CORRECTION to the paper

Geometry of Biinvariant Subsets of Complex Semisimple Lie Groups

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Statement (iii) of Theorem 5.3 in the above paper is incorrect: it has to be subdivided into three subcases. Then the correct formulation of Theorem 5.3 becomes the following:

Theorem 5.3. *Let S be a generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbit in G with base point $x_0 = n_0 \exp JX_0 \in n_0 \cdot \exp Jt^{\mathbb{R}}$. Then the Levi cone $\mathcal{C}(S)_{x_0}$ of S at x_0 can be described as follows:*

(i) If the Cartan subalgebra $\mathfrak{t}^{\mathbb{R}}$ is non-compact, then

$$\mathcal{C}(S)_{x_0} = \mathfrak{t}^{\mathbb{R}};$$

(ii) If the Cartan subalgebra $\mathfrak{t}^{\mathbb{R}}$ is compact and $\eta(n_0) \notin Z(G^{\mathbb{R}})$, then

$$\mathcal{C}(S)_{x_0} = \mathfrak{t}^{\mathbb{R}};$$

(iii) If the Cartan subalgebra $\mathfrak{t}^{\mathbb{R}}$ is compact and $\eta(n_0) \in Z(G^{\mathbb{R}})$, there are the following cases.

(a) If $\mathfrak{g}^{\mathbb{R}}$ is of hermitian type and $JX_0 \in C_{max}$, then the cone $\mathcal{C}(S)_{x_0}$ is sharp. More precisely, $\mathcal{C}(S)_{x_0}$ is isomorphic to the dual of the positive Weyl chamber defined by Δ^+.

(b) If $\mathfrak{g}^{\mathbb{R}}$ is of hermitian type and $JX_0 \notin C_{max}$, then

$$\mathcal{C}(S)_{x_0} = \mathfrak{t}^{\mathbb{R}};$$

(c) If $\mathfrak{g}^{\mathbb{R}}$ is not of hermitian type, then

$$\mathcal{C}(S)_{x_0} = \mathfrak{t}^{\mathbb{R}}.$$

Before proving the new formulation of Theorem 5.3 (iii), we explain what C_{max} is.

Resuming the notation in [FG], let G denote a simple simply connected complex Lie group, $G^{\mathbb{R}}$ a real form of G with conjugation κ, and $\mathfrak{g}$ (resp. $\mathfrak{g}^{\mathbb{R}}$) the corresponding Lie algebras. Let B denote the Killing form of $\mathfrak{g}$. Let $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k}^{\mathbb{R}} \oplus \mathfrak{p}^{\mathbb{R}}$ be the Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified decomposition of $\mathfrak{g}$. Let $\mathfrak{t}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$ be a Cartan subalgebra, $\mathfrak{t} \subset \mathfrak{g}$ its complexification, and $\Delta = \Delta^r \cup \Delta^i \cup \Delta^c$ the roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, subdivided in real, imaginary and complex roots depending on their value on $\mathfrak{t}^{\mathbb{R}}$. When $\mathfrak{t}^{\mathbb{R}}$ is a compact Cartan subalgebra of $\mathfrak{g}^{\mathbb{R}}$, all roots are imaginary and can be subdivided in compact and non-compact roots $\Delta = \Delta_c \cup \Delta_n$, depending on whether the corresponding root spaces belong to $\mathfrak{k}$ or to $\mathfrak{p}$.

Let $\{H_\alpha\}_{\alpha\in\Delta}$ be the dual roots and let $\{Z_\alpha\}_{\alpha\in\Delta}$ be a κ-stable set of root vectors. *In [FG], it was erroneously stated that after a normalization we may assume that* $[Z_\alpha, \kappa Z_\alpha] = H_\alpha$, *for all* $\alpha \in \Delta^+$. Instead, *the correct statement is: after a normalization, we may assume that*

$$[Z_\alpha, \kappa Z_\alpha] = H_\alpha,\ \forall\alpha \in \Delta^+ \setminus \Delta^i \quad \text{and} \quad [Z_\alpha, \kappa Z_\alpha] = \pm H_\alpha,\ \forall\alpha \in (\Delta^+)^i, \tag{1}$$

depending on the sign of $B(Z_\alpha, \kappa Z_\alpha)$. In the special case when $\mathfrak{t}^{\mathbb{R}}$ is a compact Cartan subalgebra of $\mathfrak{g}^{\mathbb{R}}$, condition (1) becomes

$$[Z_\alpha, \kappa Z_\alpha] = H_\alpha,\ \alpha \in \Delta_n^+ \qquad [Z_\alpha, \kappa Z_\alpha] = -H_\alpha,\ \alpha \in \Delta_c^+.$$

From classification results, one can see that for simple "equal-rank" real Lie algebras there exists a set of simple roots $\Pi = \Pi_c \cup \Pi_n$ with a unique non-compact root α (cf. [W], Lemma 4, [Kn2], Appendix C). Let Λ denote the highest root with respect to the corresponding positive system Δ^+.

