Complex Analysis

For ING3 of NHSM (2024/2025)

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Contents

1 Some topological properties in $\mathbb{C}$

Lec. 01 → It is clear that the function $d : \mathbb{C}\times\mathbb{C} \to \mathbb{R}_+$, $(z_1, z_2) \mapsto d(z_1, z_2) := |z_1 - z_2|$, defines a metric. Therefore, $(\mathbb{C}, d)$ is a metric space. Notice that $\mathbb{C} \cong \mathbb{R}^2$ (i.e., $\mathbb{C}$ and $\mathbb{R}^2$ are homeomorphic) and the above metric corresponds to the Euclidean metric on $\mathbb{R}^2$. Thus the topological structure of $\mathbb{C}$ is the same as that of $\mathbb{R}^2$.

1.1 Some fundamental point sets in $\mathbb{C}$

(a) Let $z_0 \in \mathbb{C}$ and $0 < r < R$. Then the set $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ is the *an open disc*, centered at z_0 with radius r. In particular, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is called the open unit disc. The set $A(z_0, r, R) := \{z \in \mathbb{C} : r < |z - z_0| < R\}$ is *an open annulus*, centered at z_0 with an inner radius r and an outer radius R.

(b) Let $a, b \in \mathbb{R}$ such that $a < b$. Then the sets $H^+(a) := \{z \in \mathbb{C} : \mathrm{Re}(z) > a\}$ and $H^-(a) := \{z \in \mathbb{C} : \mathrm{Re}(z) < a\}$ are, respectively, *an open right-half plane* and *an open left-half plane*. The set $T(a, b) := \{z \in \mathbb{C} : a < \mathrm{Re}(z) < b\}$ is *an open vertical strip*.
– Analogously, we can define the open upper and open lower half planes, as well as a horizontal strip, by considering $\mathrm{Im}(z)$ in place of $\mathrm{Re}(z)$ in the above sets.

(c) Let $\alpha, \beta \in]-\pi, \pi]$ such that $\alpha < \beta$. Then the set $S(\alpha, \beta) := \{z \in \mathbb{C} : \alpha < \mathrm{Arg}(z) < \beta\}$ represents an open sector.

1.2 Definitions

(a) For $\varepsilon > 0$, the ε-neighborhood of a point z_0 is the open disc $D(z_0, \varepsilon)$.

(b) Let $S \subset \mathbb{C}$. A point $z_0 \in S$ is said to be *an interior point* of S if there exists $\varepsilon > 0$ such that $D(z_0, \varepsilon) \subset S$. Moreover, S is called *an open set* of $\mathbb{C}$ if each point of S is an interior point.

Example 1.1. (i) If $S = \{z : |z| \leq 1\}$, then each point z with $|z| < 1$ is an interior point of S.

(ii) The annulus $A(z_0, r, R) := \{z \in \mathbb{C} : r < |z - z_0| < R\}$, where $z_0 \in \mathbb{C}$ and $0 < r < R$, is an open set.

(c) A point z_0 is called *an exterior point* of a set $S \subset \mathbb{C}$ if there exists $\varepsilon > 0$ such that $D(z_0, \varepsilon) \cap S = \emptyset$. Moreover, z_0 is called *a boundary point* of S if, for every $\varepsilon > 0$,

$$D(z_0, \varepsilon) \cap S \neq \emptyset \quad \text{and} \quad D(z_0, \varepsilon) \cap (\mathbb{C} \setminus S) \neq \emptyset.$$

Notice that a boundary point is neither interior nor exterior. The set of all boundary points of S is called *the boundary* of S, and denoted by ∂S.

Example 1.2. We have $\partial\mathbb{D} = \{z : |z| = 1\}$ and $\partial A(i, 1, 2) = \{z : |z - i| = 1\} \cup \{z : |z - i| = 2\}$.

(d) The closure of a subset $S \subset \mathbb{C}$, which is denoted by $\overline{S}$, is defined by $\overline{S} = S \cup \partial S$. A set S is called *a closed set* if $S = \overline{S}$, or if $\partial S \subset S$, i.e., S contains its all boundary points.

Example 1.3. The set $\{z : 1 < |z - i| \leq 2\}$ is neither open nor closed. Moreover, $\overline{\{z : 1 < |z - i| \leq 2\}} = \{z : 1 \leq |z - i| \leq 2\}$.

Remark 1.1. The set $\mathbb{C}$ is considered as open and closed in $\mathbb{C}$ at the same time. Notice that $\partial\mathbb{C} = \emptyset$ and every point $z \in \mathbb{C}$ is an interior point in $\mathbb{C}$.

(e) A point $z_0 \in \mathbb{C}$ is called *an accumulation point* (or limit point) of a set $S \subset \mathbb{C}$ if, for every $\varepsilon > 0$, $(D(z_0, \varepsilon) \setminus \{z_0\}) \cap S \neq \emptyset$. The set of all accumulation points of S is denoted by S'.

1.3 Curves and contours in $\mathbb{C}$

Definition 1.1. Let $a, b \in \mathbb{R}$ with $a < b$, and let $x, y : [a, b] \to \mathbb{R}$ be continuous functions. Then the subset $\gamma = \{z = x(t) + iy(t) : t \in [a, b]\}$ is called *a curve* (or *path*). If we define the function $\varphi : [a, b] \to \mathbb{C}$ by $\varphi(t) = x(t) + iy(t)$, then the curve γ is the image of the function φ.
The curve γ starts at its *initial point* $\varphi(a)$ and ends at its *terminal point* $\varphi(b)$.
The function φ is called a parametrization of the curve γ.

Remark 1.2.

1. We conventionally use $\gamma(t)$ (or $z(t)$) instead of $\varphi(t)$, and we write $z = \gamma(t)$ for $t \in [a, b]$ to indicate the parametrization.
2. Every curve can be expressed by several different parametrizations, as different parameterizations can trace the same geometric path in space. Any parametrization can be expressed in any form (algebraic, trigonometric, or exponential).

Example 1.4. The curve $z(t) = e^{it}$, $t \in [0, \pi]$ represents the upper-half unit circle in an anticlockwise direction. Same curve can be represented by $z(t) = e^{i\pi t}$, $t \in [0, 1]$, or by $z(t) = -t + i\sqrt{1 - t^2}$, $x \in [-1, 1]$.

Definition 1.2. – A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be *simple* if it doesn't intersect itself, i.e., $\gamma(t_1) \neq \gamma(t_2)$ for every $t_1, t_2 \in [a, b]$ with $t_1 \neq t_2$. The curve in Example 1.4 is simple.
– A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be *closed* if $\gamma(a) = \gamma(b)$.
– A *simple closed curve* (or Jordan curve) is a curve satisfying $\gamma(a) = \gamma(b)$ and $\gamma(t_1) \neq \gamma(t_2)$ for every $t_1, t_2 \in]a, b[$ with $t_1 \neq t_2$.

Example 1.5. The curve $z(t) = e^{it}$, $t \in [0, 2\pi]$, which traces the unit circle, is a simple closed curve.

Definition 1.3. – A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be *smooth* if

(i) $\dot{\gamma}(t) := x'(t) + iy'(t)$ exists and continuous on $[a, b]$.

(ii) $\dot{\gamma}(t) \neq 0$ for every $t \in [a, b]$.

– A curve $\gamma : [a, b] \to \mathbb{C}$ is called a *contour* if there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that γ is smooth on each $[t_k, t_{k+1}]$, $0 \leq k \leq n-1$. In other words, a contour is a piecewise smooth curve.

The contour can be built upon a finite sequence of smooth curves $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for $1 \leq k \leq n-1$. In this case, we write $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$. The curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ are called the smooth components of the contour γ. The operation "+" between curves is called the concatenation of curves. If the terminal point of γ_1 does not equal the initial point ofγ_2, we leave the sum $\gamma_1 + \gamma_2$ undefined.

Lec. 02 → **Example 1.6.** The curve

$$\gamma(t) = \begin{cases} t + 2it, & 0 \leq t \leq 1, \\ t + 2i, & 1 \leq t \leq 2, \end{cases}$$

is not a smooth curve, because the derivative is not continuous. However, γ is a contour, as $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1(t) = t + 2it, 0 \leq t \leq 1$, and $\gamma_2(t) = t + 2i, 0 \leq t \leq 1$, are smooth curves.

We generalize the example as follows:

– Let $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq z_2$. The curve $\ell(t) = (1-t)z_1 + tz_2$, for $t \in [0, 1]$, is called the line segment from z_1 to z_2, and is denoted by $[z_1, z_2]$ (be aware of the difference of this notation and the notation of interval). It is clear that a line segment is smooth.

– Let $z_1, \ldots, z_n \in \mathbb{C}$. The curve $L = [z_1, z_2] + \ldots + [z_{n-1}, z_n]$ is called a polygonal line from z_1 to z_n. It is clear that a polygonal line is contour.

Remark 1.3. Any point in the complex plane is a curve parameterized by a constant function. In this case, a point is called a *trivial curve*.

Homotopic curves. Let $D \subset \mathbb{C}$ be an open subset and $z_1, z_2 \in D$. Consider two curves $\gamma_1, \gamma_2 : [a, b] \to D$ (in D) with the same endpoints z_1 and z_2 (i.e., same initial point and same terminal point). Then γ_1 and γ_2 are called *homotopic* in D (or γ_1 is homotopic to γ_2 in D), if there exists a continuous function $H : [a, b] \times [0, 1] \to D$ such that

$$H(t, 0) = \gamma_1(t) \quad \text{and} \quad H(t, 1) = \gamma_2(t), \quad \forall t \in [a, b],$$

$$H(a, s) = z_1 \quad \text{and} \quad H(b, s) = z_2, \quad \forall s \in [0, 1].$$

The function H is called a *homotopy* from γ_1 to γ_2 in D. This homotopy is for curves with the same endpoints.

Consider now two Jordan curves $\gamma_1, \gamma_2 : [a, b] \to D$. Then, γ_1 and γ_2 are called homotopic in D if there exists a continuous function $H : [a, b] \times [0, 1] \to D$ such that

$$H(t, 0) = \gamma_1(t) \quad \text{and} \quad H(t, 1) = \gamma_2(t), \quad \forall t \in [a, b],$$

$$H(a, s) = H(b, s), \quad \forall s \in [0, 1].$$

This homotopy is for closed curves.
A Jordan curve in D, that is homotopic to a point in D, is said to be *null-homotopic* in D.

Example 1.7. Let $D \subset \mathbb{C}$ be a convex domain, and let $\gamma_1, \gamma_2 : [a, b] \to D$ be two curves in D with the same endpoints. Then the function $H : [a, b] \times [0, 1] \to D$, defined by $H(t, s) := (1 - s)\gamma_1(t) + s\gamma_2(t)$, is a homotopy from γ_1 to γ_2 in D.

1.4 Domains in $\mathbb{C}$

Definition 1.4. – An open set $S \subset \mathbb{C}$ is said to be *connected* if, for every $a, b \in S$, there exists a polygonal line joining a and b that lies entirely within S.
– An open connected set is called a *domain*. For example, the open annuli are domains.
– A *region* is a domain together with some, all, or none of its boundary points. For example, the set $\{z : 1 \leq |z - i| < 2\}$ is a region.

