Definition. The Γ-function $\Gamma(s)$ is defined for $s \in \mathbf{C}$, $\mathrm{Re}(s) > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}.$$

Proposition (9.1).
(i) For every $s \in \mathbf{C}$, $\mathrm{Re}(s) > 0$ one has that $\Gamma(s+1) = s\Gamma(s)$.
(ii) The Γ-function admits a meromorphic extension to $\mathbf{C}$ with poles at $0, -1, -2, \ldots$ of order 1. The residue at $-k$ is $(-1)^k/k!$.
(iii) $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ for $s \in \mathbf{C} - \mathbf{Z}$.

Proof. The first part follows easily by partial integration. Using the functional equation $\Gamma(s+1) = s\Gamma(s)$ one can extend $\Gamma(s)$ meromorphically to all of $\mathbf{C}$. For every $k \in \mathbf{Z}_{\geq 0}$ one has that

$$\Gamma(s) = \frac{1}{(s+(k-1)) \cdot \ldots \cdot (s-1)s} \Gamma(s+k)$$

which easily implies *(ii)*.
(iii) Write $F(s) = \Gamma(s)\Gamma(1-s)$. By *(ii)* the function $F(s)$ has poles of order 1 at the integers. For $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < 1$ one has that

$$F(s) = \int_0^\infty \int_0^\infty e^{-t-x} \left(\frac{t}{x}\right)^s dx \frac{dt}{t}$$

and, making the substitution $t = zx$, we find that

$$F(s) = \int_0^\infty z^s \int_0^\infty e^{-(z+1)x} dx \frac{dz}{z} = \int_0^\infty \frac{z^{s-1}}{z+1} dz.$$

Using the Residue Theorem it is rather easy to show that the last integral is equal to $\pi/\sin(\pi s)$. See Exer.9.A for the details. This proves the proposition.

Definition. For $x \in \mathbf{R}_{>0}$ we let

$$\theta(x) = \sum_{n \in \mathbf{Z}} e^{-\pi n^2 x}$$

denote the θ-function.

This θ-function is a minor modification of the well known Jacobi Θ-function $\Theta(z) = \sum_{n \in \mathbf{Z}} e^{2\pi i n^2 z} = \theta(-2iz)$ (Jacobi, German mathematician 1804–1851). The Θ-function is defined for z in the upper halfplane. It is a modular form (weight 1/2, for some $\Gamma \subset SL_2(\mathbf{Z})$). See the books by N. Koblitz [36] and S. Lang [40] for more about Θ-series and modular forms.

Proposition (9.2). *(Carl Jacobi, German mathematician 1804–1851) Let $x \in \mathbf{R}_{>0}$. Then*

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\theta(x).$$

Proof. Let $f : \mathbf{R} \longrightarrow \mathbf{C}$ be a rapidly decreasing C^∞-function i.e. for all $n \in \mathbf{Z}$ one has that $f(x)x^n \to 0$ when $x \to \pm\infty$. The basic example will be $e^{-\pi A x^2}$ for $A > 0$. We define the Fourier transform of f by

$$\hat{f}(t) = \int_{\mathbf{R}} f(x) e^{2\pi i t x} dx.$$

The proof will be a consequence of the *Poisson summation formula:*

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{n\in\mathbf{Z}} \hat{f}(n).$$

This formula can be deduced as follows: consider

$$\begin{aligned} g(x) &= \sum_{k\in\mathbf{Z}} f(k+x) \\ &= \sum_{m\in\mathbf{Z}} c_m e^{2\pi i m x} \end{aligned}$$

The second expression is the Fourier expansion of the *periodic* function $g(x)$. The Fourier coefficients c_m are given by

$$c_m = \int_0^1 g(x) e^{-2\pi i m x} dx \qquad \text{for all } m \in \mathbf{Z}.$$

The coefficients c_m can be evaluated explicitly as follows:

$$\begin{aligned} c_m &= \int_0^1 \sum_{k\in\mathbf{Z}} f(k+x) e^{-2\pi i m x} dx = \int_0^1 \sum_{k\in\mathbf{Z}} f(x) e^{-2\pi i m (x-k)} dx \\ &= \int_{\mathbf{R}} f(x) e^{-2\pi i m x} dx = \hat{f}(m) \end{aligned}$$

We conclude that

$$\sum_{k\in\mathbf{Z}} f(k) = g(0) = \sum_{m\in\mathbf{Z}} c_m = \sum_{m\in\mathbf{Z}} \hat{f}(m)$$

as required.

