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Invariant domains in complex symmetric spaces

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0. Introduction

Let X be a Stein manifold and K be a compact Lie group acting on X by holomorphic transformations. In this setting, K-invariant domains $\Omega \subset X$ are natural objects to investigate. Basic questions regard the structure of the automorphism group $\mathrm{Aut}(\Omega)$, the extendability of automorphisms of Ω beyond the boundary and the equivalence problem.

The group K is said to act on X without invariant holomorphic functions if the algebra

$$\mathcal{O}(X)^K = \{f \in \mathcal{O}(X) \mid f \circ k^{-1} = f, \forall k \in K\}$$

consists of the constant functions alone. A result by Heinzner [HN1] states that if K acts on an n-dimensional Stein manifold X with a fixed point and without invariant holomorphic functions, then there exist a linear representation $\varrho: K \to \mathrm{GL}(\mathbb{C}^n)$ and an open holomorphic embedding $\phi: X \to \mathbb{C}^n$ such that $\phi(k \cdot x) = \varrho(k)\phi(x)$ for all $x \in X$ and $k \in K$. In this way, the study of K-invariant domains Ω in manifolds X satisfying the above assumptions can be reduced to the study of domains in $\mathbb{C}^n$ which are invariant under a suitable complex linear action. For example, if K is the torus group S^1, one is led to study the domains in $\mathbb{C}^n$ which contain the origin and are invariant under the S^1-action given by

$$(z_1, \ldots, z_n) \mapsto (\lambda^{m_1} z_1, \ldots, \lambda^{m_n} z_n),$$

with $\lambda \in S^1$ and $m_i \in \mathbb{N}_{>0}$, for $i = 1, \ldots, n$. These "generalized circular domains" in $\mathbb{C}^n$ were investigated by Kaup in [KA]. Recently, his results were generalized to an arbitrary compact group by P. Heinzner. In [HN3] Heinzner proved the following theorems:

Let K be a compact Lie group acting on $\mathbb{C}^n$ by a complex linear representation and without invariant holomorphic functions.

(1) Let $\Omega \subset \mathbb{C}^n$ be a bounded K-invariant Stein domain containing the origin. Then $\mathrm{Aut}(\Omega)$ has a finite number of connected components. If G is an arbitrary closed subgroup of $\mathrm{Aut}(\Omega)$ containing K, then the G-orbit of the origin $G \cdot 0$ is a connected complex submanifold of Ω.

(2) Let Ω_1 and Ω_2 be bounded K-invariant Stein domains in $\mathbb{C}^n$ containing the origin. Then Ω_1 and Ω_2 are biholomorphically equivalent if and ony if there exists a polynomial automorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\Phi(\Omega_1) = \Omega_2$.

In this paper we study K-invariant domains in Stein manifolds of the form K^C/L^C, where K acts by left translations. Here K^C and L^C denote the universal complexifications of the compact groups K and $L \subset K$ (see [HO]). In general, due to topological obstructions, these domains are not biholomorphically equivalent to domains in $\mathbb{C}^n$. We mainly work under the assumption that K and L form a compact symmetric pair [HE1]. In this case, we call K^C/L^C a complex symmetric space. Our principal results can be summarized in the following theorems:

Theorem 0.1. *Let $\Omega \subset K^C/L^C$ be a relatively compact K-invariant Stein domain in a complex symmetric space. Then*

(a) *Ω contains a minimal orbit of type K/L;*

(b) $\mathrm{Aut}(\Omega)$ *stabilizes a minimal K-orbit of type K/L;*

(c) $\mathrm{Aut}(\Omega)$ *is a compact group.*

Theorem 0.2. *Let Ω_1 and Ω_2 be relatively compact K-invariant Stein domains in K^C/L^C. Assume that $\Omega_1^C = \Omega_2^C = K^C/L^C$ and that* $\mathrm{Aut}(\Omega_2)$ *stabilizes a minimal K-orbit in Ω_2. Then Ω_1 and Ω_2 are biholomorphic if and only if there exists $F \in \mathrm{Aut}(K^C/L^C)$ such that $F(\Omega_1) = \Omega_2$.*

Here Ω^C denotes the K-complexification of a K-manifold Ω defined by Heinzner in [HN2] (see also section 1.2).

The paper is organized as follows. In section 1 we recall some preliminary material. In section 2 we prove Theorem 0.1 for K-invariant Stein domains $\Omega \subset\subset K^C/L^C$ under the assumption that K/L is an orientable compact symmetric space. In section 3 we conclude the proof of Theorem 0.1 by discussing the non-orientable case. In section 4 we exhibit another Stein manifold $X = K^C/L^C$ for which an analogue of Theorem 0.1 holds. This is when $K \cdot [L^C]$ is the unique minimal orbit in K^C/L^C. Finally, in section 5, we consider the equivalence problem for K-invariant Stein domains in K^C/L^C.

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1. Preliminaries

1.1. Mostow Decomposition Theorem and complex symmetric spaces. Let K be a compact connected Lie group and $L \subset K$ be a closed subgroup. The coset space K/L is a compact homogeneous manifold, where K acts by left translations, and L is the isotropy subgroup in K of the point $x_0 = [L]$. In the following, we shall assume the action of K on K/L to be almost effective. If Z denotes the center of K, this implies that $L \cap Z$ is a finite group. Denote by K^C and L^C the universal complexifications of K and L (see [HO]). The complex coset K^C/L^C is a K^C-homogeneous manifold and, by a result of Matsushima [MA], it is a Stein manifold.

Denote by $\mathfrak{k}$ and $\mathfrak{l}$ the Lie algebras of K and L, by Ad_K the adjoint representation $\mathrm{Ad}_K : K \to \mathrm{GL}(\mathfrak{k})$, and by Ad_L its restriction to L. Then there exists an Ad_L-invariant vector space $\mathfrak{p} \subset \mathfrak{k}$ such that $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$. The space $\mathfrak{p}$ can be naturally identified with the tangent space to K/L at $x_0 = [L]$ and, under this identification, the isotropy representation of L at x_0 is equivalent to the adjoint representation $\mathrm{Ad}_L : L \to \mathrm{Gl}(\mathfrak{p})$. Denote by $\mathfrak{k}^C = \mathfrak{k} \oplus i\mathfrak{k}$ and $\mathfrak{l}^C = \mathfrak{l} \oplus i\mathfrak{l}$ the Lie algebras of K^C and L^C. Then one has the Ad_{L^C}-invariant decomposition $\mathfrak{k}^C = \mathfrak{l}^C \oplus \mathfrak{p}^C$, where $\mathfrak{p}^C = \mathfrak{p} + i\mathfrak{p}$. Since the subspace $\mathfrak{p} \subset \mathfrak{p}^C$ is Ad_L-invariant, the same is true for $i\mathfrak{p}$.

Define $K \times_{\mathrm{Ad}_L} i\mathfrak{p}$ to be the quotient space of $K \times i\mathfrak{p}$ by the following equivalence relation: $(k, v) \sim (k', v')$ if there exists $l \in L$ such that $(k', v') = (kl^{-1}, \mathrm{Ad}_L(l)v)$. The group K acts on $K \times_{\mathrm{Ad}_L} i\mathfrak{p}$ on the left by $h \cdot [k, v] := [hk, v]$ and with the natural projection $K \times_{\mathrm{Ad}_L} i\mathfrak{p} \to K/L$ the space $K \times_{\mathrm{Ad}_L} i\mathfrak{p}$ is a K-equivariant vector bundle.

Lemma 1.1 (Mostow Decomposition Theorem [MO]). *Let $x_0 = [L^C]$ in K^C/L^C. Then the map $\Psi : K \times_{\mathrm{Ad}_L} i\mathfrak{p} \to K^C/L^C$, given by $[k, v] \mapsto k\exp(v)x_0$ is a K-equivariant diffeomorphism.*

Remark 1.2. Recall that the K-orbit types (equivalence classes of K-orbits up to K-equivariant diffeomorphisms) in K^C/L^C can be given a partial ordering as follows: $\mathrm{Typ}_K(K/S) \leqq \mathrm{Typ}_K(K/T)$ if there exists a K-equivariant surjective map of K/T onto K/S (see [BR]). By the above Lemma, for every $x \in K^C/L^C$, the bundle projection $K \times_{\mathrm{Ad}_L} i\mathfrak{p} \to K/L$ induces a K-equivariant surjective map $K \cdot x \to K/L$. It follows that every K-orbit in K^C/L^C can be compared to K/L and that K/L is a *minimal* orbit in K^C/L^C. Furthermore, the set of minimal K-orbits in K^C/L^C is parametrized by the Ad_L-fixed points in $i\mathfrak{p}$.

Another consequence of Lemma 1.1 is that the L-invariant subset $\exp(i\mathfrak{p})x_0 \subset K^C/L^C$ intersects all the K-orbits. Moreover, two points $x = \exp(v_1)x_0$ and $y = \exp(v_2)x_0$ lie on the same K-orbit in K^C/L^C if and only if v_1 and v_2 lie on the same Ad_L-orbit in $i\mathfrak{p}$. It follows that a subset $\exp(S)x_0$ is a geometric slice for the K-action on K^C/L^C if and only if $S \subset i\mathfrak{p}$ is a geometric slice for the Ad_L-action on $i\mathfrak{p}$. We shall give an explicit description of one such slice S in the case when K/L is a compact symmetric space.

