On the Exponentiality of the Connected Isometry Group of a Damek-Ricci Space

L. Geatti and M. Moskowitz

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Abstract

In this paper we prove that the connected isometry group of a non symmetric (non compact) irreducible Damek-Ricci space has a surjective exponential map if and only if the center of the associated Heisenberg type algebra has dimension less than or equal to 5. This result is analogous to (and extends) the results proved by the second author concerning the exponential map of the connected isometry group of an irreducible, rank one, classical, symmetric space of non compact type and that of the authors of [9] in the case of the Cayley plane to all irreducile non compact DR spaces.

Let G be a non compact, connected, centerless, real rank 1, simple Lie group. With low dimensional overlaps, these are $\mathrm{SO}_0(n,1)$, $n \geq 2$, the adjoint groups of $\mathrm{SU}(n,1)$ and $\mathrm{SP}(n,1)$, $n \geq 1$ and the single exceptional adjoint group of $F_{4,-27}$. In [12] together with [13], using the geometry of the associated symmetric space G/K, where K is a maximal compact subgroup of G, the second author of the present paper proved that the *classical*, i.e. non exceptional, $\mathrm{Ad}(G)$'s have surjective exponential maps. These groups are, respectively, the isometry groups of the classical irreducible non compact symmetric spaces of constant negative curvature (real hyperbolic space) and the other two are the identity components of isometries of spaces of negative curvature which are bounded and bounded away from zero (complex hyperbolic space and quaternionic hyperbolic space), the exceptional one being the connected component of the isometries of the Cayley plane. This last was proved to have a non surjective exponential map around the same time as the positive results, by Djokovic and Thang in [9]. (For the classification of all real, simple, non compact, (higher rank) Lie groups with surjective exponential map we refer to Wüstner [18]).

This suggests there may be other non compact, simply connected Riemannian manifolds of non positive curvature whose connected isometry groups might have a surjective exponential map, if they resemble the classical irreducible non compact rank 1 symmetric spaces sufficiently closely, and a non surjective exponential map otherwise. A particularly promising such family is the irreducible Damek-Ricci (DR) spaces which actually include all non compact, rank 1, irreducible symmetric spaces (see [2]). Our purpose here is to investigate when their connected isometry groups have a surjective exponential map, with the objective of supplementing and unifying the results obtained in [12], [13] and [9] for the irreducible non compact, rank 1, symmetric spaces. We remark that the reason DR spaces are closely connected to the non compact rank 1 symmetric spaces is that both of them admit an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\dim \mathfrak{a} = 1$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K respectively, $\mathfrak{n}$ is the nilpotent part (of Heisenberg type) and $\mathfrak{a}$ is the abelian part.

A word of caution: The connected isometry groups of the irreducible, non compact, symmetric spaces G/K are *non compact real simple Lie groups*, $\mathrm{Ad}(G)$, while the connected isometry groups of the non symmetric DR spaces are *amenable groups* because (see Section 2) they admit a semidirect product decomposition with a compact connected group acting on a connected solvable group. Since the classes of connected, non compact simple groups and connected amenable groups are disjoint, our present results on the surjectivity, or not, of the exponential map of the connected isometry group of a DR space are *independent* of those concerning symmetric spaces in [12], [13] and [9] in that neither implies the other. Thus in this paper, the geometric arguments used in [12], [13] will be replaced by an algebraic study of the relevant Clifford modules (see [1] and [11]). Just as in the symmetric space case these arguments also depend on the dimension m of the center $\mathfrak{z}$ of the associated Heisenberg type algebra, whereas here m ranges over *all* the positive integers (see pg. 101 of [2]). Our present results retain a striking similarity to the symmetric space case in that there is a cut off below which the exponential map is surjective and above which it is not. Our result is the following:

Theorem. *Let G^0 be the connected isometry group of a non symmetric irreducible Damek-Ricci space. Then G^0 has a surjective exponential map if and only if the dimension of the center of the associated Heisenberg type algebra is less than or equal to 5.*

Since as remarked above, the same is also true of symmetric irreducible DR spaces, the conclusion of our theorem actually applies to all irreducible DR spaces.

Corollary. *Let G^0 be the connected isometry group of an irreducible Damek-Ricci space S. Whether S is symmetric or not, G^0 has a surjective exponential map if and only if the dimension of the center of the associated Heisenberg type algebra is less than or equal to 5.*

Because of the decomposition mentioned above, DR spaces can be modeled on S, a connected, simply connected, solvable group extension of a connected, simply connected, nilpotent group of Heisenberg type N, where N, the nil-radical of S, is of codimension 1 so that $S = NA$, and A is 1-dimensional (and of course abelian). They all have non positive sectional curvature [5] and most of them are non symmetric (for the complete picture of these spaces see [2], pg. 23).

Groups of Heisenberg type were discovered by Aroldo Kaplan in the 1980's in [10], extending earlier results on the Heisenberg group itself. An important observation of his is that while the Heisenberg Lie algebra is usually presented as a subalgebra of $M(3,\mathbb{R})$ it can also be defined by regarding $\mathbb{C}$ as an associative $\mathbb{R}$-algebra and considering its purely imaginary elements. Other important real two step nilpotent Lie algebras are based on other algebras over $\mathbb{R}$ and considering their "purely imaginary" elements, namely the quaternions and even the (non associative) Cayley numbers. DR spaces were first constructed by E. Damek and F. Ricci in [7] to disprove a conjecture of Lichnerowicz by showing that *non compact* harmonic manifolds need not be symmetric.

1 Preliminaries

The construction and the classification of irreducible DR spaces are closely related to those of Heisenberg type groups. A Heisenberg type algebra is a Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ equipped with an inner product $\langle \ , \ \rangle$ such that $\langle \mathfrak{v}, \mathfrak{z} \rangle = 0$ and whose Lie bracket satisfies $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$ and $[\mathfrak{v}, \mathfrak{z}] = [\mathfrak{z}, \mathfrak{z}] = 0$. In addition, the map $J \colon \mathfrak{z} \to End(\mathfrak{v})$, defined by $\langle J_z U, V \rangle = \langle [U, V], Z \rangle$, for $U, V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, satisfies

$$J_Z^2 = -\langle Z, Z \rangle Id_{\mathfrak{v}}. \tag{1}$$

Let $\mathfrak{n}$ be a Heisenberg type algebra and let $\mathfrak{a}$ be a 1-dimensional vector space with an inner product. Let $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ be a Lie algebra with an inner product such that $\langle \mathfrak{a}, \mathfrak{n} \rangle = 0$ and whose Lie bracket satisfies $[H, V] = \frac{1}{2} V$ and $[H, Z] = Z$, for $H \in \mathfrak{a}, V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. A Heisenberg type group (H-type group) is a simply connected Lie group N with Lie algebra $\mathfrak{n}$. A Damek-Ricci space (DR space) is a simply connected solvable group S with Lie algebra $\mathfrak{s}$ and with the left-invariant metric induced by the inner product on $\mathfrak{s}$.

The classification of H-type groups and of DR spaces follows from that of real Clifford modules. Let $m = \dim \mathfrak{z}$. The map (1) extends to a representation of the real Clifford algebra $C_m := Cl(\mathfrak{z}, q)$, where $q = \langle \ , \ \rangle|_{\mathfrak{z} \times \mathfrak{z}}$, and turns $\mathfrak{v}$ into a real C_m-module. Conversely, a real Clifford module determines a map $J \colon \mathfrak{z} \to End(\mathfrak{v})$ satisfying (1) and hence an H-type group.

If $m \not\equiv 3 \mod 4$, then, up to equivalence, there exists exactly one irreducible real C_m-module $\mathfrak{d}$ and $\mathfrak{v} \cong \sum_{i=1}^{k} \mathfrak{d}$. We denote by $N(m, k)$ the corresponding H-type group and by $S(m, k)$ the corresponding DR space.

