Entire Functions Theory

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Contents

1 Growth of entire functions

A complex function $f : \mathbb{C} \to \mathbb{C}$ that is analytic (holomorphic) on $\mathbb{C}$ is called an *entire* function. Every entire function is given by its Taylor's expansion:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad \forall z \in \mathbb{C}.$$

The maximum modulus function of an entire function f is given by:

$$M(r, f) := \max_{|z| \le r} |f(z)|, \quad r > 0.$$

By the Maximum Modulus Principle, we have:

$$M(r, f) = \max_{|z| = r} |f(z)|, \quad r > 0.$$

Moreover, $M(r, f)$ is an increasing function from a non-constant entire function f, i.e., $M(r_1, f) < M(r_2, f)$ for every $r_1, r_2 > 0$ with $r_1 < r_2$ (Again this follows from Max-Modulus principle). Since f is continuous on every compact set $S_r := \{z : |z| = r\}$, it follows that:

$$\forall r > 0, \exists z_r \in S_r : |f(z_r)| = M(r, f).$$

(In particular, we do not need to consider "sup" in defining $M(r, f)$). The behavior of the function $M(r, f)$ at infinity describes *the growth* of f.

During this section, we will classify entire functions based on the growth rates $\ln M(r, f)/\ln r$ and $\ln \ln M(r, f)/\ln r$. In particular, we will show some relation between these rates, and the coefficients of the Taylor's expansion of f.

Before doing so, we recall two main results from Cauchy's integral theory of complex functions.

Lemma 1.1 (Cauchy's integral formula)**.** *If $f : D \to \mathbb{C}$ is analytic on a simply connected domain D, then:*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi, \quad \forall n \ge 1, \forall z \in \operatorname{int}(\Gamma),$$

where Γ is any simple closed contour in D.

Lemma 1.2 (Cauchy's inequality). *Let f be an entire function given by its Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \forall z \in \mathbb{C}.$$

Then

$$|a_n| \leq \frac{M(r,f)}{r^n}, \quad \forall n \geq 1, \quad \forall r > 0.$$

Proof. we have, for any $n \geq 1$ and for any $r > 0$,

$$\begin{aligned} |a_n| = \left| \frac{f^{(n)}(0)}{n!} \right| &= \left| \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}} \, \mathrm{d}\, z \right| \\ &\leq \frac{1}{2\pi} \oint_{|z|=r} \frac{|f(z)|}{|z|^{n+1}} |\, \mathrm{d}\, z| \leq \frac{M(r,f)}{2\pi r^{n+1}} (2\pi r) = \frac{M(r,f)}{r^n}. \end{aligned} \qquad \square$$

Exercise 1.1. Let f be entire. Suppose that $\exists M > 0$ s.th. $|f(z)| \leq M e^{|z|}$, $\forall z \in \mathbb{C}$. Show that

$$|f^{(n)}(0)| \leq M \left(\frac{e}{n} \right)^n n!, \quad \forall n \geq 1.$$

Solution. One way to see this is by applying Cauchy's inequality on the circles $|z| = n$, $n \geq 1$, i.e.,

$$\left| \frac{f^{(n)}(0)}{n!} \right| \leq \frac{M(n,f)}{n^n} \leq \frac{M e^n}{n^n}, \quad \forall n \geq 1.$$

This is obviously yields the required inequality. This way doesn't show the sharpness of the upper bound in the required inequality. Thus, we proceed as follows: From Cauchy's inequality, we have:

$$|a_n| \leq \frac{M(r,f)}{r^n} \leq \frac{M e^r}{r^n}, \quad n \geq 1, \quad r > 0.$$

Since the left-hand side does not depend on r, we seek the minimum of the function:

$$\varphi(r) = \frac{M e^r}{r^n}, \quad r > 0.$$

We have:

$$\varphi'(r) = Me^r \frac{r-n}{r^{n+1}} = 0 \iff r = r_0 = n.$$

The function $\varphi(r)$ has a single stationary point at $r_0 = n$. Since $\lim\limits_{r\to 0^+} \varphi(r) = \lim\limits_{r\to +\infty} \varphi(r) = +\infty$, it follows that $r_0 = n$ is a minimum point. Substituting $r = r_0 = n$, we obtain:

$$\left| \frac{f^{(n)}(0)}{n!} \right| = |a_n| \le \frac{Me^n}{n^n} = M \left(\frac{e}{n} \right)^n, \quad n \ge 1. \qquad \square$$

1.1 Liouville's Theorems

Here we recall some generalizations of Liouville's theorem. The original version of Liouville's theorem asserts: *If an entire function is bounded in $\mathbb{C}$, then it must be constant.*

Theorem 1.1. *If f is a non-constant entire function, then $\lim\limits_{r\to+\infty} M(r,f) = +\infty$.*

Proof. Since $r \mapsto M(r,f)$ is increasing, then $\lim\limits_{r\to+\infty} M(r,f)$ exists. Assume it is finite, i.e., $\exists C > 0$ s.th. $\lim\limits_{r\to+\infty} M(r,f) = C$. Then $M(r,f) \le C$ for every $r > 0$. Let now f be written in its Taylor's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \forall z \in \mathbb{C}.$$

It follows from Cauchy's inequality that

$$|a_n| \le \frac{M(r,f)}{r^n} \le \frac{C}{r^n}, \quad \forall n \ge 1, \forall r > 0.$$

By letting $r \to +\infty$, we obtain that $a_n = 0$ for every $n \ge 1$. Thus f must be a constant, which contradicts the condition on f being non-constant. Thus $\lim\limits_{r\to+\infty} M(r,f) = +\infty$. $\square$

Theorem 1.2. *Let f be an entire function. If there exists a sequence $(r_k)_{k\geq 1}$ of positive real numbers satisfying $r_k \longrightarrow +\infty$ as $k \to +\infty$ such that*

$$M(r_k, f) \leq C r_k^{\lambda}, \quad \forall k \geq 1,$$

where $C > 0$ and $\lambda > 0$, then f is a polynomial of degree $\leq \lfloor \lambda \rfloor$.

Proof. Let f be written in its Taylor's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \forall z \in \mathbb{C}.$$

Then by Cauchy's inequality we have

$$|a_n| \leq \frac{M(r_k, f)}{r_k^n} \leq C r_k^{\lambda - n}, \quad \forall n \geq 1, \forall k \geq 1.$$

By letting $k \geq +\infty$, we obtain that $a_n = 0$, for every $n \geq \lfloor \lambda \rfloor + 1$. This completes the proof. □

The converse of Theorem 1.2 is also valid. In fact, if f is a polynomial of degree $n \geq 1$, then

$$M(r, f) \leq C r^n, \quad \forall r \geq R,$$

where $C > 0$ and $R > 0$. This follows directly from the following theorem.

Theorem 1.3. *If f is a non-constant polynomial, then*

$$\deg(f) = \lim_{r \to +\infty} \frac{\ln M(r, f)}{\ln r}.$$

Proof. Let $f(z) = a_n z^n + \cdots + a_1 z + a_0$, where $a_j \in \mathbb{C}$ and $a_n \neq 0$. Then

$$|f(z)| = |a_n||z|^n \underbrace{\left|1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}\right|}_{I(z)}, \quad |z| > 0.$$

Since $I(z) \longrightarrow 1$ as $|z| \to +\infty$, there exists $R > 0$ s.th. $1/2 < I(z) < 3/2$ for every $|z| \geq R$. Thus

$$\frac{|a_n|}{2} r^n \leq |f(z)| \leq \frac{3|a_n|}{2} r^n, \quad \forall r = |z| \geq R.$$

This yields

$$n + \frac{\ln(|a_n|/2)}{\ln r} \le \frac{\ln M(r,f)}{\ln r} \le n + \frac{\ln(3|a_n|/2)}{\ln r}, \quad \forall r \ge R,$$

from which we obtain the result. □

Combining Theorems 1.2 and 1.3 gives the characterization of polynomial functions.