We say that K/L is a *compact symmetric space* if K is a compact connected Lie group, $L \subset K$ is a closed subgroup and there exists an involutive automorphism σ of K such that

$$\mathrm{Fix}(\sigma)^0 \subset L \subset \mathrm{Fix}(\sigma).$$

Here $\mathrm{Fix}(\sigma)$ and $\mathrm{Fix}(\sigma)^0$ respectively denote the fixed point set of σ in K and its connected component of the identity. Under these conditions, a K-invariant Riemannian structure can be given on K/L, which turns it into a *Riemannian globally symmetric space* (cf. [HE1], Prop. 3.4, p. 209).

When K/L is a compact symmetric space, the decomposition $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$ coincides with the decomposition of $\mathfrak{k}$ into the ± 1 eigenspaces induced by the differential of σ at the identity. By the universality property of K^C, the automorphism σ extends to an involutive automorphism σ^C of K^C, satisfying $\mathrm{Fix}(\sigma^C)^0 \subset L^C \subset \mathrm{Fix}(\sigma^C)$. We call K^C/L^C a *complex symmetric space*.

Let $\mathfrak{z}$ denote the Lie algebra of the center Z of K. Then one has

$$\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}',$$

where $\mathfrak{k}'$ a maximal compact semisimple subalgebra of $\mathfrak{k}$. From the assumption that K acts almost effectively on K/L, it follows that $\mathfrak{l} \cap \mathfrak{z} = 0$ and one has

$$\mathfrak{z} \subset \mathfrak{p}, \quad \mathfrak{l} \subset \mathfrak{k}', \quad \mathfrak{p} = \mathfrak{z} \oplus \mathfrak{p}',$$

where $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{k}'$. Fix a maximal abelian subalgebra $\mathfrak{t}_\mathfrak{p}$ in $\mathfrak{p}$. All such subalgebras are mutually conjugate under the Ad_L-action and $\bigcup_{l \in L} \mathrm{Ad}_L(l)(\mathfrak{t}_\mathfrak{p}) = \mathfrak{p}$. The real dimension of $\mathfrak{t}_\mathfrak{p}$ is by definition the *rank* of K/L. Fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ containing $\mathfrak{t}_\mathfrak{p}$. Then

$$\mathfrak{t} = \mathfrak{t} \cap l \oplus \mathfrak{t}_\mathfrak{p} \quad \text{and} \quad \mathfrak{t}_\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{t}'_\mathfrak{p},$$

where $\mathfrak{t}'_\mathfrak{p} = \mathfrak{t}_\mathfrak{p} \cap \mathfrak{k}'$ is a maximal abelian subalgebra of $\mathfrak{p}'$. For simplicity of notations we shall write $\mathfrak{a}$, $\mathfrak{a}'$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}'_\mathfrak{p}$ for $i\mathfrak{t}$, $i\mathfrak{t}'$, $i\mathfrak{t}_\mathfrak{p}$ and $i\mathfrak{t}'_\mathfrak{p}$ respectively. Then $\mathfrak{a} = \mathfrak{a}' \oplus i\mathfrak{z}$ and $\mathfrak{a}_\mathfrak{p} = \mathfrak{a}'_\mathfrak{p} \oplus i\mathfrak{z}$. Denote by Δ the root system determined by $(\mathfrak{t}^C, \mathfrak{k}^C)$ in $\mathfrak{a}'$, identified with its dual by the Killing form. Choose a linear order on $\mathfrak{a}'$ compatible for Δ with respect to σ (see [TA], p. 450). Denote by Δ^+ the corresponding system of positive roots.

In order to determine a geometric slice for the Ad_L-action on $i\mathfrak{p}$, we need the notion of Weyl chamber associated to Δ^+. The open Weyl chamber associated to Δ^+ is the open cone

$$\mathfrak{a}'^+ := \{H \in \mathfrak{a}' \mid \alpha(H) > 0, \forall \alpha \in \Delta^+\}.$$

The inclusion $\mathfrak{a}'_\mathfrak{p} \subset \mathfrak{a}'$ yields the projection

$$\pi : (\mathfrak{a}')^* \to (\mathfrak{a}'_\mathfrak{p})^*.$$

Set $\Sigma := \pi(\Delta) \backslash \{0\}$. This is a possibly non-reduced root system in $\mathfrak{a}'_\mathfrak{p}$, called the restricted root system. Denote by Σ^+ the system of positive restricted roots induced by the above linear order. Then the positive Weyl chamber associated to Σ^+ is the (non-empty) open cone

$$\mathfrak{a}'^+_\mathfrak{p} := \{H \in \mathfrak{a}'_\mathfrak{p} \mid \alpha(H) > 0, \forall \alpha \in \Sigma^+\}.$$

Finally define

$$\mathfrak{a}^+_\mathfrak{p} := i\mathfrak{z} \oplus \mathfrak{a}'^+_\mathfrak{p} \quad \text{and} \quad \overline{\mathfrak{a}^+_\mathfrak{p}} := i\mathfrak{z} \oplus \overline{\mathfrak{a}'^+_\mathfrak{p}}.$$

Then $\overline{\mathfrak{a}^+_\mathfrak{p}} = \{H \in \mathfrak{a}'_\mathfrak{p} \mid \alpha(H) \geqq 0, \forall \alpha \in \Sigma^+\}$ and also coincides with the topological closure of $\mathfrak{a}^+_\mathfrak{p}$ in $\mathfrak{a}_\mathfrak{p}$. We call $\mathfrak{a}^+_\mathfrak{p}$ (resp. $\overline{\mathfrak{a}^+_\mathfrak{p}}$) the open (resp. closed) positive Weyl chamber in $\mathfrak{a}_\mathfrak{p}$.

The restricted Weyl group of $\mathfrak{a}_\mathfrak{p}$ is defined by $W(\mathfrak{a}_\mathfrak{p}) = N_L(\mathfrak{a}_\mathfrak{p})/Z_L(\mathfrak{a}_\mathfrak{p})$, where $N_L(\mathfrak{a}_\mathfrak{p})$ and $Z_L(\mathfrak{a}_\mathfrak{p})$ are the normalizer and the centralizer of $\mathfrak{a}_\mathfrak{p}$ in L. Observe that, $W(\mathfrak{a}_\mathfrak{p})$ a priori depends on the group L. On the other hand, $W(\mathfrak{a}_\mathfrak{p})$ turns out to be generated by the reflections in the hyperplanes determined in $\mathfrak{a}'_\mathfrak{p}$ by the root system Σ and hence only depends on the Lie algebras $\mathfrak{k}$ and $\mathfrak{l}$. Putting together the above facts with Prop. 7.2.2 in [HE1], one gets

Lemma 1.3. *Let K/L be a compact symmetric space and let $\mathrm{Ad}_L : L \to \mathrm{GL}(\mathfrak{p})$ be the isotropy representation of L on $\mathfrak{p} = \mathfrak{p}' \oplus \mathfrak{z}$. Then every L-orbit in $\mathfrak{p}$ intersects $\overline{\mathfrak{a}_\mathfrak{p}^+}$ in exactly one point.*

Now, we are able to define two slices in K^C/L^C: the *thick slice* and the *geometric slice*, both of which are basic for what follows. Let $x_0 = [L^C] \in K^C/L^C$ and let T^C (resp. $T_\mathfrak{p}^C$) be the complex torus corresponding to the Lie subalgebra $\mathfrak{t}^C = \mathfrak{t} \oplus \mathfrak{a}$ (resp. $\mathfrak{t}_\mathfrak{p}^C = \mathfrak{t}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p}$).

Definition 1.4. The "*thick slice*" in K^C/L^C associated to $\mathfrak{a}_\mathfrak{p}^C$ is defined by

$$D := T^C \cdot x_0 = T_\mathfrak{p}^C \cdot x_0 .$$

The "*geometric slice*" in K^C/L^C associated to $\overline{\mathfrak{a}_\mathfrak{p}^+} \subset \mathfrak{a}_\mathfrak{p}^C$ is defined by

$$D^+ := \exp(\overline{\mathfrak{a}_\mathfrak{p}^+}) \cdot x_0 = \overline{\exp(\mathfrak{a}_\mathfrak{p}^+) \cdot x_0} .$$

Lemma 1.1 and Lemma 1.3 justify the name "geometric slice" for D^+, for every K-orbit in K^C/L^C intersects D^+ in exactly one point. The "thick slice" D is the smallest closed complex subvariety of K^C/L^C intersecting every K-orbit; in the complex setting D is more suitable to work with than D^+. The next identities may be viewed as an improvement of Mostow Decomposition Theorem for complex symmetric spaces. If $x_0 = [L^C] \in K^C/L^C$, then

$$K^C/L^C = KD \cdot x_0 = KD^+ \cdot x_0 . \tag{1.1}$$

Remark 1.5. By Definition 1.4, one has that $D = T_\mathfrak{p}^C/T_\mathfrak{p}^C \cap L^C$. Since $\mathfrak{t}_\mathfrak{p} \cap \mathfrak{l} = 0$, the intersection $T_\mathfrak{p}^C \cap L^C = (T_\mathfrak{p} \cap L)^C$ is a finite group. Hence D is isomorphic to $(\mathbb{C}^*)^r$, where r equals the rank of K/L.

Remark 1.6. As we observed, the minimal K-orbits in K^C/L^C are parametrized by the Ad_L-fixed points in $i\mathfrak{p}$. If K^C/L^C is a complex symmetric space, they are also parametrized by the $W(\mathfrak{a}_\mathfrak{p})$-fixed points in $\mathfrak{a}_\mathfrak{p} = \mathfrak{a}'_\mathfrak{p} \oplus i\mathfrak{z} \subset i\mathfrak{p}$.

The following is a technical Lemma.