Lec. 03 Recall that a subset $S \subset \mathbb{C}$ is said to be bounded if there exists $M > 0$ such that $|z| \leq M$ for every $z \in S$. The subset S is unbounded if it is not bounded.

Theorem 1.1 (Jordan curve theorem)**.** *Any simple closed curve (Jordan curve) γ in $\mathbb{C}$ divides the complex plane into exactly two disjoint domains. One of these domains is bounded, called the interior of γ and denoted by* $\mathrm{int}(\gamma)$*. The other one is unbounded and called the exterior of γ, and denoted by* $\mathrm{ext}(\gamma)$*. The curve γ is the boundary of each domain, i.e.,* $\gamma = \partial\,\mathrm{int}(\gamma) = \partial\,\mathrm{ext}(\gamma)$*.*

It is important to be aware of the different uses of the term "interior" (resp. "exterior"). In a topological context, the interior of a subset S is the set of all interior points, or equivalently, the largest open set contained in S, and it is denoted by $\mathring{S}$. Consequently, it follows that $\mathring{\gamma} = \emptyset$.

Definition 1.5. A domain $D \subset \mathbb{C}$ is said to be *simply connected* if, for any Jordan curve γ lying in D, we have $\mathrm{int}(\gamma) \subset D$. A domain that is not simply connected is called multiply connected.

Intuitively, simply connected domains have no holes.

Remark 1.4 (Alternative Definition)**.** A domain $D \subset \mathbb{C}$ is said to be simply connected if any two curves in D with the same endpoints are homotopic in D (with homotopy of curves with the same endpoints). Or equivalently, if every Jordan curve is null-homotopic in D (with homotopy of closed curves). From Example 1.7, we deduce that any convex set is simply connected.

Definition 1.6 (Positive orientation)**.** A Jordan curve γ is said to be *positively oriented* (counterclockwise) if the interior domain lies to the left of an observer tracing the points of γ in the order they are traversed. Otherwise, γ is negatively oriented.
– A simple open curve is positively oriented if it's traced from its initial point to its terminal point.
– If the orientation of a curve γ is revered, and the roles of the endpoints are switched, then the resulting curve is called *the opposite curve* (or the reversal) of γ, and it is denoted by $-\gamma$. We say: γ and $-\gamma$ are oppositely oriented.

Example 1.8. The circle $\gamma_1(t) = e^{it}$, $t \in [0, 2\pi]$, is positively oriented, while the circle $\gamma_2(t) = e^{-it}$, $t \in [0, 2\pi]$, is negatively oriented. Notice that $\gamma_1 = -\gamma_2$.
However, the circle $\gamma_3(t) = e^{-it}$, $t \in [\pi/2, 5\pi/2]$, is not the opposite of γ_1.

Remark 1.5. (a) In Definition 1.6, we assume that curves are non-trivial, because otherwise, the trivial curves don't have orientation.

(b) In case of a simple closed curve γ that is smooth on $[a, b]$, we say that γ is positively oriented (or has the positive orientation) if, for every $t \in [a, b]$, there exists $\varepsilon > 0$ (small enough) such that $\gamma(t) + \varepsilon i\gamma'(t) \in \operatorname{int}(\gamma)$. The vector $i\gamma'(t)$ is the rotation of the tangent vector $\gamma'(t)$ by $\pi/2$, or, turning the direction to the left.

(c) Let Γ be a contour with the smooth components $\gamma_1, \dots, \gamma_{n-1}, \gamma_n$, i.e., $\Gamma = \gamma_1 + \cdots + \gamma_{n-1} + \gamma_n$. Then the opposite contour $-\Gamma$ is the contour $-\Gamma = (-\gamma_n) + (-\gamma_{n-1}) + \cdots + (-\gamma_1)$.

(d) It is understood that the opposite of a line segment $[z_1, z_2]$, where $z_1 \neq z_2$, is the segment $[z_2, z_1]$, i.e., $-[z_1, z_2] = [z_2, z_1]$.

(c) Given a curve $\gamma : [a, b] \to \mathbb{C}$, its opposite curve $-\gamma : [a, b] \to \mathbb{C}$ is given by

$$(-\gamma)(t) = \gamma(b + a - t), \quad t \in [a, b].$$

1.5 Riemann sphere and stereographic projection

Lec. 04 → In the $x_1x_2x_3$-space (i.e., $\mathbb{R}^3$), the unit sphere S^2 is called the Riemann sphere, where x_1x_2-plane corresponds to the complex plane. For the sake of simplicity, the point $(x_1, x_2, 0)$ in the x_1x_2-plane will be denoted as $z = x_1 + ix_2$. The north pole is the point $N(0, 0, 1)$ and the south pole is the point $(0, 0, -1)$.

Given $z \in \mathbb{C}$, the line passing through the north pole N and the point z intersects the Riemann sphere at exactly one point X, which is called *the stereographic projection* of z.

Lemma 1.1. *For $z \in \mathbb{C}$, the stereographic projection $X = (x_1, x_2, x_3)$, of z, is given by*

$$x_1 = \frac{2\operatorname{Re}(z)}{|z|^2 + 1}, \quad x_2 = \frac{2\operatorname{Im}(z)}{|z|^2 + 1} \quad \text{and} \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Conversely, every point $X = (x_1, x_2, x_3) \in S^2 \setminus \{N\}$ is the stereographic projection of a point $z \in \mathbb{C}$, where

$$\operatorname{Re}(z) = \frac{x_1}{1 - x_3} \quad \text{and} \quad \operatorname{Im}(z) = \frac{x_2}{1 - x_3}.$$

Proof. The proof is left as an exercise. □

– Note that the stereographic projection defines a bijection between $S^2 \setminus \{N\}$ and $\mathbb{C}$.

– The stereographic projections of points z with large modulus are close to the north pole, and as $|z| \to +\infty$, their projections tend to N. Therefore, we associate with N the extended complex number "∞" (the point at infinity, and written without "+" or "−"), and call $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the extended complex plane. Note that the point at infinity becomes unique in this way.

– Now, the stereographic projection defines a bijection between S^2 and $\widehat{\mathbb{C}}$. Because of this correspondence, $\widehat{\mathbb{C}}$ is often called the Riemann sphere.

Definition 1.7. Let X_1 and X_2 be the stereographic projections of $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}$, respectively. The Euclidean distance between X_1 and X_2 is called the *chordal distance* between z_1 and z_2, and it is denoted by $\chi[z_1, z_2]$. That is, $\chi[z_1, z_2] = \|X_1 - X_2\|_2$.

It is clear that $0 \leq \chi[z_1, z_2] \leq 2$ for all $z_1, z_2 \in \mathbb{C}$.

Lemma 1.2. *If $z_1, z_2, z \in \mathbb{C}$, then*

$$\chi[z_1, z_2] = \frac{2|z_1 - z_2|}{\sqrt{|z_1|^2 + 1}\,\sqrt{|z_2|^2 + 1}} \quad \text{and} \quad \chi[z, \infty] = \frac{2}{\sqrt{|z|^2 + 1}}.$$

Proof. The proof is left as an exercise. □

– One can easily check that the chordal distance defines a metric on $\mathbb{C}$.

– A neighborhood of ∞ is described by $\{z \in \mathbb{C} : \chi[z, \infty] < \rho\}$, where $0 < \rho < 2$, and it is the set $\{z \in \mathbb{C} : |z| > r\} = \mathbb{C} \setminus \overline{D(0, r)}$, where $r = \sqrt{(4/\rho^2) - 1} > 0$.

2 Complex functions

Let $S \subset \mathbb{C}$. A function $f : S \to \mathbb{C}$, defined on S and taking values in $\mathbb{C}$, is called a complex single-valued function, or simply, a complex function. For example, $f(z) = z^2$ and $f(z) = 1/z$ are both complex (single-valued) functions defined, respectively, on $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$. In complex analysis, we often encounter objects that, unlike regular functions, assign several (finite or infinite) values to each variable z. These are known as multi-valued functions. For example, $f(z) = \arg(z)$ is a multi-valued function.

For a complex function $f : S \to \mathbb{C}$ and for $z = x + iy \in S$, let $w = f(z)$. Then the real and imaginary parts of w are each real-valued functions of z or, equivalently, of x and y, and so we customarily write

$$w = u(x, y) + iv(x, y), \tag{2.1}$$

with u and v denoting the real and imaginary parts, respectively, of w. The common domain of the functions u and v corresponds to the domain of the function f. ***Thus, by the cartesian representation of $\mathbb{C}$, complex valued functions of a complex variable $z \mapsto f(z)$ are, in essence, a pair of real functions of two real variables***

$$f : S \subset \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto f(x, y) = (u(x, y), v(x, y))$$

We use the notation: $u(x, y) = \operatorname{Re} f(z)$ and $v(x, y) = \operatorname{Im} f(z)$. For example,

$$f(z) := z^2 + 2z = (x + iy)^2 + 2(x + iy) = \underbrace{(x^2 - y^2 + 2x)}_{=u(x,y)} + i\underbrace{(2xy + 2y)}_{=v(x,y)}.$$

Conversely, if a function f is given in the form (2.1), then we can write it in terms of z by using, e.g., the equations:

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$

Complex functions as mappings

$\xrightarrow{\text{Lec. 05}}$ If $f : S \to \mathbb{C}, z \mapsto w = f(z)$, is a complex function, then both $z, w \in \mathbb{C}$, and its graph $G(f) = \{(z, f(z)) : z \in S\}$ is a subset of 4-dimensional space. Hence, we cannot use graphs to study (or, visualize) complex functions. However, complex functions describe mappings (not necessarily one-to-one) between subsets lying in two copies of the complex plane. Each point z_0 in the z-plane *is mapped* by a function f to the unique corresponding point $w_0 = f(z_0)$ in the w-plane. We also say: f maps z_0 to w_0. In general, we say f maps a subset S to its image. For example, the function $f(z) = z/(1 + z^2)$ maps the upper open unit semi disc to the upper open half plane, see Exercise set 1 (Pb. 1).

Remark 2.1. (1) Complex functions are usually referred to as complex mappings.
(2) The geometric representation of a complex mapping $f : S \to \mathbb{C}$ consists of two figures: one representing the domain S in the z-plane, and the other showing the image of f in the w-plane.

Example 2.1. The image, by the function $f(z) = z^2$, of a vertical line is either a parabola or the non-positive real axis.
$\longrightarrow$ Let $z = x_0 + iy$, where $x_0 \in \mathbb{R}$ is fixed and $y \in \mathbb{R}$ is arbitrary. Then $u + iv := f(x_0 + iy) = (x_0 + iy)^2 = x_0^2 - y^2 + i(2x_0 y)$. This is equivalent to

$$\begin{cases} u &= x_0^2 - y^2, \\ v &= 2x_0 y. \end{cases}$$

This system represents the parabola of the equation $u = x_0^2 - \dfrac{v^2}{4x_0^2}$ in the w-plane when $x_0 \neq 0$. Otherwise, the system represents the non-positive real axis.

2.1 Limit of a complex function

Definition 2.1. Let $S \subset \mathbb{C}$ and $z_0 \in S'$. We say that $f : S \to \mathbb{C}$ has a limit $w_0 \in \mathbb{C}$ as z approaches z_0 (within S), if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall z \in S : 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon.$$

We write $\displaystyle\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = w_0$.