Now we give the proof of Prop.9.2: consider the, rapidly decreasing, function $h(y) = e^{-\pi y^2}$. It is well-known and easily checked that $\hat{h}(y) = h(y)$. It is convenient to calculate the Fourier transform of $h_b(y) = h(by)$ for $b \in \mathbf{R}$. the result is that $\hat{h_b}(y) = \frac{1}{b}\hat{h}(\frac{y}{b})$. By the Poisson summation formula we have that

$$\begin{aligned}\theta(x) &= \sum_{n\in\mathbf{Z}} e^{-\pi n^2 x} = \sum_{n\in\mathbf{Z}} h_{\sqrt{x}}(n) = \sum_{n\in\mathbf{Z}} \widehat{h_{\sqrt{x}}}(n),\\ &= \sum_{n\in\mathbf{Z}} \frac{1}{\sqrt{x}}\hat{h}(\frac{n}{\sqrt{x}}) = \frac{1}{\sqrt{x}} \sum_{n\in\mathbf{Z}} e^{-\pi(\frac{n}{\sqrt{x}})^2}\\ &= \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right),\end{aligned}$$

as required.

Proposition (9.3). *The Riemann ζ-function (G.B. Riemann, German mathematician 1826–1866) has the following properties:*

(i) (Euler product.)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $s \in \mathbf{C}$, $\operatorname{Re} s > 1$. Here the product runs over the primes p.

(ii) (Analytic continuation.) The function $\zeta(s)$ admits a meromorphic extension to $\mathbf{C}$. It has only a pole at $s = 1$. This pole is of order 1 with residue 1.

(iii) (Functional equation.) The function

$$Z(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$$

satisfies $Z(s) = Z(1-s)$.

(iv) (Zeroes) If ρ is a zero of $\zeta(s)$ then either ρ is a trivial zero, i.e. ρ is a negative even integer, or $0 \leq \rho \leq 1$

(v) (*Special values.*) Let m be an even positive integer. Then

$$\begin{aligned}\zeta(m) &= -\frac{(2\pi i)^m}{2\cdot m!}B_m,\\ \zeta(1-m) &= -\frac{B_m}{m}.\end{aligned}$$

here the B_m denote *Bernoulli numbers. They are defined by*

$$\frac{T}{e^T - 1} = \sum_{m=1}^{\infty} \frac{B_m}{m!}T^m.$$

Finally we have that $\zeta(0) = -\frac{1}{2}$.

Proof. Part *(i)* has been proved in section 4. We prove *(ii)* and *(iii)* at the same time. For $s \in \mathbf{C}$, $\operatorname{Re}(s) \geq 1$, consider the Z-function

$$Z(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$$

We can write

$$Z(s)=\int_0^\infty e^{-t}t^{s/2}\frac{dt}{t}\pi^{-s/2}\sum_{n\geq 1}n^{-s}=\int_0^\infty e^{-t}\sum_{n\geq 1}t^{s/2}\pi^{-s/2}n^{-s}\frac{dt}{t}.$$

Substituting $t=x\pi n^2$ in every term of the sum, we find

$$Z(s)=\int_0^\infty x^{s/2}\sum_{n\geq 1}e^{-x\pi n^2}\frac{dx}{x}=\int_0^\infty \frac{\theta(x)-1}{2}x^{s/2}\frac{dx}{x}.$$