Theorem 1.4. *A function f is a polynomial if and only if* $\ln M(r,f) = O(\ln r)$.

1.2 Transcendental entire functions

An entire function that is not polynomial is called a *transcendental entire function*. In fact, $f(z) = \sum_{n\ge 0} a_n z^n$ is transcendental if there exists an *infinite sequence* $(n_k)_{k\ge 1}$ of non-negative integers such that $a_{n_k} \ne 0$ for every $k \ge 1$.

Lemma 1.3. *If f is a transcendental entire function, then, for every $m \ge 1$,*

$$\lim_{r\to+\infty} \frac{M(r,f)}{r^m} = +\infty.$$

Proof. As f is transcendental, it follows that $\forall m \ge 1$, $\exists n$ s.th. $a_n \ne 0$ and $n \ge m+1$. By making use of Cauchy's inequality, we obtain

$$|a_n| \le \frac{M(r,f)}{r^n} \Longrightarrow \frac{M(r,f)}{r^m} \ge |a_n| r^{n-m} \ge |a_n| r, \quad \forall r > 0.$$

By letting $r \to +\infty$, we obtain the result. □

This lemma says that any transcendental entire function f grows faster than any non-constant polynomial (The growth of a transcendental entire function and the growth of a polynomial are not comparable). In fact, we have from Lemma 1.3 that

$$\lim_{r\to+\infty} \frac{M(r,f)}{M(r,P)} = +\infty,$$

for any transcendental entire function f and for any non-constant polynomial P.

Now, we prove the version of Theorem 1.4 for transcendental entire functions. It is clear from Theorem 1.4 that f is transcendental if and only if

$$\limsup_{r\to+\infty} \frac{\ln M(r,f)}{\ln r} + \infty.$$

In the following theorem we will prove a better version.

Theorem 1.5. *A function f is transcendental if and only if*

$$\lim_{r\to+\infty} \frac{\ln M(r,f)}{\ln r} = +\infty. \tag{1.1}$$

Proof. $\Longleftarrow$) Obvious from Theorem 1.4.

$\Longrightarrow$) Assume that f is a transcendental. By Lemma 1.3, $\forall m \geq 1, \forall C > 1$, $\exists R > 0$ s.th.

$$M(r,f) \geq Cr^m, \quad \forall r \geq R.$$

Then

$$\frac{\ln M(r,f)}{\ln r} \geq m + \frac{\ln C}{\ln r} \geq m, \quad \forall r \geq R,$$

which implies (by definition of limit) that (1.1) holds. □

Exercise 1.2. Find the maximum modulus function for the entire functions:

$$f_1(z) = e^z, \quad f_2(z) = e^{e^z}, \quad f_3(z) = \cos(z), \quad f_4(z) = e^{\cos(z)}.$$

Remark 1.1. Notice in general, for entire function g, we have

$$M(r,e^g) = \max_{|z|=r} |e^{g(z)}| = \max_{|z|=r} e^{\mathrm{Re}(g(z))} \leq \max_{|z|=r} e^{|g(z)|} = e^{\max_{|z|=r}|g(z)|} = e^{M(r,g)}.$$

Solution. **[1]** We have $M(r,f_1) \leq e^r$. At the point $z_0 = r$, we have $|f_1(z_0)| = e^r$. So, by the max-modulus principle,

$$M(r,f_1) = \max_{|z|=r} |f_1(z)| \geq |f(z_0)| = e^r.$$

Thus $M(r,f_1) = e^r$.

2 Notice that $f_2 = e^{f_1}$. Then $M(r, f_2) \le e^{M(r,f_1)} = e^{e^r}$. At the point $z_0 = r$, we have $|f_2(z_0)| = e^{e^r}$, so $M(r, f_2) = e^{e^r}$.

3 We have

$$\begin{aligned} M(r, f_3) = \max_{|z|=r} |\cos z| &= \max_{|z|=r} \left| \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{(2n)!} \right| \\ &\le \max_{|z|=r} \sum_{n=0}^{+\infty} \frac{|z|^{2n}}{(2n)!} = \sum_{n=0}^{+\infty} \frac{r^{2n}}{(2n)!} = \cosh r. \end{aligned}$$

At the point $z_0 = ir$, we have $|f_3(z_0)| = |\cos ir| = \cosh r$, so

$$M(r, f_3) = \cosh r = \frac{e^r + e^{-r}}{2} \sim \frac{e^r}{2}, \quad r \to +\infty.$$

4 Notice that $f_4 = e^{f_3}$. Then $M(r, f_4) \le e^{M(r,f_3)} = e^{\cosh r}$. At the point $z_0 = ir$, we have $|f_4(z_0)| = e^{\cosh r}$. So, $M(r, f_4) = e^{\cosh r}$. In particular, we have

$$\ln M(r, f_4) = \cosh r \sim \frac{e^r}{2}, \quad r \to +\infty.$$

□

Exercise 1.3. Estimate the maximum modulus function for $f(z) = \sin(z)$.

Solution. Note that $M(r, \sin) = \sinh r$. This can be obtained similarly to the function $M(r, \cos)$. Thus $M(r, f) \le e^{M(r,\sin)} = e^{\sinh r}$, i.e.,

$$\ln M(r, f) \le \sinh r. \tag{1.2}$$

On the other hand, we cannot use the trick replacing z by $z_0 = ir$, because in this case, we get $|f(ir)| = |e^{\sin(ir)}| = |e^{i \sinh r}| = 1$. Thus, we need another way to estimate $M(r, f)$.

We proceed as follows: We know that $\sin(\pi/2 + ir) = \cos(ir) = \cosh(r)$, and the point $z_0 = \pi/2 + ir$ is in the disc $D(0, \pi/2 + r)$ (by the triangle inequality). Then

$$M\left(\frac{\pi}{2} + r, f\right) \ge \left| f\left(\frac{\pi}{2} + ir\right) \right| = \left| e^{\cosh r} \right| = e^{\cosh r}.$$

Therefore, $\ln M(r, f) \ge \cosh(r - \pi/2)$. Combining this with (1.2) yields

$$\cosh(r - \pi/2) \le \ln M(r, f) \le \sinh r.$$

As $\cosh(r-\pi/2) \asymp e^r$ and $\sinh r \asymp e^r$ as $r \to +\infty$, we deduce that

$$\ln M(r,f) \asymp e^r, \quad r \to +\infty.$$

The Hardy's notation "$\phi \asymp \psi$" means $\phi = O(\psi)$ and $\psi = O(\phi)$. □

1.3 Order of growth

for a transcendental entire function f, we have by Theorem 1.5,

$$\lim_{r\to+\infty} \frac{\ln M(r,f)}{\ln r} = +\infty.$$

That is, $\ln M(r,f)$ cannot be comparable to $\ln r$. By reducing the growth of $\ln M(r,f)$ by introducing an additional logarithm, we expect that, at least for some functions f, $\ln\ln M(r,f)$ might become comparable to $\ln r$. This suggests that we are dealing with a concept analogous to the degree of polynomials (see Theorems 1.2, 1.3 and 1.4). This concept will later be referred to as the order of growth.

