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Geometry of Biinvariant Subsets of Complex Semisimple Lie Groups

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Introduction

Let G be a complex semisimple Lie group and let $G^{\mathbb{R}}$ be a real form of G. In this paper we investigate the CR-geometry of the $G^{\mathbb{R}} \times G^{\mathbb{R}}$-action on G given by left and right translations, i.e. $x \mapsto gxh^{-1}$. In particular, we wish to obtain information about the $G^{\mathbb{R}} \times G^{\mathbb{R}}$-invariant objects in G such as plurisubharmonic functions and domains. The motivation for this study comes from representation theory. One of the problems in representation theory is to construct geometric realizations of the irreducible unitary representations of a semisimple Lie group, possibly on natural spaces of holomorphic functions. In this context, there is a strong interplay between representation theory and complex analysis.

If $G^{\mathbb{R}}$ is a compact Lie group, every irreducible representation of $G^{\mathbb{R}} \times G^{\mathbb{R}}$ can be canonically realized in the Hilbert space $L^2(G^{\mathbb{R}})$ of square summable functions on $G^{\mathbb{R}}$. All such representations are finite dimensional and are parametrized by the positive characters on a fixed Cartan subgroup. The fact that all irreducible representations of $G^{\mathbb{R}}$ extend holomorphically to the universal complexification G yields a natural realization of such representations on spaces of holomorphic functions on G.

The situation is more complicated when the group $G^{\mathbb{R}}$ is non-compact. In this case, all unitary representations of $G^{\mathbb{R}}$ are infinite dimensional and do not extend to holomorphic representations of the group G. Moreover, there may be several non-conjugate Cartan subgroups $T_1^{\mathbb{R}}, \dots, T_r^{\mathbb{R}}$ in $G^{\mathbb{R}}$, each of them associated with a series of irreducible unitary representations of $G^{\mathbb{R}}$.

In the late seventies, Gelfand and Gindikin outlined a program for the holomorphic realization of representations by considering certain biinvariant domains in G. The idea was to realize some unitary representations of $G^{\mathbb{R}}$ either on Hilbert spaces of holomorphic functions on these domains or on their cohomology classes (cf. [GG]). Such domains are contained in the subsets of the

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form

$$G^{\mathbb{R}} \cdot \exp i\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}},$$

where $\mathfrak{t}^{\mathbb{R}}$ varies in the conjugacy classes of Cartan subalgebras of $\mathfrak{g}^{\mathbb{R}} = \mathrm{Lie}(G^{\mathbb{R}})$. When the group $G^{\mathbb{R}}$ is Hermitian and $\mathfrak{t}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$ is a compact Cartan subalgebra, the set $G^{\mathbb{R}} \cdot \exp i\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}}$ contains some natural biinvariant domains: the Ol'shanskiĭ semigroups. Ol'shanskiĭ semigroups are subsets of G of the form

$$\mathcal{S}_V = G^{\mathbb{R}} \cdot \exp iV = G^{\mathbb{R}} \cdot \exp i\mathcal{V} \cdot G^{\mathbb{R}},$$

where V is an $\mathrm{Ad}(G^{\mathbb{R}})$-invariant cone in $\mathfrak{g}^{\mathbb{R}}$ and $\mathcal{V}$ is a Weyl-invariant cone in $\mathfrak{t}^{\mathbb{R}}$. The domains $\mathcal{S}_V$ are in many respects the non-commutative analogue of tube domains in complex Euclidean space. Every such domain contains the group $G^{\mathbb{R}}$ in its Shilov boundary and the space of holomorphic functions $\mathcal{O}(\mathcal{S}_V)$ on $\mathcal{S}_V$ contains a Hardy-type Hilbert space $\mathcal{H}(\mathcal{S}_V)$, where the group $G^{\mathbb{R}}$ acts by a unitary representation. Moreover, the boundary value operator defines an isometric injection of $\mathcal{H}(\mathcal{S}_V)$ into $L^2(G^{\mathbb{R}})$. All the representations of the holomorphic discrete series of $G^{\mathbb{R}}$ can be realized in this way (see [O1], [O2], [St], [HN]).

If $G^{\mathbb{R}}$ is a compact real form of G and $\mathfrak{t}^{\mathbb{R}}$ is an arbitrary Cartan subalgebra of $\mathfrak{g}^{\mathbb{R}}$, then $\mathfrak{t}^{\mathbb{R}}$ contains no proper Weyl-invariant cones and

$$G^{\mathbb{R}} \cdot \exp i\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}} = G.$$

It was proved by Loeb [L1] that in this case there is a one-to-one correspondence between biinvariant plurisubharmonic functions on G and logarithmically convex Weyl-invariant functions on $\exp i\mathfrak{t}^{\mathbb{R}}$ (see also [Las], [FH]). Generalizing Loeb's idea, Neeb used the theory of holomorphic representations of complex semigroups to obtain a similar result for a non-compact Hermitian real form $G^{\mathbb{R}}$ of G. In [N1], [N2], he established a one-to-one correspondence between biinvariant plurisubharmonic functions on biinvariant subsets of Ol'shanskiĭ semigroups

$$\Omega \subset \mathcal{S}_V \subset G^{\mathbb{R}} \cdot \exp i\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}}$$

and logarithmically convex Weyl-invariant functions on $\Omega \cap \exp i\mathfrak{t}^{\mathbb{R}}$ satisfying a certain growing condition. Recall that, on the other hand, on a complex simple Lie group G there are no non-constant global plurisubharmonic functions which are biinvariant under a non-compact real form $G^{\mathbb{R}}$ (see [L2]).

Very little is known about the complex geometry of biinvariant objects in the subsets

$$G^{\mathbb{R}} \cdot \exp i\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}}$$

when $\mathfrak{t}^{\mathbb{R}}$ is a non-compact Cartan subalgebra in $\mathfrak{g}^{\mathbb{R}}$. One would expect biinvariant subsets related to non-conjugate Cartan subalgebras to have different geometric properties.

In this paper, we approach these problems by investigating the CR-structure and the Levi form of the generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits in G. There is a finite number of orbit types in G, of maximal dimension, whose union forms an open dense subset of G (see [Br], [St]). Orbits of these types will be called generic orbits. Our main results comprise the calculation of the Levi form of an arbitrary generic orbit S in terms of the root system of $\mathfrak{g} = \mathrm{Lie}(G)$ and the determination of the corresponding "Levi cone" (cf. Proposition 4.26, Theorem 5.3). Roughly speaking, the shape of this cone determines how smooth CR-functions defined on S locally extend to holomorphic functions on an open subset in G. Our calculations enable us to determine which generic orbits can be contained in the level sets of a biinvariant plurisubharmonic function or in the boundary of a biinvariant Stein domain in G.

If $G^{\mathbb{R}}$ is simple, generic orbits with the above properties only occur when $\mathfrak{g}^{\mathbb{R}}$ contains a compact Cartan subalgebra $\mathfrak{t}_0^{\mathbb{R}}$. In the subset $G^{\mathbb{R}} \cdot \exp i\mathfrak{g} \subset G$, they are precisely the generic orbits lying in

$$G^{\mathbb{R}} \cdot \exp i\mathfrak{t}_0^{\mathbb{R}} \cdot G^{\mathbb{R}}.$$

In other words, a biinvariant domain

$$\Omega \subset G^{\mathbb{R}} \cdot \exp i\mathfrak{g}$$

possibly admits biinvariant plurisubharmonic functions or Stein subdomains only when it is completely contained in $G^{\mathbb{R}} \cdot \exp i\mathfrak{t}_0^{\mathbb{R}} \cdot G^{\mathbb{R}}$. Examples of such domains Ω are the Ol'shanskiĭ semigroups.

The paper is organized as follows. In Section 1, we recall the basic notions about CR-structures and Levi form; in Section 2, we recall some general facts about group actions, Cartan subalgebras and Bremingan's description of the generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits in G; in Section 3, we determine the invariant CR-structure of a generic orbit; in Section 4, we calculate its Levi form; in Section 5, we describe its Levi cone and derive some consequences.

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1. – CR-structures and Levi form

Generalities about CR-structures. Let M be a complex manifold with tangent bundle TM and let $J : TM \to TM$ denote the complex structure. Let $T^{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$ denote the formal complexification of TM. Then J extends to a $\mathbb{C}$-linear morphism of $T^{\mathbb{C}}M$ and induces the decomposition

$$T^{\mathbb{C}}M = HM \oplus AM \tag{1.1}$$

into the holomorphic and antiholomorphic tangent bundles of M. The bundles HM and AM are by definition the $\pm i$-eigenspaces of J on $T^{\mathbb{C}}M$ respectively. The complex conjugation

$$\bar{\ }: T^{\mathbb{C}}M \to T^{\mathbb{C}}M, \quad X \otimes z \mapsto X \otimes \bar{z},$$

defines a $\mathbb{C}$-antilinear bundle isomorphism $\bar{\ } : HM \to AM$. The map $X \mapsto X \otimes 1$ defines a canonical embedding $TM \hookrightarrow T^{\mathbb{C}}M$ identifying TM with the real part of $T^{\mathbb{C}}M$ with respect to the complex conjugation, i.e. TM is characterized by

$$TM = \{Z \in T^{\mathbb{C}}M \mid \overline{Z} = Z\}.$$

The bundle maps $\pi^H : TM \to HM$ given by $X \mapsto \frac{1}{2}(X - iJX)$, and $\pi^A : TM \to AM$ given by $X \mapsto \frac{1}{2}(X + iJX)$ define $\mathbb{R}$-isomorphisms satisfying $\pi^H(JX) = i\pi^H(X)$ and $\pi^A(JX) = -i\pi^A(X)$ respectively.

Let S be a real submanifold of M with tangent bundle TS. In general, the subbundle $TS \subset TM$ is not J-stable. Let $x \in S$. Denote by

$$T_{\mathbb{C}}S_x := TS_x \cap JTS_x$$

the maximal J-stable subspace of the tangent space TS_x to S at x. If the complex dimension d of $T_{\mathbb{C}}S_x$ does not depend on x, then S is a CR-manifold and d is called the CR-dimension of S. Moreover, $T_{\mathbb{C}}S = TS \cap JTS$ is a well-defined J-stable subbundle of TS.

DEFINITION 1.2. A CR-manifold $S \subset M$ is called *generic* if the CR-dimension d of S is the smallest possible, i.e. $d = \dim_{\mathbb{C}}M - \operatorname{codim}_{\mathbb{R}}(S, M)$ (here $\operatorname{codim}_{\mathbb{R}}(S, M)$ denotes the real codimension of S in M).

The vector bundles $T_{\mathbb{C}}S$ and TS can be formally complexified as well. The decomposition (1.1) induces a decomposition of $T_{\mathbb{C}}^{\mathbb{C}}S$ as

$$T_{\mathbb{C}}^{\mathbb{C}}S = HS \oplus AS, \tag{1.3}$$

where $HS = T^{\mathbb{C}}S \cap HM$ and $AS = T^{\mathbb{C}}S \cap AM$. Denote by $\Gamma(M, E)$ the space of sections of a vector bundle $E \to M$ over a manifold M. A CR-submanifold S of a complex manifold is involutive, i.e.

$$[\Gamma(S, HS), \Gamma(S, HS)] \subset \Gamma(S, HS). \tag{1.4}$$

This condition implies that

$$[X, Y] - [JX, JY] \in T_{\mathbb{C}}S \quad \text{and} \quad [JX, Y] + [X, JY] \in T_{\mathbb{C}}S, \tag{1.5}$$

for all local sections $X, Y \in \Gamma(S, T_{\mathbb{C}}S)$. Observe that the condition

$$[\Gamma(S, T_{\mathbb{C}}S), \Gamma(S, T_{\mathbb{C}}S)] \subset \Gamma(S, T_{\mathbb{C}}S) \tag{1.6}$$

is much stronger than (1.4); if condition (1.6) is fulfilled for $T_{\mathbb{C}}S$, there is a foliation of S by complex submanifolds of complex dimension d.

The Levi form. Let S be a CR-submanifold of a complex manifold M. We recall the definition of the Levi form of S. For more details we refer to [Bo], [Gr], or [Tu].

Let $x \in S$. Denote by Z_x a tangent vector in $T_{\mathbb{C}}S_x$ and by $\widehat{Z}$ an arbitrary extension of Z_x to a local section in $\Gamma(S, T_{\mathbb{C}}S)$. Then the vector fields $\pi^H(\widehat{Z}) = \frac{1}{2}(\widehat{Z} - iJ\widehat{Z})$ and $\pi^A(\widehat{Z}) = \frac{1}{2}(\widehat{Z} + iJ\widehat{Z})$ are local sections of the bundles HS and AS.