If $m \equiv 3 \mod 4$, then, up to equivalence, there exist two irreducible real C_m-modules $\mathfrak{d}_+$ and $\mathfrak{d}_-$, and $\mathfrak{v} \cong \oplus_{i=1}^{k_+} \mathfrak{d}_+ \bigoplus \oplus_{i=1}^{k_-} \mathfrak{d}_-$. We denote by $N(m, k_+, k_-)$ the corresponding H-type group and by $S(m, k_+, k_-)$ the corresponding DR space. Most of DR spaces are non symmetric. Among them, the only symmetric ones are in fact

$$S(0, k), \quad S(1, k), \quad S(3, k_+, 0) \cong S(3, 0, k_-), \quad S(7, 1, 0) \cong S(7, 0, 1), \tag{2}$$

for $k, k_+, k_- \in \mathbb{Z}_{\geq 1}$. (cf.[2], pg. 23). If S is a non symmetric DR space, then the isometry group of S is given by the semidirect product $G = K \times_\phi S$, where K is the group of automorphisms of S whose differential at $e \in S$ preserves the inner product $\langle , \rangle_{\mathfrak{s}}$, the group S is identified with the group of left translations and K acts on S by conjugation (cf. [6], Thm. 4.4). Note that the differential of K at e acts trivially on $\mathfrak{a}$ and necessarily preserves $\mathfrak{n}$.

Our goal is to determine when the connected component of the identity $G^0 = K^0 \times_\phi S$ of the isometry group G of a non symmetric irreducible DR space S has a surjective exponential map. Since the roots of S are real, the roots of $\mathrm{Ad}(S)$ being quotients of those of S are also real. So by the Dixmier-Saito theorem ([8], [16], or [2], pg. 81), S is of exponential type, i.e. it is diffeomorphic to its Lie algebra via the exponential map, and we can apply some necessary and sufficient criteria for the surjectivity of the exponential map of G^0 developed by Moskowitz and Wünster in [14] and [15]. It turns out that G^0 has a surjective exponential map if and only if for every maximal torus T in K^0, the isotropy subgroupo T_x is connected for every $x \in \mathfrak{n}$.

In order to apply the above criterion, we need an explicit description of the real C_m-module $\mathfrak{v}$ and of the T-action on $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, when m varies. The *real* irreducible C_m-modules $\mathfrak{d}$ are obtained from the *complex* irreducible C_m-modules, which are constructed by an inductive procedure. For the reader's convenience, we briefly recall the main steps.

Let m be a positive integer. The Clifford algebra C_m is the algebra generated by $1, e_1, \ldots, e_m$ under the relations $e_i^2 = -1$, and $e_i e_j = -e_j e_i$, for all $i, j = 1, \ldots, m$.

For m even there is a unique (up to equivalence) *complex irreducible* C_m-module S_m; for m odd, there are two of them, namely S_m and S'_m, the latter given by the same underlying space S_m, but with the negative action. We follow [1] and [11] for an inductive construction of the modules S_m. Remark that an algebra homomorphism $\gamma_m\colon C_m \to End(S_m)$ is determined by the images $\gamma_m(e_i)$ of the generators e_i, for $i = 1, \ldots, m$.

Fix the two 1-dimensional C_1-modules S_1 and S'_1, determined by $\gamma_1(e_1) = i$ and $\gamma'(e_1) = -i$, respectively, and the 2-dimensional C_2-module S_2 determined by $\gamma_2(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\gamma_2(e_2) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$. Then for $m \geq 3$, one has

$$S_m = S_2 \otimes S_{m-2}, \quad \text{and} \quad \gamma_m(e_i) := \begin{cases} \gamma_2(e_i) \otimes 1_{S_{m-2}}, & \text{for } i = 1, 2 \\ \epsilon_2 \otimes \gamma_{m-2}(e_{i-2}), & \text{for } i = 3, \ldots, m, \end{cases} \tag{3}$$

where $\epsilon_n = (-i)^{n/2} \gamma_n(e_1) \cdot \ldots \cdot \gamma_n(e_n)$, for $n \in \mathbb{N}_{\geq 2}$.

Moreover, when m is odd, the negative algebra homomorphism $\gamma'_m\colon C_m \to End_{\mathbb{C}}(S_m)$ is defined by $\gamma'_m(e_i) := -\gamma_m(e_i)$, for $i = 1, \ldots, m$.

We follow [1] and [11] also for an inductive construction of the real and quaternionic structures on the modules S_m. Denote by $\overline{S}_m$ the module S_m with the conjugate action. For $m=1$, identify S_1 with $\mathbb{C}$ and define $J_1\colon S_1 \to \overline{S}_1'$ by $J_1(z)=\bar{z}$. One can easily verify that $J_1\gamma_1 = \gamma_1' J_1$. Then

$$\tilde{J}_1 = \begin{pmatrix} 0 & J_1^{-1} \\ J_1 & 0 \end{pmatrix} : S_1 \oplus S_1' \to \overline{S_1 \oplus S_1'}, \quad (z,w) \mapsto (\bar{w}, \bar{z})$$

satisfies $\tilde{J}_1^2 = id$ and defines a real structure on $S_1 \oplus S_1'$, commuting with the C_1-action, with C_1-invariant real form $\{(z,\bar{z})\} \cong S_1$. For $m=2$, identify S_2 with $\mathbb{C}^2$ and define a quaternionic structure $J_2\colon S_2 \to \overline{S}_2$, by $J_2(a,b) = (-\bar{b}, \bar{a})$. One has $J_2\gamma_2 = \overline{\gamma_2} J_2$. For $m=3$, define $J_3 := J_2 \otimes J_1$. Then $J_3\colon S_2 \otimes S_1 \to \overline{S_2 \otimes S_1}$ satisfies $J_3^2 = -Id$ and defines a quaternionic structure commuting with the C_3-action both on $S_3 = S_2 \otimes S_1$ and on S_3'. If $m = 2k+1$ is *odd*, with $k \geq 2$, define inductively

$$J_m := \begin{cases} J_2 \otimes J_{2k-1}, & \text{for } m \equiv 3,7 \mod 8 \text{ (for } k \equiv 1,3 \mod 4\text{, odd)} \\ \epsilon_2 J_2 \otimes J_{2k-1}, & \text{for } m \equiv 1,5 \mod 8 \text{ (for } k \equiv 0,2 \mod 4\text{, even).} \end{cases}$$

One can verify that J_m defines on S_m a quaternionic structure for $m \equiv 3 \mod 8$ and a real structure for $m \equiv 7 \mod 8$. For $m \equiv 1,5 \mod 8$, from $J_m\colon S_m \to \overline{S_m'}$, one obtains a *real* structure

$$\tilde{J}_m = \begin{pmatrix} 0 & J_m^{-1} \\ J_m & 0 \end{pmatrix}$$

on $S_m \oplus S_m'$, with real form $\{(z,\bar{z}),\ z \in S_m\} \cong S_m$. For $m = 2k+2$, with $k \geq 1$ *even*, define inductively

$$J_m := J_2 \otimes \epsilon_{2k} J_{2k}\colon S_{2k+2} \to \overline{S_{2k+2}}.$$

One can verify that, for $m \equiv 2,4 \mod 8$ (for $k \equiv 1,4 \mod 4$), J_m defines a quaternionic structure on $S_m = S_2 \otimes S_{2k}$, while it defines a real structure, for $m \equiv 6,8 \mod 8$ (for $k \equiv 2,3 \mod 4$).