This definition (as in the real case) says that the values $f(z)$ can be made *arbitrarily* close to w_0 when the values of z are chosen *sufficiently* close to z_0.

Remark 2.2. When "$z \in S$" is understood from the context, we can use the notation $\lim_{z \to z_0}$ instead of $\lim_{\substack{z \to z_0 \\ z \in S}}$.

Example 2.2. Let's show that $\lim_{z \to 1-i} (\overline{z}^2 - 2) = -2 + 2i$.
$\longrightarrow$ We have

$$\begin{aligned} |\overline{z}^2 - 2 - (-2 + 2i)| &= |\overline{z}^2 - 2i| = |z^2 + 2i| \\ &= |z - (1-i)| \times |z + (1-i)| \\ &\leq |z - (1-i)| \times \left(|z - (1-i)| + 2\sqrt{2} \right). \end{aligned} \tag{2.2}$$

Therefore, for any $\varepsilon > 0$, if we choose $\delta > 0$ to satisfy $\delta < \min\{1, \varepsilon/(1 + 2\sqrt{2})\}$, then from (2.2) we obtain

$$0 < |z - (1-i)| < \delta \implies |\overline{z}^2 - 2 - (-2 + 2i)| < \varepsilon.$$

Unlike the real case, there are infinitely many directions from which z can approach z_0 in the complex plane. For a complex limit to exist, every way by which z approaches z_0 must yield the same limiting value. The following statements are more practical.

Lemma 2.1. *If there are two different curves Γ_1 and Γ_2 passing through z_0 and $f(z)$ approaches two distinct values w_1 and w_2 as z approaches z_0 along Γ_1 and Γ_2, respectively, then $\lim_{z \to z_0} f(z)$ doesn't exist.*

Example 2.3. Let's show that $\lim_{z \to 0} (\overline{z}/z)$ doesn't exist.
$\longrightarrow$ Notice when z approaches 0 along the real axis, then $\overline{z}/z$ approaches (equals) 1. However, when z approaches 0 along the imaginary axis, then $\overline{z}/z$ approaches (equals) -1. Thus $\lim_{z \to 0} (\overline{z}/z)$ doesn't exist.

Lemma 2.2. *Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy + 0$ and $w_0 = a + ib$. Then*

$$\lim_{z \to z_0} f(z) = w_0 \iff \begin{cases} \lim_{(x,y) \to (x_0, y_0)} u(x, y) = a, \\ \text{and} \\ \lim_{(x,y) \to (x_0, y_0)} v(x, y) = b. \end{cases}$$

It is not difficult to check that the algebraic properties of the "complex" limit are similar to that of the "real" limit. In the following we mention some of these properties.

Lemma 2.3. *If* $\lim_{z\to z_0} f_1(z) = \ell_1$ *and* $\lim_{z\to z_0} f_2(z) = \ell_2$*, then*

$$\lim_{z\to z_0} (f_1(z) \pm f_2(z)) = \ell_1 \pm \ell_2, \quad \lim_{z\to z_0} (f_1(z)f_2(z)) = \ell_1\ell_2, \quad \lim_{z\to z_0} \frac{f_1(z)}{f_2(z)} = \frac{\ell_1}{\ell_2}, (\ell_2 \neq 0).$$

Remark 2.3 (Limits involving "∞")**.** Saying that z approaches the infinity point ∞, and write $z \to \infty$, means that $|z| \to +\infty$. We can handle limits involving "∞" by noting that $z \to \infty$ if and only if $1/z \to 0$. In particular, we have

$$\lim_{z\to z_0} f(z) = \infty \Longleftrightarrow \lim_{z\to z_0} \frac{1}{f(z)} = 0 \quad \text{and} \quad \lim_{z\to\infty} f(z) = \ell \Longleftrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right) = \ell.$$

2.2 Continuity of a complex function

Definition 2.2. Let $S \subset \mathbb{C}$ and $z_0 \in S$. We say that $f : S \to \mathbb{C}$ is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$. A function f is said to be continuous on S if it is continuous at each point of S.

The continuity of a function f at a point z_0 requires three actions that may not appear in the definition: (1) Existence of $\lim_{z\to z_0} f(z)$, (2) f is defined on z_0 and (3) The equality $\lim_{z\to z_0} f(z) = f(z_0)$.

Example 2.4. The function

$$f(z) = \begin{cases} z^2 & \text{if } z \neq i, \\ 0 & \text{if } z = i, \end{cases}$$

is defined at $z_0 = i$, and the limit $\lim_{z\to i} f(z)$ exists and equals -1. However, $f(i) = 0 \neq -1$, and hence f is not continuous at the point i.

From the properties of limits, we easily obtain the following properties of continuous functions.

(a) $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if $u(x, y)$ and $v(x, y)$ are both continuous at (x_0, y_0).

(b) If f and g are continuous at z_0, then $f \pm g$, fg and f/g ($g(z_0) \neq 0$) are all continuous at z_0.

(c) If f is continuous at z_0, and h is a continuous function at $f(z_0)$, then the composition $h \circ f$ is continuous at z_0.

Lec. 06 $\longrightarrow$ **Example 2.5.** Let P and Q be complex polynomials (i.e., $P, Q \in \mathbb{C}[z]$). Every polynomial is continuous on $\mathbb{C}$, and therefore, any rational function P/Q is continuous on $\mathbb{C} \setminus \{z : Q(z) = 0\}$.

Definition 2.3. We say that a complex function $f : S \to \mathbb{C}$ is uniformly continuous on S if

$$\forall\varepsilon > 0, \exists\delta > 0, \forall(z_1, z_2) \in S^2 : |z_1 - z_2| < \delta \Longrightarrow |f(z_1) - f(z_2)| < \varepsilon.$$

Example 2.6. Let's show that $f(z) = z^2$ is uniformly continuous on $D(0, r)$, where $r > 0$. $\longrightarrow$ Let $\varepsilon > 0$ and $z_1, z_2 \in D$ (be arbitrary). Then

$$\begin{aligned} |f(z_1) - f(z_2)| = |z_1^2 - z_2^2| &= |z_1 + z_2||z_1 - z_2| \\ &\leq (|z_1| + |z_2|)|z_1 - z_2| \\ &< 2r|z_1 - z_2|. \end{aligned}$$

It is easy from this to see that by taking $\delta = \varepsilon/(2r)$, we obtain

$$\forall\varepsilon > 0, \exists\delta = \frac{\varepsilon}{2r} > 0, \forall z_1, z_2 \in D(0, r) : |z_1 - z_2| < \delta \implies |f(z_1) - f(z_2)| < \varepsilon,$$

which shows that f is uniformly continuous on $D(0, r)$, $r > 0$.

Uniform continuity implies continuity, but the inverse is not true as show by the following example.

Example 2.7. The function $f(z) = 1/z$ is clearly continuous on $\mathbb{C} \setminus \{0\}$, while it is not uniformly continuous there. (Check it!)

Theorem 2.1. *Every continuous function f on a* **compact set** *S is bounded, and is uniformly continuous there. Moreover, $|f|$ attains its maximum and minimum in S.*

Some properties of complex polynomials

The aim of this paragraph is to prove the fundamental theorem of algebra.

Lemma 2.4. *If P is a non-constant polynomial with $P(0) = 1$, then for every $\varepsilon > 0$, there exists $\xi \in D(0, \varepsilon)$ such that $|P(\xi)| < 1$.*

Proof. (1) Assume first that $P(z) = 1 + a_m z^m$, where $m \geq 1$ and $a_m \neq 0$. For any $\varepsilon > 0$, we seek $\xi \in D(0, \varepsilon)$ such that $|1 + a_m\xi^m| < 1$. To do so, we may choose ξ such that $a_m\xi^m = -\alpha$, where $0 < \alpha < 1$, meaning ξ is an m-th root of $-\alpha/a_m$. Then $|1 + a_m\xi^m| = 1 - \alpha < 1$. To ensure $\xi \in D(0, \varepsilon)$, we need $|\xi| = \sqrt[m]{\alpha/|a_m|} < \varepsilon$, or equivalently, $\alpha < \varepsilon^m|a_m|$. Summarizing, for any $\varepsilon > 0$, let α satisfy $0 < \alpha < \min\{1, |a_m|\varepsilon^m\}$, and choose ξ as an m-th root of $-\alpha/a_m$. Then $|\xi| < \varepsilon$ and $|P(\xi)| = |1 + a_m\xi^m| = 1 - \alpha < 1$.

(2) Now assume $P(z) = 1 + a_m z^m + \cdots + a_n z^n$, where $m \geq 1, n \geq m+1$ and $a_n a_m \neq 0$. Here m is the least index for which $a_m \neq 0$. We write $P(z)$ in the form

$$P(z) = 1 + a_m z^m + R(z), \quad \forall z \in \mathbb{C},$$

where $R(z) = z^{m+1}\left(a_{m+1} + \cdots + a_n z^{n-m-1}\right)$, $\forall z \in \mathbb{C}$. Let

$$M = \min\left\{1, \frac{|a_m|}{|a_{m+1}| + \cdots + |a_n|}\right\}.$$

Then for $0 < |z| < M$, we obtain

$$|R(z)| < |z|^{m+1}\left(|a_{m+1}| + \cdots + |a_n|\right) < |a_m||z|^m, \tag{2.3}$$

Analogous to Case (1), for every $\varepsilon > 0$, let α satisfy $0 < \alpha < \min\{M, |a_m|\varepsilon^m\}$, and choose ξ as an m-th root of $-\alpha/a_m$. Then $|\xi| < \varepsilon$ and from (2.3) we have $|R(\xi)| < |a_m||\xi|^m = \alpha$. Consequently, we obtain

$$|P(\xi)| \leq |1 + a_m\xi^m| + |R(\xi)| = 1 \underbrace{-\alpha + |R(\xi)|}_{<0} < 1$$

This completes the proof. ☐

The following result is a consequence of Lemma 2.4.

Lemma 2.5. *If P is a non-constant polynomial with $P(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$, then for every $\varepsilon > 0$, there exists $\xi \in D(z_0, \varepsilon)$ such that $|P(\xi)| < |P(z_0)|$.*

Proof. Define the polynomial Q by

$$Q(z) = \frac{P(z + z_0)}{P(z_0)}, \quad \forall z \in \mathbb{C}.$$

Then $Q(0) = 1$. Let $\varepsilon > 0$ be arbitrary. Then, by Lemma 2.4, there exists $\xi^* \in D(0, \varepsilon)$ such that $|Q(\xi^*)| < 1$. Consequently, there exits $\xi = \xi^* + z_0 \in D(z_0, \varepsilon)$ such that

$$|P(\xi)| = |P(\xi^* + z_0)| = |P(z_0)||Q(\xi^*)| < |P(z_0)|.$$

This completes the proof. ☐

Remark 2.4. These two results don't apply for non-polynomial functions. For example, the function $f(z) = 1 + |z|^2$ is continuous on $\mathbb{C}$ with $P(0) = 1$. However, $|f(z)| \geq 1$ for every $z \in \mathbb{C}$.