Lemma 1.7. *Let K^C/L^C be a complex symmetric space and let X be a complex K^C-homogeneous manifold with $\dim_C X = \dim_C K^C/L^C$. Let $\pi : X \to K^C/L^C$ be a K^C-equivariant holomorphic mapping. Then the map π establishes a $1:1$ correspondence between K-orbits in X and K-orbits in K^C/L^C; the restriction of π to every K-orbit in X is a covering map with the same multiplicity.*

Proof. It is easy to see that $X = K^C/(L_1)^C$, for some group L_1 satisfying $L^0 \subset L_1 \subset L$ (here L^0 denotes the connected component of the identity of L). In particular, π is a covering map of multiplicity equal to the cardinality of L/L_1.

Observe that the groups L and L_1 have the same Lie algebra $\mathfrak{l}$. Therefore one can choose the same positive Weyl chamber $\overline{\mathfrak{a}_\mathfrak{p}^+}$ in $\mathfrak{a}_\mathfrak{p}$ as a geometric slice for both their adjoint actions on $i\mathfrak{p}$ (cf. Lemma 1.3). Since π is K^C-equivariant, it induces an identification between $\exp(\overline{\mathfrak{a}_\mathfrak{p}^+}) \cdot x_1$, which is a geometric K-slice in $X = K^C/(L_1)^C$, and $\exp(\overline{\mathfrak{a}_\mathfrak{p}^+}) \cdot x_0$, which is a geometric K-slice in K^C/L^C. In particular, π sets a $1:1$ correspondence between K-orbits in X and K-orbits in K^C/L^C. Now it is clear that the restriction of π to each K-orbit is a covering map with the same multiplicity as π.

1.2. General properties of K-invariant domains in K^C/L^C. Let X be a complex K-space. A complex K^C-space X^C is called a K-complexification of X if it satisfies the following properties (cf. [HN2]):

(i) there exists a K-equivariant holomorphic map $i : X \to X^C$;

(ii) for every K-equivariant holomorphic map $f : X \to Y$, where Y is a complex K^C-space, there exists a unique K^C-equivariant holomorphic map $f^C : X^C \to Y$, satisfying $f^C \circ i = f$. It is easy to see that, if X^C exists, it is unique up to K^C-equivariant biholomorphism.

Let K be a compact connected Lie group, and let $L \subset K$ be a closed subgroup. Let Ω be a K-invariant domain in K^C/L^C. It has been shown in [HN2] (sect. 4.1) that the K-complexification Ω^C of Ω exists, is K^C-homogeneous and the map $i : \Omega \to \Omega^C$ is an open embedding. In general, Ω^C does not coincide with K^C/L^C but there is a commutative diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{i} & \Omega^C \\ \downarrow j & \swarrow \pi & \\ K^C/L^C & & \end{array}$$

where i and j are K-equivariant holomorphic open embeddings and π is a K^C-equivariant holomorphic covering. In particular, $\Omega^C = K^C/L_1^C$, for some group L_1 satisfying $L^0 \subset L_1 \subset L$.

If Ω is a K-invariant Stein domain in K^C/L^C and $K^C/L^C = \Omega^C$, then Ω is orbit convex in K^C/L^C (see [HN2]). By definition, this means that for every $x \in \Omega$ and $v \in i\mathfrak{k}$ such that $\exp(v) \cdot x \in \Omega$, one has that $\exp(tv) \cdot x \in \Omega$, for $t \in [0,1]$. Orbit convexity is a property which has remarkable consequences for the K-invariant strictly plurisubharmonic functions on Ω. In the next Lemma we recall the ones which will be repeatedly used in this paper, referring to [HN2] for their proofs.

Lemma 1.8. *Let $\Omega \subset K^C/L^C$ be a K-invariant orbit convex domain and let $\phi : \Omega \to [0, +\infty[$ be a K-invariant strictly plurisubharmonic function. Then*

(i) *if $x \in \Omega$ is a critical point of ϕ, then $K \cdot x$ is a minimal K-orbit;*

(ii) *if the minimum set of ϕ is not empty, then it consists of a unique minimal K-orbit*;

(iii) *the critical set of ϕ coincides with the minimum set of ϕ.*

Remark 1.9. Let Ω be a K-invariant domain in K^C/L^C. Let $\hat{\Omega}$ denote the envelope of holomorphy of Ω and Ω^C the K-complexification of Ω. An easy generalization of some results by Rossi [RO] yields that $\hat{\Omega}$ exists and is a K-invariant Stein domain in Ω^C satisfying $\Omega \subset \hat{\Omega} \subset \Omega^C$. Furthermore, $\hat{\Omega}^C = \Omega^C$ and the automorphism group $\mathrm{Aut}(\hat{\Omega})$ is a closed subgroup of $\mathrm{Aut}(\Omega)$.

We conclude this preliminary section by a proposition collecting some general properties of K-invariant domains in K^C/L^C. As we shall see, these properties are a direct consequence of Lemma 1.8.

Proposition 1.10. *Let Ω be a K-invariant domain in K^C/L^C. Denote by $\hat{\Omega}$ the envelope of holomorphy of Ω and by Ω^C its K-complexification.*

(i) *If $\Omega^C = K^C/L^C$, then $\hat{\Omega} \subset K^C/L^C$ contains a minimal K-orbit $K \cdot x$ and the orbit $K \cdot x$ is a deformation retract of $\hat{\Omega}$.*

(ii) *If K^C/L^C is a complex symmetric space, then $\Omega^C = K^C/L^C$.*

Proof. (i) As we observed in Remark 1.9, $\hat{\Omega}$ is a K-invariant orbit convex Stein domain in K^C/L^C. Let $\phi : \hat{\Omega} \to [0, +\infty[$ be a strictly plurisubharmonic K-invariant exhaustion function of $\hat{\Omega}$. The minimum set of ϕ is not empty and consists of a unique minimal orbit $K \cdot x \subset \hat{\Omega}$ (Lemma 1.8).

Since the minimum set of ϕ coincides with its critical set (Lemma 1.8), by Morse theory [MI], Thm. 3.1, p. 12, the domain $\hat{\Omega}$ can be smoothly deformed into $K \cdot x$ along the gradient flow of ϕ.

(ii) Let Ω^C be the K-complexification of Ω. One then has the following commutative diagram

$$\begin{tikzcd} \Omega \arrow[r, "i"] \arrow[d, "j"] & \Omega^C = K^C/(L_1)^C \arrow[dl, "\pi"] \\ K^C/L^C & \end{tikzcd}$$

where π is a K^C-equivariant holomorphic covering map. Since i is injective and K^C/L^C is a complex symmetric space, by Lemma 1.7 the map π is injective and hence a biholomorphism.

Corollary 1.11. *All minimal K-orbits in $\hat{\Omega}$ are homologous. In this way, one can say that the minimal orbit K/L is a generator for the homology of $\hat{\Omega}$.*

Remark 1.12. When K^C/L^C is a complex symmetric space, part (i) of Proposition 1.10 also follows from [LA]. There, the K-invariant Stein domains $\Omega \subset K^C/L^C$ are characterized as the domains of the form $\Omega = K \cdot \exp(\omega)$, where ω is an open, linearly convex and $W(\mathfrak{a}_\mathfrak{p})$-invariant set in $\mathfrak{a}_\mathfrak{p}$.

The retraction of Ω to K/L can be constructed explicitly. It is the restriction to Ω of the map

$$F: K \times_{\mathrm{Ad}_L} i\mathfrak{p} \times [0,1] \to K \times_{\mathrm{Ad}_L} i\mathfrak{p}$$

given by $([k,v],t) \mapsto t \cdot [k,v] := [k,tv]$.

2. The case of an orientable minimal orbit

The aim of this section is to prove Theorem 0.1 for K-invariant Stein domains

$$\Omega \subset\subset K^C/L^C,$$

when K/L is an orientable compact symmetric space (Theorem 2.10). The proof of the Theorem is based on the existence of an $\mathrm{Aut}(\Omega)$-invariant strictly plurisubharmonic exhaustion function ϕ of Ω together with Lemma 1.8. Such an exhaustion function ϕ is directly related to the Bergman kernel of Ω.

2.1. A canonical K-invariant volume form on Ω and the associated Hilbert space $H^2(\Omega)$. Let K and $L \subset K$ be compact Lie groups such that K/L is a compact orientable n-dimensional manifold. The K-invariant differential k-forms on K/L (resp. K^C-invariant holomorphic $(k,0)$-forms on K^C/L^C) are in one-to-one correspondence with the Ad_L-invariant skew-symmetric $\mathbb{R}$-linear maps $\mathfrak{p} \times \ldots \times \mathfrak{p} \to \mathbb{R}$ (resp. the Ad_{L^C}-invariant skew-symmetric $\mathbb{C}$-linear maps $\mathfrak{p}^C \times \ldots \times \mathfrak{p}^C \to \mathbb{C}$). In particular, the following fact holds.

Remark 2.1. If μ is a K-invariant volume form of K/L then μ extends to a K^C-invariant holomorphic $(n,0)$-form μ^C on K^C/L^C. The form μ^C is uniquely determined by μ and is nowhere vanishing on K^C/L^C.

Observe that any two such μ (resp. μ^C) only differ by a multiplicative constant.