As a result of the above inductive constructions, the linear maps $J_m\colon S_m \to \overline{S_m'}$, for m odd, and $J_m\colon S_m \to \overline{S_m}$, for m even, commute with the respective Clifford algebra actions, i.e.

$$J_m\gamma_m = -\overline{\gamma_m} J_m \quad (m \text{ odd}), \qquad J_m\gamma_m = \overline{\gamma_m} J_m \quad (m \text{ even}).$$

The *real irreducible* C_m-modules $\mathfrak{d}$ are obtained from S_m as follows:
(a) if $m \equiv 2, 4 \mod 8$, then $\mathfrak{d} = S_m$, equipped with the quaternionic structure J_m;
(b) if $m \equiv 3 \mod 8$, then $\mathfrak{d}_+ = S_m$ and $\mathfrak{d}_- = S'_m$, both equipped with the quaternionic structure J_m;
(c) if $m \equiv 6, 8 \mod 8$, then $\mathfrak{d} = Fix(J_m, S_m)$ is the real form of S_m determined by the real structure J_m;
(d) if $m \equiv 7 \mod 8$, then $\mathfrak{d}_+ = Fix(J_m, S_m)$ and $\mathfrak{d}_- = Fix(J_m, S'_m)$ are the real forms, determined by the real structure J_m of S_m and S'_m, respectively;
(e) if $m \equiv 1, 5 \mod 8$, then $\mathfrak{d} = Fix(\tilde{J}_m, S_m \oplus S'_m)$ is the real form of $S_m \oplus S'_m$ determined by the real structure $\tilde{J}_m$.
The connected component K^0 of K was determined in [17], Thm.1: for a DR space $S(m, k)$ or $S(m, k_+, k_-)$,

$$K^0 \cong Spin(m)U^0/\Gamma,$$

where $U = O(\mathfrak{v}) \cap End_{C_m}(\mathfrak{v})$ and Γ is a finite subgroup of $Spin(m)U^0$. Precisely, from [17], Thm. 6, pg. 408 (up to some typos), one has that U^0 is one of the groups below

(a) $Sp(k)$, for $m \equiv 2, 4 \mod 8$
(b) $Sp(k_+) \times Sp(k_-)$, for $m \equiv 3 \mod 8$
(c) $SO(k)$, for $m \equiv 6, 8 \mod 8$
(d) $SO(k_+) \times SO(k_-)$, for $m \equiv 7 \mod 8$
(e) $U(k)$, for $m \equiv 1, 5 \mod 8$.

The $Spin(m)U^0$-module $\mathfrak{v}$ is respectively given by

(a) $\mathfrak{v} = \mathfrak{d} \otimes_{\mathbb{H}} \mathbb{H}^k$,
(b) $\mathfrak{v} = \mathfrak{d}_+ \otimes_{\mathbb{H}} \mathbb{H}^{k_+} \bigoplus \mathfrak{d}_- \otimes_{\mathbb{H}} \mathbb{H}^{k_-}$,
(c) $\mathfrak{v} = \mathfrak{d} \otimes_{\mathbb{R}} \mathbb{R}^k$,
(d) $\mathfrak{v} = \mathfrak{d}_+ \otimes_{\mathbb{R}} \mathbb{R}^{k_+} \bigoplus \mathfrak{d}_- \otimes_{\mathbb{R}} \mathbb{R}^{k_-}$,
(e) $\mathfrak{v} = \mathfrak{d} \otimes_{\mathbb{C}} \mathbb{C}^k$,

where $Spin(m)$ acts on $\mathfrak{d}$ by the *spin* representation, i.e. by the restriction of the C_m representation, and U^0 acts by the standard representation. On $\mathfrak{z}$, the group U^0 acts trivially, while $Spin(m)$ acts via the *vector representation*, i.e. via the double covering of $SO(m)$.

A maximal torus in K^0 is, up to finite quotient, the product $T = T_1 \times T_2$, where T_1 and T_2 are maximal tori in $Spin(m)$ and in U^0, respectively. A maximal torus T_1 of $Spin(m)$, embedded in the Clifford algebra C_m, is given by the elements of the form

$$(\cos\theta_1 + e_1e_2\sin\theta_1)\cdot\ldots\cdot(\cos\theta_r + e_{m-1}e_m\sin\theta_r), \quad \text{if } m \text{ is even} \tag{4}$$

$$(\cos\theta_1 + e_1e_2\sin\theta_1)\cdot\ldots\cdot(\cos\theta_r + e_{m-2}e_{m-1}\sin\theta_r), \quad \text{if } m \text{ is odd} \tag{5}$$

where $\theta_1, \ldots, \theta_r \in [0, 2\pi]$ and $r = [m/2]$. The action of T_1 on $\mathfrak{v}$ can be explicitly determined from (4), (5) and (3).

Notation. In what follows we fix on $\mathbb{H}$ the structure of a right $\mathbb{C}$-module and identify it with $\mathbb{C}^2$ as follows

$$q = x + iy + ju + kv \quad \mapsto \quad z + jw, \tag{6}$$

where $z = x+iy$, $w = u+iv$, $x, y, u, v \in \mathbb{R}$. Under the above identification, left multiplication by a quaternion $h = \alpha+j\beta$, with $\alpha = a+ib$ and $\beta = c+id$, is given by

$$L_h(q) = \begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}. \tag{7}$$

We identify $\mathbb{C}^k$ with $\mathbb{R}^{2k}$ equipped with the block diagonal complex structure I, consisting of k blocks $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, or equivalently via the map

$${}^t(z_1, \ldots, z_k) \mapsto {}^t(x_1, y_1, \ldots, x_k, y_k).$$

The above identification gives rise to a map $M(k, k, \mathbb{C}) \to M(2k, 2k, \mathbb{R})$, associating to an element $m_{ij} = a_{ij} + ib_{ij}$ the 2×2 matrix $\left(\begin{smallmatrix} a_{ij} & -b_{ij} \\ b_{ij} & a_{ij} \end{smallmatrix}\right)$.

2 Proof of the Theorem

We now begin the proof of our theorem which devolves to the assertion that for $m = 1, \ldots, 5$ all isotropy subgroups of $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ under the action of a maximal torus T of K^0, the maximal compact subgroup of G^0, are connected, while for $m \geq 6$ there are points with disconnected isotropy subgroup. We treat the cases $m = 1, \ldots, 6$ separately. Later we prove the existence of disconnected isotropy subgroups for $m > 6$ by reducing to the case $m = 6$.

• $m = \dim_{\mathbb{R}} \mathfrak{z} = 1$.
The real irreducible C_1-module $\mathfrak{d}$ is the real form of $S_1 \oplus S_1' \cong \mathbb{C} \oplus \mathbb{C}$ determined by the real structure $J_1(z, w) = (\bar{w}, \bar{z})$. If $\mathfrak{v}$ is the direct sum of k copies of $\mathfrak{d}$, then we coordinatize $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as $(\mathbf{v}, \mathbf{z})$, where $\mathbf{v} = (u_1, \ldots, u_k, \bar{u}_1, \ldots, \bar{u}_k) \in \mathbb{C}^{2k}$ and $\mathbf{z} \in \mathbb{R}$. A maximal torus T of K^0 is just a maximal torus in $U(k)$. If we choose as $T = T(\phi_1, \ldots, \phi_k)$ the diagonal torus in $U(k)$, then it acts on $\mathfrak{v} \oplus \mathfrak{z}$ as

$$T(\mathbf{v}, \mathbf{z}) = (e^{i\phi_1} u_1, \ldots, e^{i\phi_k} u_k, e^{-i\phi_1} \bar{u}_1, \ldots e^{-i\phi_k} \bar{u}_k, \mathbf{z}).$$

From the above formulas, one immediately obtains the next lemma.