Definition 1.1. An entire function f is said to be of **finite order of growth** if $\exists \beta > 0$ and $\exists R > 0$ (that may depend on β) s.th.

$$\ln M(r,f) \le r^\beta, \quad \forall r \ge R.$$

Otherwise, f is of infinite order. The **order of growth** of f, denoted by $\rho(f)$, is the quantity

$$\rho(f) := \inf\left\{\beta > 0 : \ln M(r,f) \le r^\beta,\ \forall r \ge R(\beta)\right\}.$$

(1) If f is of infinite order of growth, then

$$\left\{\beta > 0 : \ln M(r,f) \le r^\beta,\ \forall r \ge R(\beta)\right\} = \emptyset.$$

Here we use the convention $\inf \emptyset = +\infty$.

(2) f is of infinite order if and only if there exists a sequence $(r_n)_{n\ge1}$ of positive real numbers s.th. $r_n \longrightarrow +\infty$ as $n \to +\infty$ and

$$\ln M(r_n,f) > r_n^n, \quad \forall n \ge 1.$$

(3) Since $\ln M(r, f)$ is an increasing function for non-constant entire functions f, we may always take $R > 0$ sufficiently large so that $\ln M(r, f) > 1$ for every $r \geq R$. Hence, we have that f is of finite order if and only if

$$\ln \ln M(r, f) = O(\ln r), \quad r \geq R.$$

This is analogous to Theorem 1.4.

(4) By definition, constant functions are of zero order of growth.

Now we prove an analogous result to Theorem 1.3.

Theorem 1.6. *The order of growth of a non-constant entire function f is given by*

$$\rho(f) = \limsup_{r \to +\infty} \frac{\ln \ln M(r, f)}{\ln r}.$$

Proof. Set

$$\alpha := \limsup_{r \to +\infty} \frac{\ln \ln M(r, f)}{\ln r},$$

and we aim to show that $\alpha = \rho(f)$. By the remarks following Definition 1.1, it is clear that $\rho(f) = +\infty$ if and only if $\alpha = +\infty$. Thus we may assume that f is of finite order, i.e., α is also a finite number. By definition of limits, $\forall \varepsilon$, $\exists R = R(\varepsilon) > 0$ s.th. $\forall r \geq R$ we have $\ln M(r, f) \leq r^{\alpha + \varepsilon}$. By definition of the order of growth, $\rho(f) \leq \alpha + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we deduce that $\rho(f) \leq \alpha$. On the other hand, for any $\beta > 0$ satisfying $\ln M(r, f) \leq r^{\beta}$, $\forall r \geq R = R(\beta)$ (with $R > 0$ large enough), we have

$$\frac{\ln \ln M(r, f)}{\ln r} \leq \beta, \quad \forall r \geq R.$$

Taking the lim sup, we obtain $\rho(f) \leq \beta$. Since $\rho(f)$ is the infimum of such constants β, we deduce that $\alpha \leq \rho$. Thus $\alpha = \rho(f)$. □

Example 1.1. (1) Any non-constant polynomial is of zero order of growth.

(2) If $f(z) = e^z$, then $M(r, f) = e^r$, and consequently $\rho(f) = 1$.

(3) If $f(z) = \sin z$, then $M(r,f) \sim \dfrac{e^r}{2}$ as $r \to +\infty$, and consequently $\rho(f) = 1$.

(4) If $f(z) = e^{\sin z}$, then $\ln M(r,f) \asymp e^r$, and consequently $\rho(f) = +\infty$.

Theorem 1.7. *Let f and g be entire functions. Then*

$\boxed{1}$ $\rho(f+g) \le \max\{\rho(f), \rho(g)\}$.

$\boxed{2}$ $\rho(fg) \le \max\{\rho(f), \rho(g)\}$.

$\boxed{3}$ *If $\rho(f) > \rho(g)$, then $\rho(f+g) = \rho(f)$.*

Proof. We may clearly suppose that $\rho(f) < \infty$ and $\rho(g) < \infty$. Let $\varepsilon > 0$. Then there exists an $r(\varepsilon) > 1$ such that

$$M(r,f) \le e^{r^{\rho(f)+\varepsilon/2}} \quad \text{and} \quad M(r,g) \le e^{r^{\rho(g)+\varepsilon/2}}, \quad \forall r \ge r(\varepsilon).$$

$\boxed{1}$ By elementary estimates,

$$\begin{aligned} M(r,f+g) &= \max_{|z|\le r} |f(z)+g(z)| \le \max_{|z|\le r} |f(z)| + \max_{|z|\le r} |g(z)| \\ &= M(r,f) + M(r,g) \le e^{r^{\rho(f)+\varepsilon/2}} + e^{r^{\rho(g)+\varepsilon/2}} \\ &\le 2e^{r^{\max\{\rho(f),\rho(g)+\varepsilon/2\}}} = e^{\log 2 + r^{\max\{\rho(f),\rho(g)\}+\varepsilon/2}}, \quad r \ge r(\varepsilon). \end{aligned}$$

We may suppose that $r(\varepsilon) \ge \exp\left(\frac{2\log 2}{\varepsilon}\right)$. Then $r(\varepsilon)^{\varepsilon/2} \ge 2 \ge \log 2$ and

$$\begin{aligned} \log 2 + r^{\max\{\rho(f),\rho(g)\}+\varepsilon/2} &\le r^{\varepsilon/2} + r^{\max\{\rho(f),\rho(g)\}+\varepsilon/2} \\ &\le 2r^{\max\{\rho(f),\rho(g)\}+\varepsilon/2} \le r^{\max\{\rho(f),\rho(g)\}+\varepsilon} \end{aligned}$$

for all $r \ge r(\varepsilon)$. In particular,

$$M(r,f+g) \le e^{r^{\max\{\rho(f),\rho(g)\}+\varepsilon}}, \quad r \ge r(\varepsilon).$$

Hence $\rho(f+g) \le \max\{\rho(f), \rho(g)\} + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. Hence $\rho(f+g) \le \max\{\rho(f), \rho(g)\}$.

$\boxed{2}$ This part is proved similarly by relying on the estimates

$$M(r,fg) = \max_{|z|\le r} |f(z)g(z)| \le \max_{|z|\le r} |f(z)| \cdot \max_{|z|\le r} |g(z)|$$

$$= M(r,f)\cdot M(r,g) \le e^{r\rho(f)+\varepsilon/2}\cdot e^{r\rho(g)+\varepsilon/2}$$
$$\le e^{2r^{\max\{\rho(f),\rho(g)\}+\varepsilon/2}} \le e^{r^{\max\{\rho(f),\rho(g)\}+\varepsilon}}$$

valid for all $r \ge r(\varepsilon)$.

$\boxed{3}$ If $\rho(f) > \rho(g)$, then from $\boxed{1}$ we have $\rho(f+g) \le \rho(f)$. Note that for any entire function h we have $\rho(-h) = \rho(h)$ as $M(r,-h) = M(r,h)$. Thus from $f = (f+g) + (-g)$ we obtain by using $\boxed{1}$ that $\rho(f) \le \max\{\rho(f+g), \rho(g)\}$. Since $\rho(f) > \rho(g)$, it follows that $\rho(f) \le \rho(f+g)$. Thus $\rho(f+g) = \rho(f)$. □

Exercise 1.4. Prove $\boxed{3}$ by using the definition of the order rather than $\boxed{1}$.

Remark 1.2. (1) If $\rho(f) > \rho(g)$ and $g \not\equiv 0$, then $\rho(fg) = \rho(f)$ is also true. Similarly as above,

$$M(r,fg) \le M(r,f)\cdot M(r,g) \le e^{r^{\rho(f)+\varepsilon/2}}\cdot e^{r^{\rho(g)+\varepsilon/2}}$$
$$\le e^{2r^{\rho(f)+\varepsilon/2}} \le e^{r^{\rho(f)+\varepsilon}}, \quad r \ge r(\varepsilon),$$

and so $\rho(fg) \le \rho(f)$. However, we do not have enough tools at this point to prove the reverse inequality $\rho(fg) \ge \rho(f)$.