Set $\eta := i^{n^2} \mu^C \wedge \overline{\mu^C}$. The form η is a K-invariant volume form on K^C/L^C. If Ω is a relatively compact domain in K^C/L^C, denote by η_Ω the restriction of η to Ω and choose a normalization of μ so that $\int_\Omega \eta_\Omega = 1$. Denote by $L^2(\Omega)$ the corresponding Hilbert space of square summable functions on Ω with the inner product given by

$$\langle f, g \rangle := \int_\Omega f(z)\overline{g(z)}\, d\eta_\Omega(z).$$

Let $H^2(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$. Since Ω is a relatively compact domain in a Stein manifold, $H^2(\Omega)$ is an infinite-dimensional separable Hilbert space. Denote by $K_\Omega: \Omega \times \Omega \to \mathbb{C}$ the corresponding Bergman kernel and set $k_\Omega(z) := K_\Omega(z,z)$. If $\{e_\nu\}_{\nu \in \mathbb{N}}$ is an arbitrary orthonormal basis of $H^2(\Omega)$, then [RA]

$$k_\Omega(z) = \sum_\nu |e_\nu(z)|^2.$$

It follows that k_Ω is a smooth, real valued, non-negative plurisubharmonic function. The inclusion of $\mathcal{O}(K^C/L^C)$ into $H^2(\Omega)$ implies that k_Ω is strictly plurisubharmonic. Denote the topological boundary of Ω by $\partial\Omega$.

Lemma 2.2. *Let Ω be a relatively compact domain in a complex manifold and let $p \in \partial\Omega$. Then $\lim_{z \to p} k_\Omega(z) = +\infty$ if and only if for every sequence $x_n \to p$ in Ω, there exists a function $f \in H^2(\Omega)$ such that $\lim_{n \to +\infty} |f(x_n)| = +\infty$.*

Proof. Recall that (cf. [RA])

$$(2.1) \qquad \forall z \in \Omega, \quad k_\Omega(z) = \sup\{|f(z)|^2 \,|\, f \in H^2(\Omega), \|f\|_{L^2(\Omega)} \leqq 1\}.$$

Suppose that $\lim_{z \to p} k_\Omega(z) = +\infty$. Fix $x_n \to p$ in Ω and assume by contradiction that

$$\sup_{x_n} |f(x_n)| \leqq C_f,$$

for all $f \in H^2(\Omega)$. Here C_f is a non-negative constant depending on f. By the Banach-Steinhaus Theorem, it follows that

$$\sup_{x_n} |f(x_n)| \leqq C, \quad \forall f \in H^2(\Omega), \|f\|_{L^2(\Omega)} \leqq 1,$$

where the constant C is independent from f. The above inequality and (2.1) imply that $k_\Omega(x_n) \leqq C$, yielding a contradiction.

The converse follows directly from (2.1).

2.2. Boundary behaviour of k_Ω for Stein Reinhardt domains. Let $\mathbb{C}^*$ denote the set of non-zero complex numbers. Let $\mathrm{Re}(z)$ denote the real part of a complex number z.

Proposition 2.3. *Let Ω be a relatively compact Stein Reinhardt domain in $(\mathbb{C}^*)^r$. Then for every $p \in \partial\Omega$, one has $\lim_{z \to p} k_\Omega(z) = +\infty$.*

Proof. Let $p \in \partial\Omega$. We begin by showing that there exist an open neighbourhood U_p of p in $(\mathbb{C}^*)^r$ and a function $f \in H^2(U_p \cap \Omega)$ such that $\lim_{z \to p} |f(z)| = +\infty$. Denote by ω the logarithmic image of Ω

$$\omega := \{x = (x_1, \dots, x_r) \in \mathbb{R}^r \,|\, x_i = \log|z_i|, \, i = 1, \dots, r, \, z = (z_1, \dots, z_r) \in \Omega\}.$$

Since Ω is a relatively compact domain in $(\mathbb{C}^*)^r$ and is Stein, ω is a bounded convex region in $\mathbb{R}^r$. Let $p = (z_1^0, \dots, z_r^0) \in \partial\Omega$. Then there exists a linear function

$$h: \mathbb{R}^r \to \mathbb{R}, \quad h(x_1, \dots, x_r) = \sum_{i=1}^r a_i x_i + c,$$

such that $h(\log(|z_1^0|), \ldots, \log(|z_r^0|)) = 0$ and $h(x) < 0$, for all $x \in \omega$. Define

$$H : (\mathbb{C}^*)^r \to \mathbb{R} \quad \text{by} \quad H(z) := 1/2 \sum_{i=1}^{r} a_i \log |z_i|^2 + c .$$

It is easy to see that H is pluriharmonic, $H(p) = 0$, $dH \neq 0$ and $H(z) < 0$ for all $z \in \Omega$. Hence there exist a small neighbourhood U_p of p in $(\mathbb{C}^*)^r$ and a holomorphic function $F \in \mathcal{O}(U_p)$ such that $\operatorname{Re} F = H$ on U_p. Moreover $F(p) = 0$, $dF(p) \neq 0$ and $F(z) \neq 0$ on $U_p \cap \Omega$. Define $G(z) := \log(F(z))$. Then G is well defined and holomorphic on $U_p \cap \Omega$ and satisfies $\lim_{z \to p} |G(p)| = +\infty$. We claim that $G \in H^2(U_p \cap \Omega)$. Since $F(p) = 0$ and $dF(p) \neq 0$, one can define a biholomorphism $\psi = (F, \psi_2, \ldots, \psi_r)$ of U_p (or some $V \subset U_p$) onto a small polydisk $P(0, \varepsilon)$ around 0 in $(\mathbb{C}^*)^r$. Let $w_1, \ldots, w_r$ be the coordinates on $P(0, \varepsilon)$, where $w_1 = F$. Since the function $\log w_1$ is square integrable on $\psi(U_p \cap \Omega)$ with respect to any locally bounded volume form, the function $G(z)$ is square integrable on $U_p \cap \Omega$. Lemma 2.2 now implies that

$$\lim_{z \to p} k_{\Omega \cap U_p}(z) = +\infty .$$

By Prop. 1.1 in [DFH], if $\Omega \subset\subset \mathbb{C}^n$ is a Stein domain, $z_0 \in \partial\Omega$ and $U_1 \subset\subset U_2$ are small open neighbourhoods of z_0, there exists a constant $C > 0$ such that

$$C^{-1} k_{\Omega \cap U_2}(z) \leqq k_\Omega(z) \leqq k_{\Omega \cap U_2}(z) , \tag{2.2}$$

for all $z \in \Omega \cap U_1$. The inequality (2.2) implies the desired result.

2.3. Boundary behaviour of k_Ω for K-invariant Stein domains in complex symmetric spaces. Our next goal is to show that an analogue of Proposition 2.3 holds for K-invariant Stein domains $\Omega \subset\subset K^C/L^C$, when K/L is an orientable compact symmetric space. Note that we don't make any assumption on the boundary of Ω.

We first need to recall some prerequisites. Let r be the rank of the symmetric space K/L. Then one may identify the thick slice $D = T_{\mathfrak{p}}^C \cdot x_0 = T^C \cdot x_0$ in K^C/L^C with the group $(\mathbb{C}^*)^r$ (Remark 1.5). Denote by $X(T^C)$ (resp. $X(D)$) the characters on T^C (resp. on D). In a natural way, $X(D)$ can be considered as a subgroup of $X(T^C)$. Denote by $X^+(T^C)$ the semi-group of positive characters of T^C with respect to a given choice of the Weyl chamber $\overline{\mathfrak{a}^+}$ (Sect. 1.1) and by $X^+(D)$ the intersection $X(D) \cap X^+(T^C)$. A result of Takeuchi ([TA], Thm. 2.4) asserts the following:

Theorem (Takeuchi). *Let K and $L \subset K$ be compact Lie groups, such that K/L is a compact symmetric space, and let $L^C \subset K^C$ be their universal complexifications. Let χ be a character in $X^+(D)$. Then there exists a complex irreducible representation $\varrho_\chi : K^C \to \mathrm{GL}(V_\chi)$, with highest weight χ and such that* $\mathrm{Fix}(L^C, V_\chi)$ *is 1-dimensional.*

The representation ϱ_χ is also referred to as a "class-one representation" of the pair (K^C, L^C). Let $\chi \in X^+(D)$ and let $\varrho_\chi : K^C \to \mathrm{GL}(V_\chi)$ be the corresponding class-one representation of (K^C, L^C). The restriction of ϱ_χ to T^C decomposes V_χ into a direct sum of one-dimensional weight spaces. Fix a K-invariant hermitian inner product $(\,,\,)$ on V_χ; then fix an orthonormal basis $\{v_1, \ldots, v_d\}$ of V_χ formed by weight vectors and such that v_1 belongs to the highest weight space. Sometimes we shall write $v_\chi = v_1$. Choose a vector v^L of length one in $\mathrm{Fix}(L^C, V_\chi)$ and define $\psi_\chi \colon K^C/L^C \to V_\chi$ by

$$\psi_\chi(gL^C) := \varrho_\chi(gL^C) \cdot v^L .$$

The map ψ_χ is holomorphic and its first component $\psi_{\chi,1}(gL^C) := (\varrho_\chi(gL^C)v^L, v_\chi)$ defines a holomorphic function on K^C/L^C. Finally, define $\phi_\chi \colon K^C/L^C \to \mathbb{C}$ by

$$\phi_\chi(gL^C) := \frac{1}{(v^L, v_\chi)} \psi_{\chi,1}(gL^C) . \tag{2.3}$$

By [WA], Vol. 1, p. 211, for every $\chi \in X^+(D)$, the quantity (v^L, v_χ) is non-zero. As a consequence, the function ϕ_χ is well defined and holomorphic on K^C/L^C; furthermore

$$\phi_\chi | D = \chi . \tag{2.4}$$

Denote by $C_c(X)$ the set of continuous functions with compact support on a space X.