Lemma 1. *The isotropy subgroup of an arbitrary point in $\mathfrak{v} \oplus \mathfrak{z}$ is a subtorus of T, and it is connected.*

• $m = \dim_{\mathbb{R}} \mathfrak{z} = 2$.
The real irreducible C_2-module $\mathfrak{d}$ is just the complex irreducible module $S_2 \cong \mathbb{C}^2$, endowed with the quaternionic structure $J_2(u, v) = (-\bar{v}, \bar{u})$. Under the identification (6), J_2 corresponds to right multiplication by j. If $\mathfrak{v}$ is the direct sum of k copies of $\mathfrak{d}$, then we coordinatize $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as $(\mathbf{v}, \mathbf{z})$, where $\mathbf{v} = (u_1, v_1, \ldots, u_k, v_k) \in \mathbb{C}^{2k}$ and $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$.
We fix the maximal torus $T_1 = T_1(\theta)$ in $Spin(2)$ described in (4) and the diagonal maximal torus $T_2 = T_2(\phi_1, \ldots, \phi_k)$ in $Sp(k)$. By (7) and the fact that the T_2-action must commute with J_2, the torus $T = T_1 \times T_2$ acts on $\mathfrak{n}$ by

$$T(\mathbf{v}, \mathbf{z}) = (e^{i(\theta+\phi_1)} u_1, e^{i(-\theta+\phi_1)} v_1, \ldots, e^{i(\theta+\phi_k)} u_k, e^{i(-\theta+\phi_k)} v_k, R_{2\theta} \mathbf{z}),$$

where R_θ denotes the rotation of an angle θ in $\mathbb{R}^2$.

Lemma 2. *Let $m = 2$. For every $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$, the isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ is connected.*

Proof. Let $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$.
(a) If $\mathbf{z} \neq 0$, then either $\theta \equiv 0$ or $\theta \equiv \pi \mod 2\pi$. If for some j either $u_j \neq 0$ or $v_j \neq 0$, then $\phi_j \equiv \theta \mod 2\pi$. If $u_j = v_j = 0$, then ϕ_j is arbitrary. It

follows that $T_{(\mathbf{v},\mathbf{z})}$ is a subtorus of T, of dimension equal to the number of indices j for which $u_j = v_j = 0$.
(b) Let $\mathbf{z} = 0$. If $u_j v_j \neq 0$ for some j, then $\theta \equiv 0, \pi$ and $\phi_j \equiv \theta \mod 2\pi$. Thus we are back in case (a). If for all j, at most one of u_j and v_j is 0, then either $\phi_j \equiv \pm\theta \mod 2\pi$ or ϕ_j is arbitrary. In all cases, $T_{(\mathbf{v},\mathbf{z})}$ is a connected torus. $\square$

• $m = \dim_{\mathbb{R}} \mathfrak{z} = 3$.
There are two real irreducible C_3-modules $\mathfrak{d}_+$ and $\mathfrak{d}_-$. They are respectively given by $S_3 \cong \mathbb{C}^2$ and $S_3' \cong \mathbb{C}^2$, each equipped with the quaternionic structure $J_3(u, v) = (-\bar{v}, \bar{u})$. If the real C_3-module $\mathfrak{v}$ is the direct sum of k copies of $\mathfrak{d}_+$ and h copies of $\mathfrak{d}_-$, then we coordinatize $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as $(\mathbf{v}, \mathbf{z})$, with $\mathbf{v} = (u_1, v_1, \ldots, u_k, v_k, u_1', v_1', \ldots, u_h', v_h') \in \mathbb{C}^{2k+2h}$ and $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3$. We fix $T_1 = T_1(\theta)$ the maximal torus in $Spin(3)$ described in (5) and $T_2 = T_2(\phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_h)$ the product of the diagonal maximal tori in $Sp(k)$ and $Sp(h)$. Then by (7) and the fact that the T_2-action must commute with J_3, the torus $T = T_1 \times T_2$ acts on $\mathfrak{n}$ by

$$T(\mathbf{v}, \mathbf{z}) = \big(e^{i(\theta+\phi_1)}u_1, e^{i(-\theta+\phi_1)}v_1, \ldots, e^{i(\theta+\phi_k)}u_k, e^{i(-\theta+\phi_k)}v_k,$$
$$e^{i(\theta+\psi_1)}u_1', e^{i(-\theta+\psi_1)}v_1', \ldots, e^{i(\theta+\psi_h)}u_h', e^{i(-\theta+\psi_h)}v_h', R_{2\theta}(z_1, z_2), x_3)\big).$$

Lemma 3. *Let $m = 3$. For every $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$, the isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ is connected.*

Proof. The argument is similar to the one used for $m = 2$. Here the conditions $\mathbf{z} = 0$ or $\mathbf{z} \neq 0$ are replaced by the conditions $\mathbf{z}$ lies or $\mathbf{z}$ does not lie on the rotation axis in $\mathfrak{z}$, respectively. The conclusion is that $T_{(\mathbf{v},\mathbf{z})}$ is connected for all $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$. $\square$

• $m = \dim_{\mathbb{R}} \mathfrak{z} = 4$.
The real irreducible C_4-module $\mathfrak{d}$ is just the complex irreducible module $S_4 \cong \mathbb{C}^4$, endowed with the quaternionic structure $J_4(x, y, u, v) = (\bar{v}, \bar{u}, -\bar{y}, -\bar{x})$. If $\mathfrak{v}$ is the direct sum of k copies of $\mathfrak{d}$, then we coordinatize $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as $(\mathbf{v}, \mathbf{z})$, with

$$\mathbf{v} = (x_1, y_1, u_1, v_1, \ldots, x_k, y_k, u_k, v_k) \in \mathbb{C}^{4k} \quad \text{and} \quad \mathbf{z} = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4.$$

We take the maximal torus $T_1 = T_1(\theta_1, \theta_2)$ in $Spin(4)$ described in (4) and the diagonal maximal torus $T_2 = T_2(\phi_1, \dots, \phi_k)$ in $Sp(k)$. Then the action of $T = T_1 \times T_2$ on $\mathfrak{v}$ is given by k blocks of the form

$$T(x_j, y_j, u_j, v_j) = (e^{i(\theta_1+\theta_2+\phi_j)}x_j, e^{i(\theta_1-\theta_2+\phi_j)}y_j, e^{i(-\theta_1+\theta_2+\phi_j)}u_j, e^{i(-\theta_1-\theta_2+\phi_j)}v_j),$$

for $j = 1, \dots, k$, and on $\mathfrak{z}$ is given by

$$T(z_1, z_2, z_3, z_4) = (R_{2\theta_1}(z_1, z_2), R_{2\theta_2}(z_3, z_4)).$$

Lemma 4. *Let $m = 4$. For every $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$, the isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ is connected.*

Proof. Let $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$. We need to distinguish several subcases.
• Assume $\mathbf{z} \neq 0$.
(1) If $\mathbf{z}$ satisfies (z_1, z_2), $(z_3, z_4) \neq (0, 0)$, then $T\mathbf{z} = \mathbf{z}$ implies $\theta_1, \theta_2 \equiv 0, \pi \mod 2\pi$.
(1.a) If $\mathbf{v} = 0$, then all the ϕ_j's are arbitrary and $T_{(\mathbf{v},\mathbf{z})}$ is a connected torus.
(1.b) Assume $\mathbf{v} \neq 0$. If for an index j at least one of the coordinates x_j, y_j, u_j, v_j is non zero, then the condition $T\mathbf{v} = \mathbf{v}$ implies that one or more equations of the system below hold true

$$\begin{cases} \theta_1 + \theta_2 + \phi_j \equiv 0 \\ \theta_1 - \theta_2 + \phi_j \equiv 0 \\ -\theta_1 + \theta_2 + \phi_j \equiv 0 \\ -\theta_1 - \theta_2 + \phi_j \equiv 0 \end{cases} \mod 2\pi. \tag{8}$$

In particular, $\phi_j \equiv \theta_1 + \theta_2 \mod 2\pi$ and $T_{(\mathbf{v},\mathbf{z})}$ acts as the identity on the j^{th} irreducible component of $\mathfrak{v}$. If $x_j = y_j = u_j = u_j = 0$, then the angle ϕ_j is arbitrary and on that irreducible component $T_{(\mathbf{v},\mathbf{z})}$ acts as a 1-dimensional torus. It follows that the isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ is either trivial or a connected torus.
(2) If $\mathbf{z}$ satisfies $(z_1, z_2) \neq (0, 0)$, $(z_3, z_4) = (0, 0)$, then the condition $T\mathbf{z} = \mathbf{z}$ implies $\theta_1 \equiv 0, \pi \mod 2\pi$.
(2.a) If $\mathbf{v} = 0$, then θ_2 and all the ϕ_j's are arbitrary and $T_{(\mathbf{v},\mathbf{z})}$ is a connected torus.