Theorem 2.2 (Fundamental Theorem of Algebra)**.** *Every non-constant complex polynomial has at least one zero in $\mathbb{C}$.*

Proof. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $n \geq 1$ and $a_n \neq 0$. For $z \neq 0$, we have

$$P(z) = a_n z^n \left(1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_1}{a_n z^{n-1}} + \frac{a_0}{a_n z^n}\right).$$

Since the second factor tends to 1 as $z \to \infty$, there is an $M > 0$ for which

$$\left|1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_1}{a_n z^{n-1}} + \frac{a_0}{a_n z^n}\right| > \frac{1}{2}, \quad \text{for } |z| \geq M.$$

Consequently, we obtain

$$|P(z)| \geq \frac{|a_n|}{2}|z|^n, \quad \text{for } |z| \geq M.$$

Clearly, $\frac{|a_n|}{2}|z|^n \geq |P(0)|$ is equivalent to $|z| \geq \sqrt[n]{2|P(0)|/|a_n|}$. Then, by setting

$$M^* = \max\left\{M, \sqrt[n]{\frac{2|P(0)|}{|a_n|}}\right\},$$

we obtain

$$|P(z)| \geq |P(0)|, \quad \text{for } |z| \geq M^*. \tag{2.4}$$

From Theorem 2.1, it follows that $|P|$ attains its minimum in $\overline{D(0, M^*)}$, that is $\exists z_0 \in \overline{D(0, M^*)}$ such that

$$|P(z)| \geq |P(z_0)|, \quad \text{for } |z| \leq M^*. \tag{2.5}$$

In particular, we have $|P(0)| \geq |P(z_0)|$. This with (2.4) yields

$$|P(z)| \geq |P(z_0)|, \quad \text{for } |z| \geq M^*. \tag{2.6}$$

Combining (2.5) and (2.6) results in

$$|P(z)| \geq |P(z_0)|, \quad \forall z \in \mathbb{C}. \tag{2.7}$$

Assume that $P(z_0) \neq 0$. Then by Lemma 2.5 there exists $\xi \in \mathbb{C}$ such that $|P(\xi)| < |P(z_0)|$, which contradicts (2.7). Thus $P(z_0) = 0$, which proves the theorem. □

Remark 2.5. By factorization, every complex polynomial P of degree $n \geq 1$ has exactly n zeros (not necessarily distinct). In other words, P can always be written as

$$P(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n), \quad \forall z \in \mathbb{C},$$

where $a_n \neq 0$ and $z_1, z_2, \ldots, z_n$ are the zeros of P.

2.3 Complex differentiation (Holomorphic functions)

Lec. 07 $\longrightarrow$ Throughout this section and the next one, $D \subset \mathbb{C}$ and $U \subset \mathbb{C}$ will always denote, respectively, a domain and an open subset of $\mathbb{C}$, unless otherwise specified.

Definition 2.4. A function $f : U \to \mathbb{C}$ is said to be $\mathbb{C}$-differentiable at a point $z_0 \in U$ if the limit

$$\lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists,}$$

and in this case, we denote it as $f'(z_0)$ or $\frac{df}{dz}(z_0)$. We refer to this limit as the complex derivative of f at z_0.

By making change of variable $h = z - z_0$, one can write the derivative $f'(z_0)$, whenever it exists, in the form

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Here h is a complex number, and it belongs to the set $D - z_0 := \{z - z_0 : z \in D\}$.

Using the (ε, δ) definition of the limits, one can easily deduce that a function is $\mathbb{C}$-differentiable at z_0 if and only if there exists a constant $c \in \mathbb{C}$ such that

$$f(z_0 + h) = f(z_0) + ch + o(h), \quad \text{as} \quad h \to 0, \tag{2.8}$$

in which case, the constant c is the derivative of f at z_0, that is, $c = f'(z_0)$.

The following result is easy to establish.

Lemma 2.6. *If f is $\mathbb{C}$-differentiable at z_0, then it is continuous at z_0.*

Remark 2.6. All the real differentiation laws (addition: or linearity, product, quotient, and composition: or chain rule) still hold for complex differentiation.

We have the following known result.

Lemma 2.7. *Let $f, g : U \to \mathbb{C}$ be $\mathbb{C}$-differentiable at a point $z_0 \in U$. Then*

(i) *Linearity: $f + g$ is $\mathbb{C}$-differentiable at z_0 and $(f+g)'(z_0) = f'(z_0) + g'(z_0)$.*

(ii) *Product: fg is$\mathbb{C}$-differentiable at z_0 and*

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

(iii) *Quotient: If $g(z_0) \neq 0$, then (f/g) is $\mathbb{C}$-differentiable at z_0 and*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

(iv) *Chain rule: If $h : D\left(f(z_0), \varepsilon\right) \to \mathbb{C}$, where $\varepsilon > 0$, is $\mathbb{C}$-differentiable at $f(z_0)$, then $h \circ f$ is $\mathbb{C}$-differentiable at z_0 and*

$$\left(h \circ f\right)'(z_0) = f'(z_0)h'\left(f(z_0)\right).$$

Example 2.8. (1) Every constant function f is $\mathbb{C}$-differentiable and $f'(z) = 0$ for every $z \in \mathbb{C}$.

(2) The function $f(z) = z^n$, $n \in \mathbb{N}$, is $\mathbb{C}$-differentiable at every $z \in \mathbb{C}$. In addition, $f'(z) = nz^{n-1}$, $\forall z \in \mathbb{C}$. Indeed, we have for every $z \in \mathbb{C}$ and $h \neq 0$,

$$\frac{f(z+h) - f(z)}{h} = \frac{(z+h)^n - z^n}{h} = \sum_{k=0}^{n-1} (z+h)^k z^{n-k-1} \underset{h \to 0}{\longrightarrow} nz^{n-1}.$$

(3) By the differentiation rules, we deduce that every complex polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$, $z \in \mathbb{C}, n \geq 1$, is $\mathbb{C}$-differentiable at every $z \in \mathbb{C}$, and

$$P'(z) = na_n z^{n-1} + \cdots + 2a_2 z + a_1, \quad \forall z \in \mathbb{C}.$$

Example 2.9. The function $f(z) = \mathrm{Re}(z)$ is not $\mathbb{C}$-differentiable at all in $\mathbb{C}$.
$\longrightarrow$ Indeed, for $z = x + iy \in \mathbb{C}$ and $h = h_1 + ih_2 \neq 0$, we have

$$\frac{f(z+h) - f(z)}{h} = \frac{(x + h_1) - x}{h_1 + ih_2} = \frac{h_1}{h_1 + ih_2}. \tag{2.9}$$

The limit of (2.9) doesn't exist as $h \to 0$, since the limit along real axis ($h_2 = 0$) is 1, and the limit along imaginary axis ($h_1 = 0$) is 0. So, f is not $\mathbb{C}$-differentiable at every $z \in \mathbb{C}$.

Definition 2.5 (Holomorphic functions)**.** A function $f : U \to \mathbb{C}$ is said to be *holomorphic* at a point $z_0 \in U$ if f is $\mathbb{C}$-differentiable at z_0 and $\mathbb{C}$-differentiable at every point in some neighborhood of z_0. We say that f is holomorphic on $U \subset D$ if it is holomorphic at every point $z \in U$.

A function $f : U \to \mathbb{C}$ being holomorphic at a point $z_0 \in U$ requires checking two conditions: (1) f must be $\mathbb{C}$-differentiable at z_0, and (2) There exists $\varepsilon > 0$ such that $D(z_0, \varepsilon) \subset U$ and f is $\mathbb{C}$-differentiable at every $z \in D(z_0, \varepsilon)$.

Example 2.8 shows that every complex polynomial is holomorphic on $\mathbb{C}$, while the function $f(z) = \text{Re}(z)$ (Example 2.9) is not holomorphic on $\mathbb{C}$.

It is clear from the definition that holomorphic functions are $\mathbb{C}$-differentiable. However, a function being $\mathbb{C}$-differentiable at a point z_0 doesn't necessarily mean it is holomorphic at that point. This is clarified by the following example.

Example 2.10. The function $f(z) = |z|^2$ is $\mathbb{C}$-differentiable at 0, but it is not holomorphic at 0.

$\longrightarrow$ We first have

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(h) - f(0)}{h} = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{|h|^2}{h} = \lim_{\substack{h \to 0 \\ h \neq 0}} \overline{h} = 0.$$

So, f is $\mathbb{C}$-differentiable at 0 with $f'(0) = 0$. However, for every $z \neq 0$ and $h \neq 0$, we have

$$\frac{f(z+h) - f(z)}{h} = \frac{|z+h|^2 - |z|^2}{h} = z\left(\frac{\overline{h}}{h}\right) + \overline{h} + \overline{z}. \tag{2.10}$$

We have seen (Example 2.3) that $\overline{h}/h$ doesn't have a limit as $h \to 0$. Hence, limit of (2.10) doesn't exist as $h \to 0$, which means that f is not $\mathbb{C}$-differentiable at every $z \in \mathbb{C} \setminus \{0\}$. Consequently, the function $f(z) = |z|^2$ is not holomorphic at 0.

Definition 2.6. A function that is holomorphic on $\mathbb{C}$ is called an entire function. For example, polynomials are entire functions.

2.3.1 Complex partial derivatives.

Let $S \subset \mathbb{C}$ be any subset. By the homeomorphism $\mathbb{C} \cong \mathbb{R}^2$, the sets S and $\widetilde{S} := \{(x, y) \in \mathbb{R}^2 : x + iy \in S\} \subset \mathbb{R}^2$ can be regarded as representing the same collection of points. Therefore, in what follows, any subset S of $\mathbb{C}$ may also be interpreted as a subset of $\mathbb{R}^2$, and vice-versa. Since any complex function $f : U \to \mathbb{C}$ can be viewed as a function

$$\begin{aligned} f &: U \to \mathbb{R}^2 \\ (x, y) &\mapsto (u(x, y), v(x, y)), \end{aligned}$$

where $u(x, y) = \text{Re}\, f(z)$ and $v(x, y) = \text{Im}\, f(z)$ for $z = x + iy$, we can naturally introduced the partial derivatives for complex functions. Indeed, for $z_0 = x_0 + iy_0 \in U$, we define the complex partial derivatives (if they exist) of f at z_0 by

$$\frac{\partial f}{\partial x}(z_0) := \lim_{\substack{\mathtt{h} \to 0 \\ \mathtt{h} \in \mathbb{R}^*}} \frac{f((x_0 + \mathtt{h}) + iy_0) - f(x_0 + iy_0)}{\mathtt{h}} = \lim_{\substack{\mathtt{h} \to 0 \\ \mathtt{h} \in \mathbb{R}^*}} \frac{f(z_0 + \mathtt{h}) - f(z_0)}{\mathtt{h}},$$

$$\frac{\partial f}{\partial y}(z_0) := \lim_{\substack{\mathtt{h}\to 0\\ \mathtt{h}\in\mathbb{R}^*}} \frac{f(x_0 + i(y_0+\mathtt{h})) - f(x_0+iy_0)}{\mathtt{h}} = \lim_{\substack{\mathtt{h}\to 0\\ \mathtt{h}\in\mathbb{R}^*}} \frac{f(z_0 + i\,\mathtt{h}) - f(z_0)}{\mathtt{h}}.$$

It is easy to see that the above partial derivatives can be written in the form

$$\begin{aligned} \frac{\partial f}{\partial x}(z_0) &= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0), \\ \frac{\partial f}{\partial y}(z_0) &= \frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0), \end{aligned} \tag{2.11}$$

where $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are the usual partial derivatives of real functions of two variables.