DEFINITION 1.7. The levi form $\mathbf{L}$ of S at x is the map $\mathbf{L} : T_{\mathbb{C}}S_x \times T_{\mathbb{C}}S_x \to T^{\mathbb{C}}S_x / T^{\mathbb{C}}_{\mathbb{C}}S_x$ given by

$$\mathbf{L}(X_x, Y_x) := \frac{i}{4}[\widehat{X} - iJ\widehat{X}, \widehat{Y} + iJ\widehat{Y}]_x \mod T^{\mathbb{C}}_{\mathbb{C}}S_x .$$

REMARK 1.8. The Levi form $\mathbf{L}$ at x is well-defined, as it does not depend on the choice of the extensions $\widehat{X}$ and $\widehat{Y}$. Moreover, $\mathbf{L}$ is an $\mathbb{R}$-bilinear Hermitian form satisfying

$$\mathbf{L}(X_x, Y_x) = \mathbf{L}(JX_x, JY_x) \quad \text{and} \quad \mathbf{L}(X_x, Y_x) = \overline{\mathbf{L}(Y_x, X_x)},$$

where the conjugation on $T^{\mathbb{C}}S_x / T^{\mathbb{C}}_{\mathbb{C}}S_x$ is the restriction of the conjugation on $T^{\mathbb{C}}M_x$. It follows that $\mathbf{L}(X_x, X_x)$, is real valued, i.e. $\mathbf{L}(X_x, X_x) \in TS_x / T_{\mathbb{C}}S_x$. From (1.5), it follows that

$$\begin{aligned} \mathbf{L}(X_x, Y_x) &= \frac{i}{2}[\widehat{X}, \widehat{Y}]_x - \frac{1}{2}[\widehat{X}, J\widehat{Y}]_x \mod T^{\mathbb{C}}_{\mathbb{C}}S_x , \\ \mathbf{L}(X_x, X_x) &= \frac{1}{2}[J\widehat{X}, \widehat{X}]_x \qquad\qquad\qquad \mod T^{\mathbb{C}}_{\mathbb{C}}S_x , \end{aligned} \tag{1.9}$$

for all $X_x, Y_x \in T_{\mathbb{C}}S_x$.

The Levi form measures the degree to which condition (1.6) fails to be satisfied. The Levi form $\mathbf{L}$ defined in 1.7 is a generalization of the classical Levi form $\left(\frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_l}\right)|T_{\mathbb{C}}S$ of a real hypersurface S which is locally defined by a function ρ. A key geometric object associated with the Levi form is the real cone $\mathcal{C}_x(S)$ generated by the image of $\mathbf{L}$ in $TS_x / T_{\mathbb{C}}S_x$. The cone $\mathcal{C}_x(S)$ is the higher codimensional analogue of the signature of the classical Levi form. It will be referred to as the "Levi cone" of S at x.

DEFINITION 1.10. Let S be a CR-manifold in M and let $x \in S$. The Levi cone $\mathcal{C}_x(S)$ of S at x is defined by

$$\mathcal{C}_x(S) := \text{convex hull } \{\mathbf{L}(X_x, X_x) \mid X_x \in T_{\mathbb{C}}S_x\} \subset TS_x / T_{\mathbb{C}}S_x .$$

Observe that $\mathcal{C}_x(S)$ is a real cone which may have an empty interior. The cone $\mathcal{C}_x(S)$ governs the holomorphic extension of CR-functions defined on a neighborhood of x in S. In this regard, we mention a theorem which will be applied in Section 5 (cf. [Bo], p. 202).

THEOREM 1.11. *Let S be a generic* CR-*submanifold of a complex manifold M and let $x \in S$. Assume that the Levi cone at x satisfies the condition*

$$\mathcal{C}_x(S) = TS_x / T_{\mathbb{C}} S_x \,.$$

Then, for each neighborhood ω of x in S there exists a neighborhood Ω of x in M satisfying $\Omega \cap S \subset \omega$ and with the property that every CR-*function of class C^1 on $\Omega \cap S$ extends to a unique holomorphic function on Ω.*

2. – Generic orbits in G

Preliminaries. In this section we give a precise definition and a parametrization of the generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits in G (see [Br] and [St]). We first need some preparation.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A real subalgebra $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$ is a real form of $\mathfrak{g}$ if there exists an involutive antiholomorphic automorphism $\kappa : \mathfrak{g} \to \mathfrak{g}$ such that

$$\mathfrak{g}^{\mathbb{R}} = \{X \in \mathfrak{g} \mid \kappa(X) = X\} \,.$$

Let J denote the complex structure of $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{g}^{\mathbb{R}} \oplus J\mathfrak{g}^{\mathbb{R}} \,.$$

The automorphism κ is referred to as the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}^{\mathbb{R}}$. Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. A real form $G^{\mathbb{R}} \subset G$ is a Lie group whose Lie algebra $\mathfrak{g}^{\mathbb{R}}$ is a real form of $\mathfrak{g}$. If the complex Lie group G is not simply connected, the conjugation κ of $\mathfrak{g}$ may not induce an antiholomorphic automorphism of G (see [Ra]). To overcome this technical difficulty, we assume for the moment G to be simply connected and denote by κ the corresponding automorphism of G as well. Under this assumption, the fixed point set of κ in G is connected and

$$G^{\mathbb{R}} = \{g \in G \mid \kappa(g) = g\} \,.$$

The group $G^{\mathbb{R}} \times G^{\mathbb{R}}$ acts on G by left and right translations

$$\Phi : G^{\mathbb{R}} \times G^{\mathbb{R}} \times G \longrightarrow G \qquad (g, h), x \mapsto gxh^{-1} \,. \tag{2.1}$$

By definition, biinvariant sets and functions on G are those which are invariant under this action.

Denote by $\mathbf{Y}$ the κ-stable "complement" of $G^{\mathbb{R}}$ in G

$$\mathbf{Y} := \{x \in G \mid \kappa(x) = x^{-1}\}. \tag{2.2}$$

The set $\mathbf{Y}$ is a closed real algebraic subset of G which is invariant under the adjoint action of $G^{\mathbb{R}}$. Consider the map

$$\eta : G \to \mathbf{Y} \qquad x \mapsto x\kappa(x)^{-1}. \tag{2.3}$$

Then η has the following properties:

(i) η is equivariant, i.e. $\eta(gxh^{-1}) = g\eta(x)g^{-1}$, for all $g, h \in G^{\mathbb{R}}$. In particular, η maps $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits in G into $\mathrm{Ad}(G^{\mathbb{R}})$-orbits in $\mathbf{Y}$.
(ii) $\eta(x) = \eta(y)$ if and only if $x = yg$, for some $g \in G^{\mathbb{R}}$. In other words, η maps distinct $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits in G into distinct $\mathrm{Ad}(G^{\mathbb{R}})$-orbits in $\mathbf{Y}$, inducing an injective map $G/G^{\mathbb{R}} \hookrightarrow \mathbf{Y}$.
(iii) For $y \in \mathbf{Y}$, one has that $\eta(y) = y^2$.

In general, $\mathbf{Y}$ consists of several irreducible components; the image of η is contained in the same component $\mathbf{Y}^0$ as $\exp J\mathfrak{g}^{\mathbb{R}}$.

REMARK 2.4. If $G^{\mathbb{R}}$ is a compact real form of G, then $\mathbf{Y}^0 = \exp J\mathfrak{g}^{\mathbb{R}}$ and $G = G^{\mathbb{R}} \cdot \mathbf{Y}^0$ is just the polar decomposition of G. If $G^{\mathbb{R}}$ is a non-compact real form, the set $G^{\mathbb{R}} \cdot \mathbf{Y}$ may have a non-empty complement in G. This happens for example when $G^{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$ and $G = \mathrm{SL}(2, \mathbb{C})$. However, the interior of $G^{\mathbb{R}} \cdot \mathbf{Y}^0$ contains the group $G^{\mathbb{R}}$ as the $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbit of the identity element. For this reason, from a representation theory point of view, the most interesting biinvariant domains of G are those which are contained in $G^{\mathbb{R}} \cdot \mathbf{Y}^0$ (see [GG]).

Cartan subalgebras and Weyl groups. Cartan subalgebras in $\mathfrak{g}^{\mathbb{R}}$ play a major role in the description of the generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits in G. Recall that in a real semisimple Lie algebra there may be several, though finitely many, non-conjugate Cartan subalgebras. For a Cartan subalgebra $\mathfrak{t}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$, define

$$\mathfrak{t}^{\mathbb{R}}_{+} = \{X \in \mathfrak{t}^{\mathbb{R}} \mid \mathrm{Ad}_X \text{ has purely imaginary eigenvalues}\}$$

and

$$\mathfrak{t}^{\mathbb{R}}_{-} = \{X \in \mathfrak{t}^{\mathbb{R}} \mid \mathrm{Ad}_X \text{ has real eigenvalues}\}.$$

Then $\mathfrak{t}^{\mathbb{R}} = \mathfrak{t}^{\mathbb{R}}_{+} \oplus \mathfrak{t}^{\mathbb{R}}_{-}$. The complexification $\mathfrak{t}$ of $\mathfrak{t}^{\mathbb{R}}$ is a κ-stable Cartan subalgebra of $\mathfrak{g}$. Let $\Delta(\mathfrak{t})$ denote the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$ and let

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{t})} \mathfrak{g}_\alpha$$

be the corresponding root decomposition. Since $\mathfrak{t}$ is κ-stable, the subspace $\bigoplus \mathfrak{g}_\alpha$ is κ-stable as well.

Denote by $N_G(\mathfrak{t})$ and $Z_G(\mathfrak{t})$ the normalizer and the centralizer of $\mathfrak{t}$ in G respectively. $Z_G(\mathfrak{t})$ is also referred to as the Cartan subgroup corresponding to $\mathfrak{t}$ and denoted by T. The complex Weyl group of $\mathfrak{t}$ in G is defined by

$$W_G(\mathfrak{t}) := N_G(\mathfrak{t})/Z_G(\mathfrak{t})\,. \tag{2.5}$$

Both $Z_G(\mathfrak{t})$ and $N_G(\mathfrak{t})$ are stable under the conjugation κ. Hence there is an induced conjugation $\hat{\kappa}$ on $W_G(\mathfrak{t})$. Define

$$W_\kappa(\mathfrak{t}^{\mathbb{R}}) := N_G(\mathfrak{t}^{\mathbb{R}})/Z_G(\mathfrak{t}^{\mathbb{R}}) = \{w \in W \mid \hat{\kappa}(w) = w\}\,. \tag{2.6}$$

Then $W_\kappa(\mathfrak{t}^{\mathbb{R}})$ is the subgroup of $W_G(\mathfrak{t})$ consisting of the elements w which commute with κ as linear automorphisms of $\mathfrak{t}$. Finally, denote by $N_{G^{\mathbb{R}}}(\mathfrak{t}^{\mathbb{R}})$ and $Z_{G^{\mathbb{R}}}(\mathfrak{t}^{\mathbb{R}})$ the normalizer and the centralizer of $\mathfrak{t}^{\mathbb{R}}$ in $G^{\mathbb{R}}$. The group $Z_{G^{\mathbb{R}}}(\mathfrak{t}^{\mathbb{R}}) = G^{\mathbb{R}} \cap T$ is also referred to as the Cartan subgroup corresponding to $\mathfrak{t}^{\mathbb{R}}$ and denoted by $T^{\mathbb{R}}$. The real Weyl group of $\mathfrak{t}^{\mathbb{R}}$ is defined by

$$W_{\mathbb{R}}(\mathfrak{t}^{\mathbb{R}}) := N_{G^{\mathbb{R}}}(\mathfrak{t}^{\mathbb{R}})/Z_{G^{\mathbb{R}}}(\mathfrak{t}^{\mathbb{R}})\,. \tag{2.7}$$

Each of above Weyl groups is finite and the following inclusions hold:

$$W_{\mathbb{R}}(\mathfrak{t}^{\mathbb{R}}) \subset W_\kappa(\mathfrak{t}^{\mathbb{R}}) \subset W_G(\mathfrak{t})\,.$$

Recall that every two complex Cartan subalgebras are conjugate in $\mathfrak{g}$ and therefore have the same dimension r. The *rank* of G is by definition equal to r.

Regular elements and generic orbits. For an arbitrary smooth action of a non-compact Lie group on a manifold, there is not an obvious notion of "principal orbit". However, in the case of the $G^{\mathbb{R}} \times G^{\mathbb{R}}$-action on G defined in (2.1), there exists a finite number of orbit types with the following properties: *orbits of these types are closed, all have the same maximal dimension, and their union is an open dense subset in G.* Orbits of these types will be referred to as *generic* orbits.

Before we state Bremigan's description of the generic orbits, we recall the different notions of regularity in a semisimple Lie group G (see [Hu]). We denote by $Z_G(x)$ the centralizer of an element x in G.

DEFINITION 2.8. An element $x \in G$ is *regular* if the dimension of $Z_G(x)$ is equal to the rank of G.

DEFINITION 2.9. An element $x \in G$ is *strongly regular* if $Z_G(x)$ is a Cartan subgroup of G. Strongly regular elements in G are denoted by G'.

REMARK 2.10. *Strongly regular* (respectively *regular*) elements form an open dense subset of G which is invariant under the adjoint action of G, but which is not invariant under the $G^{\mathbb{R}} \times G^{\mathbb{R}}$-action. In general, there exist non-semisimple regular elements in G. If G is simply connected, the strongly regular elements coincide with the regular semisimple elements.

The following notion of regularity, which involves the conjugation associated to $G^{\mathbb{R}}$, characterizes the elements of G which lie on generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits (see [Br]).

DEFINITION 2.11. An element $x \in G$ is said to be *regular with respect to* κ if $\eta(x) = x\kappa(x)^{-1} \in \mathbf{Y}$ is strongly regular. Regular elements with respect to κ are denoted by $G_{\mathrm{reg},\kappa}$.

The regularity with respect to κ is different from the strong regularity defined in 2.9. In contrast to G', the set $G_{\mathrm{reg},\kappa}$ is *biinvariant* (cf. (2.3)-i).

DEFINITION 2.12. A $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbit S in G is called *generic* if it is contained in $G_{\mathrm{reg},\kappa}$ or, equivalently, if $\eta(x)$ is strongly regular for every $x \in S$.