(2.b) If $\mathbf{v} \neq 0$, fix j with the maximum number of non zero coordinates among x_j, y_j, u_j, v_j. Then $T\mathbf{v} = \mathbf{v}$ implies that one or more equations of the systems below hold true

$$\begin{cases} \theta_1 \equiv 0 \\ \theta_2 + \phi_j \equiv 0 \\ -\theta_2 + \phi_j \equiv 0 \end{cases} \quad \text{or} \quad \begin{cases} \theta_1 \equiv \pi \\ \theta_2 + \phi_j \equiv \pi \\ -\theta_2 + \phi_j \equiv \pi \end{cases} \mod 2\pi. \tag{9}$$

It follows that either $\theta_2 = 0,\, \pi$ or θ_2 is arbitrary. If $\theta_2 = 0,\, \pi$, then by arguing as in case (1.b) we conclude that $T_{(\mathbf{v},\mathbf{z})}$ is connected. If θ_2 is arbitrary, then all the angles ϕ_i are arbitrary as well, and $T_{(\mathbf{v},\mathbf{z})}$ is connected.

If $\mathbf{z}$ satisfies $(z_1, z_2) = (0,0)$, $(z_3, z_4) \neq (0,0)$, then just exchange θ_1 and θ_2 in the arguments above and analogous statements will follow.

• Assume $\mathbf{z} = 0$.

In this case the condition $T\mathbf{z} = \mathbf{z}$ imposes no restrictions on θ_1 and θ_2.

(3) Assume $\mathbf{v} \neq 0$.

(3.a) If for some j one has $x_j y_j u_j v_j \neq 0$, then on the j^{th} irreducible component of $\mathfrak{v}$ the condition $T\mathbf{v} = \mathbf{v}$ is equivalent to the system (8) and implies $(\theta_1, \theta_2) \equiv (0,0), (\pi,\pi), (\pi,0), (0,\pi) \mod 2\pi$. The same conclusion holds true if at most one of the coordinates x_j, y_j, u_j, v_j is equal to zero (any three equations out of system (8) have (θ_1, θ_2) as above as solutions). Now by arguing as in case (1.b) we conclude that $T_{(\mathbf{v},\mathbf{z})}$ is connected.

(3.b) Assume that for every $j = 1, \ldots, k$, at most two of the coordinates x_j, y_j, u_j, v_j are non zero. Fix i with two non zero coordinates. Then the condition $T\mathbf{v} = \mathbf{v}$ yields 6 possible systems of two independent equations

$$\begin{cases} \theta_1 - \theta_2 + \phi_i \equiv 0 \\ -\theta_1 - \theta_2 + \phi_i \equiv 0 \end{cases} \quad \begin{cases} \theta_1 + \theta_2 + \phi_i \equiv 0 \\ -\theta_1 + \theta_2 + \phi_i \equiv 0 \end{cases} \mod 2\pi, \tag{10}$$

$$\begin{cases} \theta_1 + \theta_2 + \phi_i \equiv 0 \\ \theta_1 - \theta_2 + \phi_i \equiv 0 \end{cases} \quad \begin{cases} -\theta_1 + \theta_2 + \phi_i \equiv 0 \\ -\theta_1 - \theta_2 + \phi_i \equiv 0 \end{cases} \mod 2\pi, \tag{11}$$

$$\begin{cases} \theta_1 - \theta_2 + \phi_i \equiv 0 \\ -\theta_1 + \theta_2 + \phi_i \equiv 0 \end{cases} \quad \begin{cases} \theta_1 + \theta_2 + \phi_i \equiv 0 \\ -\theta_1 - \theta_2 + \phi_i \equiv 0 \end{cases} \mod 2\pi. \tag{12}$$

The systems in (10) imply $\theta_1 \equiv 0, \pi$ and $\theta_2 \equiv \pm\theta_1 + \phi_i \mod 2\pi$, with ϕ_i arbitrary; the systems in (11) imply $\theta_2 \equiv 0, \pi$ and $\theta_1 \equiv \pm\theta_2 + \phi_i \mod 2\pi$, with ϕ_i arbitrary; the systems in (12) imply either $\theta_1 - \theta_2 = 0, \pi$ and $\phi_i \equiv \theta_1 -$

θ_2 or $\theta_1+\theta_2 = 0, \pi$ and $\phi_i \equiv \theta_1+\theta_2 \mod 2\pi$, with θ_2 arbitrary. Substituting the above solutions in the formulas of the T-action, we find that the isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ acts on the i^{th} irreducible component as the identity on two of the coordinates and as a rotation on the other two. If in addition, for all indices $j \neq i$, none or at most one of the coordinates x_j, y_j, u_j, v_j is non zero, then the corresponding ϕ_j's are arbitrary and $T_{(\mathbf{v},\mathbf{z})}$ is connected.
It remains to check the isotropy subgroup of points with two non zero coordinates for at least two indices. Let $i \neq j$ be a pair of such indices. Then the condition $T\mathbf{v} = \mathbf{v}$ determines a system of four equations, union of a system from (10), (11), (12) involving ϕ_i and a system from (10), (11), (12) involving ϕ_j. If the system has rank 4, then $(\theta_1, \theta_2) = (0,0), (\pi,\pi), (\pi,0), (0,\pi)$ and, by arguing as in case (1.b) one concludes that $T_{(\mathbf{v},\mathbf{z})}$ is connected. If all such systems have rank 3, then by arguing as in (2.b) one concludes that $T_{(\mathbf{v},\mathbf{z})}$ is connected. □

• $m = \dim_{\mathbb{R}} \mathfrak{z} = 5$.
The real irreducible C_5-module $\mathfrak{d}$ is the real form of $S_5 \oplus S_5' \cong \mathbb{C}^4 \oplus \mathbb{C}^4$ determined by the real structure $\widetilde{J}_5 = \begin{pmatrix} O & J_5^{-1} \\ J_5 & O \end{pmatrix}$, where $J_5 \colon S_5 \to \overline{S_5'}$ is given by $J_5(x,y,u,v) = (\bar{v}, -\bar{u}, \bar{y}, -\bar{x})$. Suppose that $\mathfrak{v}$ is the direct sum of k copies of $\mathfrak{d}$. Then we coordinatize $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as $(\mathbf{v}, \mathbf{z})$, with $\mathbf{v} = (\ldots, x_j, y_j, u_j, v_j, \bar{v}_j, -\bar{u}_j, \bar{y}_j, -\bar{x}_j, \ldots) \in \mathbb{C}^{8k}$, for $j = 1, \ldots, k$, and $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{R}^5$. We take the maximal torus $T_1 = T_1(\theta_1, \theta_2)$ in $Spin(5)$ described in (5) and the diagonal maximal torus $T_2 = T_2(\phi_1, \ldots, \phi_k)$ in $U(k)$. Then on the j^{th} irreducible component of $\mathfrak{v}$ the action of $T = T_1 \times T_2$ is given by

$$\begin{gathered} T(x_j, y_j, u_j, v_j, \bar{v}_j, -\bar{u}_j, \bar{y}_j, -\bar{x}_j) = \\ = (e^{i(\theta_1+\theta_2+\phi_j)} x_j, e^{i(\theta_1-\theta_2+\phi_j)} y_j, e^{i(-\theta_1+\theta_2+\phi_j)} u_j, e^{i(-\theta_1-\theta_2+\phi_j)} v_j, \\ e^{-i(-\theta_1-\theta_2+\phi_j)} \bar{v}_j, -e^{-i(-\theta_1+\theta_2+\phi_j)} \bar{u}_j, e^{-i(\theta_1-\theta_2+\phi_j)} \bar{y}_j, -e^{-i(\theta_1+\theta_2+\phi_j)} \bar{x}_j), \end{gathered}$$

for $j = 1, \ldots, k$, and on $\mathfrak{z}$ is given by

$$T(z_1, z_2, z_3, z_4, z_5) = (R_{2\theta_1}(z_1, z_2), R_{2\theta_2}(z_3, z_4), z_5).$$