Example 2.11. (1) The function $f(z) = \mathrm{Re}(z)$ is not $\mathbb{C}$-differentiable at all in $\mathbb{C}$, but its partial derivatives exist at every point $z \in \mathbb{C}$. Take $z = x + iy$ and $\mathtt{h} \in \mathbb{R}^*$. Then

$$\frac{f((x+\mathtt{h})+iy) - f(z)}{\mathtt{h}} = \frac{(x+\mathtt{h}) - x}{\mathtt{h}} = 1 \underset{\mathtt{h}\to 0}{\longrightarrow} 1,$$

which means $\frac{\partial f}{\partial x}(z) = 1$ for every $z \in \mathbb{C}$. Similarly, we obtain $\frac{\partial f}{\partial y}(z) = 0$ for every $z \in \mathbb{C}$.

(2) The function $f(z) = |z|^2$ is $\mathbb{C}$-differentiable at 0 but not holomorphic at all in $\mathbb{C}$. However, its complex partial derivatives exist at every $z \in \mathbb{C}$ with

$$\frac{\partial\,|z|^2}{\partial x} = 2x \quad \text{and} \quad \frac{\partial\,|z|^2}{\partial y} = 2y, \quad \forall z = x+iy \in \mathbb{C}.$$

2.3.2 Cauchy-Riemann equations

Lec. 08 $\longrightarrow$

Theorem 2.3. *If $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable at a point $z_0 \in U$, then*

$$f'(z_0) = \frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0). \tag{2.12}$$

In particular, if $f(x+iy) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$, then

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \textit{and} \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0). \tag{2.13}$$

The equations in (2.13), or the second equation in (2.12), are called the Cauchy-Riemann Equations (CRE). Clearly, the CRE in (2.13) is equivalent to the second equation in (2.12). Hence, it suffices to prove (2.12).

Proof of Theorem 2.3. The function f being $\mathbb{C}$-differentiable at z_0 means the quantity

$$f'(z_0) = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{f(z_0+h) - f(z_0)}{h}$$

exists as the limit exists as h approaches 0. When we take h approaching 0 along the real axis, i.e., $h = h_1 \in \mathbb{R}^*$, we obtain

$$f'(z_0) = \frac{\partial f}{\partial x}(z_0),$$

and when we take h approaching 0 along the imaginary axis, i.e., $h = ih_2$, where $h_2 \in \mathbb{R}^*$, we obtain

$$f'(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0),$$

from which we deduce (2.12). □

Theorem 2.3 provides a necessary conditions for the $\mathbb{C}$-differentiability, which play a role in detecting the regions where a function is not $\mathbb{C}$-differentiable. For example, we can see that the function $f(z) = \mathrm{Re}(z)$ is not $\mathbb{C}$-differentiable at all in $\mathbb{C}$ as $\frac{\partial f}{\partial x}(z) \neq -i\frac{\partial f}{\partial y}(z)$ for every $z \in \mathbb{C}$, see Example 2.11(1). Meanwhile, Example 2.11(2) shows that the function $f(z) = |z|^2$ is not $\mathbb{C}$-differentiable at every $z \neq 0$. In this example, we have $\frac{\partial f}{\partial x}(0) = 0 = -i\frac{\partial f}{\partial y}(0)$, and we proved earlier that f is $\mathbb{C}$-differentiable at 0. However, in general, we may ask the following question.

Question 2.1. If for a function $f : U \to \mathbb{C}$, the partial derivatives $\frac{\partial f}{\partial x}(z_0)$ and $\frac{\partial f}{\partial y}(z_0)$ exist at a point $z_0 \in U$, and $\frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$ (i.e., CRE hold), then is f necessarily $\mathbb{C}$-differentiable at z_0?

The answer would be NO, in general, as shown by the following example.

Example 2.12. Let f be defined as

$$f(z) = \begin{cases} \dfrac{z}{\overline{z}} - \dfrac{\overline{z}}{z}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Then the complex partial derivatives exist at 0 and they equal 0, which implies that CRE hold at 0. However, f is not $\mathbb{C}$-differentiable at 0 as it is not continuous at 0 (take z approaching 0 along the curve of equation $y = x$).

Theorem 2.4 below provides sufficient conditions on f for Question 2.1 to hold true. We first recall the following lemma from the classical real differential calculus.

Lemma 2.8. *Let $F : U \to \mathbb{R}$, $(x, y) \mapsto F(x, y)$, be a function of two real variables and let $(x_0, y_0) \in U$. If the partial derivatives $\partial F/\partial x$ and $\partial F/\partial y$ exist on an open disc $B \subset U$ centered at (x_0, y_0) and are continuous at (x_0, y_0). Then*

$$F(x_0 + h_1, y_0 + h_2) = F(x_0, y_0) + h_1\frac{\partial F}{\partial x}(x_0, y_0) + h_2\frac{\partial F}{\partial y}(x_0, y_0) + R(h),$$

where $(0, 0) \neq h = (h_1, h_2) \in B - (x_0, y_0)$, and $R : B - (x_0, y_0) \to \mathbb{R}$ is a continuous function satisfying $\lim\limits_{h\to(0,0)} |R(h)|/\|h\|_2 = 0$.

Proof. Let $(x_0, y_0) \in U$, $B \subset U$ be an open disc centered at (x_0, y_0), and let $(0,0) \neq h = (h_1, h_2) \in B - (x_0, y_0)$. We have

$$\begin{aligned} F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0) = F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0 + h_2) \\ + F(x_0, y_0 + h_2) - F(x_0, y_0). \end{aligned} \tag{2.14}$$

Since $h \neq (0,0)$, we may assume that $h_1 \neq 0$. Define the function $\varphi : [x_0, x_0 + h_1] \to \mathbb{R}$ by $\varphi(x) = F(x, y_0 + h_2)$. Existence of $\partial F / \partial x$ on B implies the existence of φ' on some interval containing $[x_0, x_0 + h_1]$, and hence the continuity of φ on $[x_0, x_0 + h_1]$. Therefore,by the mean value theorem, we deduce that there exists $\theta \in]0,1[$ such that $\varphi(x_0 + h_1) - \varphi(x_0) = h_1 \varphi'(x_0 + \theta h_1)$, that is,

$$F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0 + h_2) = h_1 \frac{\partial F}{\partial x}(x_0 + \theta h_1, y_0 + h_2). \tag{2.15}$$

Since the partial $\partial F / \partial x$ is continuous at (x_0, y_0), we may write

$$\frac{\partial F}{\partial x}(x_0 + \theta h_1, y_0 + h_2) = \frac{\partial F}{\partial x}(x_0, y_0) + r_1(h),$$

where $r_1 : B - (x_0, y_0) \to \mathbb{R}$ is continuous and $\lim_{h \to (0,0)} r_1(h) = 0$. Plugging this into (2.15) results in

$$F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0 + h_2) = h_1 \frac{\partial F}{\partial x}(x_0, y_0) + h_1 r_1(h).$$

By using the same argument, we obtain

$$F(x_0, y_0 + h_2) - F(x_0, y_0) = h_2 \frac{\partial F}{\partial y}(x_0, y_0) + h_2 r_2(h),$$

where $r_2 : B - (x_0, y_0) \to \mathbb{R}$ is continuous and $\lim_{h \to (0,0)} r_2(h) = 0$. Substituting these equations into (2.14) results in

$$F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0) = h_1 \frac{\partial F}{\partial x}(x_0, y_0) + h_2 \frac{\partial F}{\partial y}(x_0, y_0) + R(h),$$

where $R(h) = h_1 r_1(h) + h_2 r_2(h)$, which is clearly continuous on $B - (x_0, y_0)$, and

$$\frac{|R(h)|}{\|h\|_2} \leq \frac{|h_1|}{\|h\|_2} |r_1(h)| + \frac{|h_2|}{\|h\|_2} |r_2(h)| \leq |r_1(h)| + |r_2(h)| \longrightarrow 0, \quad \text{as } h \to (0,0).$$

This completes the proof of the lemma. □

Remark 2.7. (1) Lemma 2.8 asserts that F is differentiable (i.e., $\mathbb{R}^2$-differentiable) at the point (x_0, y_0).
(2) Using the little-o notation, the function $R(h)$ in Lemma 2.8 can be replaced with $o(h)$.

Theorem 2.4. *Let $f : U \to \mathbb{C}$ be a complex function, and let $z_0 \in U$. If the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ exist on an open disc $B \subset U$ centered at z_0 and are continuous at z_0, and if f satisfies CRE at z_0, then f is $\mathbb{C}$-differentiable at z_0.*

Proof. Let $f(x+iy) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$ and let $h = h_1 + ih_2 \in B - z_0$. Noting that u and v satisfy the conditions of Lemma 2.8 on U, we obtain

$$u(x_0+h_1, y_0+h_2) = u(x_0,y_0) + h_1\frac{\partial u}{\partial x}(x_0,y_0) + h_2\frac{\partial u}{\partial y}(x_0,y_0) + o(h), \quad h \to 0,$$

$$v(x_0+h_1, y_0+h_2) = v(x_0,y_0) + h_1\frac{\partial v}{\partial x}(x_0,y_0) + h_2\frac{\partial v}{\partial y}(x_0,y_0) + o(h), \quad h \to 0.$$

By using these two equations together with CRE, we obtain

$$\begin{aligned}
\frac{f(z_0+h)-f(z_0)}{h} &= \frac{u(x_0+h_1,y_0+h_2)-u(x_0,y_0)}{h} + i\frac{v(x_0+h_1,y_0+h_2)-v(x_0,y_0)}{h} \\
&= \frac{1}{h}\left(h_1\frac{\partial u}{\partial x}(x_0,y_0) + h_2\frac{\partial u}{\partial y}(x_0,y_0)\right) + \frac{i}{h}\left(h_1\frac{\partial v}{\partial x}(x_0,y_0) + h_2\frac{\partial v}{\partial y}(x_0,y_0)\right) + o(1) \\
&= \frac{1}{h}\left(h_1\frac{\partial u}{\partial x}(x_0,y_0) - h_2\frac{\partial v}{\partial x}(x_0,y_0)\right) + \frac{i}{h}\left(h_1\frac{\partial v}{\partial x}(x_0,y_0) + h_2\frac{\partial u}{\partial x}(x_0,y_0)\right) + o(1) \\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) + o(1) \\
&= \frac{\partial f}{\partial x}(z_0) + o(1), \quad h \to 0.
\end{aligned}$$

By letting $h \to 0$, we deduce that f is $\mathbb{C}$-differentiable at z_0 and $f'(z_0) = \dfrac{\partial f}{\partial x}(z_0)$. □

Lec. 09 → We have the direct consequence of Theorem 2.4.

