INVARIANT ENVELOPES OF HOLOMORPHY IN THE COMPLEXIFICATION OF A HERMITIAN SYMMETRIC SPACE

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Abstract. In this paper we investigate invariant domains in Ξ^+, a distinguished G-invariant, Stein domain in the complexification of an irreducible Hermitian symmetric space G/K. The domain Ξ^+, recently introduced by Krötz and Opdam, contains the crown domain Ξ and it is maximal with respect to properness of the G-action. In the tube case, it also contains S^+, an invariant Stein domain arising from the compactly causal structure of a symmetric orbit in the boundary of Ξ. We prove that the envelope of holomorphy of an invariant domain in Ξ^+, which is contained neither in Ξ nor in S^+, is univalent and coincides with Ξ^+. This fact, together with known results concerning Ξ and S^+, proves the univalence of the envelope of holomorphy of an arbitrary invariant domain in Ξ^+ and completes the classification of invariant Stein domains therein.

1. Introduction

Let G/K be a non-compact, irreducible, Riemannian symmetric space. Its Lie group complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a Stein manifold and left translations by elements of G are holomorphic transformations of $G^{\mathbb{C}}/K^{\mathbb{C}}$. In this situation, G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and their envelopes of holomorphy are natural objects to study.

A first example is given by the crown Ξ, introduced by D. N. Akhiezer and S. G. Gindikin in [AkGi90]. This Stein invariant domain carries an invariant Kähler structure intrinsically associated with the Riemannian structure of the symmetric space G/K and, in many respects, can be regarded as its canonical complexification. In recent years, it has been extensively studied in connection with harmonic analysis on G/K (see, e.g [KrSt04], [KrSt05]).

If G/K is a Hermitian symmetric space of tube type, two additional distinguished invariant Stein domains $S^{\pm}$ arise from the compactly casual structure of a pseudo-Riemannian symmetric space G/H lying on the boundary of Ξ. The complex geometry of $S^{\pm}$ was studied by K. H. Neeb in [Nee99]. Inside the crown Ξ, as well as inside $S^{\pm}$, an invariant domain can be described via a semisimple abelian slice, its envelope of holomorphy is univalent and Steiness is characterized by logarithmic convexity of such a slice.

One may ask how far the above results are from a complete description of envelopes of holomorphy and a classification of invariant Stein domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$. In [GeIa08], a univalence result for G-equivariant Riemann domains over $G^{\mathbb{C}}/K^{\mathbb{C}}$, and in particular for envelopes of holomorphy, was proven in the rank-one case. In addition, the complete classification of invariant Stein domains was obtained. From this result one sees that, up to finitely many exceptions, all invariant Stein domains are contained either in Ξ or, in the Hermitian case of tube type, in $S^{\pm}$. Moreover, the study the CR-structure of principal G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ (i.e. closed orbits of maximal dimension) carried out in [Gea02], suggests that the latter fact holds true

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also in the higher rank case, the exceptions being finitely many invariant domains whose boundary entirely consists of non-principal G-orbits.

Here we focus on the case of G/K irreducible of Hermitian type. In this case, B. Krötz and E. Opdam recently singled out two Stein, invariant domains Ξ^+ and Ξ^- in $G^{\mathbb{C}}/K^{\mathbb{C}}$, satisfying $\Xi^+ \cap \Xi^- = \Xi$ and maximal with respect to properness of the G-action. The relevance of the crown Ξ and of the domains Ξ^+ and Ξ^- for the representation theory of G was underlined in Theorem 1.1 in [Kro08]. Since Ξ^+ and Ξ^- are G-equivariantly anti-biholomorphic, in the sequel we simply refer to Ξ^+. If G/K is Hermitian of tube type, then Ξ^+ contains both the crown Ξ and the domain S^+ ([GeIa13], Prop. 8.7). Moreover, for $r := \text{rank}(G/K) > 1$, the complement of $\Xi \cup S^+$ in Ξ^+ has non-empty interior. Our main result is as follows.

Theorem. *Let G/K be an irreducible Hermitian symmetric space. Given a G-invariant domain D in Ξ^+, denote by $\widehat{D}$ its envelope of holomorphy.*
(i) *Assume G/K is of tube type. If D is not contained in Ξ nor in S^+, then $\widehat{D}$ is univalent and coincides with Ξ^+.*
(ii) *Assume G/K is not of tube type. If D is not contained in Ξ, then $\widehat{D}$ is univalent and coincides with Ξ^+.*

The envelopes of holomorphy of invariant domains in Ξ or S^+ are known to be univalent and their Steiness is characterized in terms of the aformentioned semisimple abelian slices. Hence, the above theorem implies the univalence of the envelope of holomorphy of an arbitrary invariant domain in Ξ^+ and yields the following classification.

Corollary. *Let G/K be an irreducible Hermitian symmetric space and let D be a Stein G-invariant proper domain in Ξ^+.*
(i) *If G/K is of tube type, then either $D \subseteq \Xi$ or $D \subseteq S^+$.*
(ii) *If G/K is not of tube type, then $D \subseteq \Xi$.*

The theorem is proved by showing that the natural G-equivariant embedding $f : D \to \widehat{D}$ admits a holomorphic extension $\hat{f} : \Xi^+ \to \widehat{D}$ to the whole Ξ^+. For this purpose, we use the unipotent, abelian slice of Ξ^+ pointed out by B. Krötz and E. Opdam in [KrOp08]. Namely, one has

$$\Xi^+ = G \cdot \Sigma ,$$

where $\Sigma := \exp i\Lambda_r^{\llcorner} \cdot x_0$ and $\Lambda_r^{\llcorner}$ is a closed hyperoctant in an r-dimensional, nilpotent, abelian subalgebra of $Lie(G)$. This sets a one-to-one correspondence

$$D \to \Sigma_D := D \cap \Sigma$$

between G-invariant domains in Ξ^+ and domains in Σ which are invariant under the action of an appropriate Weyl group (see Sect. 3).

Then a key ingredient is given by Proposition 4.7, which implies that a continuous extension of $f|_{\Sigma_D}$ to a domain $\widetilde{\Sigma}$ in Σ induces a G-equivariant, holomorphic extension of f on $G \cdot \widetilde{\Sigma}$ provided that certain compatibility conditions are satisfied. In order to obtain $\hat{f}$, we therefore construct a continuous extension of $f|_{\Sigma_D}$ to Σ satisfying such compatibility conditions.

This is done in a finite number of steps. At each step we extend $f|_{\Sigma_D}$ to a larger domain $\widetilde{\Sigma} \subset \Sigma$ properly containing Σ_D. Such extensions are obtained by equivariantly embedding in $G^{\mathbb{C}}/K^{\mathbb{C}}$ various lower dimensional complex homogenous manifolds $L^{\mathbb{C}}/H^{\mathbb{C}}$, all of whose L-invariant domains have univalent and well understood envelopes of holomorphy. The embedding of each space $L^{\mathbb{C}}/H^{\mathbb{C}}$ is carefully chosen, so that it intersects D in some L-invariant domain $T \subset L^{\mathbb{C}}/H^{\mathbb{C}}$. As a consequence, the holomorphic map $f|_T : T \to \widehat{D}$ extends L-equivariantly to

$\widehat{T} \to \widehat{D}$ and, in particular, yields a real-analytic extension of $f|_{\Sigma_D}$ along the submanifold $\widehat{T}\cap\Sigma$. It turns out that for some choices of $L^{\mathbb{C}}/H^{\mathbb{C}}$ the intersection $\widehat{T}\cap\Sigma$ is not open in Σ. In these cases, an extension of $f|_{\Sigma_D}$ to an open domain $\widetilde{\Sigma} \subset \Sigma$ is obtained by embedding in D a continuous family of copies of T.

The real homogenous manifolds L/H which play a role in our situation are: real r-dimensional vector spaces acted on by $(\mathbb{R}^r, +)$, the Euclidean plane acted on by its isometry group, and irreducible rank-one, Hermitian symmetric spaces, of both tube-type and non-tube type. In the latter case, the univalence results on equivariant Riemann domains obtained in [GeIa08] are crucial. The above strategy was inspired by the work of K. H. Neeb on bi-invariant domains in the complexification of a Hermitian semisimple Lie group ([Nee98]).

The paper is organized as follows. In section 2, we set up the notation and recall some preliminary facts which are needed in the paper. In section 3, we recall the unipotent paramentrization of Ξ^+ and of its G-invariant subdomains. In section 4, we recall some basic facts about envelopes of holomorphy and develope the main tools used in the proof of the main theorem. In section 5 we prove the main theorem.

2. Preliminaries

Let G/K be an irreducible Hermitian symmetric space of the non-compact type. We may assume G to be a connected, non-compact, real simple Lie group contained in its simple, simply connected universal complexification $G^{\mathbb{C}}$, and K to be a maximal compact subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Denote by θ both the Cartan involution of G with respect to K and the associated involution of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. The *rank* of G/K is by definition $r = \dim \mathfrak{a}$. The adjoint action of $\mathfrak{a}$ decomposes $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^{\alpha},$$

where $Z_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, the joint eigenspace $\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \text{ for every } H \in \mathfrak{a}\}$ is the α-restricted root space and $\Delta(\mathfrak{g}, \mathfrak{a})$ consists of those $\alpha \in \mathfrak{a}^*$ for which $\mathfrak{g}^{\alpha} \neq \{0\}$. A set of simple roots $\Pi_{\mathfrak{a}}$ in $\Delta(\mathfrak{g}, \mathfrak{a})$ uniquely determines a set of positive restricted roots $\Delta^+(\mathfrak{g}, \mathfrak{a})$ and an Iwasawa decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \qquad \text{where} \quad \mathfrak{n} = \bigoplus_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^{\alpha}\,.$$

The restricted root system of a Lie algebra $\mathfrak{g}$ of Hermitian type is either of type C_r (if G/K is of tube type) or of type BC_r (if G/K is not of tube type) (cf. [Moo64]), i.e. there exists a basis $\{e_1, \ldots, e_r\}$ of $\mathfrak{a}^*$ for which

$$\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm 2e_j,\ 1 \le j \le r,\ \pm e_j \pm e_k,\ 1 \le j \neq k \le r\}, \qquad \text{for type } C_r,$$

$$\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm e_j,\ \pm 2e_j,\ 1 \le j \le r,\ \pm e_j \pm e_k,\ 1 \le j \neq k \le r\}, \qquad \text{for type } BC_r.$$

Since $\mathfrak{g}$ admits a compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$, there exists a set of r positive long strongly orthogonal restricted roots $\{\lambda_1, \ldots, \lambda_r\}$ (i.e. such that $\lambda_j \pm \lambda_k \notin \Delta(\mathfrak{g}, \mathfrak{a})$, for $j \neq k$), which are restrictions of *real* roots with respect to a maximally split θ-stable Cartan subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ extending $\mathfrak{a}$.

Taking as simple roots $\Pi_{\mathfrak{a}} = \{e_1 - e_2, \ldots, e_{r-1} - e_r, 2e_r\}$, for type C_r, and $\Pi_{\mathfrak{a}} = \{e_1 - e_2, \ldots, e_{r-1} - e_r, e_r\}$, for type BC_r, one has

$$\lambda_1 = 2e_2, \ldots, \lambda_r = 2e_r\,.$$

Let Z_0 be the element in $Z(\mathfrak{k})$ defining the complex structure $J_0 = \mathrm{ad}_{Z_0}$ on G/K. For $j = 1, \ldots, r$, choose $E_j \in \mathfrak{g}^{\lambda_j}$ such that the $\mathfrak{sl}(2)$-triple

$$\{E_j,\ \theta E_j,\ A_j := [\theta E_j, E_j]\}$$

is normalized as follows

$$[A_j, E_j] = 2E_j, \quad [Z_0, E_j - \theta E_j] = A_j, \quad [Z_0, A_j] = -(E_j - \theta E_j)\,. \tag{1}$$

Then the vectors $\{A_1, \ldots, A_r\}$ form an orthogonal basis of $\mathfrak{a}$ (with respect to the restriction of the Killing form) and

$$[E_j, E_k] = [E_j, \theta E_k] = 0, \quad [A_j, E_k] = \lambda_k(A_j)E_k = 0, \quad \text{for } j \neq k\,. \tag{2}$$

That is, the above $\mathfrak{sl}(2)$-triples commute with each other. Moreover, under the above choices, the element Z_0 is given by

$$Z_0 = S + \frac{1}{2}\sum T_j, \tag{3}$$

where $T_j = E_j + \theta E_j$ and $S \in Z_{\mathfrak{k}}(\mathfrak{a})$ (see Lemma 2.4 in [GeIa13]). If G/K is of tube type one has $S = 0$.

In the sequel, we denote by $\mathfrak{g}_j$ the $\mathfrak{sl}(2)$-triple corresponding to the root $\lambda_j \in \{\lambda_1, \ldots, \lambda_r\}$, and by G_j the corresponding connected subgroup of G. In the non-tube case, to each λ_j one can also associate a connected, simple, real rank-one Hermitian subgroup $G_j^\bullet$ of G. The group $G_j^\bullet$ is by definition the connected, θ-stable subgroup of G with Lie algebra

$$\mathfrak{g}_j^\bullet = \mathbb{R}A_i \oplus \mathfrak{g}^{\pm\lambda_j/2} \oplus \mathfrak{g}^{\pm\lambda_j}$$

isomorphic to $\mathfrak{su}(m, 1)$, for some $m > 1$ (see [Kna04]).