Lemma 5. *Let $m = 5$. For every $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$, the isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ is connected.*

Proof. Let $(\mathbf{v},\mathbf{z}) \in \mathfrak{v}\oplus\mathfrak{z}$. Write $\mathbf{v} = (\mathbf{w}_1, J_5\mathbf{w}_1, \dots, \mathbf{w}_k, J_5\mathbf{w}_k)$. Then $T\mathbf{v} = \mathbf{v}$ if and only if $T\mathbf{w} = \mathbf{w}$, where $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_k)$. Moreover $T\mathbf{z} = \mathbf{z}$ if and only if $R_{2\theta_1}(z_1, z_2) = (z_1, z_2)$ and $R_{2\theta_2}(z_3, z_4) = (z_3, z_4)$. Comparing the above conditions with the one arising for $m = 4$, it is clear that the statement follows directly from Lemma 4. □

• $m = \dim_{\mathbb{R}}\mathfrak{z} = 6$.
The real irreducible C_6-module $\mathfrak{d}$ is the real form of $S_6 \cong \mathbb{C}^8$ determined the real structure J_6, which happens to coincide with $\tilde{J}_5$. Suppose that $\mathfrak{v}$ is the direct sum of k copies of $\mathfrak{d}$. We coordinatize $\mathfrak{n} = \mathfrak{v}\oplus\mathfrak{z}$ as $(\mathbf{v},\mathbf{z})$, with

$$\mathbf{v} = (\dots, x_j, y_j, u_j, v_j, \bar{v}_j, -\bar{u}_j, \bar{y}_j, -\bar{x}_j, \dots) \in \mathbb{C}^{8k},\ \mathbf{z} = (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{R}^6.$$

If $\mathfrak{v} = \mathfrak{d}$ is irreducible, then T_2 is trivial and a maximal torus T of K^0 is just a maximal torus $T_1 = T_1(\theta_1, \theta_2, \theta_3)$ of $Spin(6)$. If we choose T_1 as in (4), then the T action on $\mathfrak{d}$ is given by

$$\begin{gathered} T(x, y, u, v, \bar{v}, -\bar{u}, \bar{y}, -\bar{x}) = \\ = (e^{i(\theta_1+\theta_2+\theta_3)}x, e^{i(\theta_1+\theta_2-\theta_3)}y, e^{i(\theta_1-\theta_2+\theta_3)}u, e^{i(\theta_1-\theta_2-\theta_3)}v, \\ e^{-i(\theta_1-\theta_2-\theta_3)}\bar{v}, -e^{-i(\theta_1-\theta_2+\theta_3)}\bar{u}, e^{-i(\theta_1+\theta_2-\theta_3)}\bar{y}, -e^{-i(\theta_1+\theta_2+\theta_3)}\bar{x}). \end{gathered} \tag{13}$$

In general, if $\mathfrak{v}$ consists of k copies of the irreducible module $\mathfrak{d}$, then T_1 acts as above on each copy of $\mathfrak{d}$. If we choose a block diagonal maximal torus T_2 of $SO(k)$ with $[k/2]$ rotations as blocks, then each rotation R_ϕ acts on a block $\mathfrak{d}\oplus\mathfrak{d}$ as

$$(\mathbf{v}, \mathbf{v}') \mapsto (\cos\phi\mathbf{v} - \sin\phi\mathbf{v}', \sin\phi\mathbf{v} - \cos\phi\mathbf{v}'), \qquad \text{for } (\mathbf{v}, \mathbf{v}') \in \mathfrak{d}\oplus\mathfrak{d}. \tag{14}$$

On the j^{th} block $\mathfrak{d}\oplus\mathfrak{d}$, write

$$\mathbf{v}_j = (x_j, y_j, u_j, v_j, \bar{v}_j, -\bar{u}_j, \bar{y}_j, -\bar{x}_j),\ \mathbf{v}'_j = (x'_j, y'_j, u'_j, v'_j, \bar{v}'_j, -\bar{u}'_j, \bar{y}'_j, -\bar{x}'_j) \tag{15}$$

and set

$$X_j = \begin{pmatrix} x_j \\ x'_j \end{pmatrix},\ Y_j = \begin{pmatrix} y_j \\ y'_j \end{pmatrix},\ U_j = \begin{pmatrix} u_j \\ u'_j \end{pmatrix},\ V_j = \begin{pmatrix} v_j \\ v'_j \end{pmatrix}. \tag{16}$$

Then on each such block the T-action is determined by the maps

$$X_j \mapsto e^{i(\theta_1+\theta_2+\theta_3)}R_{\phi_j}X_j, \qquad Y_j \mapsto e^{i(\theta_1+\theta_2-\theta_3)}R_{\phi_j}Y_j \tag{17}$$

$$U_j \mapsto e^{i(\theta_1-\theta_2+\theta_3)} R_{\phi_j} U_j, \qquad V_j \mapsto e^{i(\theta_1-\theta_2-\theta_3)} R_{\phi_j} V_j. \tag{18}$$

The T action on $\mathfrak{z}$ is given by

$$(z_1, z_2, z_3, z_4, z_5, z_6) \mapsto (R_{2\theta_1}(z_1, z_2), R_{2\theta_2}(z_3, z_4), R_{2\theta_3}(z_5, z_6)).$$

Lemma 6. *Let $m = 6$. There exist points $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$ whose isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ is disconnected.*

Proof. We distinguish two cases.
(1) *Assume that $\mathfrak{v} = \mathfrak{d} \oplus \mathfrak{d} \oplus \ldots \oplus \mathfrak{d} \oplus \mathfrak{d}$ decomposes as the sum of k blocks $\mathfrak{d} \oplus \mathfrak{d}$.* Consider the point

$$(\mathbf{v}, 0) = ((\mathbf{v}_1, \mathbf{v}_1', \ldots, \mathbf{v}_k, \mathbf{v}_k'), 0) \in \mathfrak{v} \oplus \mathfrak{z},$$

where

$$(\mathbf{v}_1, \mathbf{v}_1') = ((i, -i, -i, i, -i, -i, i, i), (1, 1, 1, 1, 1, -1, 1, -1)), \quad \mathbf{v}_j = \mathbf{v}_j' = 0,$$

for all $j > 1$. In the notation (15) and (16),

$$X_1 = \begin{pmatrix} i \\ 1 \end{pmatrix},\ Y_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix},\ U_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix},\ V_1 = \begin{pmatrix} i \\ 1 \end{pmatrix},$$

$$X_j = Y_j = U_j = V_j = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{for all } j > 1.$$