Corollary 2.5. *Let $f : U \to \mathbb{C}$. If the partial derivatives exist and continuous on an open subset $B \subset U$, and if f satisfies CRE at every point $z \in B$, then f is holomorphic on B.*

Corollary 2.6. *If $f : D \to \mathbb{C}$ is holomorphic on the domain D and $f'(z) = 0$ for every $z \in D$, then f is constant on D.*

The proof of this corollary follows directly from the following equivalences:

$$f' \equiv 0 \iff \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \iff u \equiv \text{const.} \ \& \ v \equiv \text{const.} \iff f \equiv \text{const.}$$

Remark 2.8. (1) Corollary 2.6 is true only in case of D is connected. For example, the function

$$f(z) = \begin{cases} 1 & \text{if } |z| < 1 \\ 2 & \text{if } |z| > 2 \end{cases}$$

has zero derivative, but it is not constant, as f is defined on a union of two disjoint open subsets.

(2) If $f, g : D \to \mathbb{C}$ are two holomorphic functions on a domain D, and if $f'(z) = g'(z)$ for every $z \in D$, then there exists $c \in \mathbb{C}$ for which $f(z) = g(z) + c$ for every $z \in D$. This is a simple consequence of Corollary 2.6.

Remark 2.9 (General Definition)**.** Let $S \subset \mathbb{C}$ be a subset with $\overset{\circ}{S} \neq \emptyset$. A function $f : S \to \mathbb{C}$ is said to be $\mathbb{C}$-differentiable on a subset (not necessarily open) $K \subset S$ if it is $\mathbb{C}$-differentiable at each point $z \in K$. Moreover, f is said to be holomorphic on K if there exists and open subset $U \supset K$ (containing K) and f is holomorphic on U.

Example 2.13. The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(x+iy) = x^3 + 3xy^2 - 3x + i\left(y^3 + 3x^2y - 3y\right)$ is $\mathbb{C}$-differentiable on

$$K = \{z \in \mathbb{C} : \operatorname{Re}(z) = 0\} \cup \{z \in \mathbb{C} : \operatorname{Im}(z) = 0\}.$$

However, f is not holomorphic on this K.

Proof. We have $f(x+iy) = u(x,y) + iv(x,y)$, where

$$u(x,y) = x^3 + 3xy^2 - 3x \quad \text{and} \quad v(x,y) = y^3 + 3x^2y - 3y,$$

and we have

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2 + 3y^2 - 3 & 6xy \\ \\ 6xy & 3x^2 + 3y^2 - 3 \end{bmatrix}.$$

Clearly all the partial derivatives of u and v are continuous on $\mathbb{R}^2$, and therefore, f is $\mathbb{C}$-differentiable at $z = x + iy$ if and only if CRE hold at z. Well, we have $\partial u/\partial x = \partial v/\partial y$ on $\mathbb{R}^2$. However, $\partial u/\partial y = -\partial v/\partial x$ holds iff $xy = 0$, i.e., either $x = 0$ or $y = 0$. Hence, f is $\mathbb{C}$-differentiable on

$$K = \{z \in \mathbb{C} : \operatorname{Re}(z) = 0\} \cup \{z \in \mathbb{C} : \operatorname{Im}(z) = 0\}.$$

This subset is the "Largest" subset where f is $\mathbb{C}$-differentiable, since there is no other points where the CRE hold.

The function f is not holomorphic on K. Indeed, take any point $z_0 \in K$, and take any neighborhood U of z_0, and we can always find points $z_1 \in U \setminus K$, at which the function f is not $\mathbb{C}$-differentiable (because CRE don't hold true). □

2.4 Power series (Analytic functions)

A complex power series about a point $z_0 \in \mathbb{C}$ is a series of complex functions of the form

$$\sum_{n \geq 0} c_n (z - z_0)^n, \tag{2.16}$$

where $c_n \in \mathbb{C}$ for every integer $n \geq 0$, and z is a complex number in a suitable subset of $\mathbb{C}$. Note that (2.16) is absolutely convergent at z_0, and its value (sum) at z_0 would be c_0 (So, it is always possible to find a subset where a power series is well defined). We recall the following result, which specifies the exact domain in which a given power series absolutely converges.

Theorem 2.7. *For a power series* (2.16) *there exists a unique* $R \in [0, +\infty]$ *such that* (2.16) *is absolutely convergent for* $|z - z_0| < R$ *(whenever this has meaning) and divergent for* $|z - z_0| > R$*. This* R *is called the radius of convergence of the series* (2.16)*. The radius of convergence* R *can be determined by the Cauchy-Hadamard formula*

$$R = \liminf_{n\to+\infty} \frac{1}{\sqrt[n]{|c_n|}},$$

with "the convention $1/0 = \infty$*".*

Recall that if (2.16) has the radius of convergence $R > 0$, then for every $r \in]0, R[$, the series (2.16) is normally convergent in $\overline{D(z_0, r)}$. Therefore, (2.16) defines a continuous complex function $f : D(z_0, R) \to \mathbb{C}$ defined by $f(z) = \sum_{n=0}^{+\infty} c_n(z-z_0)^n$ for every $z \in D(z_0, R)$, which called the sum of the power series (2.16) in $D(z_0, R)$. Next result reveals the $\mathbb{C}$-differentiability of f on $D(z_0, R)$.

Theorem 2.8. *Consider the power series* (2.16) *with radius of convergence* $R > 0$*, and let* $f(z) = \sum_{n=0}^{+\infty} c_n(z - z_0)^n$ *for every* $z \in D(z_0, R)$*. Then* f *is holomorphic on* $D(z_0, R)$ *and*

$$f'(z) = \sum_{n=1}^{+\infty} nc_n(z - z_0)^{n-1}, \quad \forall z \in D(z_0, R). \tag{2.17}$$

Lec. 10 $\longrightarrow$ *Proof.* By the Cauchy-Hadamard formula, the power series $\sum_{n\geq1} nc_n(z-z_0)^{n-1}$ (2.17) has the radius of convergence

$$R' = \liminf_{n\to+\infty} \frac{1}{\sqrt[n]{n|c_n|}} = \liminf_{n\to+\infty} \frac{1}{\sqrt[n]{|c_n|}} = R.$$

Assume without loss of generality that $z_0 = 0$, as we can apply the same argument to the function $g(z) = f(z + z_0)$, if $z_0 \neq 0$. We proceed to prove (2.17). Let $z \in D(0, R)$, $h \in \mathbb{C}^*$ such that $z + h \in D(0, R)$. Then

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{+\infty} nc_n z^{n-1} = \sum_{n=0}^{+\infty} c_n \frac{(z+h)^n - z^n}{h} - \sum_{n=1}^{+\infty} nc_n z^{n-1}. \tag{2.18}$$

Let $\xi \in D(0, R)$ such that $|z| < |\xi|$, and take h so that $|h| \leq |\xi| - |z|$ (particularly we have $|z + h| \leq |\xi|$). Hence,

$$\left| c_n \frac{(z+h)^n - z^n}{h} \right| = |c_n| \left| \sum_{k=0}^{n-1} (z+h)^k z^{n-k-1} \right| \leq n|c_n||\xi|^{n-1}.$$

Since $\xi \in D(0,R)$, it follows that $\sum\limits_{n\geq 1} n|c_n||\xi|^{n-1}$ is convergent, and hence, the first series on the right-hand side of (2.18) is uniformly convergent for h in $D(0,|\xi|-|z|)$. The second series on the right-hand side is already convergent. Now for given $\varepsilon > 0$, there exists an integer $N \geq 1$ such that, for every $h \in D(0,|\xi|-|z|)$, we have

$$\left| \sum_{n=N+1}^{+\infty} c_n \frac{(z+h)^n - z^n}{h} - \sum_{n=N+1}^{+\infty} nc_n z^{n-1} \right| \leq \frac{\varepsilon}{2}.$$

This and (2.18) give, for every $h \in D(0,|\xi|-|z|)$,

$$\left| \frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{+\infty} nc_n z^{n-1} \right| \leq \left| \sum_{n=0}^{N} c_n \frac{(z+h)^n - z^n}{h} - \sum_{n=1}^{N} nc_n z^{n-1} \right| + \frac{\varepsilon}{2}.$$

By letting $h \to 0$, we have

$$\sum_{n=0}^{N} c_n \frac{(z+h)^n - z^n}{h} \longrightarrow \sum_{n=1}^{N} nc_n z^{n-1},$$

and consequently, there exists $\delta > 0$, such that for every h satisfying $|h| < \min\{\delta, |\xi|-|z|\}$, we have

$$\left| \sum_{n=0}^{N} c_n \frac{(z+h)^n - z^n}{h} - \sum_{n=1}^{N} nc_n z^{n-1} \right| < \frac{\varepsilon}{2}.$$

Thus, for $|h| < \min\{\delta, |\xi|-|z|\}$, we have

$$\left| \frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{+\infty} nc_n z^{n-1} \right| < \varepsilon.$$

which means that

$$f'(z) = \sum_{n=1}^{+\infty} nc_n z^{n-1}. \tag{2.19}$$

Since this true for every $z \in D(0,R)$, it follows that f is holomorphic on $D(0,R)$ and f' has the form (2.19). $\square$

By simple induction, we obtain the following consequence of Theorem 2.8.

Theorem 2.9. *Consider the power series* (2.16) *with radius of convergence* $R > 0$*, and let* $f(z) = \sum_{n=0}^{+\infty} c_n(z - z_0)^n$ *for every* $z \in D(z_0, R)$*. Then for every integer* $k \geq 1$*, the function* $f^{(k-1)}$ *(the* $(k-1)$*th derivative) is holomorphic on* $D(z_0, R)$ *and*

$$f^{(k)}(z) = \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} c_n (z - z_0)^{n-k}, \quad \forall z \in D(z_0, R).$$

Remark 2.10. From this result, we deduce the following relation between the coefficients of a power series, and the derivatives of its sum,

$$c_n = \frac{f^{(n)}(z_0)}{n!}, \quad \forall n \geq 0.$$

Definition 2.7 (Analytic functions)**.** Let $f : U \to \mathbb{C}$ be a complex function, and let $z_0 \in U$. We say that f is analytic at z_0 if there exists $r > 0$ such that $D(z_0, r) \subset U$ and f is the sum of a power series about z_0 that is absolutely convergent in $D(z_0, r)$. Moreover, we say that f is analytic on U if it is analytic at each point $z \in U$.

Example 2.14 (Exercise)**.** If a power series $\sum_{n=0}^{+\infty} c_n z^n$ has a radius of convergence $R > 0$, then its sum $f(z) = \sum_{n=0}^{+\infty} c_n z^n$ is analytic on $D(0, R)$.

Remarks.

(1) From Theorem 2.8, we see that if $f : U \to \mathbb{C}$ is analytic at $z_0 \in U$, then it is holomorphic at z_0. Consequently, if f is analytic on U, then it is holomorphic on U.

Analyticity $\Longrightarrow$ Holomorphicity.

Analytic functions are typical examples of holomorphic functions. In the following sections, we will see that holomorphic functions are, in fact, also analytic.

(2) If f is analytic on U, then all its derivatives $f^{(k)}$, $k \geq 1$, are also analytic on U.

(3) From Remark 2.10, every analytic function $f : U \to \mathbb{C}$ has, around every $\xi \in U$, the representation

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(\xi)}{n!} (z - \xi)^n, \quad z \in D(\xi, R_\xi),$$

for some $R_\xi > 0$ depending on ξ, such that $D(\xi, R_\xi) \subset U$.