Lemma 2.4 (Integral formula). *Let K/L be an orientable compact symmetric space. Then there exists a normalized left invariant measure $d(gL^C)$ on K^C/L^C such that*

$$\int_{K^C/L^C} f(gL^C)\, d(gL^C) = \int_K \int_{\mathfrak{a}_\mathfrak{p}^+} f(k \exp(a) L^C)\, \delta(a)\, da\, dk ,$$

for every function $f \in C_c(K^C/L^C)$. Here da denotes the Lebesgue measure on $\mathfrak{a}_\mathfrak{p}$, dk the Haar measure on K and $\delta(a)$ is a continuous function on $\mathfrak{a}_\mathfrak{p}$, independent from f.

Proof. Let μ be a K-invariant volume form on K/L and let μ^C be the K^C-invariant holomorphic $(n, 0)$-form on K^C/L^C extending μ (see Remark 2.1). Then $\eta = i^{n^2} \mu^C \wedge \overline{\mu^C}$ is a K^C-invariant positive volume form on K^C/L^C and, up to a constant factor, it is unique [HE2], Thm. 1.9. If K/L is a semisimple symmetric space (that is if K is semisimple), the above integral formula is contained in Theorem 8.1.1 in [SK]. Its proof is based on the fact that the map

$$\Psi \colon K/Z_L(\mathfrak{a}_\mathfrak{p}) \times \mathfrak{a}_\mathfrak{p}^+ \to K^C/L^C, \quad (kz, a) \mapsto k \exp(a) L^C$$

where $x_0 = [L^C] \in K^C/L^C$, is a diffeomorphism of $K/Z_L(\mathfrak{a}_\mathfrak{p}) \times \mathfrak{a}_\mathfrak{p}^+$ onto an open dense subset of K^C/L^C. Here $Z_L(\mathfrak{a}_\mathfrak{p})$ is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in L. The function $\delta(a)$ which appears in the right hand side of the integral formula is the jacobian determinant of Ψ at a point (kz, a). As we already saw in section 1, a map like Ψ exists in the general case as well (cf. (1.1)). In order to obtain the integral formula, we only need to compute the jacobian determinant of Ψ at $(kz, a) \in K/Z_L(\mathfrak{a}_\mathfrak{p}) \times \mathfrak{a}_\mathfrak{p}^+$.

The tangent space to $K/Z_L(\mathfrak{a}_\mathfrak{p}) \times \mathfrak{a}_\mathfrak{p}^+$ at (kz, a) can be identified with

$$\mathfrak{k}/\mathfrak{z}_L(\mathfrak{a}) \oplus \mathfrak{a}_\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{k}'/\mathfrak{z}_L(\mathfrak{a}_\mathfrak{p}) \oplus \mathfrak{a}_\mathfrak{p}' \oplus i\mathfrak{z}$$

while the tangent space to K^C/L^C at $x_0 = [L^C]$ with $\mathfrak{p}^C = \mathfrak{p} \oplus i\mathfrak{p}$. If (X, Y) is a vector in $\mathfrak{k}/\mathfrak{z}_L(\mathfrak{a}_\mathfrak{p}) \oplus \mathfrak{a}_\mathfrak{p}$, the differential $d\Psi_{(km,a)}(X, Y)$ is given by the projection of $\mathrm{Ad}_{a^{-1}} X + Y$ onto $\mathfrak{p}^C$ (cf. [SK]). Then it is easy to see that $d\Psi_{(km,a)}$ restricted to $\mathfrak{z} \subset \mathfrak{k}$ or to $i\mathfrak{z} \subset \mathfrak{a}_\mathfrak{p}$ is the identity map. Hence the jacobian determinant of Ψ only depends on the "semisimple part" of $\mathfrak{k}$ and is the same function δ on $\mathfrak{a}_\mathfrak{p}$ as in the semisimple case.

As we already observed, for every $\chi \in X_+(D)$, the function ϕ_χ defined in (2.3) is a holomorphic function globally defined on K^C/L^C. In particular, if Ω is a relatively compact domain in K^C/L^C, the restriction of ϕ_χ to Ω belongs to $H^2(\Omega)$. Let $\Omega_D = \Omega \cap D$ be the intersection of Ω with a thick slice D in K^C/L^C.

The following lemma relates the norm of $\phi_\chi | \Omega$ in $H^2(\Omega)$ with the norm of $\phi_\chi | \Omega_D$ in $H^2(\Omega_D)$.

Lemma 2.5 (Basic estimate). *Let Ω be a relatively compact K-invariant Stein domain in K^C/L^C, where K/L is an orientable compact symmetric space. Set $\Omega_D = \Omega \cap D$. Then there exists a constant $M > 0$ such that*

$$\|\Phi_\chi\|^2_{L^2(\Omega)} \leqq M \|\chi\|^2_{L^2(\Omega_D)},$$

for all $\chi \in X^+(D)$.

Proof. The main ingredient in the proof of the above estimate is the integral formula stated in Lemma 2.4. Let η_Ω again denote the volume form $i^{n^2} \mu^C \wedge \overline{\mu^C}$ of K^C/L^C restricted to Ω. Then, by Lemma 2.4, for every continuous function f on Ω, one has

$$\int_\Omega f(gL^C)\, d\eta_\Omega(gL^C) = C \int_K \int_{\omega_\Omega^+} f(k \exp(a) L^C)\, \delta(a)\, da\, dk,$$

where $\omega_\Omega^+ := \{a \in \mathfrak{a}_\mathfrak{p}^+ \mid \exp(a) \in \Omega_D\}$ and C is a constant only depending on the normalization of the measure on K^C/L^C. By a straightforward application of the preceding formula and of the definition of ϕ_χ we obtain the following identities:

$$\begin{aligned}
&|(v^L, v_\chi)|^2 \int_\Omega |\phi_\chi(gL^C)|^2\, d\eta_\Omega(gL^C) \\
&= \int_\Omega |(\varrho_\chi(gL^C) v^L, v_\chi)|^2\, d\eta_\Omega(gL^C) \\
&= C \int_{\omega_\Omega^+} \delta(a) \int_K |(\varrho_\chi(k) \varrho_\chi(\exp(a)) v^L, v_\chi)|^2\, dk\, da \\
&= C \int_{\omega_\Omega^+} \delta(a) \int_K |(\varrho_\chi(\exp(a)) v^L, \varrho_\chi(k) v_\chi)|^2\, dk\, da.
\end{aligned}$$

Recall that, up to a constant factor, there exists a unique K-invariant positive measure $d(kL)$ on K/L such that

$$\int_K f(k)\, dk = \int_{K/L} \int_L f(kl)\, dl\, d(kL),$$

for every continuous function f on K/L (see [HE2], Thm. 1.9, p. 91). Hence the above chain of identities can be continued as follows:

$$\int_{\omega_\Omega^+} \delta(a) \int_K |(\varrho_\chi(\exp(a))v^L, \varrho_\chi(k)v_\chi)|^2\, dk\, da$$
$$= C' \int_{\omega_\Omega^+} \delta(a) \int_{K/L} \int_L |(\varrho_\chi(\exp(a))v^L, \varrho_\chi([k]\,l)v_\chi)|^2\, dl\, d(kL)\, da$$
$$= C' \int_{\omega_\Omega^+} \delta(a) \int_{K/L} |(\varrho_\chi(\exp(a))v^L, \varrho_\chi([k])v^L)|^2 |(v^L, v_\chi)|^2\, d(kL)\, da\,.$$

Here we used the fact that the fixed point set $\mathrm{Fix}(L, V_\chi)$ has dimension 1 and therefore

$$\int_L \varrho_\chi(l)v_\chi\, dl = (v_\chi, v^L)v^L\,.$$

The constant C' only depends on the normalization of the measure on K/L. By the Schwarz inequality and the fact that $\|v^L\| = 1$, one has that

$$|(\varrho_\chi(\exp(a))v^L, \varrho_\chi([k])v^L)|^2 \leqq \|\varrho_\chi(\exp(a))v^L\|^2\,.$$

It follows that

$$C' \int_{\omega_\Omega^+} \delta(a) \int_{K/L} |(\varrho_\chi(\exp(a))v^L, \varrho_\chi([k])v^L)|^2 |(v^L, v_\chi)|^2\, d(kL)\, da$$
$$\leqq |(v^L, v_\chi)|^2\, C' \int_{\omega_\Omega^+} \delta(a) \int_{K/L} \|\varrho_\chi(\exp(a))v^L\|^2\, d(kL)\, da$$
$$\leqq |(v^L, v_\chi)|^2\, C' \int_{\omega_\Omega^+} \delta(a) |\chi(\exp(a))|^2\, da$$
$$\leqq |(v^L, v_\chi)|^2 \sup_{\omega_\Omega^+} |\delta(a)|\, C' \int_{\omega_\Omega^+} |\chi(\exp(a))|^2\, da\,.$$

If $\omega_\Omega := \{a \in \mathfrak{a}_\mathfrak{p} \,|\, \exp(a) \in \Omega_D\}$, then $\bigcup_{w \in W(\mathfrak{a}_\mathfrak{p})} w \cdot \omega_\Omega^+$ is an open dense subset of ω_Ω. Hence

$$\int_{\omega_\Omega^+} |\chi(\exp(a))|^2\, da = |W(\mathfrak{a}_\mathfrak{p})|^{-1} \int_{\omega_\Omega^+} |\chi(\exp(a))|^2\, da$$
$$= |W(\mathfrak{a}_\mathfrak{p})|^{-1} C'' \int_{\Omega_D} |\chi(z)|^2 \frac{dz \wedge \overline{dz}}{|z|^2}\,,$$

where $|W(\mathfrak{a}_\mathfrak{p})|$ denotes the cardinality of the Weyl group $W(\mathfrak{a}_\mathfrak{p})$ and C'' is a constant only depending on the normalization of the measure on Ω_D. Putting everything together we finally get the desired result:

$$|(v^L, v_\chi)|^2 \|\phi_\chi\|^2_{H^2(\Omega)} = |(v^L, v_\chi)|^2 \int_\Omega |\phi_\chi(gL^{\mathbb{C}})|^2\, d\eta_\Omega(gL^{\mathbb{C}})$$
$$\leqq M |(v^L, v_\chi)|^2 \int_{\Omega_D} |\chi(z)|^2 \frac{dz \wedge \overline{dz}}{|z|^2}$$
$$= M |(v^L, v_\chi)|^2 \|\chi\|^2_{H^2(\Omega_D)}\,,$$

where $M = CC'C'' \sup_{\omega_\Omega^+} |\delta(a)|\, |W(\mathfrak{a}_\mathfrak{p})|^{-1}$ is a constant not depending on χ.