Then by (17) and (18), the condition $T(\mathbf{v}, \mathbf{z}) = (\mathbf{v}, \mathbf{z})$ is equivalent to

$$TX_1 = X_1, \quad TY_1 = Y_1, \quad TU_1 = U_1, \quad TV_1 = V_1. \tag{19}$$

Observe that $e^{i\lambda} R_\phi$ has a non zero fixed point in $\mathbb{C}^2$, i.e. an eigenvector Z of eigenvalue 1, if and only if $\phi \equiv -\lambda \mod 2\pi$ and $Z \in \mathbb{C}\binom{i}{1}$ or $\phi \equiv \lambda \mod 2\pi$ and $Z \in \mathbb{C}\binom{-i}{1}$. Then conditions (19) are equivalent to the linear system mod 2π

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 + \phi_1 \equiv 0 \\ \theta_1 + \theta_2 - \theta_3 - \phi_1 \equiv 0 \\ \theta_1 - \theta_2 + \theta_3 - \phi_1 \equiv 0 \\ \theta_1 - \theta_2 - \theta_3 + \phi_1 \equiv 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \theta_1 + \theta_2 + \theta_3 + \phi_1 \equiv 0 \\ -2\theta_2 - 2\phi_1 \equiv 0 \\ -2\theta_3 - 2\phi_1 \equiv 0 \\ -4\phi_1 \equiv 0. \end{cases} \tag{20}$$

Modulo 2π and modulo ineffectivity, the solutions of the above system are

$$(\theta_1, \theta_2, \theta_3, \phi_1, \dots, \phi_r) = (0, 0, 0, 0, \phi_2, \dots, \phi_r), \quad \phi_j \in \mathbb{R},\ j > 1, \tag{21}$$

$$(\theta_1, \theta_2, \theta_3, \phi_1, \dots, \phi_r) = (\frac{\pi}{2}, -\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \phi_2, \dots, \phi_r), \quad \phi_j \in \mathbb{R},\ j > 1. \tag{22}$$

The elements of $T_{(\mathbf{v},\mathbf{z})}$ corresponding to (21) act as the identity on the first block $\mathfrak{d} \oplus \mathfrak{d}$, as

$$X_j \mapsto R_{\phi_j} X_j, \quad Y_j \mapsto R_{\phi_j} Y_j, \quad U_j \mapsto R_{\phi_j} U_j, \quad V_1 \mapsto R_{\phi_j} V_j$$

on the j^{th} block $\mathfrak{d} \oplus \mathfrak{d}$, for $j > 1$ and as the identity on $\mathfrak{z}$. The elements of $T_{(\mathbf{v},\mathbf{z})}$ coming from (22) act as

$$X_1 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} X_1, \quad Y_1 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} Y_1,$$

$$U_1 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} U_1, \quad V_1 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} V_1$$

on the first block $\mathfrak{d} \oplus \mathfrak{d}$, as

$$X_j \mapsto -iR_{\phi_j} X_j, \quad Y_j \mapsto iR_{\phi_j} Y_j, \quad U_j \mapsto iR_{\phi_j} U_j, \quad V_1 \mapsto -iR_{\phi_j} V_j,$$

on the j^{th} block $\mathfrak{d} \oplus \mathfrak{d}$, for $j > 1$, and as $-Id$ on $\mathfrak{z}$. As a result, the isotropy subgroup $T_{(\mathbf{v},\mathbf{z})} \cong \mathbb{Z}_2 \times (S^1)^{r-1}$ and it is disconnected.

(2) *Assume that $\mathfrak{v}$ is the sum of an odd number of irreducible components $\mathfrak{d}$.* Let $(0, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$ be a point satisfying $(z_1 \cdot \ldots \cdot z_6) \neq 0$. The condition $T\mathbf{v} = \mathbf{v}$ poses no restrictions, while $T\mathbf{z} = \mathbf{z}$ if and only if $\theta_i \equiv 0, \pi \mod 2\pi$, for $i = 1, 2, 3$. As a consequence, $e^{i(\pm\theta_1 \pm\theta_2 \pm\theta_3)} = \pm 1$, depending on whether $\theta_1 + \theta_2 + \theta_3$ is 0 or $\pi \mod 2\pi$. If $\mathfrak{v} = \mathfrak{d}$ is irreducible, then $T_{(\mathbf{v},\mathbf{z})} = \{(\pm Id, Id)\} \cong \mathbb{Z}_2$ and it is disconnected. If $\mathfrak{v}$ decomposes in an odd number of irreducible components, then

$$T_{(\mathbf{v},\mathbf{z})} \cong R_{\phi_1} \times \ldots \times R_{\phi_h} \times \{\pm Id\}, \qquad h = [k/2],$$

and it is disconnected. □

Finally, in the next two lemmas, we deal with the cases $m > 6$.

Lemma 7. *Let $m \equiv 1,2,3,4,5 \mod 8$. There exist points $(\mathbf{v},\mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$ whose isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ is disconnected.*

Proof. We begin with the case when $\mathfrak{v}$ is irreducible. Recall that $Spin(m)$ acts on the complex C_m-module S_m via the spin representation. We fix $T_1 = T_1(\theta_1, \dots, \theta_r)$ the maximal torus of $Spin(m)$ described in (4) and (5), where $r = [m/2]$, and T_2 the diagonal maximal torus of U^0.

• Let $m \equiv 2,3,4 \mod 8$.

The real C_m-module $\mathfrak{d}$ coincides with S_m or S'_m endowed with the quaternionic structure J_m and $T = T_1 \times T_2$ acts on S_m with weights $i(\pm\theta_1 \pm \dots \pm \theta_r + \phi)$ and one-dimensional weight spaces. Consider a point $(\mathbf{v},\mathbf{z}) \in \mathfrak{d} \oplus \mathfrak{z}$, with $\mathbf{z} = (0, \dots, 0, z_7, \dots, z_m)$ satisfying $z_7 \cdot \ldots \cdot z_m \neq 0$ and $\mathbf{v}$ all of whose coordinates are zero except the ones corresponding to the weights

$$\theta_1 + \theta_2 + \theta_3 + \omega,\ \theta_1 - \theta_2 - \theta_3 + \omega,\ \ -\theta_1 - \theta_2 + \theta_3 + \omega,\ \ -\theta_1 + \theta_2 - \theta_3 + \omega,$$

where $\omega = \sum_{j>3} \theta_j$. Then $T\mathbf{z} = \mathbf{z}$ if and only if $\theta_j \equiv 0, \pi \mod 2\pi$, for all $j \geq 4$, and $T\mathbf{v} = \mathbf{v}$ if and only if $\theta_1, \theta_2, \theta_3, \phi$ satisfy one of the linear systems mod 2π

$$\begin{cases} \phi + \theta_1 + \theta_2 + \theta_3 \equiv \alpha \\ \phi + \theta_1 - \theta_2 - \theta_3 \equiv \alpha \\ \phi - \theta_1 + \theta_2 - \theta_3 \equiv \alpha \\ \phi - \theta_1 - \theta_2 + \theta_3 \equiv \alpha, \end{cases}$$

for $\alpha \equiv 0$ or $\alpha \equiv \pi$. From the discussion of the case $m = 6$, it follows that the isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ is a finite group. In order to conclude that $T_{(\mathbf{v},\mathbf{z})}$ is disconnected, we show that it contains at least an element different from the identity: indeed, by taking $\phi = \theta_1 \equiv -\pi/2$ and $\theta_2 = \theta_3 \equiv \pi/2$, we obtain an element acting on $\mathfrak{z}$ as

$$(z_1, z_2, z_3, z_4, z_5, z_6, z_7, \dots, z_m) \mapsto (-z_1, -z_2, -z_3, -z_4, -z_5, -z_6, z_7, \dots, z_m).$$

• Let $m \equiv 1,5 \mod 8$.