Some properties of analytic functions

A subset $Z \subset D$ is called discrete in the domain D if it has no accumulation point in D.

Theorem 2.10. *Let $f : D \to \mathbb{C}$ be analytic on the domain D, and let $(z_k)_{k\geq 1} \subset D$ be a sequence of distinct complex numbers converging to $z_0 \in D$. If $f(z_k) = 0$ for every $k \geq 1$, then $f = 0$ on D.*

$\xrightarrow{\text{Lec. 11}}$ TO BE CONTINUED!

3 Elementary functions

$\xrightarrow{\text{Lec. 12}}$

3.1 The complex exponential function

Consider the power series

$$\sum_{n\geq 0} \frac{z^n}{n!}.$$

By Cauchy-Hadamard formula we have

$$R = \liminf_{n\to+\infty} \sqrt[n]{n!} = +\infty,$$

which is the radius of convergence of the above power series, and this means that the function

$$E(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}, \quad \forall z \in \mathbb{C},$$

defines an analytic (and hence a holomorphic) function on $\mathbb{C}$, i.e., an entire function.Moreover, its derivative satisfies

$$E'(z) = \sum_{n=1}^{+\infty} \frac{nz^{n-1}}{n!} = \sum_{n=0}^{+\infty} \frac{z^n}{n!} = E(z), \quad \forall z \in \mathbb{C}. \tag{3.1}$$

Note that when $z = x \in \mathbb{R}$, we clearly have $E(x) = e^x$. What can be said about $E(iy)$, where $y \in \mathbb{R}$? In fact, for $z = iy$, where $y \in \mathbb{R}$, we have

$$\begin{aligned}\sum_{n\geq 0} \frac{(iy)^n}{n!} &= \sum_{n \text{ is even}} \frac{(iy)^n}{n!} + \sum_{n \text{ is odd}} \frac{(iy)^n}{n!} \\ &= \sum_{\substack{n=2s\\ s\geq 0}} \frac{(-1)^s y^{2s}}{(2s)!} + i \sum_{\substack{n=2s+1\\ s\geq 0}} \frac{(-1)^s y^{2s+1}}{(2s+1)!} = \cos(y) + i\sin(y).\end{aligned} \tag{3.2}$$

Thus

$$E(iy) = \cos(y) + i\sin(y), \quad \forall y \in \mathbb{R}. \tag{3.3}$$

The next natural question concerns the algebraic form of $E(z)$, but first, we shall prove the property

$$E(z_1 + z_2) = E(z_1)E(z_2), \quad \forall z_1, z_2 \in \mathbb{C}. \tag{3.4}$$

This can be done by using the Cauchy product as follows:

$$\begin{aligned} E(z_1 + z_2) = \sum_{n\geq 0} \frac{(z_1 + z_2)^n}{n!} &= \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \\ &= \sum_{n\geq 0} \sum_{k=0}^{n} \left(\frac{z_1^k}{k!}\right) \left(\frac{z_2^{n-k}}{(n-k)!}\right) \\ &= \left(\sum_{n\geq 0} \frac{z_1^n}{n!}\right) \left(\sum_{n\geq 0} \frac{z_2^n}{n!}\right) = E(z_1)E(z_2). \end{aligned}$$

Plugging $z = x + iy$ into (3.4) and using (3.3), we obtain

$$E(x + iy) = E(x)E(iy) = e^x(\cos(y) + i\sin(y)). \tag{3.5}$$

From this, one can easily check that $E(z)$ satisfies CRE. In addition, (3.5) shows that $|E(z)| = e^{\mathrm{Re}(z)} > 0$ for any $z \in \mathbb{C}$. Hence, $E(z)$ never vanishes.

The restriction of E to the real line is nothing but the real exponential function. We then define the *complex exponential function*, denoted by $\exp(z)$ or more practical form e^z to be the function $E(z)$ for every $z \in \mathbb{C}$. From this definition, we will have the famous Euler's formula

$$e^{iy} = \cos(y) + i\sin(y), \quad \forall y \in \mathbb{R}.$$

Hence (3.5) yields

$$e^z = e^x(\cos(y) + i\sin(y)), \quad \forall z = x + iy \in \mathbb{C}.$$

Note that e^z is the unique solution to the differential equation (3.1) with the condition $f(0) = 1$.

The complex exponential function is periodic of the main period $2\pi i$, and we have

$$e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z, \quad \forall z \in \mathbb{C}, \forall k \in \mathbb{Z}.$$

In contrast to the real case, the complex exp is not injective, and we have

$$e^{z_1} = e^{z_2} \implies \exists k \in \mathbb{Z} : z_1 = z_2 + 2k\pi i.$$

Exercise 3.1. Solve in $\mathbb{C}$ the equation $e^z = -2$.

Solution. By setting $z = x + iy$, the equation is equivalently written as

$$\begin{cases} e^x \cos y = -2 \\ e^x \sin y = 0 \end{cases}.$$

Solving the second equation results in $y = k\pi$, $k \in \mathbb{Z}$. By substituting this into the first equation, we obtain $e^x = (-1)^{k+1}2$, $k \in \mathbb{Z}$, but this is true only when $k = 2j+1$, $j \in \mathbb{Z}$. In this case, we obtain $x = \ln 2$ and $y = (2j+1)\pi$, $j \in \mathbb{Z}$. Therefore, the solutions of the equation are $z_j = \ln 2 + i(2j+1)\pi$, $j \in \mathbb{Z}$. $\square$

Exercise 3.2. Show that, for every $z \in \mathbb{C}$, the sequence $\left\{(1+z/n)^n\right\}_{n\in\mathbb{N}}$ converges to e^z, that is,

$$e^z = \lim_{n\to+\infty}\left(1+\frac{z}{n}\right)^n.$$

3.2 Trigonometric and hyperbolic functions

Similar to the real case, we define the complex cosine and the complex sine in terms of the power series' as follows

$$\cos(z) = \sum_{n\geq 0}(-1)^n\frac{z^{2n}}{(2n)!} \quad \text{and} \quad \sin(z) = \sum_{n\geq 0}(-1)^n\frac{z^{2n+1}}{(2n+1)!}, \quad \forall z \in \mathbb{C}.$$

One can easily check that both of these functions are analytic (and hence holomorphic) on $\mathbb{C}$, i.e., they are entire functions. By replacing y with z and $-z$ in (3.2), we obtain, respectively,

$$e^{iz} = \cos(z) + i\sin(z) \quad \text{and} \quad e^{-iz} = \cos(z) - i\sin(z), \quad \forall z \in \mathbb{C}.$$

The first formula is Euler's formula for complex numbers. From these two formulas, we obtain the exponential form of cos and sin:

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i}, \quad \forall z \in \mathbb{C}. \tag{3.6}$$

The derivatives of cos and sin satisfy

$$\frac{d\cos(z)}{dz} = -\sin(z) \quad \text{and} \quad \frac{d\sin(z)}{dz} = -\cos(z), \quad \forall z \in \mathbb{C}.$$

By the periodicity of the complex exp, we deduce that cos and sin are periodic of the main period 2π.

Some properties.

P1 All trigonometric identities remain valid for complex numbers. To mention a few examples,

$$\cos^2(z) + \sin^2(z) = 1, \quad \forall z \in \mathbb{C},$$
$$\cos(z_1 \pm z_2) = \cos(z_1)\cos(z_2) \mp \sin(z_1)\sin(z_2), \quad \forall z_1, z_2 \in \mathbb{C},$$
$$\sin(z_1 \pm z_2) = \sin(z_1)\cos(z_2) \pm \cos(z_1)\sin(z_2), \quad \forall z_1, z_2 \in \mathbb{C}.$$

$\boxed{\text{P2}}$ The sets of zeros of cos and sin are $Z(\cos) = \left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}$ and $Z(\sin) = \{k\pi : k \in \mathbb{Z}\}$, respectively. Indeed, for example for the zeros of cos, we have

$$\cos(z) = 0 \implies e^{iz} = -e^{-iz} = e^{-iz-i\pi} \implies iz = -iz - i\pi + 2k\pi i,\ k \in \mathbb{Z} \implies z \in Z(\cos).$$

The converse inclusion is obvious. (check it for sin!) Here, we mention the complex tangent function, which is defined by

$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \quad \forall z \in \mathbb{C} \setminus Z(\cos).$$

$\boxed{\text{P3}}$ Recall that for every $t \in \mathbb{R}$, $\cosh(t) = \frac{e^t + e^{-t}}{2}$ and $\sinh(t) = \frac{e^t - e^{-t}}{2}$. From (3.6), we deduce for every $y \in \mathbb{R}$,

$$\cos(iy) = \cosh(y) \quad \text{and} \quad \sin(iy) = i\sinh(y).$$

Consequently, we may apply $\boxed{\text{P1}}$ to show, for every $x + iy \in \mathbb{C}$, that

$$\begin{aligned}\cos(x+iy) &= \cos(x)\cosh(y) - i\sin(x)\sinh(y),\\ \sin(x+iy) &= \sin(x)\cosh(y) + i\cos(x)\sinh(y).\end{aligned}$$

$\boxed{\text{P4}}$ It follows from $\boxed{\text{P3}}$ that, for every $x + iy \in \mathbb{C}$,

$$\begin{aligned}|\cos(x+iy)|^2 &= \cos^2(x) + \sinh^2(y),\\ |\sin(x+iy)|^2 &= \sin^2(x) + \sinh^2(y).\end{aligned}$$

In particular, cos and sin are not bounded on vertical strips. However, still bounded on horizontal strips.

Exercise 3.3. Solve in $\mathbb{C}$ the equation $\cos(z) = 2$.

Solution. The above equation is equivalently written as $e^{iz} + e^{-iz} = 4$, that is,

$$(e^{iz})^2 - 4e^{iz} + 1 = 0.$$

Clearly, e^{iz} solves the $w^2 - 4w + 1 = 0$, which has the two real solutions $2 \pm \sqrt{3}$. Hence, $e^{iz} = 2 \pm \sqrt{3}$. Since $2 \pm \sqrt{3} > 0$, we can write $e^{iz} = e^{\ln(2\pm\sqrt{3})}$, which implies that $iz = \ln(2 \pm \sqrt{3}) + 2k\pi i$, $k \in \mathbb{Z}$. Thus $z = 2k\pi - i\ln(2 \pm \sqrt{3})$, $k \in \mathbb{Z}$. We can easily check that these are the solutions of the equation.

NOTE. We can use the same method as in Exercise 3.1 . □

The hyperbolic functions cosh and sinh are now defined by

$$\cosh(z) = \cos(iz) \quad \text{and} \quad \sinh(z) = -i\sin(iz), \quad \forall z \in \mathbb{C}.$$

In addition $\tanh(z) = -i\tan(iz)$ for every $z \in \mathbb{C}$. Analogously to the trigonometric functions, the hyperbolic identities still hold for complex numbers. For example, $\cosh^2(z) - \sinh^2(z) = 1$ for every $z \in \mathbb{C}$.

Exercise 3.4. Find the algebraic form of cosh and sinh.