Lemma 2.1. *Let G/K be an irreducible Hermitian symmetric space, which is not of tube type. Let $G_j^\bullet$ be the simple real rank-one Hermitian subgroup associated to the root λ_j, for some $j \in \{1, \ldots, r\}$. Then $G_j^\bullet$ commmutes with the subgroups G_k, for every $k \neq j$.*

Proof. By relations (2), one has $[\mathfrak{g}_j, \mathfrak{g}_k] \equiv 0$, for $k \neq j$. Futhermore, since $\pm e_j \pm 2e_k$, for $j \neq i$, are not roots in $\Delta(\mathfrak{g}, \mathfrak{a})$ and $e_j(A_k) = \delta_{jk}$, one also has $[\mathfrak{g}^{\pm\lambda_j/2}, \mathfrak{g}_k] \equiv 0$. Summarizing, there is commutativity at Lie algebra level

$$[\mathfrak{g}_j^\bullet, \mathfrak{g}_k] \equiv 0, \qquad \text{for } k \neq j$$

and likewise at group level, by connectedness. □

3. Invariant subdomains of Ξ^+.

Let G/K be an irreducible Hermitian symmetric space of the non-compact type. Its complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ contains a distinguished G-invariant Stein subdomain Ξ^+, properly containing the crown Ξ, and maximal with respect to proper G-action.

A description of the domain Ξ^+ was given in [Kro08], p.286, and [KrOp08], Sect.8, via its unipotent parametrization. More precisely, fix vectors $E_j \in \mathfrak{g}^{\lambda_j}$ normalized as in (1). Then

$$\Xi^+ = G \exp i \bigoplus_{j=1}^{r} (-1, \infty) E_j \cdot x_0.$$

Define the nilpotent abelian subalgebras

$$\Lambda_r := \mathrm{span}_{\mathbb{R}}\{E_1, \ldots, E_r\} \quad \text{and} \quad \Lambda_r^{\mathbb{C}} := \mathrm{span}_{\mathbb{C}}\{E_1, \ldots, E_r\}$$

of $\mathfrak{n}$ and $\mathfrak{n}^{\mathbb{C}}$, respectively. The exponential map of $G^{\mathbb{C}}$ defines a biholomorphism between $\Lambda_r^{\mathbb{C}}$ and the unipotent abelian complex subgroup $L^{\mathbb{C}} := \exp \Lambda_r^{\mathbb{C}}$. In particular, it restricts to a diffeomorphism between Λ_r and the real unipotent subgroup $L := \exp \Lambda_r$. Since the map

$$\iota : \mathfrak{n}^{\mathbb{C}} \to N^{\mathbb{C}} \cdot x_0, \qquad Z \to \exp Z \cdot x_0, \tag{4}$$

is a biholomorphism onto its image (cf. Prop. 1.3 in [KrSt04]), so is its restriction $\iota : \Lambda_r^{\mathbb{C}} \to L^{\mathbb{C}} \cdot x_0$.

Lemma 3.1. *The intersection $\Xi^+ \cap L^{\mathbb{C}} \cdot x_0$ is a closed, r-dimensional, complex submanifold of Ξ^+, which is biholomorphic, via the map ι, to the Stein tube domain $\Lambda_r \times i \bigoplus_{j=1}^r (-1, \infty) E_j$ of $\Lambda_r^{\mathbb{C}}$.*

Proof. By a result of Rosenlicht ([Ros61], Thm. 2), the orbits of the unipotent subgroup $L^{\mathbb{C}}$ in the affine space $G^{\mathbb{C}}/K^{\mathbb{C}}$ are closed. In particular $L^{\mathbb{C}} \cdot x_0 \cap \Xi^+$ is closed in Ξ^+. Now the statement follows from the injectivity of map ι and the fact that the set $\{ X \in \Lambda_r \ : \ \exp iX \cdot x_0 \in \Xi^+ \}$ coincides with $\bigoplus_{j=1}^r (-1, \infty) E_j$ (see [Kro08], p. 286). $\square$

By Lemma 4.1 in [GeIa13], the group

$$W_K(\Lambda_r) := N_K(\Lambda_r)/Z_K(\Lambda_r)$$

is a proper subgroup of the Weyl group $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Its action on Λ_r is described by the following result.

Lemma 3.2. ([GeIa13], *Lemma 4.1) The group $W_K(\Lambda_r)$ acts on Λ_r by permutations of the basis elements $\{E_1, \dots, E_r\}$.*

One may expect that the intersection of a G-orbit in Ξ^+ with the closed slice $\exp(i \bigoplus_{j=1}^r (-1, +\infty) E_j) \cdot x_0$ is just a $W_K(\Lambda_r)$-orbit. However, as observed in [GeIa13], Remark 7.6 this is not the case. Then, when studying the G-invariant geometry of Ξ^+, it is useful to consider a smaller slice as follows. Consider the $W_K(\Lambda_r)$-invariant, closed hyperoctant

$$\Lambda_r^{\llcorner} := \operatorname{span}_{\mathbb{R}^{\geq 0}}\{E_1, \dots, E_r\}$$

of Λ_r, and the nilpotent cone in $\mathfrak{g}$ given by $\mathcal{N}^+ := \operatorname{Ad}_K(\Lambda_r^{\llcorner})$. As suggested in [KrOp08] and [Kro08], Remark 4.12, the following fact holds true.

Proposition 3.3. ([GeIa13], *Prop. 5.7) The G-equivariant map*

$$\psi \colon G \times_K \mathcal{N}^+ \to \Xi^+, \qquad [g, X] \to g \exp iX \cdot x_0$$

is a homeomorphism.

Given a G-invariant domain $D \subset \Xi^+$, define an open subset of $\bigoplus_{j=1}^r (-1, \infty) E_j$ by

$$\mathcal{D} := \{ X \in \Lambda_r \ : \ \exp iX \cdot x_0 \in D \}.$$

By the definition of $\mathcal{D}$ and Proposition 3.3, the domain D can be written as

$$D = G \exp i\mathcal{D} \cdot x_0 = G \exp i\mathcal{D}^{\llcorner} \cdot x_0,$$

where $\mathcal{D}^{\llcorner} := \mathcal{D} \cap \Lambda_r^{\llcorner}$ is a $W_K(\Lambda_r)$-invariant open subset of $\Lambda_r^{\llcorner}$.

Lemma 3.4. ([GeIa13], *Lemma 7.4) Let X be an element in $\Lambda_r^{\llcorner}$. Then the* Ad_K*-orbit of X intersects Λ_r in the $W_K(\Lambda_r)$-orbit of X in $\Lambda_r^{\llcorner}$.*

Note that the above result together with Proposition 3.3 implies that given X in $\Lambda_r^\llcorner$, one has

$$G \exp iX \cdot x_0 \bigcap \exp i\Lambda_r^\llcorner \cdot x_0 = \exp i(W_K(\Lambda_r) \cdot X) \cdot x_0,$$

i.e. every G-orbit (not just K-orbit) in Ξ^+ intersects the closed slice $\exp i\Lambda_r^\llcorner \cdot x_0$ exactly in a $W_K(\Lambda_r)$-orbit.

Consider the open Weyl chamber $(\Lambda_r^\llcorner)^+ := \big\{ \sum_{j=1}^r x_j E_j \ : \ x_1 > \cdots > x_r > 0 \big\}$. By Lemma 3.2, its topological closure

$$\overline{(\Lambda_r^\llcorner)^+} = \Big\{ \sum_{j=1}^r x_j E_j, \ : \ x_1 \geq \cdots \geq x_r \geq 0 \Big\}$$

is a perfect slice for the $W_K(\Lambda_r)$-action on $\Lambda_r^\llcorner$, implying that $\exp i\overline{(\Lambda_r^\llcorner)^+} \cdot x_0$ is a perfect slice for the G-action on Ξ^+. It follows that for a G-invariant domain D of Ξ^+ one also has

$$D = G \exp i(\mathcal{D}^\llcorner)^+ \cdot x_0, \tag{5}$$

where the subset $(\mathcal{D}^\llcorner)^+ := \mathcal{D}^\llcorner \cap \overline{(\Lambda_r^\llcorner)^+}$ is open in $\overline{(\Lambda_r^\llcorner)^+}$. In particular, $(\mathcal{D}^\llcorner)^+$ is connected if and only if D is connected.

In the sequel we also need the following fact.

Lemma 3.5. *Let X be an element in $\Lambda_r^\llcorner$. Then every connected component of $Z_K(X)$ meets $Z_K(\Lambda_r)$.*

Proof. Let X be an arbitrary element in $\Lambda_r^\llcorner$. By (i) of Lemma 4.1 and Lemma 5.6 in [GeIa13], one has

$$Z_K(\Lambda_r) \cong Z_K(\mathfrak{a}) \quad \text{and} \quad Z_K(X) \cong Z_K(\Psi(X)),$$

where $\Psi(X) = [Z_0, X - \theta X] \in \mathfrak{a}$. Thus in order to prove the lemma, it is sufficient to show that for an arbitrary element $H \in \mathfrak{a}$, every connected component of $Z_K(H)$ meets $Z_K(\mathfrak{a})$.

The centralizer $Z_G(H)$ is a θ-stable reductive subgroup of G (see [Kna04], Prop. 7.25, p. 452) of the same rank and real rank as G, with maximal compact subgroup $Z_K(H)$. The maximal abelian subspace of $Z_\mathfrak{p}(H)$ is $\mathfrak{a}$ and, as $Z_K(\mathfrak{a})$ is contained in $Z_K(H)$, one has that $Z_{Z_K(H)}(\mathfrak{a}) = Z_K(\mathfrak{a})$. Now Proposition 7.33 in [Kna04], p. 457, applied to the reductive group $Z_G(H)$, states that $Z_K(\mathfrak{a})$ meets every connected component of $Z_K(H)$, as desired. $\square$

In [GeIa13] it was shown that if G/K is of tube type, then Ξ^+ contains another distinguished Stein invariant domain, besides the crown Ξ. Such domain S^+ arises from the compactly causal structure of a pseudo-Riemannian symmetric G-orbit in the boundary of Ξ. The domain S^+ and its invariant subdomains were investigated in [Nee99]. In the unipotent parametrization of Ξ^+, the domains Ξ and S^+ are given as follows (see [KrOp08], Sect. 8, [GeIa13], Prop. 8.7).

Proposition 3.6. *Let G/K be an irreducible Hermitian symmetric space. Inside Ξ^+ the crown domain Ξ is given by*

$$G \exp i \bigoplus_{j=1}^r [0,1) E_j \cdot x_0 \, .$$

If G/K is of tube type, the domain S^+ is given by

$$G \exp i \bigoplus_{j=1}^r (1,\infty) E_j \cdot x_0 .$$

4. Envelopes of holomorphy of invariant domains in Ξ^+.

In this section we prove some preliminary results supporting the three basic ingredients of the proof of the main theorem, namely reduction 1, reduction 2 and rank-one reduction. A key result is given by Proposition 4.7, under whose assumptions one obtains G-equivariant, holomorphic extensions of the embedding $f\colon D \to \widehat{D}$ to larger invariant domains containing D.

We begin by recalling some general facts about envelopes of holomorphy. Let X be a Stein manifold and let D be a domain in X. By Rossi's results [Ros63], D admits an envelope of holomorphy $\widehat{D}$. This means that there exist an open holomorphic embedding $f\colon D \to \widehat{D}$ into a Stein manifold $\widehat{D}$ to which all holomorphic functions on D simultaneously extend. Moreover, there is a local biholomorphism q such that the diagram

$$\begin{tikzcd} & \widehat{D} \arrow[d, "q"] \\ D \arrow[ur, "f"] \arrow[r, "Id"'] & X \end{tikzcd} \tag{6}$$

commutes.

Proposition 4.1. *Let D_1 and D_2 be complex manifolds, with envelopes of holomorphy $f_1\colon D_1 \to \widehat{D}_1$ and $f_2\colon D_2 \to \widehat{D}_2$, respectively. Let $F\colon D_1 \to D_2$ be a holomorphic map. Then there exists a unique holomorphic map $\widehat{F}\colon \widehat{D}_1 \to \widehat{D}_2$ such that $\widehat{F} \circ f_1 = f_2 \circ F$.*

As a consequence of the above proposition, the following facts hold true.

Proposition 4.2. *Let X be a Stein manifold and let $D \subset X$ be a domain with envelope of holomorphy $\widehat{D}$ (cf. diagram (6)).*
(i) *Let Ω be the smallest Stein domain in X containing D. Then $q(\widehat{D})$ is contained in Ω.*
(ii) *Let Ω be a domain in X containing D. Assume there exists a holomorphic map $\hat{f}\colon \Omega \to \widehat{D}$ extending f. Then $\widehat{\Omega} = \widehat{D}$.*

If G is a Lie group acting on X by biholomorphisms and the domain D is G-invariant, then the G-action lifts to an action on $\widehat{D}$ and all the maps in diagram (6) are G-equivariant. Coming back to our case, let

$$D = G \exp i\mathcal{D} \cdot x_0 = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$$

be a G-invariant domain in Ξ^+. Since Ξ^+ is Stein, one has a commutative diagram

$$\begin{tikzcd} & \widehat{D} \arrow[d, "q"] \\ D \arrow[ur, "f"] \arrow[r, "Id"'] & \Xi^+ \end{tikzcd} \tag{7}$$

where all maps are G-equivariant. We prove that under the assumption that D is not entirely contained in Ξ nor in S^+ (in the tube case), the map $f\colon D \to \widehat{D}$ can be G-equivariantly extended to the whole Ξ^+. We gradually enlarge the domain of definition of f by iterating the following arguments.

By reduction 1, we show that f can be G-equivariantly extended to a domain $G \exp i\widetilde{\mathcal{D}}^{\llcorner} \cdot x_0$ with all the connected components of $\widetilde{\mathcal{D}}^{\llcorner}$ convex (see Prop. 4.10). By reduction 2, we show that f can be G-equivariantly extended to a domain with $\widetilde{\mathcal{D}}^{\llcorner}$ connected (see Prop. 4.13), and therefore convex.

The third basic ingredient is the rank-one reduction. It is based on the univalence and the precise description of the envelope of holomorphy of an arbitrary G-invariant domain in the complexification of a rank-one Hermitian symmetric space (cf. [GeIa08]). Finally, by applying Proposition 4.2(ii), one obtains $\widehat{D} = \Xi^+$, The strategy is similar to the one used by Neeb in [Nee98].