The following theorem, due to Bremigan [Br], gives a complete description of the regular elements with respect to κ in G. In particular, it gives a parametrization of the generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits in G. Related results are also contained in [MO] and [St].

THEOREM 2.13 [Br]. *Let G be a complex semisimple simply-connected Lie group and let $G^{\mathbb{R}} \subset G$ be a real form. Let $\mathfrak{t}_1^{\mathbb{R}}, \dots, \mathfrak{t}_m^{\mathbb{R}}$ be a maximal set of non-conjugate Cartan subalgebras in $\mathfrak{g}^{\mathbb{R}}$. Let $\{n_l^j\}_{l,j} \subset N_G(\mathfrak{t}_l^{\mathbb{R}})$, for $l = 1, \dots, m$, be a complete set of representatives of the double cosets in $W_{\mathbb{R}}(\mathfrak{t}_l^{\mathbb{R}}) \setminus W_\kappa(\mathfrak{t}_l^{\mathbb{R}}) / W_{\mathbb{R}}(\mathfrak{t}_l^{\mathbb{R}})$.*

(i) *The elements $\{n_l^j\}_{l,j}$ can be chosen so that*

$$\eta(n_l^j) \cdot \eta(n_l^j) = \mathrm{Id} \quad \textit{and} \quad \eta(n_l^j) \in \exp \mathfrak{t}_l^{\mathbb{R}}.$$

(ii) *The set $G_{\mathrm{reg},\kappa}$ decomposes as the disjoint union*

$$G_{\mathrm{reg},\kappa} = \bigcup_{j,l} G^{\mathbb{R}} \cdot (n_l^j \exp J\mathfrak{t}_l^{\mathbb{R}} \cap G_{\mathrm{reg},\kappa}) \cdot G^{\mathbb{R}}.$$

Each subset $G^{\mathbb{R}} \cdot (n_l^j \exp J\mathfrak{t}_l^{\mathbb{R}} \cap G_{\mathrm{reg},\kappa}) \cdot G^{\mathbb{R}}$ consists of a finite number of connected components. Furthermore, $G_{\mathrm{reg},\kappa}$ is an open and dense subset of G.

(iii) *Every generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbit in G intersects one of the sets $n_l^j \exp J\mathfrak{t}_l^{\mathbb{R}}$ in a finite number of points.*

Observe that the subsets $G^{\mathbb{R}} \cdot \exp J\mathfrak{t}_l^{\mathbb{R}} \cdot G^{\mathbb{R}}$ are contained in $G^{\mathbb{R}} \cdot \exp J\mathfrak{g}^{\mathbb{R}}$, while the subsets $G^{\mathbb{R}} \cdot n_l^j \exp J\mathfrak{t}_l^{\mathbb{R}} \cdot G^{\mathbb{R}}$ are contained in the complement of $G^{\mathbb{R}} \cdot \exp J\mathfrak{g}^{\mathbb{R}}$ in G, whenever the element $[n_l^j] \neq [e]$ in $W_{\mathbb{R}}(t_l^{\mathbb{R}}) \setminus W_\kappa(\mathfrak{t}_l^{\mathbb{R}}) / W_{\mathbb{R}}(\mathfrak{t}_l^{\mathbb{R}})$.

In order to compute the dimension of the generic orbits, we determine the isotropy subgroup in $G^{\mathbb{R}} \times G^{\mathbb{R}}$ of an arbitrary point $x_0 \in G$. We denote it by I_{x_0}.

LEMMA 2.14. *Let $x_0 \in G$. The isotropy subgroup I_{x_0} is given by*

$$I_{x_0} = \{(g, h) \in G^{\mathbb{R}} \times G^{\mathbb{R}} \mid g \in Z_{G^{\mathbb{R}}}(\eta(x_0)),\ h = x_0^{-1} g x_0\}.$$

PROOF. Let (g, h) be an element in I_{x_0}. By definition one has that $gx_0h^{-1} = x_0$ or, equivalently, that $h = x_0^{-1}gx_0 \in G^{\mathbb{R}}$. In particular, the element h is invariant under the conjugation κ, i.e.

$$\kappa(x_0^{-1}gx_0) = x_0^{-1}gx_0 .$$

This is equivalent to $\eta(x_0)g = g\eta(x_0)$ and the lemma follows. □

COROLLARY 2.15. *Let $x_0 = n_0t_0 \in n_0 \exp J\mathfrak{t}^{\mathbb{R}}$ be a point on a generic orbit. Then*

$$I_{x_0} = \{(g, h) \in G^{\mathbb{R}} \times G^{\mathbb{R}} \mid g \in Z_{G^{\mathbb{R}}}(\mathfrak{t}^{\mathbb{R}}),\ h = n_0^{-1}gn_0\} .$$

In particular, I_{x_0} is isomorphic to a Cartan subgroup of $G^{\mathbb{R}}$ and the codimension of the orbit is equal to the rank r of G.

REMARK 2.16. By definition, two orbits are of the same type if they have conjugate isotropy subgroups. Let $x \in n_k \exp i\mathfrak{t}_k^{\mathbb{R}}$ and $y \in n_l \exp i\mathfrak{t}_l^{\mathbb{R}}$ be points on generic orbits, where the Cartan subalgebras $\mathfrak{t}_k^{\mathbb{R}}$, $\mathfrak{t}_l^{\mathbb{R}}$ and the elements n_k, n_l are chosen as in Theorem 2.13. The isotropy subgroups I_x and I_y are conjugate in $G^{\mathbb{R}} \times G^{\mathbb{R}}$ if and only if $\mathfrak{t}_k^{\mathbb{R}} = \mathfrak{t}_l^{\mathbb{R}}$ and $[n_l] = [n_k]$ in the double coset space $W_{\mathbb{R}}(\mathfrak{t}_l^{\mathbb{R}}) \setminus W_\kappa(\mathfrak{t}_l^{\mathbb{R}})/W_{\mathbb{R}}(\mathfrak{t}_l^{\mathbb{R}})$.

REMARK 2.17. Let $\mathfrak{t}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$ be a Cartan subalgebra. Let $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$ be a representative of a double coset in $W_{\mathbb{R}}(\mathfrak{t}^{\mathbb{R}}) \setminus W_\kappa(\mathfrak{t}^{\mathbb{R}})/W_{\mathbb{R}}(\mathfrak{t}^{\mathbb{R}})$, satisfying (i) of Theorem 2.13. Then $\mathrm{Ad}(n_0)$ stabilizes $\mathfrak{g}^{\mathbb{R}}$ if and only if $\eta(n_0)$ belongs to the center $Z(G^{\mathbb{R}})$ of $G^{\mathbb{R}}$. In this case, the biinvariant subsets

$$G^{\mathbb{R}} \cdot n_0 \exp J\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}} \quad \text{and} \quad G^{\mathbb{R}} \cdot \exp J\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}}$$

are biholomorphically equivalent and the corresponding generic orbits have the same complex geometric properties. The biholomorphism is in fact left translation by n_0, which is not equivariant but maps orbits into orbits.

As well shall see in Section 5, if $\mathfrak{t}^{\mathbb{R}} \subset G^{\mathbb{R}}$ is a compact Cartan subalgebra, the generic orbits intersecting $n_0 \exp J\mathfrak{t}^{\mathbb{R}}$ have very different properties depending on whether "$\eta(n_0) \in Z(G^{\mathbb{R}})$" or "$\eta(n_0) \notin Z(G^{\mathbb{R}})$".

3. – The CR-structure of a generic orbit

In this section we determine the tangent space and the maximal complex subspace of a generic orbit at an arbitrary point. In that which follows, we fix the trivialization of the tangent bundle $TG = G \times \mathfrak{g}$ given by the left-invariant vector fields on G. Capital letters X, Y will denote both vectors in $\mathfrak{g}$ and the corresponding left-invariant vector fields on G.

The action of $G^{\mathbb{R}} \times G^{\mathbb{R}}$ on G induces global vector fields on G by the following morphism of Lie algebras

$$\xi : \mathfrak{g}^{\mathbb{R}} \times \mathfrak{g}^{\mathbb{R}} \to \Gamma(G, TG) \quad \xi_{X_1,X_2}(x) = \frac{d}{dt}\Big|_{t=0}(\exp(tX_1)x\exp(-tX_2)) = \mathrm{Ad}(x^{-1})X_1 - X_2 \in TG_x\,. \tag{3.1}$$

The vector fields ξ_{X_1,X_2} are tangent to the orbits and generate the tangent bundle TS of every orbit S. In the local coordinates induced by the given trivialization of $TG = G \times \mathfrak{g}$ one has

$$TS_x = \mathfrak{g}^{\mathbb{R}} + \mathrm{Ad}(x^{-1})\mathfrak{g}^{\mathbb{R}} \subset TG_x = \mathfrak{g}, \qquad x \in S\,. \tag{3.2}$$

The sum in (3.2) is of course not direct. Given two points x_0 and $x_1 = gx_0h^{-1}$ on the same orbit S, the following transformation formula holds:

$$TS_{x_1} = \Phi_*(g,h)TS_{x_0} = \mathrm{Ad}(h)TS_{x_0}\,. \tag{3.3}$$

Here Φ_* denotes the differential of the map Φ defined in (2.1). Since $G^{\mathbb{R}} \times G^{\mathbb{R}}$ acts on G by biholomorphic transformations, the CR-structure of S is preserved by Φ_*, i.e.

$$T_{\mathbb{C}}S_{x_1} = \Phi_*(g,h)T_{\mathbb{C}}S_{x_0} = \mathrm{Ad}(h)T_{\mathbb{C}}S_{x_0}\,. \tag{3.4}$$

Note that TS_x and $T_{\mathbb{C}}S_x$ are viewed as subspaces of $\mathfrak{g}$. By the above formulas, the tangent bundle TS and the complex tangent bundle $T_{\mathbb{C}}S$ of a generic orbit S are determined by the tangent space TS_{x_0} and the complex tangent space $T_{\mathbb{C}}S_{x_0}$ at a fixed reference point $x_0 \in S$.

In order to compute $T_{\mathbb{C}}S_{x_0}$, choose a reference point x_0 of the form

$$x_0 = n_0t_0 \in n_0 \exp J\mathfrak{t}^{\mathbb{R}}\,,$$

where $\mathfrak{t}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$ is a Cartan subalgebra and $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$ (see Theorem 2.13). Let $\mathfrak{t}$ be the complexification of $\mathfrak{t}^{\mathbb{R}}$ and $\mathfrak{g} = \mathfrak{t}\oplus\bigoplus_{\alpha\in\Delta(\mathfrak{t})}\mathfrak{g}_\alpha$ be the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{t}$. The Killing form B of $\mathfrak{g}$ is real valued on $\mathfrak{g}^{\mathbb{R}}$ and non-degenerate on $\mathfrak{t}^{\mathbb{R}}$. If $\mathfrak{q}^{\mathbb{R}}$ is the orthogonal complement of $\mathfrak{t}^{\mathbb{R}}$ in $\mathfrak{g}^{\mathbb{R}}$ with respect to B, it is easily verified that

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}}, \quad [\mathfrak{t}^{\mathbb{R}}, \mathfrak{q}^{\mathbb{R}}] = \mathfrak{q}^{\mathbb{R}}, \quad \mathfrak{q} = \mathfrak{q}^{\mathbb{R}} \oplus J\mathfrak{q}^{\mathbb{R}} = \bigoplus_{\alpha\in\Delta(\mathfrak{t})}\mathfrak{g}_\alpha\,. \tag{3.5}$$

LEMMA 3.6. *In the above notation, the tangent space TS_{x_0} and the complex tangent subspace $T_{\mathbb{C}}S_{x_0}$ at x_0 are given by*

$$TS_{x_0} = \mathfrak{t}^{\mathbb{R}} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{t})}\mathfrak{g}_\alpha \qquad T_{\mathbb{C}}S_{x_0} = \bigoplus_{\alpha\in\Delta(\mathfrak{t})}\mathfrak{g}_\alpha = \mathfrak{q}\,.$$

In particular, $TS_{x_0}/T_{\mathbb{C}}S_{x_0} \cong \mathfrak{t}^{\mathbb{R}}$ and S is generic also as a CR*-submanifold of G (cf. Definition* 1.2*).*

PROOF. According to formula (3.2), the tangent space to S at x_0 is given by

$$TS_{x_0} = \mathrm{Ad}(x_0^{-1})\mathfrak{g}^{\mathbb{R}} + \mathfrak{g}^{\mathbb{R}}\,.$$

Write $\mathrm{Ad}(x_0^{-1}) = \mathbf{DN}$, where $\mathbf{D} := \mathrm{Ad}(t_0^{-1})$ and $\mathbf{N} := \mathrm{Ad}(n_0^{-1})$. Observe that $\mathbf{D}\,|_{\mathfrak{t}} = \mathrm{Id}\,|_{\mathfrak{t}}$ and $\mathbf{D}(\mathfrak{q}) = \mathfrak{q}$. Since n_0 normalizes $\mathfrak{t}^{\mathbb{R}}$, it follows that

$$TS_{x_0} = \mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}} + \mathbf{DN}\left(\mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}}\right) = \mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}} + \mathbf{DN}(\mathfrak{q}^{\mathbb{R}})\,. \tag{3.7}$$