Lemma 2.6. *Let Ω be a relatively compact K-invariant Stein domain in K^C/L^C, where K/L is an orientable compact symmetric space. Let D be a thick slice in K^C/L^C and let $\Omega_D = \Omega \cap D$. Let $p \in \partial\Omega_D$. Then for every sequence $x_n \to p$ in Ω_D, there exists a function $F \in H^2(\Omega)$ such that*

$$\lim_{n \to +\infty} |F(x_n)| = +\infty .$$

Proof. Let $p \in \partial\Omega_D = \partial\Omega \cap D$ and let $x_n \to p$ in Ω_D. Since Ω_D is biholomorphic to a relatively compact Reinhardt domain in $(\mathbb{C}^*)^r$, by Proposition 2.3 and Lemma 2.2 there exists a function $f \in H^2(\Omega_D)$ satisfying

$$\lim_{n \to +\infty} |f(x_n)| = +\infty .$$

We are going to extend f to a function $F \in H^2(\Omega)$. Let

$$f(z) = \sum_{n \in Z^r} a_n z^n$$

be the Laurent series development of f on Ω_D. Viewing the monomials z^n as characters on D, we can rewrite the above series as $f(z) = \sum_{\chi \in X(D)} a_\chi \chi(z)$. Without loss of generality, we can actually assume

$$f(z) = \sum_{\chi \in X^+(D)} a_\chi \chi(z)$$

for a suitable choice of the positive Weyl chamber $\overline{\mathfrak{a}^+}$. Now define $F: \Omega \to \mathbb{C}$ formally by

$$F(Z) := \sum_{\chi \in X^+(D)} a_\chi \phi_\chi(Z) ,$$

where the functions ϕ_χ are those defined in (2.3). It is immediate that F satisfies

$$F | \Omega_D = f .$$

Observe also that, if $F \in L^2(\Omega)$, then automatically $F \in H^2(\Omega)$. This follows from the fact that the functions ϕ_χ belong to $H^2(\Omega)$ and $H^2(\Omega)$ is a closed subspace of $L^2(\Omega)$. Therefore, to conclude the proof of the lemma, we are left to show that F belongs to $L^2(\Omega)$.

If χ_1 and χ_2 are non-equivalent characters in $X^+(D)$, they are orthogonal elements in $H^2(\Omega_D)$. This observation and the estimate of Lemma 2.5 yield

$$\|F\|^2_{H^2(\Omega)} \leqq \sum_\chi |a_\chi|^2 \|\phi_\chi\|^2_{L^2(\Omega_D)} = M \sum_\chi |a_\chi|^2 \|\chi\|^2_{L^2(\Omega_D)} = M \|f\|^2_{L^2(\Omega_D)} ,$$

completing the proof of the lemma.

Corollary 2.7. *Let Ω be a relatively compact K-invariant Stein domain in K^C/L^C, where K/L is an orientable compact symmetric space. Let $p \in \partial\Omega$. Then for every sequence $x_n \to p$ in Ω_D*

$$\lim_{n \to +\infty} k_\Omega(x_n) = +\infty .$$

Proof. The corollary follows directly from Lemma 2.6 and Lemma 2.2.

2.4. An Aut(Ω)-invariant strictly plurisubharmonic exhaustion function of Ω. As we already mentioned, the basic ingredient in the proof of the main result of this section (Theorem 2.10) is the existence of an Aut(Ω)-invariant strictly plurisubharmonic exhaustion function of Ω. The construction of such an exhaustion function is based on a technique developed by E. Bedford [BD], which we briefly recall.

Let M be a complex manifold of complex dimension n and let $A_n(M)$ be the Hilbert space of L^2 holomorphic $(n,0)$-forms on M

$$A_n(M) = \{\eta \in \overset{n,0}{\bigwedge}(M) \mid d\eta = 0,\ \|\eta\|^2_{L^2(M)} = |\int_M \eta \wedge \bar{\eta}| < +\infty\}.$$

Let $H_n(M, \mathbb{R})$ denote the n-th homology group of M with real coefficients. Define a map $\lambda_M : H_n(M, \mathbb{R}) \to \mathbb{R}$ by

$$\lambda_M([\gamma]) = \sup_{\substack{\eta \in A_n(M) \\ \|\eta\| \leqq 1}} \operatorname{Re} \int_{\gamma'} \eta,$$

where γ' is a representative of $[\gamma]$. It is easy to check that λ_M does not depend on the choice of γ' and that it defines a seminorm on $H_n(M, \mathbb{R})$. This seminorm is a biholomorphic invariant of the manifold. If $\iota : M \to M'$ is a holomorphic injection, and

$$\iota_* : H_n(M, \mathbb{R}) \to H_n(M', \mathbb{R})$$

is the induced map between the homology groups, then for all $[\gamma] \in H_n(M, \mathbb{R})$

$$\lambda'_M(\iota_*[\gamma]) \leqq \lambda_M([\gamma]).$$

Let $[\gamma] \in H_n(M, \mathbb{R})$ with $\lambda_M([\gamma]) > 0$. Then there exists a unique holomorphic form $\omega_{[\gamma]} \in A_n(\Omega)$, with $\|\omega_{[\gamma]}\|_{L^2(M)} = 1$, such that

$$\lambda_M([\gamma]) = \int_\gamma \omega_{[\gamma]}. \tag{2.5}$$

If $f : M \to M'$ is a biholomorphic map, and f^* denotes the pull-back mapping induced by f on the differential forms, then $\lambda_{M'}(f_*[\gamma]) = \lambda_M([\gamma]) > 0$ and

$$\omega_{[\gamma]} = f^* \omega_{f_*[\gamma]}. \tag{2.6}$$

Now let Ω be a relatively compact Stein domain in $K^{\mathbb{C}}/L^{\mathbb{C}}$, where K/L is a compact orientable space. Then by the topological simplicity of Ω, the above technique yields an invariant volume form on Ω. Let η_Ω be the K-invariant volume form on Ω introduced in section 2.1.

Lemma 2.8. *Let $\Omega \subset\subset K^{\mathbb{C}}/L^{\mathbb{C}}$ be a K-invariant Stein domain, where K/L is a compact orientable space. Then η_Ω is an* Aut(Ω)*-invariant volume form on Ω.*

Proof. By Proposition 1.10 and Corollary 1.11, the domain Ω contains a minimal orbit K/L and the corresponding homology class $[\gamma] := [K/L]$ is a generator of the homology group $H_n(\Omega, \mathbb{Z}) \cong \mathbb{Z}$. Since $\lambda_\Omega(\gamma) \geqq \mathrm{Re} \int\limits_{K/L} \mu^C > 0$, there exists a unique holomorphic $(n,0)$-form $\omega_{[\gamma]} \in A_n(\Omega)$ with $\|\omega_{[\gamma]}\|_{L^2(\Omega)} = 1$ satisfying $\lambda_\Omega([\gamma]) = \int\limits_\gamma \omega_{[\gamma]}$.

Observe that $\omega_{[\gamma]}$ is K-invariant, because $[\gamma]$ is and by (2.6); moreover, for every $\phi \in \mathrm{Aut}(\Omega)$,

$$\phi^* \omega_{[\gamma]} = \pm \omega_{[\gamma]}. \tag{2.7}$$

It follows that $\omega_{[\gamma]} \wedge \overline{\omega_{[\gamma]}}$ is an $\mathrm{Aut}(\Omega)$-invariant volume form on Ω. By our choice of the normalization, $\omega_{[\gamma]} = \mu^C | \Omega$ and η_Ω is an $\mathrm{Aut}(\Omega)$-invariant volume form on Ω, as required.

As in section 2.1, let $H^2(\Omega)$ be the space of square summable holomorphic functions with respect to the volume form η_Ω. Denote again by $k_\Omega(z) := K_\Omega(z, z)$ the corresponding Bergman kernel function. As the next lemma shows, k_Ω is the $\mathrm{Aut}(\Omega)$-invariant exhaustion function of Ω we needed.

Lemma 2.9. *Let $\Omega \subset\subset K^C/L^C$ be a K-invariant Stein domain, where K/L is a compact orientable space. Then k_Ω is a smooth* $\mathrm{Aut}(\Omega)$*-invariant non-negative strictly plurisubharmonic function of Ω. If furthermore K/L is symmetric, then k_Ω is an exhaustion function.*

Proof. As we already observed in section 2.1, the function k_Ω is a smooth non-negative strictly plurisubharmonic function on Ω. In order to show the $\mathrm{Aut}(\Omega)$-invariance of k_Ω, observe that by Lemma 2.8 the group $\mathrm{Aut}(\Omega)$ acts isometrically on $H^2(\Omega)$ by

$$f \mapsto f \circ \phi^{-1},$$

where $f \in H^2(\Omega)$ and $\phi \in \mathrm{Aut}(\Omega)$. Then, the property of being an orthonormal basis of $H^2(\Omega)$ is invariant under pullbacks by elements of $\mathrm{Aut}(\Omega)$ and this implies the $\mathrm{Aut}(\Omega)$-invariance of k_Ω. If K/L is symmetric, the exhaustion property of k_Ω follows from Corollary 2.7.