The rank-one case. For the reader's convenience we recall the rank-one case, in the formulation which is needed in this paper.

We begin with the tube case $G/K = SL(2,\mathbb{R})/SO(2,\mathbb{R}) \cong SU(1,1)/U(1)$. Let $\{E, \theta E, A\}$ be the basis of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ defined in (1). Then $\Xi^+ = G \exp i[0,\infty)E \cdot x_0$ and every G-invariant domain in Ξ^+ is of the form $D = G \exp iIE \cdot x_0$, where I is an open, connected interval in $[0,\infty)$.

The curve $\ell : [0,+\infty) \to \Xi^+$, given by $t \to \exp itE \cdot x_0$, starts at x_0 and intersects every G-orbit in Ξ^+ precisely once. For every $t > 0$ the orbit $G \cdot \ell(t)$ is a real hypersurface in Ξ^+. Denote by $T^{CR}_{\ell(t)}(G \cdot \ell(t)) := T_{\ell(t)}(G \cdot \ell(t)) \cap J_{\ell(t)} T_{\ell(t)}(G \cdot \ell(t))$ the complex tangent space to $G \cdot \ell(t)$ at $\ell(t)$. The quadratic Levi form of $G \cdot \ell(t)$ at $\ell(t)$ is given by

$$\mathcal{L}_{\ell(t)}(W,W) = \frac{1}{8} \frac{(t^2-1)}{t} |W|^2 \, \dot{\ell}(t) \,, \quad \text{for } W \in T^{CR}_{\ell(t)}(G \cdot \ell(t)) \,.$$

The above formula shows that for every $t \neq 1$ the hypersurface $G \cdot \ell(t)$ has non-degenerate definite Levi form. Hence it bounds a Stein G-invariant domain in Ξ^+. Note that the concavities of the hypersurfaces $G \cdot \ell(t)$, for $t < 1$, and $G \cdot \ell(t)$, for $t > 1$, point in opposite directions. The hypersurface $G \cdot \ell(1)$ is Levi-flat. All proper Stein, G-invariant subdomains of Ξ^+ are given by (cf. Lemma 8.1 in [GeIa13] and Ex. 6.2 [GeIa08])

$$\begin{aligned} &G \exp i[0,b)E \cdot x_0, \quad && \text{for} \quad 0 < b \leq 1 \\ &G \exp i(a,\infty)E \cdot x_0, \quad && \text{for} \quad 1 \leq a < \infty \,. \end{aligned} \tag{8}$$

Moreover one has the following description of envelopes of holomorphy in Ξ^+.

Proposition 4.3. *Let $G = SL(2,\mathbb{R})$ and let D be a G-invariant domain in Ξ^+. Then the envelope of holomorphy $\widehat{D}$ of D is univalent and given as follows.*
(i) *If $D = G \exp i(a,b)E \cdot x_0$ or $D = G \exp i[0,b)E \cdot x_0$, with $b \leq 1$, then*

$$\widehat{D} = G \exp i[0,b)E \cdot x_0;$$

(ii) *If $D = G \exp i(a,b)E \cdot x_0$ or $D = G \exp i(a,\infty)E \cdot x_0$, with $1 \leq a$, then*

$$\widehat{D} = G \exp i(a,\infty)E \cdot x_0;$$

(iii) *If D contains the orbit $G \cdot \ell(1)$, then $\widehat{D} = \Xi^+$.*

Proof. The center Z of $SL(2,\mathbb{R})$ acts trivially on $D \subset G^{\mathbb{C}}/K^{\mathbb{C}}$ and, by the analytic continuation principle, on $\widehat{D}$. Thus the projection $q : \widehat{D} \to \Xi^+$ is $PSL(2,\mathbb{R})$-equivariant and, by Theorem 7.6 in [GeIa08], is injective. Then, by Proposition 4.2, the envelope of holomorphy $\widehat{D}$ coincides with the smallest Stein, G-invariant domain in Ξ^+ containing D. The rest of the statement follows from the classification of all Stein, G-invariant domains in Ξ^+ given in (8). □

Consider now the rank-one, Hermitan symmetric space $G/K = SU(n,1)/U(n)$, for $n > 1$, which is not of tube type. The difference with the previous case lies in the fact that, for $t > 1$, the hypersurface $G \cdot \ell(t)$ has non-degenerate, indefinite Levi form. As a consequence it cannot lie on the boundary of a Stein G-invariant domain in Ξ^+. The hypersurface $G \cdot \ell(1)$ has semidefinite Levi form and lies on the boundary of the crown domain Ξ, which is Stein. In this case all proper, Stein, G-invariant subdomains of Ξ^+ are given by

$$G\exp i[0,b), \quad \text{for} \quad 0 < b \leq 1\,,$$

(cf. Lemma 8.1 in [GeIa13], and Ex. 6.3 [GeIa08]) and similar arguments as in Proposition 4.3 give the description of the envelopes of holomorphy in this case.

Proposition 4.4. *Let $G = SU(n,1)$ and let D be a G-invariant domain in Ξ^+. Then the envelope of holomorphy $\widehat{D}$ is univalent and given as follows.*
(i) *If $D = G\exp i(a,b)E\cdot x_0$ or $D = G\exp i[0,b)E\cdot x_0$, with $b \leq 1$, then*

$$\widehat{D} = G\exp i[0,b)E\cdot x_0\,.$$

(ii) *If D contains an orbit $G\cdot \ell(t)$, for some $t \geq 1$, then $\widehat{D} = \Xi^+$.*

The extension lemma. The goal of this subsection is to prove the "extension lemma", which provides sufficient conditions for a continuous lift $f\colon \exp i\mathcal{C}\cdot x_0 \to \widehat{D}$ to extend to a G-equivariant holomorphic map $\hat{f}\colon G\exp i\mathcal{C}\cdot x_0 \to \widehat{D}$. One of the conditions involves the isotropy subgroups of points $z \in D$ and $f(z) \in \widehat{D}$.

Since the projection $q\colon \widehat{D} \to \Xi^+$ is a G-equivariant local biholomorphism, the isotropy subgroup of $z \in \widehat{D}$ is the union of connected components of the isotropy subgroup of $q(z) \in \Xi^+$. In addition, since $f\colon D \to \widehat{D}$ is a G-equivariant biholomorphism onto its image and $q|_{f(D)} \circ f = Id_D$, there is actually an identity of isotropy subgroups $G_z = G_{q(z)}$, for all $z \in f(D)$. In the sequel it will be crucial to have such an identity of isotropy subgroups for points lying in suitable submanifolds of $q(\widehat{D})$ intersecting D, to which the map f extends holomorphically.

Lemma 4.5. *Let $\mathcal{C}$ be an open subset of $\Lambda_r^{\llcorner}$ and let $f : \exp i\mathcal{C}\cdot x_0 \to \widehat{D}$ be a continuous map such that $q\circ f = Id$. Assume that there exists an open subset $\mathcal{F}$ of $\mathcal{C}$ such that*
(i) *$G_{f(\exp iX'\cdot x_0)} = G_{\exp iX'\cdot x_0}$ for all X' in $\mathcal{F}$,*
(ii) *for every $X \in \mathcal{C}$, there exist an element $X' \in \mathcal{F}$ such that the segment $\{\, X' + t(X - X') \;:\; t \in [0,1]\}$ is contained in $\mathcal{C}$, and a holomorphic extension of f to the submanifold $\mathcal{S} = \{\, \exp(i(X' + \lambda(X - X')))\cdot x_0 \;: \mathrm{Re}\lambda \in [0,1]\,\}$.*
Then $G_{f(\exp iX\cdot x_0)} = G_{\exp iX\cdot x_0}$, for every X in $\mathcal{C}$.

Proof. Since q is G-equivariant and $q\circ f = Id$ on $\exp i\mathcal{C}\cdot x_0$, it is clear that $G_{f(\exp iX\cdot x_0)} \subset G_{\exp iX\cdot x_0}$ for all $X \in \mathcal{C}$. In order to prove the opposite inclusion, we consider first generic elements in $\mathcal{C}$.

By definition, generic elements $X \in \Lambda_r^{\llcorner}$ are those for which $Z_K(X) = Z_K(\Lambda_r)$, and by Lemma 7.3 in [GeIa13], they are dense in $\Lambda_r^{\llcorner}$. Let X be a generic element in $\mathcal{C}$ and let g be an element in $G_{\exp iX\cdot x_0} = Z_K(\Lambda_r)$. The fixed point set of g in $\widehat{D}$

$$Fix(g,\widehat{D}) := \{z \in \widehat{D} \mid g\cdot z = z\}$$

is a complex analytic subset of $\widehat{D}$. Let $X' \in \mathcal{F}$ be an element satisfying condition (ii) of the lemma. Since both $\mathcal{C}$ and $\mathcal{F}$ are open, X' can be chosen generic as well. Consider the strip $S := \{\lambda \in \mathbb{C} \;:\; \mathrm{Re}\lambda \in [0,1]\}$ and define the function

$$\phi\colon S \to \widehat{D}, \qquad \phi(\lambda) := f(\exp i(X' + \lambda(X - X')\cdot x_0))\,.$$

We are going to show that the set

$$A := \{\lambda \in S \;:\; g\cdot\phi(\lambda) = \phi(\lambda)\}$$

contains the element 1: this implies that $f(\exp iX\cdot x_0) \in Fix(g,\widehat{D})$ and proves the statement for X generic.

Since both X and X' are generic in $\Lambda_r^{\llcorner}$, one has that $G_{\exp iX'\cdot x_0} = G_{\exp iX\cdot x_0} = Z_K(\Lambda_r)$. Therefore $g \in G_{\exp iX'\cdot x_0}$ and, by condition (i), it follows that $f(\exp iX' \cdot x_0) \in Fix(g, \widehat{D})$. Consequently $0 \in A$. Since $\mathcal{F}$ is open, there exists $\varepsilon > 0$ such that $[0, \varepsilon) \subset A$. Let $[0, b)$ be the maximal open interval in $A \cap \mathbb{R}$ containing 0 and assume by contradiction that $b \neq 1$. Since A is closed, it follows that $b \in A$ and, by the definition of A, one has that $\phi(b) \in Fix(g, \widehat{D})$. Locally, in a neighbourhood U of $\phi(b)$ in $\widehat{D}$, the analytic set $Fix(g, \widehat{D})$ is given as

$$Fix(g, \widehat{D}) \cap U = \{z \in U \mid \psi_1(z) = \ldots = \psi_k(z) = 0\},$$

for some $\psi_1, \ldots, \psi_k \in \mathcal{O}(U)$. Thus, for each $j = 1, \ldots r$, the holomorphic function

$$\psi_j \circ \phi : \phi^{-1}(U) \to \mathbb{C}, \quad \lambda \mapsto \psi_j(f(\exp i(X' + \lambda(X - X')) \cdot x_0))$$

vanishes identically on $[0, b]$. Since $\phi^{-1}(U)$ is open in S, there exists $\epsilon' > 0$ such that the restriction $\psi_j \circ \phi_{(b-\epsilon', b+\epsilon')}$ is real analytic and identically zero on $(b - \epsilon', b]$. Hence it is identically zero on the whole interval $(b - \epsilon', b + \epsilon')$, contradicting the maximality of b. Thus $b = 1$ and $b \in A$, as claimed. This concludes the case of generic elements in $\mathcal{C}$.

Consider now a non-generic element $X \in \mathcal{C}$. Since generic elements form an open dense subset of $\mathcal{C}$, there exists a sequence of generic elements $\{X_n\} \subset \mathcal{C}$ converging to X. Recall that all generic elements in $\mathcal{C}$ have the same isotropy subgroup $Z_K(\Lambda_r)$. Therefore, by the previous step, one has

$$g \cdot f(\exp iX_n \cdot x_0) = f(\exp iX_n \cdot x_0), \quad \text{for all } \ g \in Z_K(\Lambda_r).$$

Passing to the limit, one obtains that $g \cdot f(\exp iX \cdot x_0) = f(\exp iX \cdot x_0)$, for all $g \in Z_K(\Lambda_r)$. This fact together with Lemma 3.5 implies that $G_{\exp iX\cdot x_0} \subset G_{f(\exp iX\cdot x_0)}$ for all $X \in \mathcal{C}$, and concludes the proof of the lemma. □

Lemma 4.6. *Let $D = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$ be a G-invariant domain in Ξ^+ and let X be a G-space. A G-equivariant map $f : D \to X$ is continuous if and only if its restriction to $\exp i\mathcal{D}^{\llcorner} \cdot x_0$ is continuous.*

Proof. One implication is clear. For the converse, we first prove that f is continuous on $K \exp i\mathcal{D}^{\llcorner} \cdot x_0 = \exp i\mathrm{Ad}_K\mathcal{D}^{\llcorner} \cdot x_0$. Consider the identification $\mathrm{Ad}_K\mathcal{D}^{\llcorner} \to \exp i\mathrm{Ad}_K\mathcal{D}^{\llcorner} \cdot x_0$ defined by $X \to \exp iX \cdot x_0$ (see Lemma 3.3) and let $X_n \to X_0$ be a converging sequence in $\mathrm{Ad}_K\mathcal{D}^{\llcorner}$. Choose elements k_n in K such that $\mathrm{Ad}_{k_n}X_n \in \mathcal{D}^{\llcorner}$. Since K compact, we can assume that the sequence $\{k_n\}_n$ converges to an element $k_0 \in K$ and that $\mathrm{Ad}_{k_n}X_n \to \mathrm{Ad}_{k_0}X_0$.