Define $\mathfrak{d} = \mathfrak{d}(x_0) := \mathbf{DN}(\mathfrak{q}^{\mathbb{R}}) \subset \mathfrak{q}$. By Corollary 2.15, one has that $\mathrm{codim}_{\mathbb{R}}(S, G) = \dim_{\mathbb{R}} \mathfrak{t}^{\mathbb{R}} = r$. A dimension argument shows that the sum $\mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}} + \mathbf{DN}(\mathfrak{q}^{\mathbb{R}})$ is direct. Since $\mathbf{DN}$ stabilizes $\mathfrak{q}$, it follows that

$$TS_{x_0} = \mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{q} \quad \text{and} \quad T_{\mathbb{C}}S_{x_0} = \left(\mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{q}\right) \cap J\left(\mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{q}\right) = \mathfrak{q}\,. \qquad \square$$

4. – The Levi form of a generic orbit

The extension $\widehat{Z}$. Let S be a generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbit in G. In this section we compute the Levi form of S at a base point x_0. Given arbitrary tangent vectors Z, W in $T_{\mathbb{C}}S_{x_0}$, it is necessary to extend them to local sections of the subbundle $T_{\mathbb{C}}S$ of the tangent bundle TS and to compute their brackets at x_0 (see Definition 1.7). Observe that the left-invariant vector fields on G are not tangent to the $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits. By Theorem 2.13, the base point $x_0 \in S$ can be taken of the form

$$x_0 = n_0 t_0 \in n_0 \exp J\mathfrak{t}^{\mathbb{R}}\,,$$

where $\mathfrak{t}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$ is a Cartan subalgebra and $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$. Then by Lemma 3.6, the tangent space and the complex tangent subspace to S at x_0 are given by

$$TS_{x_0} = \mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{q} \quad \text{and} \quad T_{\mathbb{C}}S_{x_0} = \mathfrak{q}\,.$$

Let $x = gx_0h^{-1}$ be a point in a neighborhood of x_0 in S. Transformation formula (3.4), given by

$$T_{\mathbb{C}}S_{gx_0h^{-1}} = \mathrm{Ad}(h)T_{\mathbb{C}}S_{x_0} = \mathrm{Ad}(h)\mathfrak{q}\,,$$

identifies the complex tangent subspace at x with a subspace of $\mathfrak{g}$. However, such a formula cannot be used directly to define an extension $\widehat{Z}(x)$ of a vector $Z \in \mathfrak{q}$. Since the group elements g and h are not uniquely determined by x in the presentation $x = gx_0h^{-1}$, an extension given by "$\widehat{Z}(gx_0h^{-1}) := \mathrm{Ad}(h)Z$" would not be well defined. One has in fact that

$$x = gx_0h^{-1} = \tilde{g}x_0\tilde{h}^{-1} \tag{4.1}$$

for every $\tilde{g} = gc$ and $\tilde{h} = hc$, with $c \in Z_{G^{\mathbb{R}}}(\mathfrak{t}^{\mathbb{R}})$, and in general $\mathrm{Ad}(hc)Z \neq \mathrm{Ad}(h)Z$ for $Z \neq 0$.

The ambiguity in the presentation (4.1) can be eliminated if the element "h" is taken in an appropriate small neighborhood of the identity in $\exp \mathfrak{q}^{\mathbb{R}} \subset G^{\mathbb{R}}$. Let U_{x_0} be a neighborhood of x_0 in G, such that every $x \in U_{x_0} \cap S$ can be written in a unique way as

$$x = g x_0 q^{-1}, \quad \text{with } g = g(x) \in G^{\mathbb{R}} \text{ and } q = q(x) \in \exp \mathfrak{q}^{\mathbb{R}}.$$

This determines a well-defined projection

$$q : U_{x_0} \to \exp \mathfrak{q}^{\mathbb{R}}, \quad x \to q(x), \tag{4.2}$$

and an extension $\widehat{Z}(x)$ of $Z \in \mathfrak{q}$ can be defined as follows:

$$\widehat{Z}(x) := \mathrm{Ad}(q(x)^{-1})Z. \tag{4.3}$$

The brackets of the extensions $\widehat{Z}, \widehat{W}$. Let $Z, W \in \mathfrak{q}$ and let $\widehat{Z}, \widehat{W}$ be local extensions of Z, W to a neighborhood of x_0. To compute the brackets $[\widehat{Z}, \widehat{W}]_{x_0}$ mod $\mathfrak{q}$, select a complex basis of $\mathfrak{g}$ as follows: choose an arbitrary complex basis $\{Z_\lambda\}$ of $\mathfrak{q}$ and complete it by a basis $\{H_1, \dots, H_r\}$ of $\mathfrak{t}$, orthonormal with respect to the Killing form B and formed by vectors $H_j \in \mathfrak{t}_{\mathbb{R}} := J\mathfrak{t}_+^{\mathbb{R}} \oplus \mathfrak{t}_-^{\mathbb{R}}$. This is possible since the restriction of the Killing form $B\,|_{\mathfrak{t}_{\mathbb{R}}}$ is positive definite. Extend the vectors $\{Z_\lambda, H_j\}_{\lambda,j}$ to global left-invariant vector fields on G. A local extension $\widehat{Z} = \widehat{Z}(x)$ of a vector $Z \in \mathfrak{q}$ can be written as a linear combination with non-constant coefficients of the vector fields $\{Z_\lambda, H_j\}_{\lambda,j}$

$$\widehat{Z}(x) = \sum_\lambda \phi_Z^\lambda(x) Z_\lambda + \sum_j \psi_Z^j(x) H_j. \tag{4.4}$$

The coefficients $\phi_Z^\lambda(x)$, $\psi_Z^j(x)$ are complex valued functions satisfying

$$\psi_Z^j(x_0) = 0 \quad \forall j \quad \text{and} \quad Z = \sum_\lambda \phi_Z^\lambda(x_0) Z_\lambda. \tag{4.5}$$

LEMMA 4.6. *Let $\widehat{Z}, \widehat{W}$ be local extensions of $Z, W \in \mathfrak{q}$ to some neighborhood of x_0. Then*

$$\left[\widehat{Z}, \widehat{W}\right]_{x_0} \equiv \left([Z, W] + \Sigma_j (Z(\psi_W^j) - W(\psi_Z^j))H_j\right) \qquad \operatorname{mod} \mathfrak{q}$$

$$\left[\widehat{Z}, J\widehat{W}\right]_{x_0} \equiv \left([Z, JW] + \Sigma_j (Z(\psi_W^j)\cdot JH_j - JW(\psi_Z^j)\cdot H_j)\right) \quad \operatorname{mod} \mathfrak{q}.$$

PROOF. Decompose $\widehat{Z}$ and $\widehat{W}$ according to (4.4) and evaluate the brackets at the reference point x_0. From (4.5) it follows that

$$\begin{aligned}
\left[\widehat{Z}, \widehat{W}\right]_{x_0} &\equiv \left[\Sigma_\lambda \phi_Z^\lambda(x) Z_\lambda + \Sigma_j \psi_Z^j(x) H_j,\ \Sigma_\lambda \phi_W^\lambda(x) Z_\lambda + \Sigma_j \psi_W^j(x) H_j\right]_{x_0} \\
&\equiv \Big(\Sigma_\lambda \phi_W^\lambda(x_0)[Z, Z_\lambda] + \Sigma_j Z(\psi_W^j) H_j - \Sigma_j W(\psi_Z^j) H_j \\
&\qquad + \Sigma_{j,k} \psi_Z^j(x_0) H_j(\psi_W^k) H_k - \Sigma_{j,k} \psi_W^j(x_0) H_j(\psi_Z^k) H_k\Big) \\
&\equiv \left([Z, W] + \Sigma_j (Z(\psi_W^j) - W(\psi_Z^j))H_j\right) \qquad \operatorname{mod} \mathfrak{q}.
\end{aligned}$$

The second formula follows in a similar way. □

REMARK 4.7. By Lemma 4.6, the computation of the brackets $[\widehat{Z}, \widehat{W}]_{x_0}$ $\bmod \mathfrak{q}$ involves the Lie brackets of $\mathfrak{g}$ and the first derivatives $Z(\psi_W^j)$ of the functions ψ_W^j with respect to Z at x_0. The explicit description of these derivatives is the main technical difficulty in the computation of the Levi form and requires several preparatory lemmas. Observe that the functions ψ_W^j are *not* holomorphic, and in general $(JZ)(\psi_W^j) \neq i(Z(\psi_W^j))$.

If $\widehat{Z}$, $\widehat{W}$ are the local extensions defined in (4.3), the functions ψ_W^j are defined in terms of the projection q. Hence, to calculate the derivatives $Z(\psi_W^j)$, we need to describe the projection q quite explicitly.

The projection q. In order to construct the projection $q : U_{x_0} \to \exp \mathfrak{q}^{\mathbb{R}}$ consider the map

$$\Psi : \mathfrak{g}^{\mathbb{R}} \times J\mathfrak{t}^{\mathbb{R}} \times \mathfrak{q}^{\mathbb{R}} \to G \qquad (X, H, S) \mapsto \exp X \cdot x_0 \cdot \exp H \cdot \exp S,$$
$$\text{for } X \in \mathfrak{g}^{\mathbb{R}}, \quad H \in J\mathfrak{t}^{\mathbb{R}}, \quad S \in \mathfrak{q}^{\mathbb{R}}.$$

LEMMA 4.8. *The map Ψ is a local diffeomorphism at* 0.

PROOF. The identity

$$\exp X \cdot x_0 \cdot \exp H \cdot \exp S = x_0 \cdot \exp \mathrm{Ad}(x_0^{-1})X \cdot \exp H \cdot \exp S \tag{4.9}$$

implies that the differential $\Psi_* : \mathfrak{g}^{\mathbb{R}} \times J\mathfrak{t}^{\mathbb{R}} \times \mathfrak{q}^{\mathbb{R}} \to \mathfrak{g}$ at 0 is given by

$$(X, H, S) \mapsto \mathrm{Ad}(x_0^{-1})X + H + S\,.$$

If $x_0 \in n_0 \exp J\mathfrak{t}^{\mathbb{R}}$ is a point on a generic orbit, arguing as in the proof of Lemma 3.6, one obtains that

$$\mathrm{Ad}(x_0^{-1})\mathfrak{g}^{\mathbb{R}} = \mathfrak{t}^{\mathbb{R}} \oplus \mathrm{Ad}(x_0^{-1})\mathfrak{q}^{\mathbb{R}} \quad \text{and} \quad \mathrm{Ad}(x_0^{-1})\mathfrak{q}^{\mathbb{R}} + \mathfrak{q}^{\mathbb{R}} = \mathrm{Ad}(x_0^{-1})\mathfrak{q}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}} = \mathfrak{q}\,.$$

Hence the differential Ψ_* at 0 is surjective and the claim follows. □

The map Ψ defines local coordinates on a small neighborhood U_{x_0} of x_0 in G which locally parametrize the $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbits near x_0. Every point $x \in U_{x_0}$ can be written as

$$x = \exp X(x) \cdot x_0 \cdot \exp H(x) \cdot \exp S(x)\,,$$

for suitable vectors $X(x) \in \mathfrak{g}^{\mathbb{R}}$, $H(x) \in i\mathfrak{t}^{\mathbb{R}}$ and $S(x) \in \mathfrak{q}^{\mathbb{R}}$, depending on x. Then the projection q of (4.2) is *by definition*

$$q : U_{x_0} \to \exp \mathfrak{q}^{\mathbb{R}} \qquad q(x) := \exp S(x)\,.$$

Retaining the notation introduced in Section 3, write $\mathrm{Ad}(x_0^{-1}) = \mathbf{DN}$, where $\mathbf{D} = \mathrm{Ad}(t_0^{-1})$ and $\mathbf{N} = \mathrm{Ad}(n_0^{-1})$, and set $\mathfrak{d}^{\mathbb{R}} = \mathfrak{d}^{\mathbb{R}}(x_0) := \mathbf{DN}(\mathfrak{q}^{\mathbb{R}}) \subset \mathfrak{q}$. Denote by $\pi^{\mathfrak{q}}$, $\pi^{\mathfrak{d}}$ and $\pi^{\mathfrak{t}}$ the linear projections induced by the direct sum decomposition

$$\mathfrak{g} = \mathbf{DN}(\mathfrak{q}^{\mathbb{R}}) \oplus \mathfrak{t} \oplus \mathfrak{q}^{\mathbb{R}} = \mathfrak{d}^{\mathbb{R}} \oplus \mathfrak{t} \oplus \mathfrak{q}^{\mathbb{R}}$$

onto $\mathfrak{q}^{\mathbb{R}}$, $\mathfrak{d}^{\mathbb{R}}$ and $\mathfrak{t}$ respectively. Since (4.9) holds and the maps

$$(A, B, C) \mapsto x_0 \exp A \cdot \exp B \cdot \exp C \ \text{ and } \ (A, B, C) \mapsto x_0 \exp(A + B + C)$$

have the same differential at 0, the projection q can be approximated at infinitesimal level by the linear projection

$$\pi^{\mathfrak{q}} : \mathfrak{g} \to \mathfrak{q}^{\mathbb{R}} .$$

Observe that both q and $\pi^{\mathfrak{q}}$ *depend on the point* x_0. Let Z be a vector in a small neighborhood of 0 in $\mathfrak{g}$, such that $x_0 \cdot \exp Z \in U_{x_0}$. It follows that

$$\begin{aligned} q : U_{x_0} \to \exp \mathfrak{q}^{\mathbb{R}} \quad q(x_0 \exp Z) &= \exp S(x_0 \exp Z) \\ &= \exp S\big(x_0 \exp(\pi^{\mathfrak{d}}(Z) + \pi^{\mathfrak{t}}(Z) + \pi^{\mathfrak{q}}(Z))\big) \\ &\approx \exp \pi^{\mathfrak{q}}(Z) \quad (\text{up to } 1^{\text{st}} \text{ order}) . \end{aligned} \tag{4.10}$$

It turns out that the linear projection $\pi^{\mathfrak{q}}$ is precisely what is needed to compute the derivatives in Lemma 4.6, when the extensions $\widehat{Z}$, $\widehat{W}$ are defined by (4.3).