If in addition $\mathfrak{g}^{\mathbb{R}}$ is of hermitian type, the positive system Δ^+ can be assumed to have a good ordering. This means that every positive non-compact root is larger than an arbitrary compact root or, equivalently, that the set Δ_n^+ is invariant under the Weyl group W_{Δ_c}. In this case, $\mathfrak{t}_{\mathbb{R}}$ contains two natural W_{Δ_c}-invariant cones $C_{min} \subset C_{max}$, defined as

$$C_{min} = cone(\{[Z_\alpha, \kappa Z_\alpha],\ Z_\alpha \in \mathfrak{g}^\alpha,\ \alpha \in \Delta_n^+\}),$$

and

$$C_{max} = (C_{min})^* = \{Y \in \mathfrak{t}_{\mathbb{R}} \mid B(X,Y) \geq 0,\ \forall X \in C_{min}\}.$$

Proof of (iii). By Remark 2.7 in [FG], it is sufficient to consider the case when $\eta(n_0) = 1$. Recall that the Levi cone of S at x_0 is given by

$$C_{x_0}(S) = -cone(\{(\coth(\alpha(JX_0) + 1)J[Z_\alpha, \kappa Z_\alpha]\}_{\alpha\in\Delta^+}) \subset \mathfrak{t}^{\mathbb{R}}. \tag{2}$$

For simplicity, we identify $\mathfrak{t}^{\mathbb{R}}$ with $\mathfrak{t}_{\mathbb{R}}$ via the map $X \mapsto JX$. We consider the image of $\mathcal{C}(S)_{x_0}$ in $\mathfrak{t}_{\mathbb{R}}$ and denote it by the same symbol. Since S intersects $\exp J\mathfrak{t}^{\mathbb{R}}$ in the orbit of the Weyl group $W_G(\mathfrak{t}^{\mathbb{R}})$, we may assume $\alpha(JX_0) > 0$, for all $\alpha \in \Delta_c^+$. By (2), we have that $-H_\alpha \in \mathcal{C}(S)_{x_0}$ for all $\alpha \in \Delta_c^+$, while $\pm H_\alpha \in \mathcal{C}(S)_{x_0}$ for $\alpha \in \Delta_n^+$, depending on whether $\alpha(JX_0)$ is positive or negative.

• *Hermitian case.*

Without loss of generality, we may assume $\Lambda(JX_0) > 0$.

(a) If $JX_0 \in W_{max}$, then $\alpha(JX_0) > 0$, for all roots $\alpha \in \Delta^+$. Hence

$$C_{x_0}(S) = cone(\{H_\alpha\}_{\alpha\in\Delta_n^+}, \{-H_\alpha\}_{\alpha\in\Delta_c^+}).$$

This cone is sharp: it is the image of the dual of the positive Weyl chamber, under the reflection with respect to the highest root of each simple factor of $\mathfrak{k}$.

(b) If $X_0 \notin W_{max}$, then $\alpha(JX_0) < 0$, where α is the simple non-compact root. By the good ordering of Δ^+, all positive non-compact roots are obtained from α by adding simple compact roots. Since Λ is non-compact and $\Lambda(JX_0) > 0$, there exists a non-compact root μ, such that $\mu(JX_0) > 0$. Assume μ is a root of minimal order with this property. Write

$$\mu = \alpha + \sum_{s=1}^{p} n_s \beta_s \quad n_s > 0,$$

with $\beta_s \in \Pi_c$. Let β be a root in Π_c such that $\mu - \beta$ is a non-compact root, with negative value on JX_0. Consider the triplet of roots $\beta,\ \mu - \beta,\ \mu$. By (2),

$$cone(H_\mu, -H_\beta, -H_{\mu-\beta}) = \operatorname{span}_{\mathbb{R}}\{H_\mu, H_\beta\} \subset \mathcal{C}(S)_{x_0}.$$