(2) The inequality $\rho(f+g) \le \max\{\rho(f), \rho(g)\}$ can be strict. E.g., $f(z) = e^z$ and $g(z) = z - e^z$. Similarly, the inequality $\rho(fg) \le \max\{\rho(f), \rho(g)\}$ can be strict. $f(z) = e^z$ and $g(z) = ze^{-z}$. E.g.,

1.4 Growth in terms of Taylor's coefficients

In this section, besides establishing the relationship between the growth of the modulus of an entire function and the rate of decay of its Taylor coefficients, we will also construct entire functions with a pre-given order of growth.

Theorem 1.8. *Let f be an entire function given by its Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \forall z \in \mathbb{C},$$

and let $\alpha \geq 0$. Then $\rho(f) \leq \alpha$ if and only if, $\forall \varepsilon > 0$, the sequence

$$\left(n^{\frac{1}{\alpha+\varepsilon}} \sqrt[n]{|a_n|} \right)_{n \geq 1} \tag{1.3}$$

is bounded.

Proof. $\Longrightarrow$) Assume that $\rho(f) \leq \alpha$. Let $\varepsilon > 0$ be arbitrary. Then $\exists R > 0$ s.th.

$$\ln M(r, f) \leq r^{\alpha+\varepsilon}, \quad \forall r \geq R.$$

To show that (1.3) is bounded, we may assume that $a_n \neq 0$ for every $n \geq 1$. By Cauchy's inequality we obtain, for every $n \geq 1$, that

$$\ln |a_n| \leq \ln M(r, f) - n \ln r \leq r^{\alpha+\varepsilon} - n \ln r, \quad \forall r \geq R,$$

and hence

$$\frac{\ln n}{\alpha+\varepsilon} + \frac{\ln |a_n|}{n} \leq \frac{\ln n}{\alpha+\varepsilon} + \frac{r^{\alpha+\varepsilon}}{n} - \ln r, \quad \forall r \geq R. \tag{1.4}$$

The function $\phi(r) = \dfrac{r^{\alpha+\varepsilon}}{n} - \ln r$ is differentiable on $]0, +\infty[$, and

$$\phi'(r) = \frac{\alpha+\varepsilon}{n} r^{\alpha+\varepsilon} - \frac{1}{r} = 0 \Longleftrightarrow r = \left(\frac{n}{\alpha+\varepsilon} \right)^{\frac{1}{\alpha+\varepsilon}}.$$

Since $\lim\limits_{r \to +\infty} \phi(r) = \lim\limits_{r \to 0^+} \phi(r) = +\infty$, it follows that ϕ takes its minimum at $r_n = \left(\dfrac{n}{\alpha+\varepsilon} \right)^{\frac{1}{\alpha+\varepsilon}}$. We take $N \geq 1$ large enough so that $r_n \geq R$ for every $n \geq N$. In particular, we may choose $N = \lfloor (\alpha+\varepsilon) R^{\alpha+\varepsilon} \rfloor$. Plugging r_n in (1.4), we obtain, every $n \geq N$,

$$\frac{\ln n}{\alpha+\varepsilon} + \frac{\ln |a_n|}{n} \leq \frac{1}{\alpha+\varepsilon} \left(1 + \ln(\alpha+\varepsilon) \right) =: K.$$

Taking the exponential of both sides, we deduce that $\forall \varepsilon$, $\exists C > 0$ s.th. $n^{\frac{1}{\alpha+\varepsilon}} \sqrt[n]{|a_n|} \leq C$, $\forall n \geq 1$.

$\Longleftarrow$) Conversely, let $\varepsilon > 0$. Then $\exists C > 0$ s.th. $0 \leq n^{\frac{1}{\alpha+\varepsilon}} \sqrt[n]{|a_n|} \leq C$, $\forall n \geq 1$. Hence

$$|a_n| \leq n^{\frac{-n}{\alpha+\varepsilon}} C^n, \quad \forall n \geq 1. \tag{1.5}$$

From the Taylor's expansion of f we have

$$M(r, f) \leq |a_0| + \sum_{n\geq 1} |a_n| r^n, \quad r > 0.$$

Using (1.5) yields, for $r > 0$,

$$\begin{aligned} M(r, f) &\leq |a_0| + \sum_{n\geq 1} n^{\frac{-n}{\alpha+\varepsilon}} C^n r^n \\ &\leq |a_0| + \underbrace{\sum_{n\geq (2Cr)^{\alpha+\varepsilon}} \left(n^{\frac{-1}{\alpha+\varepsilon}} Cr\right)^n}_{S_1} + \underbrace{\sum_{n< (2Cr)^{\alpha+\varepsilon}} n^{\frac{-n}{\alpha+\varepsilon}} (Cr)^n}_{S_2}. \end{aligned}$$

For the sum S_1, we have

$$n \geq (2Cr)^{\alpha+\varepsilon} \Longleftarrow n^{\frac{-1}{\alpha+\varepsilon}} Cr \leq \frac{1}{2},$$

and hence

$$S_1 \leq \sum_{n\geq (2Cr)^{\alpha+\varepsilon}} \frac{1}{2^n} \leq \sum_{n\geq 1} \frac{1}{2^n} = 1.$$

For the sum S_2, we have, for sufficiently large $r > 0$,

$$\begin{aligned} S_2 &\leq \left(\sum_{n< (2Cr)^{\alpha+\varepsilon}} n^{\frac{-n}{\alpha+\varepsilon}} \right) (Cr)^{(2Cr)^{\alpha+\varepsilon}} \\ &\leq \left(\sum_{n\geq 1} n^{\frac{-n}{\alpha+\varepsilon}} \right) (Cr)^{(2Cr)^{\alpha+\varepsilon}} \leq K (Cr)^{(2Cr)^{\alpha+\varepsilon}}. \end{aligned}$$

The series $\sum\limits_{n\geq 1} n^{\frac{-n}{\alpha+\varepsilon}}$ converges (By Cauchy's test). Therefore, for sufficiently large $r > 0$, we have

$$\ln M(r, f) = O\left(r^{\alpha+\varepsilon} \ln r\right),$$

which clearly implies that $\rho(f) \leq \alpha + \varepsilon$. Again, as $\varepsilon > 0$ is arbitrary, we deduce that $\rho(f) \leq \alpha$. $\square$

Theorem 1.9. *Let f be an entire function given by its Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \forall z \in \mathbb{C}.$$

Set, for $n \geq 0$,

$$b_n = \begin{cases} 0, & \text{if } a_n = 0, \\ \dfrac{n \ln n}{-\ln |a_n|}, & \text{if } a_n \neq 0. \end{cases}$$

Then $\rho(f) = \limsup_{n \to +\infty} b_n$.