Now observe that $\mathcal{D}^{\llcorner} = \Lambda_r^{\llcorner} \cap \mathrm{Ad}_K\mathcal{D}^{\llcorner}$ (see Lemma 3.4). It follows that $\mathcal{D}^{\llcorner}$ is closed in $\mathrm{Ad}_K\mathcal{D}^{\llcorner}$, implying that $\mathrm{Ad}_{k_0}X_0$ is contained in $\mathcal{D}^{\llcorner}$ (and not just in $\mathrm{Ad}_K\mathcal{D}^{\llcorner}$). Then one has

$$f(\exp iX_n \cdot x_0) = k_n^{-1} \cdot f(\exp i(\mathrm{Ad}_{k_n}X_n) \cdot x_0) \to k_0^{-1} \cdot f(\exp i(\mathrm{Ad}_{k_0}X_0) \cdot x_0) =$$
$$= f(\exp iX_0 \cdot x_0)\,,$$

which says that f is continuous on $\exp i\mathrm{Ad}_K\mathcal{D}^{\llcorner} \cdot x_0$, as claimed.

Next, consider the following commutative diagram

$$\begin{tikzcd} G \times \mathrm{Ad}_K\mathcal{D}^{\llcorner} \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ D \arrow[r, "f"] & \Xi^+ \end{tikzcd},$$

where π is the map given by $(g, X) \to g \exp iX \cdot x_0$ and $\tilde{f}$ is the lift of f to $G \times \mathrm{Ad}_K\mathcal{D}^{\llcorner}$. As a consequence of Proposition 3.3, the map f is continuous if and only if so is

$\tilde{f}$. So let $(g_n, X_n) \to (g_0, X_0)$ be a converging sequence in $G \times \mathrm{Ad}_K \mathcal{D}^{\llcorner}$. Since f is continuous on $\exp i\mathrm{Ad}_K \mathcal{D}^{\llcorner} \cdot x_0$, one has

$$\tilde{f}(g_n, X_n) = f(g_n \exp iX_n \cdot x_0) =$$

$$g_n \cdot f(\exp iX_n \cdot x_0) \to g_0 \cdot f(\exp iX_0 \cdot x_0) = f(g_0 \exp iX_0 \cdot x_0) = \tilde{f}(g_0, X_0)\,.$$

Thus $\tilde{f}$ is continuous, implying that f is continuos. □

Lemma 4.7. (Extension lemma). *Let $\mathcal{C}$ be an open subset of $\Lambda_r^{\llcorner}$ and let $f\colon \exp i\mathcal{C}\cdot x_0 \to \widehat{D}$ be a continuous map such that $q \circ f = Id$ and $G_{\exp iX\cdot x_0} = G_{f(\exp iX\cdot x_0)}$, for every $X \in \mathcal{C}$. Assume that for every pair $X, X' \in \mathcal{C}$ on the same $W_K(\Lambda_r)$-orbit there exists $n \in N_K(\Lambda_r)$ such that*

$$X' = \mathrm{Ad}_n X \quad \text{and} \quad f(\exp iX' \cdot x_0) = n \cdot f(\exp iX \cdot x_0).$$

Then there exists a unique G-equivariant holomorphic map $\hat{f}\colon G\exp i\mathcal{C}\cdot x_0 \to \widehat{D}$ which extends f.

We point out that the domain $G\exp i\mathcal{C}\cdot x_0$ coincides with $G\exp i(W_K(\Lambda_r)\cdot\mathcal{C})\cdot x_0$.

Proof. If one such $\hat{f}$ exists, it is uniquely determined by the relation

$$\hat{f}(g\exp iX \cdot x_0) := g \cdot f(\exp iX \cdot x_0), \qquad \text{for } X \in \mathcal{C} \text{ and } g \in G.$$

First we show that $\hat{f}$ is well defined. Assume that $g'\exp iX' \cdot x_0 = g\exp iX \cdot x_0$, for some other $X' \in \mathcal{C}$ and $g' \in G$. By Proposition 3.3, there exists $k \in K$ such that

$$g' = gk^{-1} \qquad \text{and} \qquad X' = \mathrm{Ad}_k X\,.$$

In addition, by Lemma 3.4, two such elements $X,\ X' \in \Lambda_r^{\llcorner}$, lie on the same $W_K(\Lambda_r)$-orbit. Then, by the compatibility assumption, there exists $n \in N_K(\Lambda_r)$ such that

$$X' = \mathrm{Ad}_k X = \mathrm{Ad}_n X \quad \text{and} \quad f(\exp iX' \cdot x_0) = n \cdot f(\exp iX \cdot x_0).$$

In view of the above relations, we obtain

$$g' \cdot f(\exp iX' \cdot x_0) = gk^{-1}n \cdot f(\exp iX \cdot x_0).$$

Now observe that $k^{-1}n \in Z_K(X)$ and that $Z_K(X) = G_{\exp iX\cdot x_0} = G_{f(\exp iX\cdot x_0)}$, where the first identity follows from Proposition 3.3 and the second one from the assumptions. It follows that

$$g' \cdot f(\exp iX' \cdot x_0) = g \cdot f(\exp iX \cdot x_0)\,,$$

proving that $\hat{f}$ is well defined.

Next we show that $\hat{f}$ is continuous. By Proposition 3.3 and Lemma 3.4, one has $G\exp i\mathcal{C}\cdot x_0 = G\exp(iW_K(\Lambda_r)\cdot\mathcal{C})\cdot x_0$. Then, by Lemma 4.6, it is sufficient to show that $\hat{f}$ is continuous on $\exp(iW_K(\Lambda_r)\cdot\mathcal{C})\cdot x_0$, i.e. on each set $\exp(i\gamma\cdot\mathcal{C})\cdot x_0$, for γ in $W_K(\Lambda_r)$. By assumption, $\hat{f}$ is continuous on $\exp i\mathcal{C}\cdot x_0$. This settles the case when γ is the neutral element in $W_K(\Lambda_r)$. Otherwise, write $\gamma = nZ_K(\Lambda_r)$, for some $n \in N_K(\Lambda_r)$. Then by the G-equivariance of $\hat{f}$ one has

$$\hat{f}(\exp(i\gamma\cdot X)\cdot x_0) = \hat{f}(\exp i\mathrm{Ad}_n X \cdot x_0) = n \cdot \hat{f}(\exp iX \cdot x_0)\,,$$

for every $X \in \mathcal{C}$, proving that $\hat{f}$ is continuous on $\exp(i\gamma\cdot\mathcal{C})\cdot x_0$, as wished.

Finally we show that $\hat{f}$ is holomorphic. Note that $q \circ \hat{f} = Id$, since by assumption such equality holds true on $\exp i\mathcal{C}\cdot x_0$ and $\hat{f}$ is G-equivariant. Let x be an element of $G\exp i\mathcal{C}\cdot x_0$ and choose a connected open neighborhood U of $\hat{f}(x)$ such that the restriction $q|_U : U \to \hat{f}(U)$ is a biholomorphism. Then, given a neighborhood V of x such that $\hat{f}(V) \subset U$, one has $\hat{f}|_V = (q|_U)^{-1} \circ Id$, implying that $\hat{f}$ is holomorphic. □

Reduction 1. Let

$$D = G \exp i\mathcal{D} \cdot x_0 = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$$

be a G-invariant domain in Ξ^+. The first reduction reduces to the case where all connected components of $\mathcal{D}^{\llcorner}$ are convex. It consists of showing that the map f in diagram (18) has a G-equivariant holomorphic extension to a domain $G \exp i\widetilde{\mathcal{D}}^{\llcorner} \cdot x_0$, with $\widetilde{\mathcal{D}}^{\llcorner}$ a set containing $\mathcal{D}^{\llcorner}$, all of whose connected components are convex.

We need some preliminary remarks. Recall that $(\widetilde{\mathcal{D}}^{\llcorner})^+ = \widetilde{\mathcal{D}}^{\llcorner} \cap (\Lambda_r^{\llcorner})^+$ is a perfect slice for D and that it is connected (cf. (5)).

Definition 4.8. *Denote by $\mathcal{D}_\circ$ (resp. by $\mathcal{D}_\circ^{\llcorner}$ the connected component of $\mathcal{D}$ (resp. of $\mathcal{D}^{\llcorner}$) containing $(\mathcal{D}^{\llcorner})^+$.*

Note that the set $\mathcal{D}_\circ$ is open in Λ_r; the set $\mathcal{D}_\circ^{\llcorner}$ is open in $\Lambda_r^{\llcorner}$, while it need not be open in Λ_r. Both $\mathcal{D}_\circ$ and $\mathcal{D}_\circ^{\llcorner}$ need not be $W_K(\Lambda_r)$-invariant.

For $k \in \{1, \ldots, r-1\}$, denote by γ_{kk+1} the reflection flipping the k^{th} and the $(k+1)^{th}$ coordinates in $\Lambda_r^{\llcorner}$. By Lemma 3.2 such reflections generate the Weyl group $W_K(\Lambda_r)$. Denote by Γ^0 the set of those γ_{kk+1} for which there exists a non-zero element in $Fix(\gamma_{kk+1}) \cap (\mathcal{D}^{\llcorner})^+$, i.e. whose fixed point hyperplane intersects $(\mathcal{D}^{\llcorner})^+$ non-trivially. Consider the subgroup of $W_K(\Lambda_r)$

$$W^0 := \langle \{ \gamma_{kk+1} \in \Gamma^0 \} \rangle ,$$

generated by the elements of Γ^0.

Lemma 4.9. $W^0 \cdot (\mathcal{D}^{\llcorner})^+ = \mathcal{D}_\circ^{\llcorner}$.

Proof. Set $\mathcal{C} := W^0 \cdot (\mathcal{D}^{\llcorner})^+$. We first show that $\mathcal{C}$ is contained in $\mathcal{D}_\circ^{\llcorner}$. For this note that $(\mathcal{D}^{\llcorner})^+ \cap \gamma_{kk+1} \cdot (\mathcal{D}^{\llcorner})^+ \neq \emptyset$, for all $\gamma_{kk+1} \in \Gamma^0$. Thus $\gamma_{kk+1} \cdot (\mathcal{D}^{\llcorner})^+ \subset \mathcal{D}_\circ^{\llcorner}$ and γ_{kk+1} stabilizes $\mathcal{D}_\circ^{\llcorner}$. Then the whole group W^0 stabilizes $\mathcal{D}_\circ^{\llcorner}$, implying that $\mathcal{C} \subset \mathcal{D}_\circ^{\llcorner}$.

Next, we claim that for $\gamma \in W_K(\Lambda_r)$, one has that $\gamma \cdot (\mathcal{D}^{\llcorner})^+ \cap \mathcal{C} \neq \emptyset$ if and only if $\gamma \in W^0$. One implication is clear, since $\gamma \cdot (\mathcal{D}^{\llcorner})^+ \subset \mathcal{C}$ if $\gamma \in W^0$. Conversely, if $\gamma \cdot (\mathcal{D}^{\llcorner})^+ \cap \mathcal{C} \neq \emptyset$ there exists γ_1 in W^0 such that

$$\gamma_1 \gamma \cdot (\mathcal{D}^{\llcorner})^+ \cap (\mathcal{D}^{\llcorner})^+ \neq \emptyset .$$

Since $(\mathcal{D}^{\llcorner})^+$ is a fundamental region for the action of $W_K(\Lambda_r)$ on $\mathcal{D}^{\llcorner}$, it follows that there exists X in the boundary of $(\mathcal{D}^{\llcorner})^+$ such that $\gamma_1\gamma \cdot X = X$. In other words, $\gamma_1\gamma$ lies in the stabilizer subgroup $W_K(\Lambda_r)_X$ of X in $W_K(\Lambda_r)$. Since $W_K(\Lambda_r)_X$ is generated by the elements γ_{kk+1} in $\Gamma^0 \cap W_K(\Lambda_r)_X$ (see [BrTD85], Thm.4.1, p. 202), one has that $\gamma_1\gamma \in W^0$. Then $\gamma \in W^0$, as claimed.

It follows that $\mathcal{D}^{\llcorner}$ is the union of the two disjoint subsets

$$\mathcal{C} \qquad \text{and} \qquad \bigcup_{\gamma \in W_K(\Lambda_r) \setminus W^0} \gamma \cdot (\mathcal{D}^{\llcorner})^+ \, .$$

As $(\mathcal{D}^{\llcorner})^+$ is closed in $\mathcal{D}^{\llcorner}$, both subsets are closed in $\mathcal{D}^{\llcorner}$. Thus $\mathcal{C}$ must be the union of connected components of $\mathcal{D}^{\llcorner}$. Since we already showed that $\mathcal{C} \subset \mathcal{D}_\circ^{\llcorner}$, it follows that $\mathcal{C} = \mathcal{D}_\circ^{\llcorner}$, as stated. □

Proposition 4.10. (Reduction 1) *The inclusion $f \colon D \hookrightarrow \widehat{D}$ extends holomorphically and G-equivariantly to the G-invariant domain $G \exp i\mathrm{Conv}(\mathcal{D}_\circ^{\llcorner}) \cdot x_0$.*

Proof. Let $\mathcal{D}_\circ \subset \mathcal{D}$ be the connected component defined in Definition 4.8. By Lemma 3.1, the intersection $D \cap L^{\mathbb{C}} \cdot x_0$ is a closed r-dimensional L-invariant complex submanifold of D, biholomorphic, via the map ι, to the tube domain $\Lambda_r \times i\mathcal{D}$.

By Bochner's tube theorem, its envelope of holomorphy is univalent and given by $L\exp i\mathrm{Conv}(\mathcal{D}_\circ)\cdot x_0 \subset \Xi^+$. Then, by Proposition 4.1, the map f admits a holomorphic extension to an L-equivariant map

$$L\exp i\mathrm{Conv}(\mathcal{D}_\circ)\cdot x_0 \to \widehat{D}.$$

Note that the convexification $\mathrm{Conv}(\mathcal{D}_\circ)$ contains $\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)$, which is an open subset of $\Lambda_r^\llcorner$ and coincides with $\mathrm{Conv}(\mathcal{D}_\circ)\cap\Lambda_r^\llcorner$. Moreover, given $X\in\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)$ and $X'\in\mathcal{D}_\circ^\llcorner$, the one-dimensional complex manifold

$$\mathcal{S}=\{\,\exp(i(X'+\lambda(X-X')))\cdot x_0 \ :\mathrm{Re}\lambda\in[0,1]\,\}=$$
$$=\{\,\exp s(X-X')\exp(i(X'+t(X-X')))\cdot x_0 \ : s\in\mathbb{R},\ t\in[0,1]\,\}$$

is contained in $L\exp i\mathrm{Conv}(\mathcal{D}_\circ)\cdot x_0$. Then by applying Lemma 4.5, with $\mathcal{F}=\mathcal{D}_\circ^\llcorner$ and $\mathcal{C}=\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)$, we obtain that $G_{f(\exp iX\cdot x_0)}=G_{\exp iX\cdot x_0}$, for every X in $\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)$.