The linear projection $\pi^{\mathfrak{q}}$. We want to obtain explicit formulas for the linear projection $\pi^{\mathfrak{q}}$. Recall that the base point x_0 is of the form

$$x_0 = n_0 t_0 = n_0 \exp J T_0 , \tag{4.11}$$

where $T_0 \in \mathfrak{t}^{\mathbb{R}}$, $\mathfrak{t}^{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{R}}$ and $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$. Let $\mathfrak{t} \subset \mathfrak{g}$ be the complexification of $\mathfrak{t}^{\mathbb{R}}$ and let

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{t})} \mathfrak{g}_\alpha$$

be the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Write $\overline{Z}$ for the conjugate $\kappa(Z)$ of a vector $Z \in \mathfrak{g}$. Since $\mathfrak{t}$ is κ-stable, κ acts on the set of roots by the rule

$$\kappa(\alpha)(H) = \overline{\alpha(\kappa(H))} .$$

A basis $\{Z_\alpha\}_{\alpha \in \Delta}$ of $\mathfrak{q}$, which is formed by root vectors $Z_\alpha \in \mathfrak{g}_\alpha$, is said to be κ-stable if

$$\overline{Z_\alpha} = Z_{\kappa(\alpha)}, \qquad \alpha \in \Delta .$$

The complex Weyl group $W_G(\mathfrak{t})$ acts on the root system $\Delta(\mathfrak{t})$ by

$$\gamma(\alpha)(H) = \alpha(\gamma^{-1}(H)), \qquad \gamma \in W_G(\mathfrak{t}).$$

The linear operators

$$\mathbf{D} = \mathrm{Ad}(t_0^{-1}) \quad \mathbf{N} = \mathrm{Ad}(n_0^{-1}) \quad \mathbf{C} := \mathrm{Ad}(\eta(n_0)) \tag{4.12}$$

are associated to the basepoint x_0. Note that $\mathbf{N}$ and $\mathbf{C}$ stabilize the root decomposition. Denote by w the element in $W_G(t)$ corresponding to the restriction $\mathbf{N} \mid \mathfrak{t}$. Write

$$\eta(n_0) = \exp C, \quad C \in \mathfrak{t}^{\mathbb{R}} \text{ and } c(\alpha) := e^{\alpha(C)}. \tag{4.13}$$

Since $\mathbf{C}^2 = \mathrm{Id}$ (cf. Theorem 2.13), it follows that $c(\alpha) \in \{-1; +1\}$ and $\mathbf{C}|\mathfrak{g}_\alpha = c(\alpha)\mathrm{Id}$.

Define A_R and A_I the "real" and the "imaginary" part of a linear operator $A : \mathfrak{q} \to \mathfrak{q}$, as the compositions of the following maps:

$$A_R : \mathfrak{q}^{\mathbb{R}} \xrightarrow{A} J\mathfrak{q}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}} \xrightarrow{\mathrm{pr}_2} \mathfrak{q}^{\mathbb{R}} \qquad A_I : \mathfrak{q}^{\mathbb{R}} \xrightarrow{A} J\mathfrak{q}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}} \xrightarrow{\mathrm{pr}_1} J\mathfrak{q}^{\mathbb{R}}.$$

Extend these $\mathbb{R}$-linear maps to $\mathbb{C}$-linear maps $\mathfrak{q} \to \mathfrak{q}$. Denote by $\bar{A}$ the conjugate operator $\bar{A} = \kappa \circ A \circ \kappa$. Then one has that $A_R = \frac{1}{2}(A + \bar{A})$ and $A_I = \frac{1}{2}(A - \bar{A})$. To simplify the notation, in the next lemma we write $\mathbf{A} := \mathbf{DN}$.

Lemma 4.14 (Projection formula). *Let W be an arbitrary vector in $\mathfrak{q}$. Then the linear projection $\pi^{\mathfrak{q}}$ of W is given by*

$$\pi^{\mathfrak{q}}(W) = \frac{W + \overline{W}}{2} - \mathbf{A}_R\left((\mathbf{A}_I)^{-1}\left(\frac{W - \overline{W}}{2}\right)\right) = \frac{1}{2}\left(\mathbf{A}(\mathbf{A}_I)^{-1}\overline{W} - \bar{\mathbf{A}}(\mathbf{A}_I)^{-1}W\right).$$

Proof. An arbitrary vector $W \in \mathfrak{q}$ can be decomposed both with respect to $\mathfrak{q} = J\mathfrak{q}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}}$ and with respect to $\mathfrak{q} = \mathfrak{d}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}}$, yielding

$$W = W^I + W^R = \frac{W - \overline{W}}{2} + \frac{W + \overline{W}}{2}$$

(the decomposition with respect to $J\mathfrak{q}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}}$) and

$$W = \mathbf{A}(X) + Y = \pi^{\mathfrak{d}}(W) + \pi^{\mathfrak{q}}(W), \; X, Y \in \mathfrak{q}^{\mathbb{R}}$$

(the decomposition with respect to $\mathfrak{d}^{\mathbb{R}} \oplus \mathfrak{q}^{\mathbb{R}}$).

By the uniqueness of these decomposition, one obtains the following system of linear equations

$$W^I = \mathbf{A}_I(X) \qquad W^R = Y + \mathbf{A}_R(X).$$

Observe that, by Lemma 3.6, the operator $\mathbf{A}_I$ is invertible. The solution of the above system yields the first identity of the lemma. The second one is an elementary reformulation of the former. □

For $\alpha \in \Delta(\mathfrak{t})$ define the κ-stable subset

$$\mathfrak{q}[\alpha] := \mathfrak{g}_\alpha + \mathfrak{g}_{\kappa(\alpha)}\,.$$

The next lemma is a collection of elementary facts which are need later.

LEMMA 4.15. *In the above notation* (*cf.* (4.11) (4.12) (4.13)), *the following identities hold:*

$$\begin{array}{rl} \text{i)} & \mathfrak{g}_\alpha = \operatorname{Eig}\left(\mathbf{D}, e^{-i\alpha(T_0)}\right) = \operatorname{Eig}\left(\overline{\mathbf{D}}, e^{i\alpha(T_0)}\right) \\ \text{ii)} & \mathfrak{g}_{\kappa(\alpha)} = \operatorname{Eig}\left(\mathbf{D}, e^{-i\,\overline{\alpha(T_0)}}\right) = \operatorname{Eig}\left(\overline{\mathbf{D}}, e^{i\overline{\alpha(T_0)}}\right) \\ \text{iii)} & \mathbf{N}\circ\kappa = \kappa\circ\mathbf{N}\circ\mathbf{C} \\ \text{iv)} & \overline{\mathbf{N}} = \mathbf{N}\circ\mathbf{C} \\ \text{v)} & \mathbf{N}(\mathfrak{q}[\alpha]) = \mathfrak{q}\left[w^{-1}(\alpha)\right] \\ \text{vi)} & c(\alpha) = c(\kappa(\alpha)) = c(-\alpha),\ \textit{hence}\ \mathbf{C}\mid_{\mathfrak{q}[\alpha]} = c(\alpha)\,\mathrm{Id}\,. \end{array}$$

PROOF. The proof of the lemma is elementary and is omitted. □

Now we can calculate the linear projection $\pi^{\mathfrak{q}}$ on arbitrary root vectors $W_\alpha \in \mathfrak{g}_\alpha$.

LEMMA 4.16 (Projection formula). *Let W_α be an arbitrary root vector in $\mathfrak{g}_\alpha$. Then*

$$2\pi^{\mathfrak{q}}(W_\alpha) = \begin{cases} \big(1 - i\cot(\alpha(T_0))\big)W_\alpha + \big(1 + i\cot(\overline{\alpha(T_0)}\,)\big)\overline{W}_\alpha & \textit{if } c(\alpha) = 1 \\ \big(1 + i\tan(\alpha(T_0))\big)W_\alpha + (1 - i\tan(\overline{\alpha(T_0)}\,))\overline{W}_\alpha & \textit{if } c(\alpha) = -1\,. \end{cases}$$

PROOF. We compute the projection $\pi^{\mathfrak{q}}(W_\alpha)$ using the formulas of Lemma 4.14. In order to do this, we need to compute $\mathbf{DN}$ and $\overline{\mathbf{DN}}$ on the elements of a κ-stable basis $\{Z_\alpha\}_{\alpha\in\Delta(\mathfrak{t})}$ of $\mathfrak{q}$ formed by root vectors $Z_\alpha \in \mathfrak{g}_\alpha$. For simplicity, we write α^w for $w^{-1}(\alpha)$, and W_{α^w} for $\mathbf{N}(Z_\alpha)$. By definition, $\alpha^w(H) = \alpha(w(H))$, for every $H \in \mathfrak{t}$. Using the identities i)-vi) of Lemma 4.15, we derive the following formulas

$$\begin{aligned} \mathbf{DN}(Z_\alpha) &= \mathbf{D}(\mathbf{N}(Z_\alpha)) = e^{-i\alpha^w(T_0)}\mathbf{N}(Z_\alpha) = e^{-i\alpha^w(T_0)}W_{\alpha^w} \\ \mathbf{DN}(\overline{Z}_\alpha) &= \mathbf{D}(\overline{\mathbf{NC}(Z_\alpha)}) = \mathbf{D}(c(\alpha)\overline{\mathbf{N}(Z_\alpha)}) = c(\alpha)e^{-i\,\overline{\alpha^w(T_0)}}\overline{\mathbf{N}(Z_\alpha)} \\ &= c(\alpha)e^{-i\,\overline{\alpha^w(T_0)}}\overline{W}_{\alpha^w} \\ \overline{\mathbf{DN}}(Z_\alpha) &= \overline{\mathbf{D}}(\mathbf{NC}(Z_\alpha)) = c(\alpha)e^{i\alpha^w(T_0)}\mathbf{N}(Z_\alpha) = c(\alpha)e^{i\alpha^w(T_0)}W_{\alpha^w} \\ \overline{\mathbf{DN}}(\overline{Z}_\alpha) &= \overline{\mathbf{D}}(\mathbf{NC}(\overline{Z}_\alpha)) = c(\alpha)\overline{\mathbf{D}}(\mathbf{N}(\overline{Z}_\alpha)) = c(\alpha)\overline{\mathbf{D}}(\overline{\mathbf{NC}(Z_\alpha)}) \\ &= c^2(\alpha)e^{i\,\overline{\alpha^w(T_0)}}\overline{\mathbf{N}(Z_\alpha)} = e^{i\,\overline{\alpha^w(T_0)}}\overline{W}_{\alpha^w}\,. \end{aligned}$$

Since $\mathbf{A}_I = \frac{\mathbf{A}-\bar{\mathbf{A}}}{2}$, from the above equations we obtain

$$(\mathbf{DN})_I(Z_\alpha) = \frac{e^{-i\alpha^w(T_0)} - c(\alpha)e^{i\alpha^w(T_0)}}{2} W_{\alpha^w}$$
$$= \begin{cases} -i\sin\alpha^w(T_0)W_{\alpha^w} & \text{if } c(\alpha) = \ \ 1 \\ \ \ \cos\alpha^w(T_0)W_{\alpha^w} & \text{if } c(\alpha) = -1\,, \end{cases}$$

$$(\mathbf{DN})_I(\overline{Z}_\alpha) = c(\alpha)\frac{e^{-i\,\overline{\alpha^w(T_0)}} - c(\alpha)e^{i\,\overline{\alpha^w(T_0)}}}{2}\overline{W}_{\alpha^w}$$
$$= \begin{cases} -i\sin\overline{\alpha^w(T_0)}\,\overline{W}_{\alpha^w} & \text{if } c(\alpha) = \ \ 1 \\ -\cos\overline{\alpha^w(T_0)}\,\overline{W}_{\alpha^w} & \text{if } c(\alpha) = -1\,. \end{cases}$$

In particular, we have that $(\mathbf{DN})_I(\mathfrak{q}[\alpha]) \subset \mathfrak{q}[\alpha^w]$. The inverse map $(\mathbf{DN})_I^{-1}$: $\mathfrak{q}[\alpha] \to \mathfrak{q}[w(\alpha)]$ is given by

$$(\mathbf{DN})_I^{-1}(W_\alpha) = \begin{cases} \dfrac{1}{-i\sin\alpha(T_0)}Z_{w(\alpha)} & \text{if } c(\alpha) = \ \ 1 \\ \dfrac{1}{\cos\alpha(T_0)}Z_{w(\alpha)} & \text{if } c(\alpha) = -1 \end{cases}$$

$$(\mathbf{DN})_I^{-1}(\overline{W}_\alpha) = \begin{cases} \dfrac{1}{-i\sin\overline{\alpha(T_0)}}\,\overline{Z}_{w(\alpha)} & \text{if } c(\alpha) = \ \ 1 \\ \dfrac{1}{-\cos\overline{\alpha(T_0)}}\,\overline{Z}_{w(\alpha)} & \text{if } c(\alpha) = -1\,. \end{cases}$$