Proof. Set $\sigma := \limsup_{n \to +\infty} b_n$. Assume first that $\sigma < +\infty$.
Let $\varepsilon > 0$. Then $\exists n_0 \geq 1$ s.th. $\forall n \geq n_0$, we have $b_n \leq \sigma + \varepsilon$. Thus

$$\frac{n \ln n}{-\ln |a_n|} \leq \sigma + \varepsilon, \quad \forall n \geq n_0,\ a_n \neq 0.$$

Taking $n_0 \geq 1$ sufficiently large so that $-\ln |a_n| > 0$ yields

$$n^{\frac{1}{\sigma+\varepsilon}} \sqrt[n]{|a_n|} \leq 1, \forall n \geq n_0.$$

From Theorem 1.8, and since $\varepsilon > 0$ is arbitrary, we directly deduce that $\rho(f) \leq \sigma$. If $\sigma = 0$, then $\rho(f) = 0$, and hence the theorem is proved. If $\sigma > 0$, then assume that $\rho(f) < \sigma$. Therefore, $\exists \xi > 0$ s.th. $\rho(f) \leq \sigma - \xi$. Again, making use of Theorem 1.8 (for $\alpha = \sigma - \xi$ and $\varepsilon = \xi/2$), $\exists C > 0$ s.th.

$$n^{\frac{1}{\sigma - \xi/2}} \sqrt[n]{|a_n|} \leq C, \quad \forall n \geq 1.$$

Taking $n_0 \geq 1$ sufficiently large so that $|a_n| < 1$, for every $n \geq n_0$, yields

$$\frac{1}{-\ln |a_n|} \leq \frac{1}{\frac{n \ln n}{\sigma - \xi/2} - n \ln C}, \quad \forall n \geq n_0,\ a_n \neq 0.$$

Hence

$$\sigma = \limsup_{n \to +\infty} b_n = \limsup_{n \to +\infty} \frac{n \ln n}{-\ln |a_n|} \leq \limsup_{n \to +\infty} \frac{n \ln n}{\frac{n \ln n}{\sigma - \xi/2} - n \ln C} = \sigma - \frac{\xi}{2},$$

which is a contradiction. Thus $\rho(f) = \sigma$.

Now, if $\sigma = +\infty$, then assume that $\rho(f) < +\infty$, i.e., there exists $\alpha \geq 0$ s.th. $\rho(f) \leq \alpha < +\infty$. From Theorem 1.8, $\exists C > 0$ s.th. $n^{1/(2\alpha)} \sqrt[n]{|a_n|} \leq C$, $\forall n \geq 1$. Hence as in previous case, we obtain

$$+\infty = \limsup_{n \to +\infty} b_n \leq 2\alpha,$$

which is a contradiction. Thus $\rho(f) = +\infty$. This completes the proof. □

Using this theorem, one may construct any entire function with pre-given order of growth.

Theorem 1.10. *Let $\alpha \in [0, +\infty]$. Then there exists an entire function f whose order of growth is $\rho(f) = \alpha$.*

Proof. The functions

$$f_1(z) = \sum_{n \geq 0} e^{-n^2} z^n, \quad f_2(z) = \sum_{n \geq 0} \frac{1}{(\ln n)^{\ln n}} z^n$$

$$\text{and} \quad f_3(z) = \sum_{n \geq 2} \left(\frac{\ln n}{n} \right)^{\frac{n}{\alpha}} z^n,\ 0 < \alpha < +\infty,$$

are all entire functions (By Cauchy-Hadamard formula for the radius of convergence), and are of orders $\rho(f_1) = 0$, $\rho(f_2) = +\infty$ and $\rho(f_3) = \alpha$, respectively. One can use Theorem 1.9 to check the orders. □

Remark 1.3. In Theorem 1.8, $\varepsilon > 0$ cannot be replaced with 0. Take, e.g., the function f_3 in previous result. we have $\rho(f_3) = \alpha$, but $n^{1/\alpha} \sqrt[n]{|a_n|} = \ln n$ is not bounded.

1.5 Hadamard 3-circles Theorem

Let ψ and ϕ be two real functions, where ϕ is increasing on an interval I. The function ψ is said to be convex in ϕ (or with respect to ϕ) on I if the function $\psi \circ \phi^{-1}$ is convex on $\phi(I)$. In other words, ψ is said to be convex in ϕ if for every $x_1, x_2, x_3 \in \phi(I)$ with $x_1 < x_2 < x_3$ the following inequalities hold

$$\psi(x_2) \leq \frac{\phi(x_3) - \phi(x_2)}{\phi(x_2) - \phi(x_1)} \psi(x_1) + \frac{\phi(x_3) - \phi(x_1)}{\phi(x_2) - \phi(x_1)} \psi(x_2).$$

This subsection is devoted to showing, for non-constant entire functions f, that $\ln M(r,f)$ is convex in $\ln r$. That is, for every $r_1, r_2 > 0$ with $r_1 < r_2$, and for every $r \in]r_1, r_2[$ the following hold

$$\ln M(r,f) \le \frac{\ln r_2 - \ln r}{\ln r_2 - \ln r_1} \ln M(r_1, f) + \frac{\ln r - \ln r_1}{\ln r_2 - \ln r_1} \ln M(r_2, f).$$

Theorem 1.11 (Hadamard 3-circles Theorem)**.** *Let f be a non-constant entire function. Then $\ln M(r,f)$ is convex in $\ln r$.*

Proof. (1) Consider the case $f(z) = cz^n$, $c \in \mathbb{C}^*$ and $n \ge 1$ (integer). Then

$$\ln M(r,f) = \ln |c| + n \ln r, \quad r > 0,$$

which is clearly convex in $\ln r$.
(2) Assume now that f is not of the form cz^n. Define the function $\phi : \mathbb{C}^* \to \mathbb{R}$ by

$$z \mapsto \phi(z) = \frac{|f(z)|}{|z|^\alpha}, \quad \alpha > 0.$$

Clearly the function ϕ is continuous on $\mathbb{C}^*$.

Lemma 1.4. *On any closed annulus $\mathfrak{C} = \overline{A_{r_1,r_2}(0)}$, $0 < r_1 < r_2$, the function ϕ achieves its maximum on the boundary $\partial\mathfrak{C}$, i.e., either on $\{|z| = r_1\}$ or on $\{|z| = r_2\}$.*

Proof of Lemma 1.4. Assume there exists z_0 with $r_1 < |z_0| < r_2$ such that $\phi(z) \le \phi(z_0)$ for every $z \in \mathfrak{C}$. Let $U \subset \mathfrak{C}$ be any neighborhood of z_0. Then there exists an analytic branch of $\log z$ such that

$$F(z) := \frac{f(z)}{z^\alpha} = f(z) e^{-\alpha \log z}$$

is analytic on $\overline{U}$. By maximum modulus principle, the function F is constant on U, and hence $f(z) = cz^\alpha$ for every $z \in U$ for some nonzero constant c. Consequently from the identity theorem we deduce that $f(z) = cz^\alpha$ in a slit complex plane $\mathbb{C} \setminus L_\vartheta$, where

$$L_\vartheta := \left\{ s e^{i\vartheta} : s \ge 0 \right\}.$$

Therefore, $\max_{\theta \neq \vartheta} |f(re^{i\theta})| = |c|r^\alpha$, $r > 0$. Notice by the continuity of $|f|$, we have

$$|f(re^{i\vartheta})| = \lim_{\theta \to \vartheta} |f(re^{i\theta})| = |c|r^\alpha, \quad r > 0.$$

Thus $M(r, f) = |c|r^\alpha$, for every $r > 0$. This, in fact, implies that $\alpha \in \mathbb{N}$ and f is of the form cz^n, which contradicts our assumption. This completes the proof of Lemma 1.4. □

Now we choose α so that $\max_{|z|=r_1} \phi(z) = \max_{|z|=r_2} \phi(z)$, where $0 < r_1 < r_2$. In this way, we get

$$\max_{z \in \mathfrak{C}} \phi(z) = \max_{z \in \partial\mathfrak{C}} \phi(z) = \max_{|z|=r_1} \phi(z) = \max_{|z|=r_2} \phi(z).$$

This is always possible by choosing

$$\alpha = \frac{\ln M(r_2, f) - \ln M(r_1, f)}{\ln r_2 - \ln r_1}.$$

Therefore, for every $r \in]r_1, r_2[$ we get

$$\ln M(r, f) \leq \frac{\ln r_2 - \ln r}{\ln r_2 - \ln r_1} \ln M(r_1, f) + \frac{\ln r - \ln r_1}{\ln r_2 - \ln r_1} \ln M(r_2, f),$$

and this shows that $\ln M(r, f)$ is convex in $\ln r$. □

Remark 1.4. This theorem also shows that f is a continuous function on $[0, +\infty[$.