2.5. The proof of Theorem 0.1 for orientable minimal orbit.

Theorem 2.10. *Let $\Omega \subset K^C/L^C$ be a relatively compact K-invariant Stein domain. Assume K/L is an orientable compact symmetric space. Then*

(a) *Ω contains a minimal K-orbit of type K/L;*

(b) $\mathrm{Aut}(\Omega)$ *stabilizes a minimal K-orbit of type K/L;*

(c) $\mathrm{Aut}(\Omega)$ *is a compact group;*

(d) *Ω is complete for the Bergman metric.*

Proof. The proof of (a) is contained in Proposition 1.10 or Remark 1.12.

(b) Let ϕ denote the $\mathrm{Aut}(\Omega)$-invariant strictly plurisubharmonic exhaustion function of Ω constructed in Lemma 2.9. Denote by M the minimum set of ϕ. By Lemma 1.8, the set M consists of a unique $\mathrm{Aut}(\Omega)$-invariant minimal K-orbit of type K/L in Ω.

(c) Since Ω is hyperbolic, $\mathrm{Aut}(\Omega)$ acts properly on Ω. Then the compactness of $\mathrm{Aut}(\Omega)$ follows directly from (b).

The assertion (d) follows directly from Theorem B in [DFH] and the fact that $\mathcal{O}(K^C/L^C)$ embeds in $H^2(\Omega)$ as a dense subset.

3. The case of a non-orientable minimal orbit

The aim of this section is to complete the proof of Theorem 0.1, that is to extend the results of Theorem 2.10 to the case when K/L is a non-orientable compact symmetric space (Theorem 3.2). Since a non-orientable manifold admits no nowhere vanishing volume form, the techniques introduced in section 2 cannot be applied directly. What we shall do in this case is to reduce ourselves to the orientable situation.

We begin by briefly recalling some facts about the $2:1$ orientable covering of a non-orientable manifold. Let M be a non-orientable manifold of real dimension n and let $p: F(M) \to M$ be the associated principal frame bundle. The fibre of $F(M)$ over a point $m \in M$ consists of all ordered bases of $T(M)_m$, the tangent space to M at m. The group $\mathrm{GL}(n, \mathbb{R})$ acts on the fibers of $F(M)$ by right translations. Denote by $\mathrm{GL}(n, \mathbb{R})^0$ the connected component of the identity of $\mathrm{GL}(n, \mathbb{R})$ and define $\tilde{M} := F(M)/\mathrm{GL}(n, \mathbb{R})^0$. Then $\tilde{M}$ is a connected orientable manifold and, with the projection $p: \tilde{M} \to M$, is a $2:1$ covering of M. Moreover, $\tilde{M}$ has a nice lifting property. If ϕ is a diffeomorphism of M, then the differential $d\phi$ of ϕ defines a bundle map of $F(M)$. Since $d\phi$ commutes with the right $\mathrm{GL}(n, \mathbb{R})^0$-action on $F(M)$, it induces a diffeomorphism of $\tilde{M}$, which we denote by $\tilde{\phi}$. The map $\phi \mapsto \tilde{\phi}$ is an injective group homomorphism $\mathrm{Diff}(M) \to \mathrm{Diff}(\tilde{M})$, and for all $\phi \in \mathrm{Diff}(M)$ the corresponding element $\tilde{\phi} \in \mathrm{Diff}(\tilde{M})$ satisfies $p \circ \tilde{\phi} = \phi \circ p$. If $M = K/L$ is a K-homogeneous manifold, the same is true for $\tilde{M}$ and there exists a subgroup $L_1 \subset L$ such that $\tilde{M} = K/L_1$. In particular, if $M = K/L$ is a compact symmetric space, $\tilde{M} = K/L_1$ is symmetric as well. The projection $p: K/L_1 \to K/L$ is K-equivariant and likewise $p^C: K^C/L_1^C \to K^C/L^C$ is a K^C-equivariant $2:1$ holomorphic covering.

If Ω is a relatively compact K-invariant Stein domain in K^C/L^C, then $\Omega_1 := (p^C)^{-1}(\Omega)$ is a relatively compact K-invariant Stein domain in K^C/L_1^C and $p^C: \Omega_1 \to \Omega$ is a K-equivariant holomorphic $2:1$ covering.

Lemma 3.1. *Let $\Omega \subset\subset K^C/L^C$ be a K-invariant Stein domain, where K/L is a non-orientable compact symmetric space. Let K/L_1 be the $2:1$ orientable covering of K/L and let Ω_1 be the domain in K^C/L_1^C covering Ω. Then there exists a subgroup $G \subset \mathrm{Aut}(\Omega)$ of finite index, with the property that for all $g \in G$ there exists $\tilde{g} \in \mathrm{Aut}(\Omega_1)$ such that*

$$p^C \circ \tilde{g} = g \circ p^C.$$

Proof. Consider $p^C : \Omega_1 \to \Omega$ as a principal $\mathbb{Z}_2$-bundle and denote by $[\Omega_1]$ the corresponding class in $H^1(\Omega, \mathbb{Z}_2)$. For $g \in \mathrm{Aut}(\Omega)$, denote by g^* the pull-back isomorphism $g^* : H^1(\Omega, \mathbb{Z}_2) \to H^1(\Omega, \mathbb{Z}_2)$. Then the map $g \mapsto g^*$ defines a representation

$$* : \mathrm{Aut}(\Omega) \to \mathrm{GL}(H^1(\Omega, \mathbb{Z}_2)).$$

Denote by H the image of $\mathrm{Aut}(\Omega)$ in $\mathrm{GL}(H^1(\Omega, \mathbb{Z}_2))$ and by $H_{[\Omega_1]}$ the stabilizer of $[\Omega_1]$ in H. Set $G := (*)^{-1}(H_{[\Omega_1]})$. By definition, G is the group formed by those elements $g \in \mathrm{Aut}(\Omega)$ such that the pull-back bundle $g^*\Omega_1$ is equivalent to Ω_1. Therefore, for all $g \in G$ there exists $\tilde{g} \in \mathrm{Aut}(\Omega_1)$ such that $p^C \circ \tilde{g} = g \circ p^C$. Since $H^1(\Omega, \mathbb{Z}_2) \cong H^1(K/L, \mathbb{Z}_2)$ is a finite dimensional vector space over $\mathbb{Z}_2$, the group G is of finite index in $\mathrm{Aut}(\Omega)$, and this completes the proof of the lemma.

Theorem 3.2. *Let $\Omega \subset K^C/L^C$ be a relatively compact K-invariant Stein domain in a complex symmetric space. Then*

(a) *Ω contains a minimal orbit of type K/L;*

(b) $\mathrm{Aut}(\Omega)$ *stabilizes a minimal K-orbit of type K/L;*

(c) $\mathrm{Aut}(\Omega)$ *is a compact group.*

Proof. The proof of (a) is already contained in Proposition 1.10 or Remark 1.12.

If K/L is a compact orientable symmetric space, the asserts (b), (c) and (d) have been proved in Theorem 2.10. So it remains to discuss the case when K/L is non-orientable. Let K/L_1 be the $2:1$ orientable covering of K/L and let Ω_1 be the domain in K^C/L_1^C covering Ω. Recall that Ω_1 is a relatively compact K-invariant Stein domain in K^C/L_1^C and hence satisfies the hypotheses of Theorem 2.10.

By Lemma 3.1, there exists a subgroup $G \in \mathrm{Aut}(\Omega)$ of finite index whose elements all lift to automorphisms of Ω_1. Since $\mathrm{Aut}(\Omega_1)$ stabilizes a minimal K-orbit in Ω_1 (by Thm. 2.10), so does G. It follows that G is compact and $\mathrm{Aut}(\Omega)$ itself is compact. To prove the existence of an $\mathrm{Aut}(\Omega)$-invariant minimal K-orbit in Ω, take an arbitrary strictly plurisubharmonic exhaustion function of Ω. Since $\mathrm{Aut}(\Omega)$ is compact, an $\mathrm{Aut}(\Omega)$-invariant strictly plurisubharmonic exhaustion function ϕ' of Ω can be obtained from ϕ by a standard averaging process. By Lemma 1.8, the minimum set of ϕ' is an $\mathrm{Aut}(\Omega)$-invariant minimal K-orbit in Ω.

Corollary 3.3. *Let $\Omega \subset K^C/L^C$ be a relatively compact K-invariant domain in a complex symmetric space. Then* $\mathrm{Aut}(\hat{\Omega})$ *is a compact group* (*cf. Remark* 1.9).

4. The case of a unique non-symmetric minimal orbit

In this section we exhibit another situation where an analogue of Theorem 3.2 holds. It is when the manifold K^C/L^C satisfies the following condition:

(C) *$K \cdot x_0$, for $x_0 = [L^C]$, is the unique minimal K-orbit in K^C/L^C.*

Again, we first consider the case when the minimal orbit $K \cdot x_0 = K/L$ is orientable. In this case, if Ω is a relatively compact K-invariant Stein domain in K^C/L^C, we can apply Bedford techniques to construct a smooth Aut(Ω)-invariant strictly plurisubharmonic function on Ω. In contrast with the symmetric case, this function may not be an exhaustion function. So we need to show directly that its critical set is not empty.