Next take $\gamma \in \Pi_c$ such that $\mu - \beta - \gamma$ is a non-compact root, with negative value on JX_0. By the result of the previous step and the same argument, one has that

$$\mathbb{R} \cdot H_\gamma, \quad \mathbb{R} \cdot H_{\mu-\beta}, \quad \mathbb{R} \cdot H_{\mu-\beta-\gamma} \subset \mathcal{C}(S)_{x_0}.$$

Subtracting simple roots from μ in this way, we finally obtain that $\mathbb{R} \cdot H_{\beta_s}$, $\mathbb{R} \cdot H_\alpha$ are contained in $\mathcal{C}(S)_{x_0}$, for all simple roots which appear in μ. To obtain the same result for the remaining simple roots, we add them one by one to μ, until we obtain the highest root. Observe that the value of the non-compact roots obtained in this way remains positive on JX_0. If γ is a root in Π_c, such that $\mu + \gamma \in \Delta$, consider the triplet of roots $\gamma,\ \mu,\ \mu + \gamma$. By the results of the previous steps, we have that

$$\mathbb{R} \cdot H_\gamma, \quad \mathbb{R} \cdot H_\mu, \quad \mathbb{R} \cdot H_{\mu+\gamma} \subset \mathcal{C}(S)_{x_0}.$$

Iterating this argument until all simple roots are exhausted, we obtain statement (3.b).

• *Non-Hermitian case.*

(c) In this case, the highest root Λ is compact and the coefficient of the root $\alpha \in \Pi_n$ in Λ is equal to 2 (cf. [Kn2], Appendix C).

Assume first that all non-compact roots are positive on JX_0. Since the coefficient of α in Λ is equal to 2, there exists a compact root ν which is sum of precisely two non-compact roots $\nu = \lambda + \mu$. Observe that

$$cone(-H_\nu, H_\lambda, H_\mu) = \operatorname{span}\{H_\lambda, H_\mu\} \subset \mathcal{C}(S)_{x_0}.$$

The root λ (resp. μ) contains α with coefficient one and from λ one can construct the highest root by adding simple roots. If $\lambda + \beta \in \Delta$, for some $\beta \in \Pi_c$, then

$$cone(\mathbb{R} \cdot H_\lambda, -H_\beta, H_{\lambda+\beta}) = \operatorname{span}\{H_\lambda, H_\beta\} \subset C(S)_{x_0}.$$

When non-compact root α is added, yielding a compact root, we obtain $\mathbb{R} \cdot H_\alpha \subset C(S)_{x_0}$.

Claim: If $\mathbb{R} \cdot H_\alpha \in \mathcal{C}(S)_{x_0}$, then $\mathcal{C}(S)_{x_0} = \mathfrak{t}^{\mathbb{R}}$.

Proof of Claim: Let β be a root in Π_c, such that $\alpha + \beta \in \Delta$. For the triplet of roots $\alpha, \beta, \alpha + \beta$, we have that

$$cone(\mathbb{R} \cdot H_\alpha, -H_\beta, H_{\alpha+\beta}) = \mathrm{span}_{\mathbb{R}}\{H_\alpha, H_\beta\} \subset \mathcal{C}(S)_{x_0}.$$

If γ is a root in Π_c such that $\alpha+\beta+\gamma \in \Delta$, then consider the triplet of roots $\alpha+\beta, \gamma, \alpha+\beta+\gamma$. By the previous step and the same argument, one has that

$$cone(\pm H_{\alpha+\beta}, -H_\gamma, H_{\alpha+\beta+\gamma}) = \mathrm{span}_{\mathbb{R}}\{H_{\alpha+\beta}, H_\gamma\} \subset \mathcal{C}(S)_{x_0}.$$

By iterating this argument until all the simple roots are exhausted, the claim follows.

Assume now that $\alpha \in \Pi_n$ is negative on JX_0. Since there exists a compact root which is sum of non-compact roots, there exists a non-compact root which is positive on JX_0. Let λ be a root of minimal order with this property. Then λ is of the form

$$\lambda = \alpha + \sum_{s=1}^{p} n_s \beta_s, \quad n_s > 0,$$

i.e. it is obtained by adding simple compact roots to α. From now on the proof continues as in case (b).

[FG] G. Fels, L. Geatti, *Geometry of Biinvariant Subsets of Complex Semisimple Lie Groups*, Ann. SNS Pisa, Vol. XXVI (1998) 329–346.