Theorem 1.12 (Clunie)**.** *Let $\phi :]r_0, +\infty[\to \mathbb{R}_+$ be an increasing function and convex in $\ln r$ such that $\phi(r)/\ln r \to +\infty$ as $r \to +\infty$. Then there exists an entire function f such that*

$$\ln M(r, f) \sim \phi(r), \quad \text{as } r \to +\infty.$$

1.6 Type of growth

Let f be entire. If $0 < \rho(f) < +\infty$, then the growth of f can be expressed more precisely in terms of its type.

Definition 1.2. Let f be an entire function of order $0 < \rho = \rho(f) < +\infty$. We say that f is of finite type if there exist $\beta > 0$ and $R > 0$ (R may depend on β) such that

$$\ln M(r, f) \leq \beta r^{\rho}, \quad \forall r \geq R.$$

Otherwise, f has an infinite type. The type of f, denoted by $\tau(f)$, is defined by

$$\tau(f) := \inf \left\{ \beta > 0 : \ln M(r, f) \leq \beta r^{\rho}, \ \forall r \geq R(\beta) \right\}.$$

Clearly $\tau(f) \in [0, +\infty]$. According to as $\tau = \infty$, $0 < \tau < \infty$ or $\tau = 0$, the function f is said to be of maximum (or infinite), mean (or finite) or minimum (or zero) type of order ρ.

Theorem 1.13. *Let f be an entire function of order $0 < \rho = \rho(f) < +\infty$. Then*

$$\tau(f) = \limsup_{r \to +\infty} \frac{\ln M(r, f)}{r^{\rho}}.$$

Proof. Left as an exercise! □

Example 1.2. The entire function $f(z) = \cos(az)$, where $a \in \mathbb{C}^*$, satisfies

$$M(r, f) = \cosh(|a|r) \sim \frac{e^{|a|r}}{2}, \quad r \to +\infty.$$

Thus $\rho(f) = 1$ and $\tau(f) = |a|$.

Lemma 1.5. *Let f be analytic in a neighborhood of 0 having the Taylor expansion*

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n. \tag{1.6}$$

Suppose there exist $\lambda > 0$, $\mu > 0$ and $N = N(\lambda, \mu) \in \mathbb{N}$ such that

$$|a_n| \le \left(\frac{e\mu\lambda}{n}\right)^{n/\mu}, \quad \forall n \ge N. \tag{1.7}$$

Then f is an entire function. Moreover, for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\ln M(r, f) \le (\lambda + \varepsilon) r^\mu, \quad \forall r \ge R. \tag{1.8}$$

Proof. From (1.7), the radius of convergence of the series in (1.6) equals

$$\liminf_{n\to+\infty} \frac{1}{\sqrt[n]{|a_n|}} \ge \liminf_{n\to+\infty} \left(\frac{n}{e\mu\lambda}\right)^{1/\mu} = +\infty.$$

Thus f is an entire function. To prove (1.8), for a fixed $r > 0$, we write

$$M(r, f) \le \sum_{n=0}^{+\infty} |a_n| r^n = \sum_{n=0}^{N} |a_n|\, r^n + \sum_{n=N+1}^{M(r)} |a_n|\, r^n + \sum_{n=M(r)+1}^{+\infty} |a_n|\, r^n =: \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where $M(r)$ satisfies

$$\begin{cases} M(r) > N + 1, \\ |a_n| r^n \le 1/2^n, \quad \forall n \ge N(r). \end{cases}$$

If we choose $M(r) \ge \max\{N, e\mu\lambda(2r)^\mu\}$, then from (1.7), we obtain

$$|a_n| r^n \le \left(\frac{e\mu\lambda}{n}\right)^{n/\mu} r^n \le \frac{1}{2^n}, \quad \forall n \ge M(r).$$

Since $e\mu\lambda(2r)^\mu \to +\infty$ as $r \to +\infty$, there exists $R_1 > 0$ such that $e\mu\lambda(2r)^\mu > N + 1$ for every $r > R_1$. So, we can choose $M(r) = \lfloor e\mu\lambda(2r)^\mu \rfloor$ and $r > R_1$. So, in particular, $\Sigma_2 \ne 0$. Therefore, for $r > R_1$, we have

$$\Sigma_1 \le C r^N \quad \text{and} \quad \Sigma_2 \le \sum_{n=M(r)+1}^{+\infty} \frac{1}{2^n} \le 1,$$

where $C = \sum_{n=0}^{N} |a_n|$. To estimate Σ_2, define

$$H(x) = \left(\frac{e\mu\lambda}{x}\right)^{x/\mu} r^x, \quad x > 0.$$

Then

$$H'(x) = H(x)\left(\frac{1}{\mu}\left(\ln\left(\frac{e\mu\lambda}{x}\right) - 1\right) + \ln r\right) = 0$$

if and only if $x = \mu\lambda r^\mu$. Notice that H reaches its maximum at this point. Therefore,

$$\left(\frac{e\mu\lambda}{x}\right)^{x/\mu} r^x \leq H(\mu\lambda r^\mu) = e^{\lambda r^\mu}, \quad \forall x > 0.$$

Hence, for $r > R_1$,

$$\begin{aligned}\Sigma_2 &\leq (M(r) - N) \max_{N+1 \leq n \leq M(r)} |a_n| r^n \\ &\leq (M(r) - N) \max_{n \geq N} \left(\left(\frac{e\mu\lambda}{n}\right)^{n/\mu} r^n\right) \\ &\leq (M(r) - N) e^{\lambda r^\mu}.\end{aligned}$$

Putting all together, the estimate (1.8) follows. □

Theorem 1.14. *Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function of order ρ $(0 < \rho < +\infty)$. Then*

$$\tau(f) = \frac{1}{e\rho} \limsup_{n \to +\infty} \left(n \sqrt[n]{|a_n|^\rho}\right).$$

Proof. Left as an exercise! □

Question. For any $(\alpha, \beta) \in]0, +\infty[\times [0, +\infty]$, is there always an entire function f with $\rho(f) = \alpha$ and $\tau(f) = \beta$?

1.7 Order and type of a derivative

Exercise 1.5. Let f be an entire function.

(1) Show that

$$\frac{M(r, f) - |f(0)|}{r} \leq M(r, f') \leq \frac{M(R, f)}{R - r}, \quad \text{for every } 0 < r < R < +\infty. \tag{1.9}$$

(2) Deduce that f and f' have the same order of growth and the same type (when it's defined).

Solution. (1) Integrating along a line segment $[0, z]$ gives

$$f(z) = \int_0^z f'(\xi)\,\mathrm{d}\,\xi + f(0).$$

Hence,

$$|f(z)| \leq M(r, f') \int_0^z |\,\mathrm{d}\,\xi| + |f(0)| = rM(r, f') + |f(0)|, \quad \forall z \text{ s.th. } |z| = r > 0.$$

This gives the first inequality in (1.9). Now, let z with $|z| = r > 0$ and let $R > r$. By using Cauchy's integral's formula along the circle $|\xi - z| = R - r$, we obtain

$$|f'(z)| \leq \frac{1}{2\pi} \oint_{|\xi - z| = R - r} \frac{|f(\xi)|}{|\xi - z|^2}\,\mathrm{d}\,|\xi| \leq \frac{1}{2\pi} \frac{M(R, f)}{(R - r)^2} 2\pi(R - r) = \frac{M(R, f)}{R - r},$$

which gives the second inequality in (1.9).