Next, we check that the extension of f to $\exp i\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)\cdot x_0$ satisfies the compatibility condition of Lemma 4.7. As a consequence of Lemma 4.9, the convexification $\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)$ is W^0-invariant. Denote by N^0 the preimage of W^0 in $N_K(\Lambda_r)$ under the canonical projection $\pi\colon N_K(\Lambda_r)\to W_K(\Lambda_r)$. Since both Λ_r and $\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)$ are Ad_{N^0}-invariant, the domain $L\exp i\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)\cdot x_0$ is N^0-invariant. Moreover, the map $f\colon L\exp i\mathcal{D}_\circ^\llcorner\cdot x_0\to\widehat{D}$ is N^0-equivariant and so is its extension to $L\exp i\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)\cdot x_0$, by Proposition 4.1. Hence the extension of f to $\exp i\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)\cdot x_0$ satisfies all the assumptions of Lemma 4.7 and f extends to a holomorphic, G-equivariant map

$$G\exp i\mathrm{Conv}(\mathcal{D}_\circ^\llcorner)\cdot x_0\to\widehat{D},$$

as claimed. □

Reduction 2. Given a domain $D=G\exp i\mathcal{D}^\llcorner\cdot x_0$, the second reduction consists of showing that the map $f\colon D\to\widehat{D}$ has a G-equivariant holomorphic extension to the domain $\widetilde{D}=G\exp i\widetilde{\mathcal{D}}^\llcorner\cdot x_0$, where the set $\widetilde{\mathcal{D}}^\llcorner$ is the convex envelope of $\mathcal{D}^\llcorner$.

We first need to recall some properties of the universal covering of the isometry group of the Euclidean plane. Namely, let $\widetilde{S}:=\mathbb{R}\ltimes\mathbb{R}^2$ be the semidirect product Lie group with the multiplication defined by

$$\left(t,\begin{pmatrix}a\\b\end{pmatrix}\right)\cdot\left(t',\begin{pmatrix}a'\\b'\end{pmatrix}\right):=\left(t+t',\begin{pmatrix}\cos t&-\sin t\\ \sin t&\cos t\end{pmatrix}\begin{pmatrix}a'\\b'\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}\right).$$

Its Lie algebra $\mathfrak{s}$ is isomorphic to $\mathbb{R}^3$. If $\{\widetilde{L},\,\widetilde{M},\,\widetilde{N}\}$ denotes the canonical basis of $\mathbb{R}^3$, then the Lie algebra structure is defined by

$$[\widetilde{L},\widetilde{M}]=\widetilde{N}\,,\quad[\widetilde{L},\widetilde{N}]=-\widetilde{M}\,,\quad[\widetilde{M},\widetilde{N}]=0\,.$$

In particular, $\widetilde{S}$ is a solvable Lie group.

The universal complexification of $\widetilde{S}$ is given by $\widetilde{S}^{\mathbb{C}}:=\mathbb{C}\ltimes\mathbb{C}^2$, endowed with the extended multiplication law. Consider the subgroup

$$\widetilde{H}^{\mathbb{C}}:=\left\{\left(t+is,\begin{pmatrix}0\\0\end{pmatrix}\right)\ :\ t+is\in\mathbb{C}\right\}$$

of $\widetilde{S}^{\mathbb{C}}$ with Lie algebra $\mathbb{C}\widetilde{L}$. In order to perform the second reduction we embed $\widetilde{S}$-invariant subdomains of $\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}}$ into D. We use the following facts, which can be easily verified.

Lemma 4.11.
(i) *The map* $\mathbb{C}^2\to\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}}$, *defined by* $(z,w)\to\left(0,\begin{pmatrix}z\\w\end{pmatrix}\right)\widetilde{H}^{\mathbb{C}}$, *is a biholomorphism.*

(ii) *The* $\widetilde{S}$-*invariant domains in* $\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}}$ *correspond to tube domains in* $\mathbb{C}^2$ *whose bases are annuli.*

(iii) *Any such tube domain* $\mathbb{R}^2 + i\Omega$ *is Stein if and only if the base* Ω *is convex, i.e. a disc. In particular, if* $\mathbb{R}^2 + i\Omega$ *is Stein, then* Ω *contains the origin.*
(iv) *The orbit of the base point* $e\widetilde{H}^{\mathbb{C}}$ *under the one-parameter subgroup* $\exp i\mathbb{R}\widetilde{M}$ *is a slice for the left* $\widetilde{S}$*-action on* $\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}}$. *There is a homeomorphism*

$$\widetilde{S} \setminus \widetilde{H}^{\mathbb{C}}/\widetilde{S}^{\mathbb{C}} \cong \mathbb{R}\widetilde{M}/\mathbb{Z}_2,$$

where the $\mathbb{Z}_2$*-action on* $\mathbb{R}\widetilde{M}$ *is generated by the restriction of* $\mathrm{Ad}_{\exp \pi \widetilde{L}}$ *to* $\mathbb{R}\widetilde{M}$*, namely the reflection given by* $\widetilde{M} \to -\widetilde{M}$.

The crucial step of reduction 2 deals with the case of two convex connected components of $\mathcal{D}^{\llcorner}$ symmetrically placed with respect to the fixed point set of a reflection $\gamma \in W_K(\Lambda_r) \setminus W^0$. The action of γ decomposes Λ_r into the direct sum

$$\Lambda_r = Fix(\gamma) \oplus Fix(\gamma)^{\perp}.$$

Denote by $Z_G(Fix(\gamma))$ the centralizer of $Fix(\gamma)$ in G, and by $Z_{\mathfrak{g}}(Fix(\gamma))$ its Lie algebra.

Lemma 4.12. *The Lie algebra* $Z_{\mathfrak{g}}(Fix(\gamma))$ *contains a 3-dimensional solvable subalgebra isomorphic to the Lie algebra* $\mathfrak{s} = \mathrm{Lie}(\widetilde{S})$.
(*i*) *There exists a Lie group morphism* $\psi : \widetilde{S}^{\mathbb{C}} \to G^{\mathbb{C}}$ *mapping* $\widetilde{H}^{\mathbb{C}}$ *to* $K^{\mathbb{C}}$;
(*ii*) *the group morphism* ψ *induces a closed embedding* $\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$.

Proof. (i) Recall that the restricted root system of $\mathfrak{g}$ is either of type C_r or of type BC_r (see Sect. 2). For simplicity of exposition we assume $\gamma := \gamma_{12}$, the reflection flipping the first and the second coordinates (the remaining cases can be dealt in the same way). Then $Fix(\gamma)^{\perp} = \mathbb{R}(E_1 - E_2)$ and $Fix(\gamma) = \mathrm{span}\{E_1 + E_2, E_3, \dots E_r\}$. Take an arbitrary element $Q \in \mathfrak{g}^{e_1 - e_2}$ and set

$$L := Q + \theta Q, \quad M := E_1 - E_2, \quad N := [L, M].$$

We first show that L, M, N lie in the centralizer $Z_{\mathfrak{g}}(Fix(\gamma))$. By construction, one has that

$$L \in \mathfrak{k}, \quad M \in \mathfrak{g}^{2e_1} \oplus \mathfrak{g}^{2e_2}, \quad N \in \mathfrak{g}^{e_1 + e_2}.$$

In order to see that $[L, E_1 + E_2] = 0$, let $Z_0 = \frac{1}{2}\sum_j T_j + S$, with $T_j = E_j + \theta E_j$ and $S \in Z_{\mathfrak{k}}(\mathfrak{a})$, be the central element in $\mathfrak{k}$ given in (3). Since $[L, T_j] = 0$ for $j = 3, \dots, r$, and the terms $[L, T_1 + T_2]$ and $[L, S]$ are linearly independent, the relation $[L, Z_0] = 0$ implies $[L, T_1 + T_2] = [L, S] = 0$. From $[L, T_1 + T_2] = 0$ and the identity $\theta L = L$, it follows that $[L, E_1 + E_2] + \theta[L, E_1 + E_2] = 0$. This is equivalent to $[L, E_1 + E_2] \in \mathfrak{g}^{e_1 + e_2} \cap \mathfrak{p}$ and implies $[L, E_1 + E_2] = 0$, as desired. The remaining bracket relations

$$[L, E_j] = [M, E_j] = [N, E_j] = 0, \text{ for } j \geq 3, \qquad [M, E_1 + E_2] = [N, E_1 + E_2] = 0,$$

are all straightforward.

Next we prove that the vectors $\{L, M, N\}$ generate a 3-dimensional solvable subalgebra of $\mathfrak{g}$ isomorphic to the algebra $\mathfrak{s} := Lie(\widetilde{S})$, discussed above. In order to see this, observe that $[M, N] = 0$. Then it remains to show that, by normalizing Q if necessary, one has $[L, N] = -M$. Endow the 3-dimensional subspace of $\mathfrak{g}$

$$V := \mathfrak{g}^{2e_1} \oplus \mathfrak{g}^{2e_2} \oplus \mathbb{R}N,$$

with the restriction of the Ad_K-invariant inner product of $\mathfrak{g}$, defined by $B_\theta(X, Y) := -B(X, \theta Y)$, for $X, Y \in \mathfrak{g}$. One can easily verify that the vectors $\{E_1 + E_2, M = E_1 - E_2, N = [L, M]\}$ form an orthogonal basis of V with respect to B_θ. Since ad_L is a skew-symmetric operator and $[L, E_1 + E_2] = 0$, the 2-dimensional subspace $Span\{M, N\}$ is ad_L-stable in V. Thus one can normalize Q so that $\mathrm{ad}_L(N) = -M$, as desired.

(ii) Under the identification of $\mathbb{C}^2$ with $\widetilde{S}^{\mathbb{C}}/\widetilde{H}^{\mathbb{C}}$ given in Lemma 4.11, the induced map is given by $(z,w) \to \exp(zM + wN) \cdot x_0$. Its image can be viewed as the orbit through the base point x_0 of the abelian subgroup with Lie algebra $\mathrm{span}_{\mathbb{C}}\{M,\, N\}$. Now the result follows from the injectivity of the map ι defined in (4) and Theorem 2 in [Ros61], stating that the orbits of a unipotent subgroup in the affine space $G^{\mathbb{C}}/K^{\mathbb{C}}$ are closed. $\square$

Example. As an example take $G = Sp(r, \mathbb{R})$. Fix

$$Q = \begin{pmatrix} \check{Q} & O & O & O \\ O & O & O & O \\ O & O & -\check{Q}^t & O \\ O & O & O & O \end{pmatrix} \in \mathfrak{g}^{e_1 - e_2}, \qquad \text{with} \quad \check{Q} = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}.$$

The Lie subalgebra of $\mathfrak{g}$ generated by the matrices

$$L = \begin{pmatrix} \check{L} & O & O & O \\ O & O & O & O \\ O & O & \check{L} & O \\ O & O & O & O \end{pmatrix}, \quad M = \begin{pmatrix} O & O & \check{M} & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix}, \quad N = \begin{pmatrix} O & O & \check{N} & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix},$$

where

$$\check{L} = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \check{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \check{N} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is isomorphic to $\mathfrak{s}$. The corresponding group is closed in $Sp(r, \mathbb{R})$ and given by

$$\begin{pmatrix} U & O & B & O \\ O & I_{r-2} & O & O \\ O & O & U & O \\ O & O & O & I_{r-2} \end{pmatrix}, \qquad U \in SO(2),\ B = {}^tB,\ tr(B) = 0.$$

Proposition 4.13. (Reduction 2) *Let γ be a reflection in $W_K(\Lambda_r) \setminus W^0$. The map*

$$f \colon G \exp i(\mathcal{D}_\circ^{\llcorner} \cup \gamma \cdot \mathcal{D}_\circ^{\llcorner}) \cdot x_0 \to \widehat{D}$$

has a G-equivariant, holomorphic extension to the domain

$$\widetilde{D} = G \exp i \ \mathrm{Conv}(\mathcal{D}_\circ^{\llcorner} \cup \gamma \cdot \mathcal{D}_\circ^{\llcorner}) \cdot x_0.$$

Proof. Again for simplicity of exposition we assume $\gamma = \gamma_{12}$. Then $Fix(\gamma) = \mathrm{span}\{E_1 + E_2, E_3, \ldots, E_r\}$ and $Fix(\gamma)^{\perp} = \mathbb{R}(E_1 - E_2)$. Now set $M := E_1 - E_2$ and let N and L be as in the proof of Lemma 4.12. Denote by $\mathfrak{s}$ the Lie subalgebra of $Z_{\mathfrak{g}}(Fix(\gamma))$ generated by $\{L,\ M,\ N\}$ and by S the corresponding subgroup in $Z_G(Fix(\gamma))$. Denote by $\mathfrak{m}$ the abelian subalgebra of $\mathfrak{s}$ generated by $\{M,\ N\}$, and by H the (possibly non-closed) subgroup of $Z_G(Fix(\gamma)) \cap K$ with Lie algebra $\mathbb{R}L$.