Observe that $Z_{w(\alpha)} = \mathbf{N}^{-1}(W_\alpha)$. Putting everything together, we obtain then

$$2\pi^{\mathfrak{q}}(W_\alpha) = \begin{cases} \dfrac{e^{i\alpha(T_0)}}{i\sin\alpha(T_0)}W_\alpha - \dfrac{e^{-i\,\overline{\alpha(T_0)}}}{i\sin\overline{\alpha(T_0)}}\,\overline{W}_\alpha & \text{if } c(\alpha) = \ \ 1 \\ \dfrac{e^{i\alpha(T_0)}}{\cos\alpha(T_0)}W_\alpha + \dfrac{e^{-i\,\overline{\alpha(T_0)}}}{\cos\overline{\alpha(T_0)}}\,\overline{W}_\alpha & \text{if } c(\alpha) = -1\,. \end{cases}$$ □

The first derivatives of the coefficient functions. Recall that the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{q}$ is orthogonal with respect to the Killing form B of $\mathfrak{g}$ and that $\{H_j\}$ is an orthogonal basis of $\mathfrak{t}$ with respect to B. Hence the coefficient functions $\{\psi_W^j\}$ defined in (4.4) can be expressed as

$$\psi_W^j(x) = B\big(\mathrm{Ad}\,(q^{-1}(x))W, H_j\big)\,. \tag{4.17}$$

Let $Z \in \mathfrak{q}$. Then the first derivatives of the functions $\{\psi^j_W\}$ with respect to Z at x_0 are given by

$$\begin{aligned} Z(\psi^j_W)(x_0) &= \frac{d}{dt}\Big|_{t=0} B\big(\operatorname{Ad}(q^{-1}(x_0 \exp tZ))W, H_j\big) \\ &= B\left(\left[\frac{d}{dt}\Big|_{t=0}(q^{-1}(x_0 \exp tZ)), W\right], H_j\right). \end{aligned}$$

By (4.10), one obtains

$$\frac{d}{dt}\Big|_{t=0}(q^{-1}(x_0 \exp tZ)) = -\pi^{\mathfrak{q}}(Z). \tag{4.18}$$

Finally, (4.18) and Lemma 4.6 yield the following bracket formula

$$[\widehat{Z}, \widehat{W}]_{x_0} \equiv \big([Z, W] - [\pi^{\mathfrak{q}}(Z), W] + [\pi^{\mathfrak{q}}(W), Z]\big) \mod \mathfrak{q}. \tag{4.19}$$

The Levi form of root vectors. We calculate the Levi form of a generic orbit S at the reference point x_0. Recall that the point x_0 is of the form

$$x_0 = n_0 \cdot \exp JT_0,$$

where $T_0 \in \mathfrak{t}^{\mathbb{R}}$, $\mathfrak{t}^{\mathbb{R}}$ is a Cartan subalgebra in $\mathfrak{g}^{\mathbb{R}}$, and $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$. Then the complex tangent space $T_{\mathbb{C}}S_{x_0}$ is identified with $\mathfrak{q}$ and $T^{\mathbb{C}}S/T^{\mathbb{C}}_{\mathbb{C}}S = (\mathfrak{q} \oplus \mathfrak{t}^{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}/\mathfrak{q} \otimes_{\mathbb{R}} \mathbb{C}$ is identified with the complex Cartan subalgebra $\mathfrak{t}$ by the map

$$(\mathfrak{q} \oplus \mathfrak{t}^{\mathbb{R}}) \otimes \mathbb{C} \to \mathfrak{t}, \quad X \otimes 1 + Y \otimes i \mapsto X + JY.$$

In particular, the Levi form of S at x_0 is a map

$$\mathbf{L} : \mathfrak{q} \times \mathfrak{q} \to \mathfrak{t} \tag{4.20}$$

(cf. Lemma 3.6, Definition 1.7). Let $\{Z_\alpha\}_{\alpha \in \Delta(\mathfrak{t})}$ be a κ-stable basis of $\mathfrak{q}$ formed by roots vectors with respect to $\mathfrak{t}$. The next proposition gives explicit formulas for the Levi form $\mathbf{L}(Z_\alpha, Z_\beta)$.

PROPOSITION 4.21. *Let $Z_\alpha \in \mathfrak{g}_\alpha$ and $Z_\alpha \in \mathfrak{g}_\beta$. Then the following identities hold modulo* $\mathfrak{q}$

$$\text{for } c(\alpha) = 1 \quad 2\mathbf{L}(Z_\alpha, Z_\beta) \equiv \begin{cases} -(\cot\alpha(T_0) + i)[Z_\alpha, Z_{-\alpha}] & \text{if } \beta = -\kappa(\alpha) \\ 0 & \text{if } \beta \neq -\kappa(\alpha), \end{cases}$$

$$\text{for } c(\alpha) = -1 \quad 2\mathbf{L}(Z_\alpha, Z_\beta) \equiv \begin{cases} (\tan\alpha(T_0) - i)[Z_\alpha, Z_{-\alpha}] & \text{if } \beta = -\kappa(\alpha) \\ 0 & \text{if } \beta \neq -\kappa(\alpha). \end{cases}$$

PROOF. Combining (1.9) and the bracket formula (4.19), one obtains

$$\begin{aligned} 2\mathbf{L}(Z_\alpha, Z_\beta) &\equiv -[\widehat{Z}_\alpha, J\widehat{Z}_\beta]_{x_0} + i[\widehat{Z}_\alpha, \widehat{Z}_\beta]_{x_0} \\ &\equiv -[Z_\alpha, iZ_\beta] + [\pi^{\mathfrak{q}}(Z_\alpha), iZ_\beta] - [\pi^{\mathfrak{q}}(JZ_\beta), Z_\alpha] \\ &\quad + i\big([Z_\alpha, Z_\beta] - [\pi^{\mathfrak{q}}(Z_\alpha), Z_\beta] + [\pi^{\mathfrak{q}}(Z_\beta), Z_\alpha]\big) \\ &\equiv i[\pi^{\mathfrak{q}}(Z_\beta), Z_\alpha] - [\pi^{\mathfrak{q}}(JZ_\beta), Z_\alpha] \mod \mathfrak{q}. \end{aligned}$$

By substituting the projection formulas (4.16) in the above expression, the desired result follows. □

If $\mathfrak{t} \subset \mathfrak{g}$ is an arbitrary κ-stable Cartan subalgebra, the roots in $\Delta = \Delta(\mathfrak{t})$ can be subdivided depending on their behaviour under the action of the conjugation κ.

DEFINITION 4.22. Denote by

$$\begin{array}{lll} \Delta_r := \{\alpha \in \Delta \mid \kappa(\alpha) = \alpha\} & \text{i.e. } \alpha(\mathfrak{t}^{\mathbb{R}}) \subset \mathbb{R} & \text{the real roots}, \\ \Delta_i := \{\alpha \in \Delta \mid \kappa(\alpha) = -\alpha\} & \text{i.e. } \alpha(\mathfrak{t}^{\mathbb{R}}) \subset i\mathbb{R} & \text{the imaginary roots}, \\ \Delta_c := \{\alpha \in \Delta \mid \kappa(\alpha) \neq \pm\alpha\} & \text{i.e. } \alpha(\mathfrak{t}^{\mathbb{R}}) \not\subset \mathbb{R}, i\mathbb{R} & \text{the complex roots}. \end{array}$$

Let $\mathfrak{q}[\alpha] := \mathfrak{g}_\alpha + \mathfrak{g}_{\kappa(\alpha)}$. Then $\mathfrak{q}[\alpha] = \mathfrak{g}_\alpha$, for $\alpha \in \Delta_r$ and $\mathfrak{q}[\alpha] = \mathfrak{q}[-\alpha] = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, for $\alpha \in \Delta_i$.

COROLLARY 4.23. *Let $\Delta = \Delta_r \cup \Delta_i \cup \Delta_c$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Then*

$$\mathbf{L}(\mathfrak{q}[\alpha], \mathfrak{q}[\beta]) = 0 \quad \textit{if} \quad \begin{cases} \alpha \in \Delta_r & \textit{and} \quad \beta \neq -\alpha \\ \alpha \in \Delta_i & \textit{and} \quad \beta \neq \ \ \alpha \\ \alpha \in \Delta_c & \textit{and} \quad \beta \neq -\alpha, -\kappa(\alpha). \end{cases}$$

PROOF. The Levi form $\mathbf{L}(\mathfrak{q}[\alpha], \mathfrak{q}[\beta])$ is determined by the terms

$$\mathbf{L}(Z_\alpha, Z_\beta), \quad \mathbf{L}(Z_\alpha, \overline{Z}_\beta), \quad \mathbf{L}(\overline{Z}_\alpha, Z_\beta), \quad \mathbf{L}(\overline{Z}_\alpha, \overline{Z}_\beta).$$

Then, by Proposition 4.21, one has that $\mathbf{L}(\mathfrak{q}[\alpha], \mathfrak{q}[\beta])$ is identically zero, unless $\beta = -\alpha$ or $\beta = -\bar{\alpha}$. □

Quadratic Levi form. We conclude this section by giving an explicit formula for the quadratic Levi form of a generic orbit S at the base point x_0. As usual, the point x_0 is of the form $x_0 = n_0 \exp JT_0$, where $T_0 \in \mathfrak{t}^{\mathbb{R}}$, $\mathfrak{t}^{\mathbb{R}}$ is a Cartan subalgebra and $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$. Then, the quadratic Levi form is the map $\widehat{\mathbf{L}} : \mathfrak{q} \to \mathfrak{t}^{\mathbb{R}}$ defined by $\widehat{\mathbf{L}}(Z) := \mathbf{L}(Z, Z)$. Write

$$\mathfrak{q} = \bigoplus_{\Delta_r} \mathfrak{q}[\alpha] \oplus \mathfrak{q}[-\alpha] \oplus \bigoplus_{\Delta_i} \mathfrak{q}[\alpha] \oplus \bigoplus_{\Delta_c} \mathfrak{q}[\alpha] \oplus \mathfrak{q}[-\alpha], \tag{4.24}$$

with the convention that each term $\mathfrak{q}[\alpha] = \mathfrak{q}[\kappa(\alpha)]$ appears in the above direct sum exactly once. By Corollary 4.23, decomposition (4.24) is orthogonal with respect to the Levi form. Therefore the quadratic Levi form is given by

$$\widehat{\mathbf{L}}\,|_{\mathfrak{q}} = \sum_{\Delta_r} \widehat{\mathbf{L}}\,|_{\mathfrak{q}[\alpha]\oplus\mathfrak{q}[-\alpha]} + \sum_{\Delta_i} \widehat{\mathbf{L}}\,|_{\mathfrak{q}[\alpha]} + \sum_{\Delta_c} \widehat{\mathbf{L}}\,|_{\mathfrak{q}[\alpha]\oplus\mathfrak{q}[-\alpha]}. \tag{4.25}$$

In the next proposition we explicitly compute each of the terms appearing in (4.25). By $\operatorname{Re} X$ and $\operatorname{Im} X$ we denote the real and imaginary part of a vector $X \in \mathfrak{t}$ with respect to the decomposition $\mathfrak{t} = \mathfrak{t}^{\mathbb{R}} \oplus J\mathfrak{t}^{\mathbb{R}}$.

PROPOSITION 4.26. *Let Z be a vector in $\mathfrak{q}$ and let*

$$Z = \sum_{\alpha\in\Delta_r} (a_\alpha Z_\alpha + a_{-\alpha} Z_{-\alpha}) + \sum_{\alpha\in\Delta_i} (a_\alpha Z_\alpha + a_{-\alpha} Z_{-\alpha}) + \sum_{\alpha\in\Delta_c} (a_\alpha Z_\alpha + b_\alpha \overline{Z}_\alpha + a_{-\alpha} Z_{-\alpha} + b_{-\alpha} \overline{Z}_{-\alpha}),$$

for $\{a_\alpha, b_\alpha\} \in \mathbb{C}$, be the decomposition of Z with respect to a κ-stable basis $\{Z_\alpha\}_{\alpha\in\Delta(\mathfrak{t})}$ of $\mathfrak{q}$ formed by root vectors. Then:

(1) *For $\alpha \in \Delta_r$*

$$\widehat{\mathbf{L}}(a_\alpha Z_\alpha + a_{-\alpha} Z_{-\alpha}) = \begin{cases} \operatorname{Re}\big(a_\alpha \bar{a}_{-\alpha}(-\cot\alpha(T_0) - i)\big)[Z_\alpha, Z_{-\alpha}] & \textit{for } c(\alpha) = 1 \\ \operatorname{Re}\big(a_\alpha \bar{a}_{-\alpha}(\tan\alpha(T_0) - i)\big)[Z_\alpha, Z_{-\alpha}] & \textit{for } c(\alpha) = -1; \end{cases}$$

(2) *For $\alpha \in \Delta_i$ and $c(\alpha) = 1$,*

$$2\widehat{\mathbf{L}}(a_\alpha Z_\alpha + a_{-\alpha} Z_{-\alpha}) = \big(\,|\,a_\alpha\,|^2\,(\coth \operatorname{Im}\alpha(T_0) - 1) + |\,a_{-\alpha}\,|^2\,(1 + \coth \operatorname{Im}\alpha(T_0))\big) \quad J[Z_\alpha, Z_{-\alpha}];$$

If $\operatorname{Im}\alpha(T_0) > 0$, *then*

$$\coth \operatorname{Im}\alpha(T_0) > 1 \textit{ and } \coth \operatorname{Im}\alpha(T_0) - 1 > 0.$$