(2) Left as an exercise! □

2 Zeros of entire functions

If f is a non-constant polynomial, then from the first lecture, and by FTA, we have

$$\limsup_{r \to +\infty} \frac{\ln M(r, f)}{\ln r} = \#\{z : f(z) = 0\}.$$

Question. Does the relation between the number of zeros of f and the growth of $M(r, f)$ extend to transcendental entire functions? and how ?

In what follows, f will be always a transcendental entire function. The zeros of f cannot accumulate in $\mathbb{C}$. In particular, if f has infinitely many zeros, then they are accumulating to ∞.

Exercise 2.1. If f is an entire function with no zeros, then show that there exists an entire function g such that $f = e^g$.

Solution. Since f has no zeros, it follows that f'/f is an entire function. Hence f'/f has a primitive in $\mathbb{C}$, i.e., there exists an entire function h for which $h' = f'/f$ in $\mathbb{C}$. Define now $\varphi = fe^{-h}$. Then $\varphi' = (f' - fh')e^{-h} \equiv 0$, which yields φ is a constant function, say $\varphi(z) = e^a$ for every $z \in \mathbb{C}$. Hence, $f(z) = e^{h(z)+a}$ for every $z \in \mathbb{C}$. This completes the proof. $\square$

An analytic function f that has no zeros in a domain D is called a zero-free function in D.

The previous exercise can be generalized to any zero-free analytic function f on simply connected domain D. That, if f is zero-free in a simply connected domain D, then $f = e^g$, where g is an analytic function on D.

2.1 Jensen's formula and its consequences

Theorem 2.1 (Jensen's Theorem)**.** *Let f be an entire function such that $f(0) \neq 0$, and let $r > 0$ and $a_1, \ldots, a_k$ be the zeros of f in the disc $|z| < r$ (each is repeated according to its multiplicity), and suppose that f has no zeros on the circle $|z| = r$. Then*

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})| \,\mathrm{d}\,\theta - \sum_{j=1}^{k} \ln\frac{r}{|a_j|}.$$

Proof. Consider the function

$$F(z) = f(z)\prod_{j=1}^{k}\frac{r^2 - \bar{a}_j z}{r\,(z - a_j)}.$$

Notice that F is analytic on the disc $D(0, r)$, and has no zeros there. In addition, F has no zeros on the circle $|z| = r$, since

$$\left|\frac{r^2 - \bar{a}_k z}{r\,(z - a_k)}\right| = \left|\frac{r^2 - \bar{a}_k re^{i\varphi}}{r\,(re^{i\varphi} - a_k)}\right| = \left|\frac{r - \bar{a}_k e^{i\varphi}}{r - a_k e^{-i\varphi}}\right| = \left|\frac{r - \overline{a_k e^{-i\varphi}}}{r - a_k e^{-i\varphi}}\right| = 1, \quad \text{for } |z| = r.$$

That is, $|F(re^{i\theta})| = |f(re^{i\theta})| \neq 0$ for every $\theta \in [0, 2\pi]$. Then there exists $R > r$ for which F is zero-free and analytic on $D(0, R)$. This implies that $\ln|F(z)|$

is harmonic on the disc $D(0,R)$ (*Hint.* Use similar technique as Exercise 2.1). By the mean property of harmonic functions, we have

$$\ln|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|F(re^{i\theta})|\,\mathrm{d}\,\theta. \tag{2.1}$$

Since

$$|F(0)| = |f(0)|\prod_{j=1}^{k}\frac{r}{|a_j|} \quad \text{and} \quad |F(re^{i\theta})| = |f(re^{i\theta})|,\ \forall\theta\in[0,2\pi],$$

the conclusion of Jensen's theorem follow from (2.1). □

Denote by $n_f(r)$ the number of zeros z_n of f in $|z|\le r$, where each zero is counted according to its multiplicity. We define *the integrated counting function* $N_f(r)$ for f by

$$N_f(r) = \int_0^r \frac{n_f(t)-n_f(0)}{t}\,\mathrm{d}\,t + n_f(0)\ln r, \quad r>0.$$

Notice that if $f(0)\neq 0$, then $N_f(r) = \int_0^r \frac{n_f(t)}{t}\,\mathrm{d}\,t$.

Theorem 2.2 (Jensen's Theorem - Version 2)**.** *Let f be an entire function such that $f(0)\neq 0$, and suppose that f has no zeros on the circle $|z|=r>0$. Then*

$$N_f(r) = \frac{1}{2\pi}\int_0^{2\pi}\ln\left|f\left(re^{i\theta}\right)\right|d\theta - \log|f(0)|.$$

Proof. Let $a_1,a_2,\ldots,a_k$ be the zeros of f in the disc $D(0,r)$ (each is repeated according to its multiplicity). Then by Jensen's Theorem (Thm 2.1) we have

$$\sum_{j=1}^{k}\ln\frac{r}{|a_j|} = \frac{1}{2\pi}\int_0^{2\pi}\ln\left|f\left(re^{i\theta}\right)\right|d\theta - \log|f(0)|.$$

It remains to show that

$$\sum_{j=1}^{k}\ln\frac{r}{|a_j|} = N_f(r).$$

Assume without loss of generality that the zeros are ordered according to their moduli, i.e., $|a_1| \leq |a_2| \leq \cdots \leq |a_k|$. Then

$$\begin{aligned}
\sum_{j=1}^{k} \ln \frac{r}{|a_j|} &= \ln \frac{r^k}{|a_1| \times |a_2| \times \cdots \times |a_{k-1}| \times |a_k|} \\
&= \ln \left(\frac{|a_2|}{|a_1|} \times \frac{|a_3|^2}{|a_2|^2} \times \cdots \times \frac{|a_k|^{k-1}}{|a_{k-1}|^{k-1}} \times \frac{r^k}{|a_k|^k} \right) \\
&= \sum_{j=1}^{k-1} j \left(\ln |a_{j+1}| - \ln |a_j| \right) + k \left(\ln r - \ln |a_k| \right) \\
&= \sum_{j=1}^{k-1} \int_{|a_j|}^{|a_{j+1}|} \frac{j}{t} \, \mathrm{d}\, t + \int_{|a_k|}^{r} \frac{k}{t} \, \mathrm{d}\, t = \int_0^r \frac{n_f(t)}{t} \, \mathrm{d}\, t,
\end{aligned}$$

where we have used

$$n_f(t) = \begin{cases} 0, & 0 \leq t < |a_1|, \\ j, & |a_j| \leq t < |a_{j+1}|, \quad (1 \leq j \leq k-1), \\ k, & |a_k| \leq t < r. \end{cases}$$

This completes the proof. □

Exercise 2.2. [1] The restriction regarding zeros on $|z| = r$ can be removed by constructing a suitable function F that is zero-free and analytic on $D(0, r)$. (Address the zeros on the circle $|z| = r$.)
[2] By applying the Poisson formula for harmonic functions, derive an analogous expression for $\ln |f(z)|$ when $f(z) \neq 0$ and $z \in D(0, r)$.
[3] What is the analogous statement for the case when $f(0) = 0$?

Corollary 2.3. *If f is an entire function such that $f(0) \neq 0$, then*

$$n_f(r) \leq \ln M(er, f) - \ln |f(0)|, \quad \forall r > 0.$$

Proof. Let $r > 0$. From Theorem 2.2, we have

$$\int_0^{er} \frac{n_f(t)}{t} \, \mathrm{d}\, t \leq \ln M(er, f) - \ln |f(0)|.$$

In addition, since $n_f(r)$ is non-decreasing function, we have

$$\int_0^{er} \frac{n_f(t)}{t}\,\mathrm{d}\,t \geq \int_r^{er} \frac{n_f(t)}{t}\,\mathrm{d}\,t \geq n_f(r)\int_r^{er} \frac{\mathrm{d}\,t}{t} = n_f(r).$$

Combining these two inequalities, we obtain the conclusion. □

This corollary may be helpful in estimating the number of zeros of f is a certain disc, or detecting the zero-free discs of f. E. g., if $f(0) \neq 0$ and let $r_0 > 0$ such that $M(er_0, f) < e|f(0)|$, then by the corollary we have

$$n_f(r_0) \leq \ln M(er_0, f) - \ln|f(0)| < 1.$$

So, f has no zeros in the disc $|z| \leq r_0$.