By reduction 1 we may assume that $\mathcal{D}_\circ^{\llcorner}$ is convex. Let X be an arbitrary element in $\mathcal{D}_\circ^{\llcorner}$. Then X decomposes in a unique way as

$$X = Y + Z,$$

where $Y = Y(X) \in Fix(\gamma)$ and $Z = Z(X) \in Fix(\gamma)^{\perp} = \mathbb{R}M$ depend continuously on X. For $X \in \mathcal{D}_\circ^{\llcorner}$, define

$$\Sigma_Y := \mathbb{R}M \bigcap ((\mathcal{D}_\circ^{\llcorner} \cup \gamma \cdot \mathcal{D}_\circ^{\llcorner}) - Y),$$

and

$$A_Y := \mathrm{Ad}_H \Sigma_Y.$$

Since the Adjoint action of H on $\mathfrak{m}$ is by rotations, the set A_Y is an annulus in $\mathfrak{m}$. Denote by

$$T_Y := \exp(iA_Y + \mathfrak{m}) \cdot x_0 = S \exp i\Sigma_Y \cdot x_0 \tag{9}$$

the image of the tube domain $iA_Y + \mathfrak{m}$ in $\mathfrak{m}^{\mathbb{C}} \cong \mathbb{C}^2$ under the embedding

$$\iota : \mathfrak{m}^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}, \quad W \to \exp W \cdot x_0 \tag{10}$$

(see Lemma 4.11(i) and Lemma 4.12(ii)). Note that $Y + \Sigma_Y$ is contained in $\mathcal{D}_\circ^{\llcorner} \cup \gamma \cdot \mathcal{D}_\circ^{\llcorner}$. Since $Y \in Fix(\gamma)$ and S centralizes $Fix(\gamma)$, right-translation by $\exp iY$

$$T_Y \to D, \qquad \exp W \cdot x_0 \mapsto \exp iY \, \exp W \cdot x_0, \qquad W \in iA_Y + \mathfrak{m}$$

is S-equivariant, and so is the holomorphic map

$$f_Y : T_Y \to \widehat{D}, \qquad \exp W \cdot x_0 \to f(\exp iY \exp W \cdot x_0)\,.$$

Now recall that by Bochner's tube theorem, the envelope of holomorphy of T_Y is univalent and given by

$$\widehat{T}_Y = \exp(i\mathrm{Conv}(A_Y) + \mathfrak{m}) \cdot x_0 = S \exp i\mathrm{Conv}(\Sigma_Y) \cdot x_0$$

(note that $\mathrm{Conv}(\Sigma_Y) = \mathrm{Conv}(A_Y) \cap \mathbb{R}M$). In particular, it is contained in Ξ^+. Hence, by Lemma 4.1, the map f_Y extends holomorphically and S-equivariantly to

$$\hat{f}_Y : \widehat{T}_Y \to \widehat{D}\,.$$

As X varies in $\mathcal{D}_\circ^{\llcorner}$, one obtains a family of S-equivariant holomorphic maps $\hat{f}_Y$, parametrized by Y. Set

$$\widetilde{\mathcal{D}} := \bigcup_{X \in \mathcal{D}_\circ^{\llcorner}} Y + \mathrm{Conv}(\Sigma_Y)\,.$$

Then an argument similar to the one of Lemma 7.7 (iv) in [Nee98], shows that

$$\widetilde{\mathcal{D}} = \mathrm{Conv}(\mathcal{D}_\circ^{\llcorner} \cup \gamma \cdot \mathcal{D}_\circ^{\llcorner}).$$

We define a candidate for the desired extension $\hat{f}\colon \exp i\widetilde{\mathcal{D}} \cdot x_0 \to \widehat{D}$ as follows

$$\hat{f}(\exp iX \cdot x_0) := \hat{f}_Y(\exp iZ \cdot x_0)\,. \tag{11}$$

First of all, the map $\hat{f}$ coincides with f on $\exp i(\mathcal{D}_\circ^{\llcorner} \cup \gamma \cdot \mathcal{D}_\circ^{\llcorner}) \cdot x_0$, since for $X \in \mathcal{D}_\circ^{\llcorner} \cup \gamma \cdot \mathcal{D}_\circ^{\llcorner}$ one has that $Z \in \Sigma_Y$ and

$$\hat{f}(\exp iX \cdot x_0) = \hat{f}_Y(\exp iZ \cdot x_0) = f_Y(\exp iZ \cdot x_0) =$$
$$= f(\exp iY \exp iZ \cdot x_0) = f(\exp i(Y+Z) \cdot x_0) = f(\exp iX \cdot x_0)\,.$$

In order to apply the extension Lemma 4.7 and obtain a G-equivariant holomorphic extension of f to $G \exp i\widetilde{\mathcal{D}} \cdot x_0$, we need to check that $\hat{f}$ defined in (11) meets all the necessary assumptions.

• The map $\hat{f}$ is a lift of the natural inclusion $\exp i\widetilde{\mathcal{D}} \cdot x_0 \hookrightarrow \Xi^+$.
Since $\hat{f}$ extends f, one has $q \circ \hat{f}(\exp iX \cdot x_0) = \exp iX \cdot x_0$, for all $X \in \mathcal{D}_\circ^{\llcorner} \cup \gamma \cdot \mathcal{D}_\circ^{\llcorner}$. In particular, from (9), the S-equivariance of $q \circ f_Y$ and the fact that S centralizes Y, one has

$$q \circ f_Y(\exp iZ \cdot x_0) = \exp iY \exp iZ \cdot x_0\,, \quad \text{for all } Z \in \Sigma_Y.$$

By applying the analytic continuation principle to each $q \circ \hat{f}_Y : \widehat{T}_Y \to G^{\mathbb{C}}/K^{\mathbb{C}}$, one obtains

$$q \circ \hat{f}(\exp iX \cdot x_0) = q \circ \hat{f}_Y(\exp iZ \cdot x_0) = \exp iY \exp iZ \cdot x_0 = \exp iX \cdot x_0$$

for all $X \in \widetilde{\mathcal{D}}$.

• The map $\hat{f}$ is continuous.
The Stein Riemann domain $\widehat{D}$ admits a holomorphic embedding in some $\mathbb{C}^N$. Then, in order to prove that $\hat{f}$ is continuous, it is sufficient to show that given an arbitrary

holomorphic function $F : \widehat{D} \to \mathbb{C}$, the composition $F \circ \hat{f} : \exp i\widetilde{\mathcal{D}} \cdot x_0 \to \mathbb{C}$ is continuous. Since the map ι in (10) is an embedding, this is equivalent to checking that the map

$$F \circ \hat{f} \circ \iota|_{i\widetilde{\mathcal{D}}} : i\widetilde{\mathcal{D}} \to \mathbb{C}, \qquad iX \to F \circ \hat{f}(\exp iX \cdot x_0)$$

is continuous.

Choose an open set U in $Fix(\gamma)$ and an open γ-invariant subset Σ in $\mathbb{R}M = Fix(\gamma)^{\perp}$ such that $U + \Sigma \subset \mathcal{D}_{\circ}^{\llcorner} \cup \gamma \cdot \mathcal{D}_{\circ}^{\llcorner}$. By the definition of Σ, for $Y \in U$ the functions f_Y are all defined on the tube domain $T_\Sigma = S \exp i\Sigma \cdot x_0$. Moreover, the map

$$U \to \mathcal{O}(T_\Sigma, \mathbb{C}), \quad Y \to F \circ f_Y|_{T_\Sigma}$$

is continuous with respect to the compact-open topology on the Fréchet algebra $\mathcal{O}(T_\Sigma, \mathbb{C})$ of holomorphic functions on T_Σ. Indeed, for $W \in iA_Y + \mathfrak{m}$ and $Y \in U$, one has

$$F \circ f_Y(\exp W \cdot x_0) = F \circ f(\exp iY \exp W \cdot x_0) = F \circ f(\exp(iY + W) \cdot x_0) .$$

Thus, if $Y_n \to Y_0$, then $F \circ f_{Y_n} \to F \circ f_{Y_0}$ uniformly on compact subsets of T_Σ. Since the extension map $\mathcal{O}(T_\Sigma, \mathbb{C}) \to \mathcal{O}(\widehat{T}_\Sigma, \mathbb{C})$ is continuous (see cap. I in [Gun90]), it follows that also the map

$$U \to \mathcal{O}(\widehat{T}_\Sigma, \mathbb{C}), \quad Y \to F \circ \hat{f}_Y|_{\widehat{T}_\Sigma}$$

is continuous with respect to the compact-open topology on $\mathcal{O}(\widehat{T}_\Sigma, \mathbb{C})$. As we already remarked, one has $\widehat{T}_\Sigma = S \exp i\mathrm{Conv}(\Sigma) \cdot x_0$. As a consequence, the map

$$F \circ \hat{f} \circ \iota|_{i(U + \mathrm{Conv}(\Sigma))} : i(U + \mathrm{Conv}(\Sigma)) \to \mathbb{C} ,$$

defined by

$$iX \to F \circ \hat{f}(\exp iX \cdot x_0) = F \circ \hat{f}_Y(\exp iZ \cdot x_0)$$

is continuous. Since the domains of the form $i(U + \mathrm{Conv}(\Sigma))$ cover $i\widetilde{\mathcal{D}}$, the map $\hat{f}$ is continuous.

• For all $X \in \widetilde{\mathcal{D}}$, one has $G_{\hat{f}(\exp iX \cdot x_0)} = G_{\exp iX \cdot x_0}$.
We apply Lemma 4.5, with $\mathcal{C} = \widetilde{\mathcal{D}}$ and $\mathcal{F} = \mathcal{D}_{\circ}^{\llcorner} \cup \gamma \cdot \mathcal{D}_{\circ}^{\llcorner}$. In order to check condition (ii) of the lemma, let $X = Y + Z \in Y + \mathrm{Conv}(\Sigma_Y)$ be an arbitrary element of $\mathcal{C} \setminus \mathcal{F}$. Then there exists $Z' \in \Sigma_Y$ such that $X' = Y + Z' \in Y + \Sigma_Y \subset \mathcal{F}$ and the one dimensional complex submanifold in (ii) of Lemma 4.5 is given by

$$\begin{aligned} \mathcal{S} &:= \{ \exp i(X' + \lambda(X - X')) \cdot x_0 \ : \mathrm{Re}\lambda \in [0, 1] \} = \\ &= \{ \exp i(Y + Z' + \lambda(Z - Z')) \cdot x_0 \ : \mathrm{Re}\lambda \in [0, 1] \} . \end{aligned}$$

Note that $Z - Z'$ belongs to $\mathbb{R}M$ and that the strip

$$\{ i(Z' + \lambda(Z - Z'))) \cdot x_0 \ : \mathrm{Re}\lambda \in [0, 1] \}$$

is contained in $i\mathrm{Conv}(\Sigma_Y) + \mathfrak{m}$. Thus $\exp i(Z' + \lambda(Z - Z')) \cdot x_0 \in T_Y$ and one has a natural holomorphic extension of f to the one dimensional complex submanifold $\mathcal{S}$, namely

$$\hat{f}(\exp i(Y + Z' + \lambda(Z - Z')) \cdot x_0) = \hat{f}_Y(\exp i(Z' + \lambda(Z - Z')) \cdot x_0) .$$

This shows that we can apply Lemma 4.5, as claimed.

• The map $\hat{f}$ satisfies the compatibility condition.
Let $k_\gamma \in H$ be the element inducing the reflection with respect to the origin in $\mathbb{R}M$. Since H centralizes $Fix(\gamma)$, the element k_γ belongs to $N_K(\Lambda_r^{\llcorner})$ and induces the reflection γ given in the statement. Hence, for every $X \in \widetilde{\mathcal{D}}$ one has $\gamma \cdot X = \mathrm{Ad}_{k_\gamma} X$. Moreover, by the H-equivariance of the maps $\hat{f}_Y$, one obtains the identity

$$\hat{f}(\exp(i\gamma \cdot X) \cdot x_0) = \hat{f}(\exp i(Y + \gamma \cdot Z) \cdot x_0) = \hat{f}_Y(\exp i\mathrm{Ad}_{k_\gamma} Z \cdot x_0) =$$

$$= \hat{f}_Y(k_\gamma \exp iZ \cdot x_0) = k_\gamma \cdot \hat{f}_Y(\exp iZ \cdot x_0) = k_\gamma \cdot \hat{f}(\exp iX \cdot x_0)\,,$$

which is the desired compatibility condition.

In conclusion, since the map $\hat{f}$ defined in (11) meets all the assumptions of Lemma 4.7, it extends to a G-equivariant, holomorphic map

$$\hat{f}\colon G \exp i\mathrm{Conv}(\mathcal{D}^{\llcorner}_{\circ} \cup \gamma \cdot \mathcal{D}^{\llcorner}_{\circ}) \cdot x_0 \to \widehat{D}\,,$$

as claimed. □

Corollary 4.14. *By iterating the above reduction 2 finitely many times, we obtain a G-equivariant holomorphic extension of $f\colon D = G \exp i\mathcal{D}^{\llcorner} \cdot x_0 \to \widehat{D}$ to*

$$\hat{f}\colon G \exp i\mathrm{Conv}(\mathcal{D}^{\llcorner}) \cdot x_0 \to \widehat{D}.$$

5. The main theorem

Let D be a G-invariant domain in Ξ^+. Assume that D is not entirely contained in the crown Ξ nor in the domain S^+ (in the tube case). In this section we prove our main theorem, namely that the envelope of holomorphy $\widehat{D}$ of D is univalent and coincides with Ξ^+ (Thm. 5.1). As a by-product we obtain that every Stein G-invariant subdomain of Ξ^+ is either contained in Ξ or, in the tube case, in S^+ (Thm. 5.2).

Theorem 5.1. *Let G/K be an irreducible Hermitian symmetric space. Given a G-invariant domain D in Ξ^+, denote by $\widehat{D}$ its envelope of holomorphy.*
(i) *Assume G/K is of tube type. If D is not contained in Ξ nor in S^+, then $\widehat{D}$ is univalent and coincides with Ξ^+.*
(ii) *Assume G/K is not of tube type. If D is not contained in Ξ, then $\widehat{D}$ is univalent and coincides with Ξ^+.*

Proof. The proof of the theorem consists of a sequence of rank-one reductions and convexifications (reduction 1), until an extension of the lift $f\colon \exp i\mathcal{D}^{\llcorner} \cdot x_0 \to \widehat{D}$ to the whole $\exp i\Lambda_r^{\llcorner} \cdot x_0$ is obtained. Such an extension is constructed so that it satisfies the assumptions of Lemma 4.7 and yields a G-equivariant holomorphic extension of the map $f\colon D \to \widehat{D}$ to the whole Ξ^+. Then the theorem follows from (ii) of Proposition 4.2.