Lemma 4.1. *Let L be a compact Lie group acting linearly on $\mathbb{C}^n$ with the origin as a unique fixed point. Let $B \subset \mathbb{C}^n$ be an L-invariant ball centered at 0. If $\phi : B \to [0, +\infty[$ is a smooth L-invariant function, then 0 is a critical point of ϕ.*

Proof. Identify the tangent space to B at 0 with $\mathbb{C}^n$ and denote by $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$ the space of $\mathbb{R}$-linear functionals on $\mathbb{C}^n$. The L-action on $\mathbb{C}^n$ induces an L-action on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$ by $l \cdot \chi(v) := \chi(l \cdot v)$. The fact that 0 is the unique L-fixed point in $\mathbb{C}^n$ implies that the zero functional is the only L-fixed point in $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$. Since the differential of the function ϕ at 0 defines an L-invariant element in $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$, the Lemma follows.

Theorem 4.2. *Let $\Omega \subset\subset K^C/L^C$ be a K-invariant Stein domain with $\Omega^C = K^C/L^C$. Set $x_0 = [L^C]$ and assume that $K \cdot x_0 = K/L$ is the unique minimal orbit in K^C/L^C. Then*

(a) *Ω contains the minimal orbit K/L;*

(b) Aut(Ω) *stabilizes K/L;*

(c) Aut(Ω) *is a compact group.*

Proof. The proof of (a) is contained in Prop. 1.10. We proof (b) and (c) in the case where the minimal orbit K/L is orientable. The non-orientable case follows from the orientable one as in Theorem 3.2.

(b) Let ϕ be the Aut(Ω)-invariant strictly plurisubharmonic function on Ω constructed in Lemma 2.9. We claim that the minimal orbit $K/L \subset \Omega$ coincides with the minimum set of ϕ and in particular it is Aut(Ω)-invariant. We begin by showing that the critical set of ϕ is not empty; more precisely, that $x_0 = [L^C]$ is a critical point of ϕ. If we consider K^C/L^C as an L-space, then x_0 is an isolated fixed point of L. The L-action can be locally linearized around x_0, that is there exist an L-invariant neighbourhood U_0 of x_0 in K^C/L^C, a ball $B \subset \mathbb{C}^n$ centered at 0 and a biholomorphic map $F: U_0 \to B$, such that $F(x_0) = 0$ and $F(l \cdot x) = i_0(l)F(x)$, for all $x \in U_0$ and $l \in L$. Here i_0 denotes the isotropy representation of L at x_0. Since the function

$$\phi \,|\, U_0 \circ F^{-1}$$

satisfies the assumptions of Lemma 4.1, it has a critical point at 0, which is equivalent to ϕ having a critical point at x_0. Since Ω is orbit convex in K^C/L^C, the point x_0 is also a minimum point of ϕ and the minimal orbit $K \cdot x_0 = K/L$ is precisely the minimum set of ϕ.

(c) follows from (b) as in Thm. 2.10.

5. The equivalence problem

In this section we consider the equivalence problem for relatively compact K-invariant Stein domains in K^C/L^C. As a by-product we obtain a result on the extendability of automorphisms of such domains beyond the boundary.

Lemma 5.1. *Let K, L, G, H be compact Lie groups such that $K \subset G$, $H \subset G$ and $L = K \cap H$. Assume that G/H is connected and $G/H = K/L$. Then G^C/H^C is connected and $G^C/H^C = K^C/L^C$.*

Proof. Since G^C and G have the same number of connected components, the connectedness of G/H implies that of G^C/H^C. The inclusion $K \subset G$ induces the inclusion $K^C \subset G^C$. So K^C acts naturally on G^C/H^C by left translations. Set $x_0 = [H^C] \in G^C/H^C$. Since $G \cdot x_0 = G/H$ is a totally real submanifold of G^C/H^C of maximal dimension and $K \cdot x_0 = K/L = G/H$, one has that $\dim_{\mathbb{C}} K^C \cdot x_0 = \dim_{\mathbb{C}} G^C/H^C$. It follows that $K^C \cdot x_0$ is open in G^C/H^C. In order to conclude that $K^C/L^C = G^C/H^C$ we need to show that $K^C \cdot x_0$ is closed in G^C/H^C. This can be done using an argument by [HN2] (page 649) which we recall for the sake of completeness. By a consequence of the Peter and Weyl theorem, there exist a real vector space V and a linear action of G on V, such that $G \cdot v = G/H$ for some $v \in V$ (see [BR]). The complexification G^C acts holomorphically on the complexification $V \otimes \mathbb{C}$ and there exists a G-invariant hermitian inner product on $V \otimes \mathbb{C}$ whose restriction to V is a G-invariant scalar product on V. Denote by $F: V \otimes \mathbb{C} \to \mathbb{R}$ the corresponding norm on $V \otimes \mathbb{C}$, which is a G-invariant strictly plurisubharmonic function. Since $G \cdot v$ is perpendicular to the line $\mathbb{R}v$, the point v is a critical point for the restriction of F to the G^C-orbit $G^C \cdot v$. By [KN] and [PS], the orbit $G^C \cdot v$ is closed in $V \otimes \mathbb{C}$ and is biholomorphic to G^C/H^C.

Consider now the restriction of the G^C-action on $V \otimes \mathbb{C}$ to K^C. By the same argument, the orbit $K^C \cdot v$ is closed in $V \otimes \mathbb{C}$ and is biholomorphic to K^C/L^C. Since G^C/H^C is closed in $V \otimes \mathbb{C}$, the orbit $K^C \cdot v$ is closed in G^C/H^C as well.

Theorem 5.2. *Let Ω_1, Ω_2 be relatively compact K-invariant Stein domains in K^C/L^C. Assume that $\Omega_1^C = \Omega_2^C = K^C/L^C$ and that $\mathrm{Aut}(\Omega_2)$ stabilizes a minimal K-orbit. Let $F: \Omega_1 \to \Omega_2$ be a biholomorphic map. Then there exists $F^C \in \mathrm{Aut}(K^C/L^C)$ such that $F^C | \Omega_1 = F$.*

Proof. Let $G := \mathrm{Aut}(\Omega_2)$ and let K/L be the minimal K-orbit stabilized by G in Ω_2. Then G is a compact Lie group and $K/L = G/H$ for some compact subgroup $H \subset G$. Since K/L is connected, by Lemma 5.1 one has that $K^C/L^C = G^C/H^C$ and the domain Ω_2 can be viewed as a G-invariant domain in G^C/H^C. The biholomorphic map $F: \Omega_1 \to \Omega_2$ is K-equivariant, if we let K act on Ω_2 by

$$k \cdot w := F \circ k \circ F^{-1}(w)\,, \tag{5.1}$$

for $k \in K$ and $w \in \Omega_2$. Since G is the full group of automorphisms of Ω_2, the K-action given in (5.1) is defined on the whole space G^C/H^C. In this way, G^C/H^C can be considered both as a K and a K^C-space for such an action. Since $K^C/L^C = \Omega_1^C$ and the map

$$F: \Omega_1 \to \Omega_2 \subset G^C/H^C$$

is K-equivariant, there exists a unique K^C-equivariant holomorphic map

$$F^C : \Omega_1^C = K^C/L^C \to G^C/H^C,$$

which makes the following diagram commutative.

$$\begin{array}{ccc} \Omega_1 & \xrightarrow{\iota} & \Omega_1^C = K^C/L^C \\ \downarrow F & \swarrow F^C & \\ G^C/H^C & & \end{array}$$

Similarly, the K-equivariant map

$$F^{-1} : \Omega_2 \subset \Omega_2^C = K^C/L^C = G^C/H^C \to \Omega_1 \subset K^C/L^C$$

yields a unique K^C-equivariant holomorphic map $(F^{-1})^C : G^C/H^C \to K^C/L^C$, satisfying

$$(F^{-1})^C | \Omega_2 = F^{-1}.$$

By the identity principle $(F^{-1})^C = (F^C)^{-1}$ and the map F^C is an automorphism of K^C/L^C satisfying $F^C | \Omega_1 = F$.

Remark 5.3. Let Ω_1 and Ω_2 be relatively compact K-invariant Stein domains in K^C/L^C. Assume that $\Omega_1^C = \Omega_2^C = K^C/L^C$ and that $\mathrm{Aut}(\Omega_2)$ stabilizes a minimal K-orbit. Then, given a biholomorphic map $F : \Omega_1 \to \Omega_2$, there exist $\mathrm{Aut}(\Omega_i)$-invariant minimal K-orbits M_i in Ω_i, for $i = 1, 2$, such that $F(M_1) = M_2$.

Proof. Since $\mathrm{Aut}(\Omega_2)$ stabilizes a minimal K-orbit, it is a compact group. Let ϕ be an $\mathrm{Aut}(\Omega_2)$-invariant strictly plurisubharmonic exhaustion function of Ω_2. Then $\Psi := \phi \circ F$ is an $\mathrm{Aut}(\Omega_1)$-invariant strictly plurisubharmonic exhaustion function of Ω_1. Its minimum set is not empty and, by Lemma 1.8, it is a minimal K-orbit M_1 in Ω_1. Clearly, $F(M_1)$ is the minimum set of ϕ and is an $\mathrm{Aut}(\Omega_2)$-invariant minimal K-orbit in Ω_2.

Applying Theorem 5.2 to the case when $\Omega_1 = \Omega_2$, we obtain the following extension result for automorphisms of domains in K^C/L^C:

Corollary 5.4. *Let $\Omega \subset\subset K^C/L^C$ be a K-invariant Stein domain. Assume that $\Omega^C = K^C/L^C$ and that* $\mathrm{Aut}(\Omega)$ *stabilizes a minimal orbit. Then every automorphism of Ω extends to an automorphism of K^C/L^C.*

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