(3) *For $\alpha \in \Delta_i$ and $c(\alpha) = -1$,*

$$2\widehat{\mathbf{L}}(a_\alpha Z_\alpha + a_{-\alpha} Z_{-\alpha}) = \big(\,|\,a_\alpha\,|^2\,(\tanh \operatorname{Im}\alpha(T_0) - 1) + |\,a_{-\alpha}\,|^2\,(1 + \tanh \operatorname{Im}\alpha(T_0))\big) \quad J[Z_\alpha, Z_{-\alpha}].$$

If $\operatorname{Im}\alpha(T_0) > 0$, *then* $0 < \tanh \operatorname{Im}\alpha(T_0) < 1$; *in particular*

$$\tanh \operatorname{Im}\alpha(T_0) - 1 < 0 \quad \textit{and} \quad \tanh \operatorname{Im}\alpha(T_0) + 1 > 0.$$

(4) *For $\alpha \in \Delta_c$*

$$\widehat{\mathbf{L}}(a_\alpha Z_\alpha + b_\alpha \overline{Z}_\alpha + a_{-\alpha} Z_{-\alpha} + b_{-\alpha} \overline{Z}_{-\alpha})$$

$$= \begin{cases} \operatorname{Re}\big((-a_\alpha \bar{b}_{-\alpha}(i + \cot\alpha(T_0)) + a_{-\alpha}\bar{b}_\alpha(i - \cot\alpha(T_0)))\,[Z_\alpha, Z_{-\alpha}]\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for } c(\alpha) = 1 \\ \operatorname{Re}\big((-a_\alpha \bar{b}_{-\alpha}(i - \tan\alpha(T_0)) + a_{-\alpha}\bar{b}_\alpha(i + \tan\alpha(T_0)))\,[Z_\alpha, Z_{-\alpha}]\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for } c(\alpha) = -1. \end{cases}$$

PROOF. 1) If $\alpha \in \Delta_r$, then $-\alpha \in \Delta_r$ and $[Z_\alpha, Z_{-\alpha}] \in \mathfrak{t}$. For $c(\alpha) = 1$, applying Proposition 4.21 we obtain

$$\begin{aligned}\widehat{\mathbf{L}}(a_\alpha Z_\alpha + a_{-\alpha} Z_{-\alpha}) &= 2\operatorname{Re}(a_\alpha \bar{a}_{-\alpha} \mathbf{L}(Z_\alpha, Z_{-\alpha})) \\ &= \operatorname{Re}(a_\alpha \bar{a}_{-\alpha}(-\cot\alpha(T_0) - i)) \quad [Z_\alpha, Z_{-\alpha}].\end{aligned}$$

If $c(\alpha) = -1$, the proof of the corresponding statement follows in a similar way.

2) Let $\alpha \in \Delta_i$ and $c(\alpha) = 1$. Recall that in this case $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are orthogonal with respect to the Levi form $\mathbf{L}$. Applying Proposition 4.21, we obtain

$$\begin{aligned}
2\widehat{\mathbf{L}}(a_\alpha Z_\alpha + a_{-\alpha} Z_{-\alpha}) &= \mid a_\alpha \mid^2 2\widehat{\mathbf{L}}(Z_\alpha) + \mid a_{-\alpha} \mid^2 2\widehat{\mathbf{L}}(Z_{-\alpha}) \\
&= \mid a_\alpha \mid^2 (-\cot\alpha(T_0) - i)\,[Z_\alpha, Z_{-\alpha}] \\
&\quad + \mid a_{-\alpha} \mid^2 (\cot\alpha(T_0) - i)\,[Z_{-\alpha}, Z_\alpha] \\
&= \big(\mid a_\alpha \mid^2 (-\coth i\alpha(T_0) - 1) \\
&\quad + \mid a_{-\alpha} \mid^2 (1 - \coth i\alpha(T_0))\big)\, i\,[Z_\alpha, Z_{-\alpha}] \\
&= \big(\mid a_\alpha \mid^2 (\coth \operatorname{Im}\alpha(T_0) - 1) \\
&\quad + \mid a_{-\alpha} \mid^2 (1 + \coth \operatorname{Im}\alpha(T_0))\big)\, i\,[Z_\alpha, Z_{-\alpha}]\,.
\end{aligned}$$

3) Let $\alpha \in \Delta_i$ and $c(\alpha) = -1$. Then we have

$$\begin{aligned}
2\widehat{\mathbf{L}}(a_\alpha Z_\alpha + a_{-\alpha} Z_{-\alpha}) &= \mid a_\alpha \mid^2 2\widehat{\mathbf{L}}(Z_\alpha) + \mid a_{-\alpha} \mid^2 2\widehat{\mathbf{L}}(Z_{-\alpha}) \\
&= \mid a_\alpha \mid^2 (\tan\alpha(T_0) - i)\,[Z_\alpha, Z_{-\alpha}] \\
&\quad + \mid a_{-\alpha} \mid^2 (-\tan\alpha(T_0) - i)\,[Z_{-\alpha}, Z_\alpha] \\
&= \big(\mid a_\alpha \mid^2 (-\tanh i\alpha(T_0) - 1) \\
&\quad + \mid a_{-\alpha} \mid^2 (1 - \tanh i\alpha(T_0))\big)\, i\,[Z_\alpha, Z_{-\alpha}] \\
&= \big(\mid a_\alpha \mid^2 (\tanh \operatorname{Im}\alpha(T_0) - 1) \\
&\quad + \mid a_{-\alpha} \mid^2 (1 + \tanh \operatorname{Im}\alpha(T_0))\big)\, i\,[Z_\alpha, Z_{-\alpha}]\,.
\end{aligned}$$

4) Finally, if $\alpha \in \Delta_c$ and $c(\alpha) = 1$, we have

$$\begin{aligned}
\widehat{\mathbf{L}}(a_\alpha Z_\alpha + b_\alpha \overline{Z}_\alpha + a_{-\alpha} Z_{-\alpha} + b_{-\alpha} \overline{Z}_{-\alpha}) &= 2\operatorname{Re}(a_\alpha \bar{b}_{-\alpha} \mathbf{L}(Z_\alpha, \overline{Z}_{-\alpha})) \\
&\quad + 2\operatorname{Re}(a_{-\alpha} \bar{b}_\alpha \mathbf{L}(Z_{-\alpha}, \overline{Z}_\alpha)) \\
&= \operatorname{Re}\big(- a_\alpha \bar{b}_{-\alpha}(i + \cot\alpha(T_0))[Z_\alpha, Z_{-\alpha}]\big) \\
&\quad + \operatorname{Re}\big(a_{-\alpha} \bar{b}_\alpha (i - \cot\alpha(T_0))[Z_\alpha, Z_{-\alpha}]\big)\,.
\end{aligned}$$

If $c(\alpha) = -1$, the corresponding statement follows in a similar way. □

5. – Levi cone and applications

In this section, we compute the Levi cone $C_{x_0}(S)$ of a generic orbit S at a base point x_0. In this way, we give a geometric meaning to the Levi form which we calculated in Section 4. As usual, the point $x_0 \in S$ is of the form

$$x_0 = n_0 \exp JT_0\,,$$

where $T_0 \in \mathfrak{t}^{\mathbb{R}}$, $\mathfrak{t}^{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{R}}$, and $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$. Then, $\mathcal{C}_{x_0}(S)$ is a real cone in $\mathfrak{t}^{\mathbb{R}}$. Let $\mathfrak{t}$ be the complexification of $\mathfrak{t}^{\mathbb{R}}$ and $\Delta = \Delta(\mathfrak{t})$ denote the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Then a system of positive roots Δ^+ can be chosen such that

$$\operatorname{Im}\alpha(T_0) > 0, \qquad \text{for all} \quad \alpha \in \Delta_i \cap \Delta^+ . \tag{5.1}$$

REMARK. For $x_0 \in G_{\mathrm{reg},\kappa}$ one may have that $\alpha(T_0) = 0$, for some $\alpha \in \Delta(\mathfrak{t})$. However $\alpha(T_0) \neq 0$ for all $\alpha \in \Delta_i$.

Let $\{Z_\alpha\}_{\alpha \in \Delta}$ be a κ-stable basis of $\mathfrak{q}$ consisting of root vectors; let the operator $\mathbf{C} = \operatorname{Ad}(\eta(n_0))$ and the numbers $c(\alpha)$, for $\alpha \in \Delta$, be the ones defined in (4.12) and (4.13).

LEMMA 5.2. *The Levi cone $\mathcal{C}_{x_0}(S)$ of a generic orbit S at x_0 is a real cone in $\mathfrak{t}^{\mathbb{R}}$, generated by the vectors*

$$\begin{array}{rlr} \pm[Z_\alpha, Z_{-\alpha}] & \text{for } \alpha \in \Delta_r, & \\ J[Z_\alpha, Z_{-\alpha}] & \text{for } \alpha \in \Delta_i^+ & \text{and } c(\alpha) = \ \ 1, \\ \pm J[Z_\alpha, Z_{-\alpha}] & \text{for } \alpha \in \Delta_i^+ & \text{and } c(\alpha) = -1, \\ \pm\operatorname{Re}[Z_\alpha, Z_{-\alpha}], \quad \pm\operatorname{Im}[Z_\alpha, Z_{-\alpha}] & \text{for } \alpha \in \Delta_c . & \end{array}$$

PROOF. The lemma follows directly from (5.1) and Proposition 4.26. □

The next theorem gives a precise description of the Levi cone $\mathcal{C}_{x_0}(S)$. It turns out that if $\mathfrak{t}^{\mathbb{R}}$ is a compact Cartan subalgebra and $x_0 \in n_0 \exp J\mathfrak{t}^{\mathbb{R}}$, the size of the cone $\mathcal{C}_{x_0}(S)$ depends on the element $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$. The case when $\mathfrak{t}^{\mathbb{R}}$ is compact and $\eta(n_0) \in Z(G^{\mathbb{R}})$ is the only case when the Levi cone $\mathcal{C}_{x_0}(S)$ does not coincide with the whole vector space $\mathfrak{t}^{\mathbb{R}}$. For simplicity, we state the theorem for a simple and simply connected Lie group G. In some later remarks, we explain how to deal with an arbitrary semisimple group.

THEOREM 5.3. *Let G be a simply connected complex simple Lie group and let $G^{\mathbb{R}} \subset G$ be a real form. Let S be a generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbit with base point $x_0 \in n_0 \cdot \exp J\mathfrak{t}^{\mathbb{R}}$. Let $\Delta^+ = \Delta^+(\mathfrak{t})$ be a system of positive roots for $\mathfrak{g}$ chosen as in* (5.1). *Then the Levi cone $\mathcal{C}(S)_{x_0}$ of S at x_0 can be described as follows:*

(i) *If the Cartan subalgebra $\mathfrak{t}^{\mathbb{R}}$ is non-compact, then*

$$\mathcal{C}(S)_{x_0} = \mathfrak{t}^{\mathbb{R}} ;$$

(ii) *If the Cartan subalgebra $\mathfrak{t}^{\mathbb{R}}$ is compact and $\eta(n_0) \notin Z(G^{\mathbb{R}})$ then*

$$\mathcal{C}(S)_{x_0} = \mathfrak{t}^{\mathbb{R}} ;$$

(iii) *If the Cartan subalgebra $\mathfrak{t}^{\mathbb{R}}$ is compact and $\eta(n_0) \in Z(G^{\mathbb{R}})$, then $\mathcal{C}(S)_{x_0}$ is a proper subset of $\mathfrak{t}^{\mathbb{R}}$. More precisely, $\mathcal{C}(S)_{x_0}$ is isomorphic to the dual $\mathcal{W}^\vee$ of the Weyl chamber $\mathcal{W}$ defined by Δ^+.*

PROOF. Let $\{H_\alpha\}_{\alpha\in\Delta}$ be the set of dual roots in $\mathfrak{t}_{\mathbb{R}} = J\mathfrak{t}^{\mathbb{R}}_+ \oplus \mathfrak{t}^{\mathbb{R}}_-$. We can assume, perhaps after a renormalization, that the κ-stable basis $\{Z_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{q}$ satisfies $[Z_\alpha, Z_{-\alpha}] = H_\alpha$, for all $\alpha \in \Delta$. In the proof of the theorem, it is convenient to identify $\mathfrak{t}^{\mathbb{R}} = \mathfrak{t}^{\mathbb{R}}_+ \oplus \mathfrak{t}^{\mathbb{R}}_-$ with $\mathfrak{t}_{\mathbb{R}} = J\mathfrak{t}^{\mathbb{R}}_+ \oplus \mathfrak{t}^{\mathbb{R}}_-$ via the map

$$(X_+, X_-) \mapsto (-JX_+, X_-)\,.$$

We denote by $\mathcal{C}(S)_{x_0}$ as well the image in $\mathfrak{t}_{\mathbb{R}}$ of the cone $\mathcal{C}(S)_{x_0}$ by the above map. By Lemma 5.2, after this identification, $\mathcal{C}(S)_{x_0}$ is generated by the vectors

$$\begin{array}{lll} \pm H_\alpha & \text{for } \alpha \in \Delta\backslash\Delta_i\,, & \\ H_\alpha & \text{for } \alpha \in \Delta_i^+ & \text{and } c(\alpha) = 1\,, \\ \pm H_\alpha & \text{for } \alpha \in \Delta_i^+ & \text{and } c(\alpha) = -1\,. \end{array}$$

Let $\Pi \subset \Delta^+$ be a maximal set of positive simple roots. Since the vectors $\{H_\alpha\}_{\alpha\in\Pi}$ form a basis of the vector space $\mathfrak{t}_{\mathbb{R}}$, the interior of $\mathcal{C}_{x_0}(S)$ is non-empty.