We need to distinguish several cases.

Case 1. We first consider a domain $D = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$ in Ξ^+ satisfying the condition

$$\mathcal{D}^{\llcorner} \bigcap \Lambda_r^{\llcorner} \setminus \left(\bigoplus_{j=1}^{r} [0,1)E_j \bigcup \bigoplus_{j=1}^{r} (1,\infty)E_j \right) \neq \emptyset.$$

The above condition implies that D is not contained in the crown Ξ nor, in the tube case, in the domain S^+. By reductions 1 and 2, we can assume that $\mathcal{D}^{\llcorner}$ is a $W_K(\Lambda_r)$-invariant, open convex subset of $\Lambda_r^{\llcorner}$. We claim that $\mathcal{D}^{\llcorner}$ contains a point X with exactly one coordinate equal to 1, and the other ones either all < 1 (Case 1.a) or all > 1 (Case 1.b). This follows from the fact that an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$ intersects at least one of the open sets

$$\bigoplus_j [0,1)E_j \quad \text{or} \quad \bigoplus_j (1,+\infty)E_j.$$

In particular, it contains an open piece of its boundary. Since the points with exactly one coordinate equal to 1 form an open dense subset of the boundary of each set, the claim follows.

For the rank-one reduction, denote by G_j the rank-one subgroup of G associated to the root $\lambda_j \in \{\lambda_1, \dots, \lambda_r\}$, and by K_j the intersection $G_j \cap K$ (see Sect.2). The quotient G_j/K_j is a rank-one Hermitian symmetric space of tube-type. The envelope of holomorphy of an invariant domain in $G_j^{\mathbb{C}}/K_j^{\mathbb{C}}$ is univalent and described by Theorem 4.3.

Case 1.a. By the $W_K(\Lambda_r)$-invariance of $\mathcal{D}^{\llcorner}$, we can assume that $(\mathcal{D}^{\llcorner})^+$ contains a point

$$X = (1, x_2, \dots, x_r), \qquad \text{with} \quad 1 > x_2 > \dots x_r > 0. \tag{12}$$

Our first goal is to obtain an extension of f to a set $\exp i\widetilde{\mathcal{D}} \cdot x_0$, where $\widetilde{\mathcal{D}}$ is an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$ containing $\mathcal{D}^{\llcorner}$ and the point $(1, 0, \dots, 0)$. This requires a number of steps aimed at gradually extending f to $W_K(\Lambda_r)$-invariant larger sets $\exp i\mathcal{C} \cdot x_0$, with $\mathcal{C}$ containing $\mathcal{D}^{\llcorner}$ and, in order, the points

$$(1, x_2, \dots, x_{r-2}, x_{r-1}, 0), \quad (1, x_2, \dots, x_{r-2}, 0, 0), \ \dots \ , \ (1, 0, \dots, 0).$$

Denote by

$$\operatorname{int}(\overline{(\Lambda_r^{\llcorner})^+})$$

the interior of $\overline{(\Lambda_r^{\llcorner})^+}$ (in the relative topology), which coincides with $\overline{(\Lambda_r^{\llcorner})^+} \setminus \mathcal{H}$, where $\mathcal{H} := \cup_{\gamma \in W_K(\Lambda_r)}\{Fix(\gamma)\}$ denotes the set of reflection hyperplanes in $\Lambda_r^{\llcorner}$. Likewise, denote by

$$\operatorname{int}((\mathcal{D}^{\llcorner})^+) = (\mathcal{D}^{\llcorner})^+ \cap \operatorname{int}(\overline{(\Lambda_r^{\llcorner})^+})$$

the interior of $(\mathcal{D}^{\llcorner})^+$ in the relative topology. Under the above assumption (12), the interior of $(\mathcal{D}^{\llcorner})^+$ contains an open set of the form

$$U + V, \quad \text{with} \quad (1, x_2, \dots x_{r-1}, 0) \in U \subset E_r^{\perp}, \quad \text{and} \quad V = (a_r, b_r)E_r.$$

Write an arbitrary element $W \in U + V$ as

$$W = Y + Z, \quad \text{with} \quad Y = Y(W) \in U \text{ and } Z = Z(W) \in V$$

depending continuously on W. Define

$$D_r := G_r \exp iV \cdot x_0.$$

Since G_r commutes with the subgroups G_j, for $j \neq r$, right translation by $\exp iY$

$$D_r \to D, \quad g \exp iZ \cdot x_0 \mapsto \exp iY g \exp iZ \cdot x_0$$

is G_r-equivariant and so is the holomorphic map

$$f_Y \colon D_r \to \widehat{D}, \quad g \exp iZ \cdot x_0 \to f(\exp iY g \exp iZ \cdot x_0).$$

Recall that the envelope of holomorphy of D_r is univalent and given by $\widehat{D}_r = G_r \exp i\widehat{V} \cdot x_0$, with $\widehat{V} = [0, b_r)E_r$ (see Prop. 4.3(i)). As W varies in $U + V$, one obtains a family of G_r-equivariant holomorphic maps

$$\hat{f}_Y \colon \widehat{D}_r \longrightarrow \widehat{D},$$

parametrized by $Y = Y(W) \in U$.

Define a map

$$\hat{f} \colon \exp i(U + \widehat{V}) \cdot x_0 \to \widehat{D}, \quad \text{by} \quad \exp iW \cdot x_0 \to \hat{f}_Y(\exp iZ \cdot x_0).$$

Observe that $U + \widehat{V}$ is an open set in $\Lambda_r^{\llcorner}$, since it is entirely contained in the interior of $\overline{(\Lambda_r^{\llcorner})^+}$. Next we show that $\hat{f}$ satisfies all the assumptions of the extension Lemma 4.7.

- $\hat{f}$ concides with f on the set $\exp i(U + V) \cdot x_0$, since for every $Y \in U$,

$$\hat{f}_Y(\exp iZ \cdot x_0) = f_Y(\exp iZ \cdot x_0), \quad \text{for all} \quad Z \in V.$$

- $\hat{f}$ is a lift of the natural inclusion of $\exp i(U+\widehat{V})\cdot x_0$ into Ξ^+. The analytic continuation principle, applied to each holomorphic map $q\circ \hat{f}_Y\colon \widehat{D}_r\to\Xi^+$, implies

$$q\circ \hat{f}_Y|_{\exp i\widehat{V}\cdot x_0} = Id|_{\exp i\widehat{V}\cdot x_0}, \qquad W\in U+\widehat{V}. \tag{13}$$

- $\hat{f}$ is continuous, by similar arguments as the ones used in Lemma 4.11.
- for every $X\in U+\widehat{V}$, one has $G_{\exp iX\cdot x_0} = G_{\hat{f}(\exp iX\cdot x_0)}$. This follows from Lemma 4.5, by taking $\mathcal{C}=U+\widehat{V}$ and $\mathcal{F}=U+\widehat{V}$, and arguing as in Lemma 4.11.
- since $U+\widehat{V}$ is entirely contained in the perfect slice $\overline{(\Lambda_r^{\llcorner})^+}$, the compatibility conditions of Lemma 4.7 are automatically satisfied.

As a consequence $\hat{f}\colon \exp i(U+\widehat{V})\cdot x_0\to\widehat{D}$ extends to a G-equivariant holomorphic map

$$\hat{f}\colon G\exp i(U+\widehat{V})\cdot x_0\to\widehat{D}. \tag{14}$$

Note that $G\exp i(U+\widehat{V})\cdot x_0$ is an open set in Ξ^+, which has open intersection with D and coincides with $G\exp i\left(W_K(\Lambda_r)\cdot(U+\widehat{V})\right)\cdot x_0$. By the analytic continuation principle, the map (14) coincides with $f\colon D\to\widehat{D}$ on the points of D. As a result, we have obtained a G-equivariant holomorphic extension of f to the larger domain $G\exp i\widetilde{\mathcal{D}}\cdot x_0$, where

$$\widetilde{\mathcal{D}} = \mathcal{D}^{\llcorner}\bigcup W_K(\Lambda_r)\cdot(U+\widehat{V}).$$

The set $\widetilde{\mathcal{D}}$ contains the point $(1,x_2,\dots x_{r-1},0)$, the projection of the initial point X onto the hyperplane $x_r=0$, and by reduction 1, may be assumed to be convex.

So set $\mathcal{D}^{\llcorner}=\widetilde{\mathcal{D}}$ and, for the second step, take an open subset of $\operatorname{int}((\mathcal{D}^{\llcorner})^+)$ of the form $U+V$, with

$$(1,x_2,\dots x_{r-2},0,0)\in U\subset E_{r-1}^{\perp}, \quad \text{and} \quad V=(a_{r-1},b_{r-1})E_{r-1}.$$

Write an arbitrary element $W\in U+V$ as

$$W=Y+Z, \quad \text{with} \quad Y=Y(W)\in U \quad \text{and} \quad Z=Z(W)\in V$$

depending continuously on W.

Define $D_{r-1}:=G_{r-1}\exp iV\cdot x_0$. The map

$$D_{r-1}\to D, \qquad g\exp iZ\cdot x_0\to \exp iY\, g\exp iZ\cdot x_0$$

defines a G_{r-1}-equivariant holomorphic embedding of D_{r-1} into D and induces a G_{r-1}-equivariant holomorphic map $\widehat{D}_{r-1}\longrightarrow\widehat{D}$. As in the previous step, consider the family of G_{r-1}-equivariant holomorphic maps

$$\hat{f}_Y\colon \widehat{D}_{r-1}\longrightarrow\widehat{D},$$

obtained by letting W vary in $U+V$, and define a map

$$\hat{f}\colon\ \exp i(U+\widehat{V})\cdot x_0\to\widehat{D}, \quad \text{by} \quad \hat{f}(\exp iW\cdot x_0):=\hat{f}_Y(\exp iZ\cdot x_0).$$

Recall that the envelope of holomorphy of D_{r-1} is univalent and given by $\widehat{D}_{r-1} = G_{r-1}\exp i\widehat{V}\cdot x_0$, where $\widehat{V}:=[0,b_{r-1})E_{r-1}$ (see Prop. 4.3(i)). Since this time $U+\widehat{V}$ is not entirely contained in the interior of $\overline{(\Lambda_r^{\llcorner})^+}$, we restrict $\hat{f}$ to the set $\exp i\mathcal{C}\cdot x_0$, where

$$\mathcal{C}=(U+\widehat{V})\bigcap \operatorname{int}(\overline{(\Lambda_r^{\llcorner})^+}). \tag{15}$$

The same arguments used in the previous step show that the restricted map $\hat{f}|_{\exp i\mathcal{C}\cdot x_0}$, coincides with f on $D\cap\exp i\mathcal{C}\cdot x_0$, it is a lift of the natural inclusion of $\exp i\mathcal{C}\cdot x_0$ into Ξ^+, and it is continuous.

Moreover, because of (15), the compatibility conditions of Lemma 4.7 are automatically satisfied on $\exp i\mathcal{C}\cdot x_0$. As a consequence, $\hat{f}|_{\exp i\mathcal{C}\cdot x_0}$ extends to a G-equivariant holomorphic map

$$\hat{f}\colon G\cdot\exp i\mathcal{C}\cdot x_0\to\widehat{D}.$$

By the analytic continuation principle, the above map coincides with $f\colon D \to \widehat{D}$ on the points of D. As a result, we have obtained an extension of f to the domain $G \exp i\widetilde{\mathcal{D}} \cdot x_0$, where

$$\widetilde{\mathcal{D}} = \mathcal{D}^{\llcorner} \bigcup W_K(\Lambda_r) \cdot (U + \widehat{V}).$$

The above set contains the point $(1, x_2, \ldots x_{r-2}, 0, 0)$, the projection of the initial point X onto the linear subspace $x_r = x_{r-1} = 0$, and, by reduction 1, it may be assumed to be convex. By applying the procedure, used for the coordinate x_{r-1}, to the coordinates $x_{r-2}, \ldots x_2$, we obtain an extension of f to a domain $G \exp i\widetilde{\mathcal{D}} \cdot x_0$, where $\widetilde{\mathcal{D}}$ is an open, $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$ containing $\mathcal{D}^{\llcorner}$ and the point $(1, 0, \ldots, 0)$.

Set $\mathcal{D}^{\llcorner} = \widetilde{\mathcal{D}}$ and, for the final step, take an open subset of $\mathrm{int}((\mathcal{D}^{\llcorner})^+)$ of the form $U + V$, with

$$U \subset E_1^{\perp}, \text{ and } \quad V = (a_1, b_1)E_1, \quad a_1 < 1 < b_1.$$

This time $D_1 = G_1 \exp iV \cdot x_0$ is a G_1-invariant complex submanifold $G^{\mathbb{C}}/K^{\mathbb{C}}$ whose envelope of holomorphy is univalent and given by $\widehat{D}_1 = G_1 \exp i\widehat{V} \cdot x_0$, with $\widehat{V} := [0, \infty)E_1$ (see Prop. 4.3(iii)).

The usual procedure yields a G-equivariant extension of $f\colon D \to \widehat{D}$ to the whole $\Xi^+ \setminus G/K$. Since the orbit G/K is a totally real submanifold of Ξ^+ (of maximal dimension), f extends to the whole Ξ^+ (see [Fie82]), as desired.