The real irreducible C_m-module $\mathfrak{d}$ is given by S_m embedded in $S_m \oplus S'_m$ as $\mathfrak{d} = \{(\mathbf{v}, J_m\mathbf{v}),\ \mathbf{v} \in S_m\}$. In this case, take a point $(\mathbf{w},\mathbf{z}) = ((\mathbf{v}, J_m\mathbf{v}), \mathbf{z}) \in \mathfrak{d} \oplus \mathfrak{z}$ with $\mathbf{v}$ and $\mathbf{z}$ as in the previous case. Since $T\mathbf{w} = \mathbf{w}$ if and only if $T\mathbf{v} = \mathbf{v}$, the above discussion implies that $T_{(\mathbf{w},\mathbf{z})}$ is disconnected.

If $\mathfrak{v} = \oplus_{j=1}^{k} \mathfrak{d}$ is the direct sum of k irreducible components, then take a point $(\mathbf{w},\mathbf{z}) = ((\mathbf{w}_1, 0 \dots, 0), \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$ with $\mathbf{w}$ all of whose components in the irreducible summands are zero except for $\mathbf{w}_1$. In addition, take $\mathbf{w}_1$ and $\mathbf{z}$ as $\mathbf{w}$ and $\mathbf{z}$ in the irreducible case. Since the T_2 action preserves the irreducible components of $\mathfrak{v}$, by the above discussion, we can conclude that $T_{(\mathbf{w},\mathbf{z})}$ is disconnected. $\square$

Lemma 8. *Let* $m \equiv 6, 7, 8 \mod 8$. *Then there exist points* $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$ *whose isotropy subgroup* $T_{(\mathbf{v},\mathbf{z})}$ *is disconnected.*

Proof. If $m \equiv 6, 7, 8 \mod 8$, then the real irreducible C_m-module $\mathfrak{d}$ is the real form of S_m or S'_m determined by the real structure J_m. The maximal torus T_1 of $Spin(m)$, described in (4) and (5), acts on the complex irreducible C_m-module S_m with weights $e^{i(\pm\theta_1 \pm \ldots \pm\theta_r)}$, for $r = [m/2]$, and 1-dimensional weight spaces. This action commutes with the real structure J_m and leaves invariant the real irreducible C_m-module $\mathfrak{d}$.

Assume $\mathfrak{v}$ is reducible and sum of $k \geq 2$ irreducible components. Then, as we already saw for $m = 6$, the block diagonal maximal torus T_2 of $SO(k)$, with $[k/2]$ rotations as blocks, acts on a block $\mathfrak{d} \oplus \mathfrak{d}$ as

$$(\mathbf{v}, \mathbf{v}') \mapsto (\cos\phi\mathbf{v} - \sin\phi\mathbf{v}', \sin\phi\mathbf{v} - \cos\phi\mathbf{v}'), \qquad \text{for } (\mathbf{v}, \mathbf{v}') \in \mathfrak{d} \oplus \mathfrak{d}. \quad (23)$$

If we write $\mathbf{v} = (x_1, \ldots x_s) \in \mathfrak{d}$ and $\mathbf{v}' = (x'_1, \ldots x'_s) \in \mathfrak{d}$ and

$$X^1 = \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}, \quad \ldots, X^s = \begin{pmatrix} x_s \\ x'_s \end{pmatrix}, \qquad s = \dim_{\mathbb{R}}\mathfrak{d}, \quad (24)$$

then the $T_1 \times T_2$-action can be written more conveniently as

$$X^l \mapsto e^{i(\pm\theta_1 \pm \ldots \pm\theta_r)} R_\phi X^l, \quad l = 1, \ldots, s. \quad (25)$$

We exhibit points $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$ with disconnected isotropy subgroup, by reducing to the case $m = 6$.

If $\mathfrak{v} = \oplus_{odd}\mathfrak{d}$ is the direct sum of an odd number of irreducible components, then one can easily see that the point $(0, \mathbf{z})$ with $z_1 \cdot \ldots \cdot z_m \neq 0$ has disconnected isotropy subgroup. In fact, $T(0, \mathbf{z}) = (0, \mathbf{z})$ if and only if $\theta_j \equiv 0, \pi$. Then $e^{i(\pm\theta_1 \pm \ldots \pm\theta_r)} = \pm 1$ and from (25) it follows that the isotropy subgroup

$$T_{(0,\mathbf{z})} \cong \mathbb{R}_{\phi_1} \times \ldots \times R_{\phi_h} \times \{\pm Id\} \times \{Id\}, \quad h = [k/2]$$

is disconnected.

If $\mathfrak{v} = \oplus_{j=1}^{k} \mathfrak{d} \oplus \mathfrak{d}$ is the direct sum of k blocks $\mathfrak{d} \oplus \mathfrak{d}$, then we take a point $(\mathbf{v}, \mathbf{z}) \in \mathfrak{v} \oplus \mathfrak{z}$ as follows: the vector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}'_1, \ldots, \mathbf{v}_k, \mathbf{v}'_k) \in \mathfrak{v}$ satisfies $\mathbf{v}_j = \mathbf{v}'_j = 0$, for all $j > 1$, and the vector $\mathbf{z} = (0, \ldots, 0, z_7, \ldots, z_m) \in \mathfrak{z}$ satisfies $z_7 \cdot \ldots \cdot z_m \neq 0$. In addition, if we write the components of $\mathbf{v}$ in the first block $\mathfrak{d} \oplus \mathfrak{d}$ as in (24), then we take all $X^j = \binom{0}{0}$ except for

$$\begin{pmatrix} i \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad \begin{pmatrix} i \\ 1 \end{pmatrix},$$

in correspondence with the weights

$$\begin{aligned} &\theta_1 + \theta_2 + \theta_3 + \omega, \quad \theta_1 + \theta_2 - \theta_3 + \omega, \\ &\theta_1 - \theta_2 + \theta_3 + \omega, \quad \theta_1 - \theta_2 - \theta_3 + \omega, \end{aligned}$$

with $\omega = \sum_{j>3} \theta_j$, and for their conjugates $J_m X^j$. Then $T(\mathbf{v}, \mathbf{z}) = (\mathbf{v}, \mathbf{z})$ if and only if $\theta_4, \ldots, \theta_r \equiv 0, \pi \mod 2\pi$ and $\theta_1, \theta_2, \theta_3, \phi_1$ satisfy one of the systems

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 + \phi_1 \equiv \alpha \\ \theta_1 + \theta_2 - \theta_3 - \phi_1 \equiv \alpha \\ \theta_1 - \theta_2 + \theta_3 - \phi_1 \equiv \alpha \\ \theta_1 - \theta_2 - \theta_3 + \phi_1 \equiv \alpha, \end{cases}$$

where $\alpha \equiv 0$ or $\alpha \equiv \pi \mod 2\pi$. At this point the discussion of Lemma 7 applies, and implies that the isotropy subgroup $T_{(\mathbf{v},\mathbf{z})}$ is disconnected. $\square$

We conclude this section with the proof of our main theorem. It is based on the results of the previous lemmas.

Proof of the Theorem. Since S is a simply connected solvable group of exponential type, in order to determine whether G^0 has a surjective exponential map, or not, we first apply the criteria developed by Moskowitz and Wustner in [14]. If T is a maximal torus in K^0, then it acts linearly on $\mathfrak{s}$ preserving $\mathfrak{n}$. By Corollary 7 of [14], when $m \leq 5$, G^0 has a surjective exponential map since for all $x \in \mathfrak{n}$, the isotropy groups T_x are connected. On the other hand, suppose $m \geq 6$. If $K^0 \times S$ had a surjective exponential map, by Corollary 3 of [15], $K^0 \times N$ would also have a surjective exponential map. However, by Theorem 5 of [14], $K^0 \times N$ has a surjective exponential map iff T_x is connected for each $x \in \mathfrak{n}$. Since when $m \geq 6$ this is incorrect, this contradiction tells us that for $m \geq 6$, $K^0 \times S$ can never have a surjective exponential map and completes the proof of the theorem. $\square$

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