in \mathbb{B}_1(\mathbf{0})$. Therefore $\widehat{u}_\varepsilon^+(\mathbf{y}) > \widehat{u}_\varepsilon(\mathbf{y}) > -\ln 3\varepsilon$, implying that the boundary of $\widetilde{\Omega}_\varepsilon$ is contained in Ω_ε and it is given by $\partial\widetilde{\Omega}_\varepsilon = \{\mathbf{y} \in \Omega_\varepsilon \ : \ \widehat{u}_\varepsilon^+(\mathbf{y}) = \delta_\varepsilon\}$. Concerning the smoothness of $\partial\widetilde{\Omega}_\varepsilon$, the rank one case is trivial. So assume $r > 1$.

Let $\widehat{\mathbf{y}} \in \partial\widetilde{\Omega}_\varepsilon$. Set $\mathbf{v} := (1, \ldots, 1)$, in the non-tube case, and $\mathbf{v} := (1, \ldots, 1, 0)$, in the tube case. Since $\mathbf{v}$ lies in the cone C and $\widehat{u}_\varepsilon^+$ is strictly $\overline{C}$-decreasing, for γ small enough the real function $g : (-\gamma, \gamma) \to \mathbb{R}$, defined by $g(t) := \widehat{u}_\varepsilon^+(\widehat{\mathbf{y}} + t\mathbf{v})$, is strictly decreasing. By the stable convexity of $\widehat{u}_\varepsilon^+$, it is also stricltly convex and $g'(0) < 0$. As $g'(0)$ is a directional derivative of $\widehat{u}_\varepsilon^+$ in $\widehat{\mathbf{y}}$, the differential $d\widehat{u}_\varepsilon^+|_{\widehat{\mathbf{y}}}$ does not vanish and the boundary of $\widetilde{\Omega}_\varepsilon$ is smooth.

(iii) For $m > n$, the inclusion $\Omega_{1/n} \subset \Omega_{1/m}$ and the inequality $\widehat{u}_{1/n}^+ > \widehat{u}_{1/m}^+$ imply that $\widetilde{\Omega}_{1/n} \subset \widetilde{\Omega}_{1/m}$. This concludes the proof of the lemma. □

Proof of Theorem 3.4: the general case. Let D be an arbitrary Stein, N-invariant domain in G/K. By Remark 3.2 (ii), the base Ω of the associated tube domain is necessarily convex. Assume by contradiction that Ω is not C-invariant, i.e. there exist $\mathbf{y} \in \Omega$ and $\mathbf{z} \in (\mathbf{y} + C) \cap \partial\Omega$. By the convexity of Ω, the open segment from $\mathbf{y}$ to $\mathbf{z}$ is contained in Ω. Moreover, the vector $\mathbf{v} = \mathbf{z} - \mathbf{y}$ lies in the cone C and points to the exterior of Ω. Let $\mathbb{B}_\varepsilon(\mathbf{y})$ be a relatively compact ball in Ω and define

$$t_{\max} := \max\{\, t > 0 \ : \ \mathbb{B}_\varepsilon(\mathbf{y} + t\mathbf{v}) \subset \Omega \,\}\,.$$

Then there exists $\mathbf{w} \in \partial\mathbb{B}_\varepsilon(\mathbf{y} + t_{\max}\mathbf{v}) \cap \partial\Omega$, and by construction

$$\langle \mathbf{w} - (\mathbf{y} + t_{max}\mathbf{v}), \mathbf{v} \rangle \ > \ 0.$$

Let $\mathbf{n} = (n_1, \ldots, n_r)$ be the outer normal to $\partial\mathbb{B}_\varepsilon(\mathbf{y} + t_{\max}\mathbf{v})$, given by $\mathbf{n} := \mathbf{w} - (\mathbf{y} + t\mathbf{v})$. Then $n_j > 0$, for some $j \in \{1, \ldots, r\}$ in the non-tube case and $n_j > 0$, for some $j \in \{1, \ldots, r-1\}$, in the tube case. From the result of the theorem in the smooth case, it follows that the N-invariant subdomain $N\exp(L^{-1}(\mathbb{B}_\varepsilon(\mathbf{y} + t_{\max}\mathbf{v}))) \cdot eK$, with smooth boundary, is not Levi pseudoconvex in $\exp(L(\mathbf{w}))K$. Then Lemma 7.1 implies that D is not Stein, contradicting the assumption.

Conversely, assume that Ω is convex and C-invariant. By Lemma 7.2, the domain Ω can be realized as the increasing union of the convex C-invariant sets with smooth boundary $\widetilde{\Omega}_{1/n}$. The the domain D can be realized as the increasing union of the N-invariant domains $D_{1/n} := N\exp(L^{-1}(\widetilde{\Omega}_{1/n}))\cdot eK$. By the result of the theorem in the smooth case, the domains $D_{1/n}$ are Stein and so is their increasing union D. This completes the proof of the theorem. □

References

[Car73] Carmignani R. *Envelopes of holomorphy and holomorphic convexity.* Trans. of the AMS **179** (1973) 415-431.

[CoLo86] Coeuré G., Loeb J.-J. *Univalence de certaines enveloppes d'holomorphie.* C. R. Acad. Sci. Paris Sér. I Math. **302** (1986) 59–61.

[Fle78] Flensted-Jensen M. *Spherical functions of real semisimple Lie groups. A method of reduction to the complex case.* J. Funct. Anal. (1) **30** (1978) 106–46.

[GeIa22] Geatti L., Iannuzzi A. *Invariant plurisubharmonic functions on non-compact Hermitian symmetric spaces.* Math. Zeit. **300**, 1 (2022) 57–80.

[GPSV68] Gindikin S., Pyatetskii-Shapiro I., Vinberg E. *Homogeneous Kähler manifolds.* In Geometry of bounded domains, CIME 1968, Ed. Cremonese, Roma 1968, 3–87.

[Gun90] Gunning R. C. Introduction to Holomorphic Functions of Several Variables, Vol I: Function Theory. Wadsworth & Brooks/Cole, 1990.

[HeSc07] Heinzner, P., Schwarz G. W. *Cartan decomposition of the moment map.* Math. Ann. **337** (2007) 197–232.

[Hör94] Hörmander L. Notions of convexity. Birkhäuser, Basel–Boston–Berlin, 1994.

[PS69] Pyatetskii-Shapiro I. I. Automorphic Functions and the Geometry of Classical Domains Gordon and Breach, New York, 1969.

[Ran86] Range R. M. Holomorphic Functions and Integral Representations in Several Complex Variables. GTM Vol. 108, Springer-Verlag, New York, 1986.

[Ros63] Rossi H. *On envelopes of holomorphy.* Comm. Pure Appl. Math. **16** (1963) 9–17.

[RoVe73] Rossi H., Vergne M. *Representations of Certain Solvable Lie Groups On Hilbert Spaces of Holomorphic Functions and the Application to the Holomorphic Discrete Series of a Semisimple Lie Group.* J. Funct. Anal. **13** (1973) 324–389.

[Wol72] Wolf J.A. *Fine structure of Hermitian symmetric spaces.* in Boothby, W., Weiss, G. Eds., Symmetric spaces. Short Courses, Washington University, St. Luis (MO), 1969-1970, Pure and App. Math. Vol. 8, Dekker, New York, 1972, pp. 271-357.

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Roma, Italy

Email address: geatti@mat.uniroma2.it, iannuzzi@mat.uniroma2.it

Roma, 26 ottobre 2022