Corollary 2.4. *If f is an entire function of order $\rho < +\infty$. Then for every $\varepsilon > 0$, we have $n_f(r) \lesssim r^{\rho+\varepsilon}$ for sufficiently large $r > 0$.*

The proof left as an exercise.

If we define the quantity $\lambda(f)$ for an entire function f by

$$\lambda(f) = \limsup_{r\to+\infty} \frac{\ln n_f(r)}{\ln r}, \tag{2.2}$$

then from Corollary 2.4, we directly deduce that $\lambda(f) \leq \rho(f)$.

Exercise 2.3. Show that

$$\lambda(f) = \limsup_{r\to+\infty} \frac{\ln N_f(r)}{\ln r}. \tag{2.3}$$

2.2 Exponent of convergence

This section is devoted to showing that $\lambda(f)$, introduced in (2.2) or (2.3), is, in fact, the exponent of convergence of the sequence of moduli of the zeros of an entire function f.

Definition 2.1. Let f be an entire function, and let $(z_n)_{n\geq 1}$ be the sequence of nonzero zeros of f, repeated according to their multiplicities and ordered according to their moduli, i.e., $0 < |z_1| \leq |z_2| \leq \cdots$. The exponent of convergence $\mu(f)$ of $(z_n)_{n\geq 1}$ is defined by

$$\mu(f) = \inf\left\{\alpha > 0 : \sum_{n=1}^{+\infty} |z_n|^{-\alpha} < +\infty\right\}.$$

In particular, if $\sum_{n=1}^{+\infty} |z_n|^{-\alpha} = +\infty$ for every $\alpha > 0$, then $\mu(f) = +\infty$.

Lemma 2.1. *Given $\alpha > 0$. Then*

$$\sum_{n=1}^{+\infty} |z_n|^{-\alpha} < +\infty \Longleftrightarrow \int_0^{+\infty} \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t < +\infty.$$

Proof. We may assume that $f(0) \neq 0$. Notice that we can always replace $n_f(r)$ with $n_f(r) - n_f(0)$ when $f(0) = 0$. In addition, finitely many zeros at the origin have no significant impact on the growth of $n_f(r)$ if it is unbounded. Let $(r_j)_{j\geq 1}$ be the sequence of radii at which all the zeros of f are located. Clearly the sequence $(r_j)_{j\geq 1}$ must be increasing, as the zeros don't have accumulation points in $\mathbb{C}$. The number of zeros located on a circle $|z| = r_j$ is given by $n_f(r_j^+) - n_f(r_j^-)$ counting the multiplicities. Notice that $t \mapsto n_f(t)$ is a step function. By Riemann-Stieltjes integration, we have

$$\int_0^r \frac{dn_f(t)}{t^\alpha} = \sum_{r_j<r} \frac{n_f(r_j^+) - n_f(r_j^-)}{r_j^\alpha} = \sum_{|z_n|<r} \frac{1}{|z_n|^\alpha}.$$

Integration by parts yields

$$\sum_{|z_n|<r} \frac{1}{|z_n|^\alpha} = \frac{n_f(r)}{r^\alpha} + \alpha \int_0^r \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t. \tag{2.4}$$

Assume that $\sum_{n\geq 1} |z_n|^\alpha < +\infty$. Then from (2.4), we have

$$0 < \int_0^r \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t \leq \sum_{n\geq 1} |z_n|^\alpha < +\infty,$$

which means the integral is convergent. Now, assume that $\displaystyle\int_0^{+\infty} \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t < +\infty$. Since $t \mapsto n_f(t)$ is non-decreasing, it follows that

$$\frac{n_f(r)}{r^\alpha} = \alpha n_f(r) \int_r^{+\infty} \frac{\mathrm{d}\,t}{t^{\alpha+1}} \leq \alpha \int_r^{+\infty} \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t \longrightarrow 0, \quad r \to +\infty. \tag{2.5}$$

From (2.4), we deduce that the series $\sum\limits_{n\geq 1} |z_n|^\alpha$ is convergent. □

Remark 2.1. From (2.4) and (2.5), we dedeuce that

$$\sum_{n\geq 1} |z_n|^\alpha = \alpha \int_0^{+\infty} \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t.$$

From this we directly obtain that if f is an entire function such that $f(0) \neq 0$, then

$$\mu(f) = \inf\left\{\alpha > 0 : \int_0^{+\infty} \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t < +\infty\right\}. \tag{2.6}$$

Theorem 2.5. *Let f be an entire function, then $\lambda(f) = \mu(f)$.*

Proof. Since finitely many possible zeros at the origin have no a affect on $\lambda(f)$ or on $\mu(f)$, we may assume that $f(0) \neq 0$. We will denote $\lambda(f)$ and $\mu(f)$ by λ and μ respectively.
We prove first that $\mu \leq \lambda$. This inequality is obvious if $\lambda = +\infty$, and so we assume that $\lambda < +\infty$. Recall that

$$\lambda = \limsup_{r\to+\infty} \frac{\ln n_f(r)}{\ln r}.$$

For any given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$n_f(r) \leq r^{\lambda+\varepsilon}, \quad \forall r \geq r_\varepsilon.$$

From this we have, for every $r > r_\varepsilon$ and for every $\alpha > 0$,

$$\int_0^r \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t = \int_0^{r_\varepsilon} \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t + \int_{r_\varepsilon}^r \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t \leq \int_0^{r_\varepsilon} \frac{n_f(t)}{t^{\alpha+1}}\,\mathrm{d}\,t + \underbrace{\int_{r_\varepsilon}^r \frac{1}{t^{\alpha+1-\lambda-\varepsilon}}\,\mathrm{d}\,t}_{I(r)}.$$

If $\alpha > \lambda + \varepsilon$, then $I(r)$ converges as $r \to +\infty$, and hence $\int_0^{+\infty} \frac{n_f(t)}{t^{\alpha+1}} \, \mathrm{d}\, t < +\infty$. This shows that

$$]\lambda + \varepsilon, +\infty[\subset \left\{ \alpha > 0 : \int_0^{+\infty} \frac{n_f(t)}{t^{\alpha+1}} \, \mathrm{d}\, t < +\infty \right\}.$$

Therefore, (2.6) yields $\mu \leq \lambda + \varepsilon$. As $\varepsilon > 0$ is arbitrary, we deduce that $\mu \leq \lambda$.

Now we prove $\mu \geq \lambda$. By definition of μ, we have, for every $\varepsilon > 0$,

$$\int_0^{+\infty} \frac{n_f(t)}{t^{\mu+\varepsilon+1}} \, \mathrm{d}\, t < +\infty.$$

Similarly to (2.5) we obtain

$$\frac{n_f(r)}{r^{\mu+\varepsilon}} \leq (\mu+\varepsilon) \int_r^{+\infty} \frac{n_f(t)}{t^{\mu+\varepsilon+1}} \, \mathrm{d}\, t \leq (\mu+\varepsilon) \int_0^{+\infty} \frac{n_f(t)}{t^{\mu+\varepsilon+1}} \, \mathrm{d}\, t = \text{constant}, \quad \forall r > 0.$$

Thus $n_f(r) \lesssim r^{\mu+\varepsilon}$ for $r > 0$, which yields $\lambda \leq \mu$. This completes the proof. $\square$