Next, we split the integral in two pieces: a piece from 0 to 1 and another from 1 to ∞. In the first piece we change the variable x to $1/x$ and we find, using Prop.9.2, that

$$\begin{aligned}
Z(s) &= \int_1^\infty \frac{\theta(x)-1}{2}x^{s/2}\frac{dx}{x}+\int_\infty^1 \frac{\theta(1/x)-1}{2}x^{-s/2}\frac{d(1/x)}{1/x},\\
&= \int_1^\infty \frac{\theta(x)-1}{2}x^{s/2}\frac{dx}{x}+\int_1^\infty \frac{\sqrt{x}\theta(x)-1}{2}x^{-s/2}\frac{dx}{x},\\
&= \int_1^\infty \frac{\theta(x)-1}{2}\left(x^{\frac{s}{2}}+x^{(1-s)/2}\right)\frac{dx}{x}+\int_1^\infty \frac{x^{(1-s)/2}-x^{-s/2}}{2}\frac{dx}{x},\\
&= \int_1^\infty \frac{\theta(x)-1}{2}\left(x^{s/2}+x^{(1-s)/2}\right)\frac{dx}{x}-\frac{1}{s}-\frac{1}{1-s}.
\end{aligned}$$

Now we have an expression for $Z(s)$ which converges for all $s\in\mathbf{C}\setminus\{0,1\}$. We clearly have that $Z(s)=Z(1-s)$. The Z function has poles at 0 and 1, with residues $Res_Z(0)=-1$ and $Res_Z(1)=1$. Since the function $\Gamma(s/2)=(2/s)\Gamma(s/2+1)$ has a simple pole at 0, but not at 1, it follows that the ζ-function has only a pole at 1. From Prop.9.1*(iii)* we see that $\Gamma(1/2)=\sqrt{\pi}$. Therefore the residue of $\zeta(s)$ at 1 is 1.

(iv) Suppose ρ is a zero of the Riemann ζ-function. By *(i)* we have that $\mathrm{Re}(s)\leq 1$. Suppose $\mathrm{Re}(s)<0$ and that ρ is *not* an even negative integer. We have that $Z(\rho)=\Gamma(\rho/2)\pi^{-\rho/2}\zeta(\rho)$. By Prop.9.1, the function $\Gamma(s/2)$ does not have a pole at ρ. Therefore $Z(\rho)=0$ and by *(iii)* we see that $Z(1-\rho)=0$. It is immediate from Prop.9.1*(iii)* that the Γ-function has no zeroes. We conclude that $\zeta(1-\rho)=0$. This contradicts the fact that $\mathrm{Re}(\rho)>1$ and the result follows.

(v) Let $R\in\frac{1}{2}+\mathbf{Z}$ be a large number and let C_R be a big square in $\mathbf{C}$ with corners $\pm R\pm iR$. The contour integral

$$\int_{C_R}\frac{z^{-m}}{e^z-1}dz \qquad \text{for } m\geq 2$$

approaches 0 as $R\to\infty$, yielding

$$0=2\pi i\sum_{n\in\mathbf{Z}}Res_F(2\pi i n),\qquad F=\frac{z^{-m}}{e^z-1}.$$

One has

$$Res_F(0) = \frac{B_m}{m!}, \qquad Res_F(2\pi i n) = (2\pi i n)^{-m}, \quad n \in \mathbf{Z}_{\neq 0},$$

and summing up over $n \in \mathbf{Z}$, one finds for *even* values of m that

$$\sum_{n=1}^{\infty} \frac{1}{(2\pi i n)^m} + \frac{1}{2}\frac{B_m}{m!} = 0.$$

This gives

$$\zeta(m) = \sum_{n \geq 1} \frac{1}{n^s} = -\frac{(2\pi i)^m}{2 \cdot m!} B_m,$$

for even positive integers m. The values of $\zeta(1-m)$ follow from the functional equation proved in *(ii)*. Finally one obtains that $\zeta(0) = -1/2$ by observing that both $Z(s)$ and $\Gamma(s/2)$ have a simple pole at 0. The Z-function has a residue equal to -1 and the function $\Gamma(s/2)$ has a residue 2, because $\Gamma(s/2) = (2/s)\Gamma(s/2+1)$. This proves Proposition 9.3.