Case 1.b. By the $W_K(\Lambda_r)$-invariance of $\mathcal{D}^{\llcorner}$, we can assume that $(\mathcal{D}^{\llcorner})^+$ contains a point

$$X = (x_1, x_2, \ldots, 1), \qquad \text{with} \quad x_1 > x_2 > \ldots x_{r-1} > 1. \tag{16}$$

Our goal is to contruct an extension of f to a set $\exp i\widetilde{\mathcal{D}} \cdot x_0$, where $\widetilde{\mathcal{D}}$ is an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$ containing $\mathcal{D}^{\llcorner}$ and the point $(1, 0, \ldots, 0)$. Then the result follows from the last step in Case 1.a.

By the above assumption (16), the interior of $(\mathcal{D}^{\llcorner})^+$ contains an open set of the form

$$U + V, \quad \text{with} \quad (x_1, x_2, \ldots x_{r-1}, 0) \in U \subset E_r^{\perp}, \quad \text{and} \quad V = (a_r, b_r)E_r,\ a_r < 1 < b_r.$$

Consider $D_r = G_r \exp iV \cdot x_0$ and recall that the envelope of holomorphy of D_r is univalent and given by $\widehat{D}_r = G_r \exp i\widehat{V} \cdot x_0$, with $\widehat{V} = [0, \infty)E_r$ (see Prop. 4.3(iii)). The usual procedure yields an extension of f to a set $G \exp i\widetilde{\mathcal{D}} \cdot x_0$, where $\widetilde{\mathcal{D}}$ is an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$, containing $(x_1, x_2, \ldots x_{r-1}, 0)$.

We claim that $\widetilde{\mathcal{D}}$ can be assumed to contain $(x_1, \ldots, x_{r-2}, 1, 0)$. Since $\widetilde{\mathcal{D}}$ is $W_K(\Lambda_r)$-invariant and convex, it contains the point $(x_1, x_2, \ldots, 0, x_{r-1})$ and the segment

$$(x_1, x_2, \ldots, t x_{r-1}, (1-t)x_{r-1}), \quad \text{for } t \in [0, 1].$$

In particular it contains the point $(x_1, x_2, \ldots, 2/3\, x_{r-1}, 1/3\, x_{r-1})$, which lies in $(\mathcal{D}^{\llcorner})^+$ and has a smaller $(r-1)^{th}$ coordinate than $(x_1, x_2, \ldots x_{r-1}, 0)$. By iterating the procedure, we obtain a convex set $\widetilde{\mathcal{D}}$ containing $(x_1, x_2, \ldots, x'_{r-1}, 0)$, for some $x'_{r-1} < 1$. Then the claim follows from the convexity of $\widetilde{\mathcal{D}}$ and the inequality $x_{r-1} > 1$.

Acting in the same way on the coordinates $x_{r-1}, x_{r-2}, \ldots, x_1$ we obtain a convex set $\mathcal{D}^{\llcorner}$ containing $(1, 0, \ldots, 0)$, as desired.

Case 2. It remains to consider the case of G/K not of tube-type and $D = G \exp i\mathcal{D}^{\llcorner} \cdot x_0$ a domain entirely contained in Ω^+. By Proposition 3.6, this is equivalent to requiring

$$\mathcal{D}^{\llcorner} \subset \bigoplus_{j=1}^{r} (1, \infty)E_j. \tag{17}$$

By the $W_K(\Lambda_r)$-invariance of $\mathcal{D}^{\llcorner}$ and (5), we can assume that $(\mathcal{D}^{\llcorner})^+$ contains a point

$$X = (x_1, x_2, \dots x_r), \qquad \text{with} \quad x_1 > x_2 > \dots > x_r > 1.$$

Our goal is to show that the map f extends to a set $\exp i\widetilde{\mathcal{D}} \cdot x_0$, where $\widetilde{\mathcal{D}}$ is an open $W_K(\Lambda_r)$-invariant convex set in $\Lambda_r^{\llcorner}$ containing $\mathcal{D}^{\llcorner}$ and the point $E_1 = (1, 0, \dots, 0)$. Then the result follows from the last step in Case 1.a.

By the above assumption (17), the interior of $(\mathcal{D}^{\llcorner})^+$ contains an open set of the form

$$U + V, \quad \text{with} \quad U \subset E_r^{\perp},\ (x_1, x_2, \dots x_{r-1}, 0) \in U, \quad V = (a_r, b_r)E_r, \quad a_r > 1.$$

Since we are in the non-tube case, we can consider the θ-stable real rank-one subgroup $G_r^\bullet$ of G associated to the root λ_r (see Sect. 2). Then the intersection $K_r^\bullet := G_r^\bullet \cap K$ is a maximal subgroup of $G_r^\bullet$ and the quotient $G_j^\bullet/K_j^\bullet$ is a rank-one Hermitian symmetric space, not of tube-type.

Define $D_r := G_r^\bullet \exp iV \cdot x_0$. Since $G_r^\bullet$ commutes with the rank-one subgroups G_i associated to the roots λ_j, for $j \neq r$ (Lemma 2.1), the map

$$D_r \to D, \quad g \exp iZ \cdot x_0 \longrightarrow \exp iY g \exp iZ \cdot x_0$$

defines a $G_r^\bullet$-equivariant holomorphic embedding and induces a $G_r^\bullet$-equivariant holomorphic map $\widehat{D}_r \longrightarrow \widehat{D}$. Recall that the envelope of holomorphy of D_r is univalent and given by $\widehat{D}_r = G_r^\bullet \exp i\widehat{V} \cdot x_0$, with $\widehat{V} = [0, \infty)E_r$ (see Prop. 4.4(ii)). The same procedure used in the previous cases, yields a G-equivariant holomorphic map

$$\hat{f} \colon G \exp i\mathcal{C} \cdot x_0 \to \widehat{D},$$

where $\mathcal{C}$ is an open convex $W_K(\Lambda_r)$-invariant subset of $\Lambda_r^{\llcorner}$ containing $(x_1, \dots, x_{r-1}, 0)$. By applying the same arguments to the remaining variables $x_{r-1}, x_{r-2}, \dots, x_2$, we achieve the desired extension. This concludes the proof of the theorem. $\square$

Stein G-invariant domains in Ξ and S^+ were classified in [GiKr02] and [Nee99], respectively. Inside the crown Ξ, as well as inside S^+, an invariant domain can be described via a semisimple abelian slice, and Steiness is characterized by logarithmic convexity of such a slice. These results together with the above theorem conclude the classification of Stein G-invariant domains in Ξ^+.

Corollary 5.2. *Let G/K be a Hermitian symmetric space and let D be a Stein G-invariant proper subdomain of Ξ^+.*
(i) *If G/K is of tube type, then either $D \subseteq \Xi$ or $D \subseteq S^+$.*
(ii) *If G/K is not of tube type, then $D \subseteq \Xi$.*

Remark 5.3. Let G/K be an arbitrary irreducible, non-compact, Riemannian symmetric space. The crown domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ is given by $\Xi = G \exp i\Omega_{AG} \cdot x_0$, where $\Omega_{AG} := \{H \in \mathfrak{a} \ : \ |\alpha(H)| < \frac{\pi}{2}, \ \text{ for all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{a})\}$. Invariant domains in Ξ can be written as $D = G \exp i\Omega \cdot x_0$, for some $W_K(\mathfrak{a})$-invariant open set $\Omega \subset \Omega_{AG}$. Stein invariant domains have been characterized in [GiKr02] as the ones for which the slice $\exp i\Omega \cdot x_0$ is logarithmically convex. However, we are not aware of an explicit univalence statement for the envelope of holomorphy of an arbitrary invariant domain $D \subset \Xi$. For the sake of completeness, we outline the proof of this fact here.

Let $D = G \exp i\Omega \cdot x_0$ be an invariant domain in Ξ. As Ξ is Stein, one has a commutative diagram

$$\begin{tikzcd} & \widehat{D} \arrow[d, "q"] \\ D \arrow[ur, "f"] \arrow[r, "Id"'] & \Xi \end{tikzcd} \tag{18}$$

Define $\Omega^+ := \Omega \cap \overline{\mathfrak{a}^+}$, where $\overline{\mathfrak{a}^+}$ is a closed Weyl chamber in $\mathfrak{a}$, and let $\Omega_\circ$ be the connected component of Ω containing Ω^+. Consider the set Γ^0 of simple reflections in $\mathfrak{a}$ whose fixed point hyperplanes contain a non-zero element of $\overline{\mathfrak{a}^+}$ and let W^0 be the subgroup of $W_K(\mathfrak{a})$ generated by Γ^0. As in Lemma 4.9, one can show that $W^0 \cdot \Omega^+ = \Omega_\circ$.

Set $A := \exp \mathfrak{a}$ and consider the r-dimensional complex submanifold $A \exp i\Omega_\circ \cdot x_0$ of D, which is biholomorphic to a tube domain in $\mathbb{C}^r$ with base $\Omega_\circ$. The restriction $f|_{A \exp i\Omega_\circ \cdot x_0} : A \exp i\Omega_\circ \cdot x_0 \to \widehat{D}$ of f to $A \exp i\Omega_\circ \cdot x_0$ extends to an A-equivariant holomorphic map $A \exp i\mathrm{Conv}(\Omega_\circ) \cdot x_0 \to \widehat{D}$. Then the same arguments as in Proposition 4.10 show that the inclusion $f \colon D \to \widehat{D}$ admits a G-equivariant extension to the domain $G \exp i\mathrm{Conv}(\Omega_\circ) \cdot x_0$. Thus, without loss of generality, we may assume that all connected components of Ω are convex.

The second part of the proof consists of showing that the map f admits a G-equivariant holomorphic extension to the domain $G \exp i\mathrm{Conv}(\Omega) \cdot x_0$. For this purpose, we first consider the case where Ω consists of two connected components $\Omega_\circ$ and $s_\alpha \cdot \Omega_\circ$, simmetrically placed with respect to the fixed point hyperplane $Fix(s_\alpha)$ of a reflection $s_\alpha \in W_K(\mathfrak{a}) \setminus W^0$. Let H_α be a generator of $Fix(s_\alpha)^\perp$. Choose $X_\alpha \in \mathfrak{g}^\alpha$ so that the vectors $\{X_\alpha, \theta X_\alpha, H_\alpha\}$ generate a θ-stable $\mathfrak{sl}(2)$-subalgebra. Denote by G_α the corresponding subgroup of G and by $K_\alpha = G_\alpha \cap K$. The quotient G_α/K_α is a Hermitian rank-one symmetric space of tube type.

Let X be an arbitrary element in $\Omega_\circ$. Then X decomposes in a unique way as $X = Y + Z$, where $Y = Y(X) \in Fix(s_\alpha)$ and $Z = Z(X) \in \mathbb{R}H_\alpha$ depend continuosly on X. Define

$$\Sigma_Y := \mathbb{R}H_\alpha \cap ((\Omega_\circ \cup s_\alpha \cdot \Omega_\circ) - Y) \quad \text{and} \quad D_Y = G_\alpha \exp i\Sigma_Y \cdot x_0.$$

Then D_Y is biholomorphic to a G_α-invariant domain inside the crown $\Xi_\alpha \subset G_\alpha^{\mathbb{C}}/K_\alpha^{\mathbb{C}}$. By Proposition 4.3, the envelope of holomorphy of D_Y is univalent and it is given by $\widehat{D}_Y = G_\alpha \exp i\mathrm{Conv}(\Sigma_Y) \cdot x_0$.

Note that $Y + \Sigma_Y \subset \Omega_\circ \cup s_\alpha \cdot \Omega_\circ$, and that $\alpha(Y) = 0$, for all $Y \in Fix(s_\alpha)$. It follows that $\exp iY$ commutes with G_α and the map

$$D_Y \to D, \qquad g \exp iZ \cdot x_0 \to \exp iY g \exp iZ \cdot x_0$$

is a G_α-equivariant embedding.

As X varies in $\Omega_\circ$, one obtains a family of G_α-equivariant holomorphic maps $f_Y \colon D_Y \to \widehat{D}$, defined by $g \exp iZ \cdot x_0 \mapsto f(\exp iY g \exp iZ \cdot x_0)$, equivariantly extending to $\hat{f}_Y \colon \widehat{D}_Y \to \widehat{D}$ (cf. Lemma 4.1).

Define $\widetilde{\Omega} := \bigcup_{X \in \Omega_\circ} Y + \mathrm{Conv}(\Sigma_Y)$ and

$$\hat{f} : \exp i\widetilde{\Omega} \cdot x_0 \to \widehat{D}, \quad \exp iX \cdot x_0 \to \hat{f}_Y(\exp iZ \cdot x_0).$$

By Lemma 7.7 in [Nee98], the set $\widetilde{\Omega}$ coincides with $\mathrm{Conv}(\Omega_\circ \cup s_\alpha \Omega_\circ)$. Arguments analogous to the ones used in the proof of Proposition 4.13 show that $\hat{f}$ is a continuos extension of the lift $f|_{\exp i\Omega \cdot x_0} \colon \exp i\Omega \cdot x_0 \to \widehat{D}$ and that it satisfies the assumptions of Lemma 4.7. As a consequence, $\hat{f}$ further extends to a G-equivariant holomorphic map $\hat{f} \colon G \exp i\widetilde{\Omega} \cdot x_0 \to \widehat{D}$. By iterating the above procedure if necessary, one eventually obtains a G-equivariant holomorphic extension

$$\hat{f} \colon G \exp i\mathrm{Conv}(\Omega) \cdot x_0 \to \widehat{D}.$$

Since the domain $G \exp i\mathrm{Conv}(\Omega) \cdot x_0$ is Stein (see [GiKr02]), it coincides with the envelope of holomorphy $\widehat{D}$ of D. This shows that the envelope of holomorphy of $D = G \exp i\Omega \cdot x_0$ is univalent and given by $G \exp i\mathrm{Conv}(\Omega) \cdot x_0$.

Remark 5.4. The univalence result of Theorem 5.1 does not hold true for equivariant Riemann domains $p\colon X \to G^{\mathbb{C}}/K^{\mathbb{C}}$, which are not envelopes of holomorphy. For a Hermitian symmetric space G/K of tube type one can construct a non-trivial G-equivariant Stein covering of the domain $S^+ \subset \Xi^+$.

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