(i) Let $\mathfrak{t}^{\mathbb{R}}$ a be non-compact Cartan subalgebra. Then $\Delta \neq \Delta_i$ and, regardless whether $c(\alpha) = \pm 1$, the cone $\mathcal{C}(S)_{x_0}$ contains at least the vectors $\{\pm H_\alpha\}$ for $\alpha \in \Delta\backslash\Delta_i$ and $\{H_\alpha\}$ for $\alpha \in \Delta_i^+$. Statement (i) is equivalent to showing that the vectors $\{-H_\alpha\}$ belong to $\mathcal{C}(S)_{x_0}$ as well, for every $\alpha \in \Pi \cap \Delta_i^+$.

Note that if $\alpha,\ \beta \in \Delta_i^+$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta_i^+$. Then every maximal set of simple roots $\Pi \subset \Delta^+$ contains a root $\beta \in \Pi\backslash\Delta_i^+$. If none of the simple roots is imaginary, the theorem follows directly from Lemma 5.2. If there are imaginary simple roots in Π, let $\alpha_1, \ldots \alpha_s$ be consecutive simple imaginary roots next to β in the Dynkin diagram of $\mathfrak{g}$. For the Killing form B this means that

$$B(\beta, \alpha_1) < 0, \quad B(\alpha_i, \alpha_{i+1}) < 0, \quad i = 1, \ldots, s-1\,,$$

while

$$B(\beta, \alpha_i) = 0 \text{ for } i \neq 1, \quad \text{and} \quad B(\alpha_i, \alpha_j) = 0 \text{ for } j \neq i \pm 1\,.$$

As a consequence, $\beta + \alpha_1 + \ldots + \alpha_i$ is a root, for every $i = 1, \ldots, s$, and is not imaginary. It follows that $\pm H_{\beta+\alpha_1+\ldots+\alpha_i} \in \mathcal{C}(S)_{x_0}$. Writing

$$(\beta + \alpha_1 + \ldots + \alpha_i) + \alpha_{i+1} = (\beta + \alpha_1 + \ldots + \alpha_i + \alpha_{i+1})\,,$$

one gets a similar relation among the corresponding dual roots

$$H_{\beta+\alpha_1+\ldots+\alpha_i} + H_{\alpha_{i+1}} = H_{\beta+\alpha_1+\ldots+\alpha_i+\alpha_{i+1}}\,.$$

This implies that $\pm H_{\alpha_i} \in \mathcal{C}(S)_{x_0}$. Iterating his argument eventually yields that $\{\pm H_\alpha\}$ belong to $\mathcal{C}(S)_{x_0}$, for every simple root α. Hence $\mathcal{C}(S)_{x_0} = \mathfrak{t}_{\mathbb{R}}$.

(ii) Let $\mathfrak{t}^{\mathbb{R}}$ be a compact Cartan subalgebra. Assume that the operator $\mathbf{C} = \mathrm{Ad}_{\eta(n_0)}$ does not act on the root spaces as the identity i.e. $c(\beta) = -1$, for some root $\beta \in \Delta$ (see (4.12) (4.13)). In this case $\Delta = \Delta_i$ and, by Lemma 5.2, the cone $\mathcal{C}(S)_{x_0}$ contains the vectors $\{H_\alpha\}$, for $\alpha \in \Delta_i^+$ with $c(\alpha) = 1$, and the vectors $\{\pm H_\alpha\}$, for $\alpha \in \Delta_i^+$ with $c(\alpha) = -1$. We need to show that the vectors $\{-H_\alpha\}$ belong to $\mathcal{C}(S)_{x_0}$ as well, for all simple roots $\alpha \in \Delta_i^+$.

Recall that the operator $\mathbf{C}$ is a Lie algebra automorphism, i.e. $\mathbf{C}[Z_\alpha, Z_\beta] = [\mathbf{C}Z_\alpha, \mathbf{C}Z_\beta]$. In particular, if α, β are simple roots such that $c(\alpha) = 1$, $c(\beta) = -1$ and $\alpha + \beta \in \Delta$, then $c(\alpha + \beta) = -1$. It follows that there exists a simple root $\beta \in \Delta_i^+$ with $c(\beta) = -1$. If $c(\beta) = -1$ for every simple root $\beta \in \Delta_i^+$, the statement follows directly from Lemma 5.2. Otherwise, let $\alpha_1, \dots, \alpha_s$ be consecutive simple roots, with $c(\alpha_j) = 1$, next to β in the Dynkin diagram of $\mathfrak{g}$. Then $\beta+\alpha_1+\ldots+\alpha_i$ is a root, for every $i = 1, \dots, s$, and $c(\beta+\alpha_1+\ldots+\alpha_i) = -1$. Arguing as in (i), we obtain that $\mathcal{C}(S)_{x_0} = \mathfrak{t}^{\mathbb{R}}$.

(iii) Let $\mathfrak{t}^{\mathbb{R}}$ be a compact Cartan subalgebra and assume that the operator $\mathbf{C} = \mathrm{Ad}_{\eta(n_0)}$ acts on the root spaces as the identity. By Lemma 5.2, the cone $\mathcal{C}(S)_{x_0}$ is generated by the vectors $\{H_\alpha\}_{\alpha\in\Delta^+}$. Then it coincides with the dual $\mathcal{W}^\vee$ of the Weyl chamber

$$\mathcal{W} = \{T \in \mathfrak{t}_{\mathbb{R}} \mid B(T, H_\alpha) = \alpha(T) \geq 0, \quad \text{for all } \alpha \in \Delta^+\}$$

as stated in the theorem. □

REMARK 5.4. Let G be a simply connected complex semisimple Lie group and $G^{\mathbb{R}}$ a real form of G. Then G decomposes as a direct product of simply connected and κ-stable complex subgroups $G = G_1\times\ldots\times G_n$ such that either G_j is simple or $G_j = H\times H$ is a product of two copies of a simple Lie group H. In the second case, the restriction of the conjugation $\kappa\,|_{H\times H}$ is given by $(h_1, h_2) \mapsto (\tau(h_2), \tau(h_1))$, where τ is a conjugation on H, and $G_j^{\mathbb{R}} = (H \times H)^{\mathbb{R}} \cong H$. In this way, the real form $G^{\mathbb{R}}$ of G decomposes as $G^{\mathbb{R}} = G_1^{\mathbb{R}} \times \ldots \times G_n^{\mathbb{R}}$, where each $G_j^{\mathbb{R}}$ is a real form of G_j, for $j = 1, \dots, n$. Similarly, a generic $G^{\mathbb{R}} \times G^{\mathbb{R}}$-orbit S in G decomposes as the product

$$S = S_1 \times \ldots \times S_n$$

of generic $G_j^{\mathbb{R}} \times G_j^{\mathbb{R}}$-orbits S_j in G_j.

The complex tangent space $T_{\mathbb{C}}S_{x_0}$ to S at a base point is the direct sum of the complex tangent spaces of all factors. By Corollary 4.23, such a sum is orthogonal with respect to the Levi form. Therefore the Levi cone $\mathcal{C}(S)_{x_0}$ of S at x_0 is the direct product of the Levi cones of all factors. Observe that the real form of a non-simple factor $G_j = H \times H$ has, up to conjugation, only one Cartan subalgebra $\mathfrak{h}$. Since $\mathfrak{h}$ is the realification of a complex Cartan subalgebra, it is non-compact.

REMARK 5.5. Assuming G simply connected in Theorem 5.3 is not an essential restriction. Let G be an arbitrary complex semisimple Lie group and

$G^{\mathbb{R}}$ a real form of G. Let $\tilde{G}$ denote the universal covering group of G and $\pi : \tilde{G} \to G$ the corresponding covering map. Define $\tilde{G}^{\mathbb{R}} := \pi^{-1}(G^{\mathbb{R}})$. Then $\tilde{G}^{\mathbb{R}}$ is a real form of G, consisting of finitely many connected components. Denote by $\tilde{G}_0^{\mathbb{R}}$ the connected component of the identity of $\tilde{G}^{\mathbb{R}}$. Denote by $\tilde{S}$ (respectively by $\tilde{S}_0$) a generic orbit of $\tilde{G}^{\mathbb{R}} \times \tilde{G}^{\mathbb{R}}$ (respectively of $\tilde{G}_0^{\mathbb{R}} \times \tilde{G}_0^{\mathbb{R}}$) in $\tilde{G}$. Theorem 5.3 applies to the orbit $\tilde{S}_0$, Clearly, all the connected components of $\tilde{S}$ have the same geometric properties as $\tilde{S}_0$. Since the map π is locally biholomorphic and equivariant, the same is true for the orbit S in G, covered by $\tilde{S}$ and $\tilde{S}_0$.

We conclude this section by stating the main results of the paper. By Theorem 5.3 and Remarks 5.4 and 5.5, we are able to determine which generic orbits can be contained in the level sets of a biinvariant plurisubharmonic function or in the boundary of a biinvariant Stein domain in G.

Corollary 5.6. *Let Ω be a biinvariant domain in G. Assume Ω contains in its boundary a generic orbit S_0 satisfying condition* (i) *or* (ii) *of Theorem 5.3. Then Ω cannot be Stein.*

Proof. We show that all holomorphic functions defined on Ω extend to a strictly larger domain $\Omega' \supset \Omega$. Let $S_0 \subset G$ be a generic orbit in G satisfying condition (i) or (ii) of Theorem 5.3. Then, for every point $x \in S_0$, the Levi cone $\mathcal{C}(S_0)_x$ coincides with vector space $T(S_0)_x / T_{\mathbb{C}}(S_0)_x$. By the extension Theorem 1.11, there exists an open neighborhood $U(S_0)$ of S_0 in G with the property that every CR-function f on S_0 extends to a holomorphic function $\hat{f}$ on $U(S_0)$.

The important fact is that, roughly speaking, a generic orbit sufficiently close to S_0 admits an open neighborhood with the extension property which cannot be too small with respect to $U(S_0)$. Fix a base point $x_0 \in S_0$ lying in the submanifold

$$A := n_0 \cdot \exp J\mathfrak{t}^{\mathbb{R}},$$

where $\mathfrak{t}^{\mathbb{R}} \subset \mathfrak{g}^{\mathbb{R}}$ is a Cartan subalgebra and $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$. Fix a distance function d on A. Then $U(S_0)$ can be assumed to be biinvariant of the form

$$U_R(S_0) = G^{\mathbb{R}} \cdot B_R(x_0) \cdot G^{\mathbb{R}},$$

where $B_R(x_0)$ is a ball of center x_0 and radius R in A. Moreover, if R is sufficiently small, for every $x \in B_R(x_0)$ the orbit $G^{\mathbb{R}} \cdot x \cdot G^{\mathbb{R}}$ is generic, intersects $B_R(x_0)$ transversally and exacly once (cf. Theorem 2.13).

Recall that every orbit S is real analytic and that for S in $U(S_0)$ the CR-structure depends in a real analytic way on the reference point $x = S \cap B_R(x_0)$ (see Section 3 and Section 4). Then, by a continuity argument (cf. [Bo], ch. 15), every orbit $S = G^{\mathbb{R}} \cdot x \cdot G^{\mathbb{R}}$ sufficiently close to S_0 admits a biinvariant neighborhood with the extension property of the form $U_r(S) = G^{\mathbb{R}} \cdot B_r(x) \cdot G^{\mathbb{R}}$, with radius $r > R/2$.

Let $\Omega \subset G$ be a biinvariant domain containing the orbit S_0 in its boundary. Let $U_R(S_0)$ be a biinvariant neighborhood of S_0 with the extension property. By the above discussion, there exists an orbit S contained in $\Omega \cap U_{R/4}(S_0)$ and such that $U_{R/2}(S)$ has the extension property. Then every holomorphic function on Ω extends to $\Omega' := \Omega \cup U_{R/2}(S)$.

COROLLARY 5.7. *Let S be a generic orbit in G satisfying condition* (i) *or* (ii) *of Theorem 5.3. Then S cannot be contained in a level set of a non-constant biinvariant plurisubharmonic function.*

REMARK 5.8. On a complex simple Lie group G there exist no non-constant global plurisubharmonic functions which are biinvariant by a non-compact real form $G^{\mathbb{R}}$ (see [L2]). Corollary 5.7 excludes the existence of biinvariant plurisubharmonic functions on all biinvariant domains which are not completely contained in the subsets

$$G^{\mathbb{R}} \cdot \exp J\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}} \quad \text{or} \quad G^{\mathbb{R}} \cdot n_0 \exp J\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}},$$

where $\mathfrak{t}^{\mathbb{R}}$ is a compact Cartan subalgebra and $n_0 \in N_G(\mathfrak{t}^{\mathbb{R}})$ satisfies $\eta(n_0) \in Z(G^{\mathbb{R}})$.

If $G^{\mathbb{R}}$ is a non-compact simple Hermitian real form of G, then $\mathfrak{g}^{\mathbb{R}}$ contains a compact Cartan subalgebra $\mathfrak{t}^{\mathbb{R}}$ and the subset

$$G^{\mathbb{R}} \cdot \exp J\mathfrak{t}^{\mathbb{R}} \cdot G^{\mathbb{R}}$$

admits both biinvariant Stein subdomains and biinvariant plurisubharmonic functions. It contains for example the Olsanskiĭ-semigroups (cf. [N1], [N2]).

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