3.3 Complex logarithmic function

Lec. 13 → In the real case of the exponential function $y = e^x$, $x \in \mathbb{R}$, we know that its inverse function is $x = \ln y$, $y > 0$. This is well-defined as $\exp : \mathbb{R} \to]0, +\infty[$, $\exp(x) = e^x$, is bijective. Let us now investigate the complex case. To do so, for a given $z \in \mathbb{C} \setminus \{0\}$, let us solve, for w, the equation

$$z = e^w. \tag{3.7}$$

Set $w = u + iv$, where $u, v \in \mathbb{R}$ don't vanish at the same time. Then the equation in question will written as $e^u e^{iv} = |z|e^{i\theta_z}$, where θ_z is an argument of z. Therefore,

$$\begin{cases} e^u = |z| \\ v = \theta_z + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

The first equation involves real variables only, and solving it for u yields

$$\begin{cases} u = \ln |z| \\ v = \theta_z + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

From this, we deduce that the solutions of (3.7) are given by

$$w_k = \ln |z| + i\theta_z + 2k\pi i, \quad k \in \mathbb{Z}.$$

Note that there are infinitely many solutions of (3.7) for any given $z \in \mathbb{C} \setminus \{0\}$. Hence, if we define a multi-valued function to be the "inverse function" to the exponential function, then it would be analogous to the real logarithmic function.

Definition 3.1 (The complex logarithm)**.** Let $z \in \mathbb{C} \setminus \{0\}$. The complex logarithm is a multi-valued function, denoted by $\log(z)$, and is defined by

$$\log(z) = \ln |z| + i\theta_z + 2k\pi i, \quad k \in \mathbb{Z},$$

where θ_z is an argument of z. This equality indicates that all possible values solving equation (3.7), for any given nonzero z, are collectively represented by the single notation $\log(z)$. In particular, $e^{\log(z)} = z$ for any nonzero z.

The equality in the definition also represents all possible values that the multi-valued function $\log(z)$ may assume for a given $z \neq 0$. Note that for any $z \neq 0$, $e^{\log(z)}$ assumes only one value, which is z. Hence the expression $e^{\log(z)}$ is a single-values function.

Since $\theta_z \equiv \operatorname{Arg}(z) \mod (2\pi)$, we can replace θ_z with $\operatorname{Arg}(z)$ in Definition 3.1, that is, the complex logarithm of $z \in \mathbb{C} \setminus \{0\}$ would be defined by

$$\log(z) = \ln |z| + i \operatorname{Arg}(z) + 2k\pi i, \quad k \in \mathbb{Z}.$$

Recall that $\operatorname{Arg}(z)$ is the principal value of the argument, that is, the value of the argument that lies within the interval $]-\pi, \pi]$.

Example 3.1. We have

$$\log(i) = \ln|i| + i\operatorname{Arg}(i) + 2k\pi i = \left(\frac{1}{2} + 2k\right)\pi i, \quad k \in \mathbb{Z}.$$

Some properties. For $z, z_1, z_2 \in \mathbb{C} \setminus \{0\}$ and for an integer $n \in \mathbb{Z} \setminus \{0\}$, we have

$\boxed{\text{L1}}$ $\log(z_1 z_2) = \log(z_1) + \log(z_2)$.

$\boxed{\text{L2}}$ $\log(z^n) = n\log(z)$.

$\boxed{\text{L3}}$ $\log(e^z) = z + 2k\pi i$, $k \in \mathbb{Z}$.

3.3.1 The principal value of the complex logarithm

Definition 3.2. The principal value (the principal branch) of the complex logarithm is a single-valued function, denoted by $\operatorname{Log}(z)$ (with uppercase L), and is defined by

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z), \quad \forall z \in \mathbb{C} \setminus \{0\}.$$

In particular, $\operatorname{Log}(x) = \ln(x)$ for every $x > 0$.

The properties of $\log(z)$ do not necessarily work for $\operatorname{Log}(z)$. Take for example, $z_1 = z_2 = -1$. Then $\operatorname{Log}(z_1) = \operatorname{Log}(z_2) = \operatorname{Log}(-1) = \ln|-1| + i\operatorname{Arg}(-1) = \pi i$. On the other hand, we have $\operatorname{Log}(z_1 z_1) = \operatorname{Log}(1) = \ln(1) = 0$. So, $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$.

Proposition 3.1. *The exponential function $f(z) = e^z$ is injective in the strip*

$$S = \{z \in \mathbb{C} : -\pi < \operatorname{Im}(z) \leq \pi\},$$

and its inverse is $f^{-1}(z) = \operatorname{Log}(z)$ for every $z \neq 0$.

Proof. The fact that e^z is injective in S follows from the properties of the exponential function (the periodicity of period $2\pi i$). From one side we have $e^{\operatorname{Log}(z)} = z$, for every $z \neq 0$. On the other hand, for every $z \in S$, we have $\operatorname{Im}(z) \in]-\pi, \pi]$, and therefore,

$$\operatorname{Log} e^z = \ln|e^z| + i\operatorname{Arg}(e^z) = \operatorname{Re}(z) + i\operatorname{Im}(z) = z, \quad \forall z \in S.$$

This shows that $f : S \to \mathbb{C} \setminus \{0\}$ is bijective, and $f^{-1}(z) = \operatorname{Log}(z)$. □

Theorem 3.2. *The principal branch* $\mathrm{Log}(z)$ *of the complex logarithm is holomorphic on the* slit complex plane $\mathbb{C} \setminus \{z : \mathrm{Im}(z) = 0, \mathrm{Re}(z) \leq 0\}$, *and there we have*

$$\frac{d\,\mathrm{Log}(z)}{dz} = \frac{1}{z}.$$

Proof. The proof will be discussed in the tutorial sessions. □

Exercise 3.5. Determine the domain of holomorphicity of the function $f(z) = \mathrm{Log}(3z - i)$.

Solution. From Theorem 3.2, f is holomorphic for $3z - i \notin \{z : \mathrm{Im}(z) = 0, \mathrm{Re}(z) \leq 0\}$. We have

$$\mathrm{Im}(3z - i) = 0 \text{ and } \mathrm{Re}(3z - i) \leq 0 \iff \mathrm{Im}(z) = \frac{1}{3} \text{ and } \mathrm{Re}(z) \leq 0.$$

Thus f is holomorphic on $\mathbb{C} \setminus \left\{z : \mathrm{Im}(z) = \frac{1}{3}, \mathrm{Re}(z) \leq 0\right\}$. □

3.3.2 Complex powers

Let $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{0\}$. Then z^α is, in general, a multi-valued function defined by $z^\alpha = e^{\alpha \log(z)}$. In particular, if $\alpha = n \in \mathbb{N}$, then $z^n = e^{n \log(z)} = e^{n \ln|z| + in\,\mathrm{Arg}(z) + 2nk\pi i} = e^{n \ln|z| + in\,\mathrm{Arg}(z)} = e^{n\,\mathrm{Log}(z)}$, which is a single-valued function.

For $\alpha_1, \alpha_2 \in \mathbb{C}$ and $z \neq 0$ we have (as multi-valued functions)

$$z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}, \quad \frac{z^{\alpha_1}}{z^{\alpha_2}} = z^{\alpha_1 - \alpha_2}, \quad (z^{\alpha_1})^k = z^{k\alpha_1},\ k \in \mathbb{Z}.$$

For $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$ we have

$$(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha.$$

In general, $(z^{\alpha_1})^{\alpha_2} \neq z^{\alpha_1 \alpha_2}$. For example

$$\left[(-i)^2\right]^i = (-1)^i = e^{i \log(-1)} = e^{-(2k+1)\pi}, \quad k \in \mathbb{Z},$$
$$(-i)^{2i} = e^{2i \log(-i)} = e^{(1-4k)\pi}, \quad k \in \mathbb{Z}.$$

The principal value (the principal branch) of z^α is a single-valued function given by $e^{\alpha\,\mathrm{Log}(z)}$. For example, the principal value of $(-i)^i$ is $e^{i\,\mathrm{Log}(-i)} = e^{\pi/2}$. The principal value of z^α is holomorphic on the slit plane $\mathbb{C} \setminus \{z : \mathrm{Im}(z) = 0, \mathrm{Re}(z) \leq 0\}$, and there we have

$$\frac{dz^\alpha}{dz} = \alpha z^{\alpha - 1}.$$

A particular case of the complex power function is the nth root of a complex number $z \neq 0$. Let $n \geq 1$ be an integer, then the nth root of $z \neq 0$ is given by

$$z^{1/n} = \exp\left\{\frac{\ln|z|}{n} + i\frac{\mathrm{Arg}(z)}{n} + \frac{2k\pi}{n} i\right\}, \quad k \in \mathbb{Z}.$$

Note that $z^{1/n}$ is a multi-valued function with exactly n values for each $z \neq 0$ corresponding to the cases $k = 0, 1, \ldots, n-1$. We say that $z^{1/n}$ is n-valued function. The principal value of $z^{1/n}$ is given by $\sqrt[n]{|z|} e^{i \operatorname{Arg}(z)}$ for every $z \neq 0$.

By choosing the principal value of z^α, the identity $(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha$ may not hold. Take for example $\alpha = 1/2$. Then the principal value of $(-1)^{1/2}$ is $e^{i\pi/2}$ and the principal value of $(i)^{1/2}$ is $e^{i\pi/4}$. However, the principal value of $(-i)^{1/2}$ is $e^{-i\pi/4} \neq e^{3i\pi/4} = (-1)^{1/2}(i)^{1/2}$.

4 Complex Integration

4.1 Complex-valued functions of real variable

Lec. 14

Definition 4.1. Let $\varphi : [a, b] \to \mathbb{C}$, where $\operatorname{Re}(\varphi)$ and $\operatorname{Im}(\varphi)$ are integrable on $[a, b]$. Then φ is integrable on $[a, b]$ and

$$\int_a^b \varphi(t)\,\mathrm{d}\,t = \int_a^b u(t)\,\mathrm{d}\,t + i\int_a^b v(t)\,\mathrm{d}\,t.$$

Remark 4.1. (1) Clearly, the integration of complex-valued functions of real variable is a "linear operator".

(2) From the fundamental theorem of calculus for the real-valued functions, we easily deduce the the result: *If $\varphi : [a, b] \to \mathbb{C}$ is integrable, and if there exists a differentiable function $\Phi : [a, b] \to \mathbb{C}$ such that $\Phi' = \varphi$, then*

$$\int_a^b \varphi(t)\,\mathrm{d}\,t = \Phi(b) - \Phi(a).$$

Lemma 4.1 (Triangle inequality)**.** *Let $\varphi : [a, b] \to \mathbb{C}$ be integrable. Then*

$$\left|\int_a^b \varphi(t)\,\mathrm{d}\,t\right| \le \int_a^b |\varphi(t)|\,\mathrm{d}\,t.$$

Proof. Since the quantity $\int_a^b \varphi(t)\,\mathrm{d}\,t$ is complex, we may write it in the polar form, i.e.,

$$\int_a^b \varphi(t)\,\mathrm{d}\,t = \rho e^{i\theta}, \quad \rho >\geq 0, \theta \in \mathbb{R}.$$

Therefore,

$$\int_a^b e^{-i\theta}\varphi(t)\,\mathrm{d}\,t = \rho.$$

The LHS of this equality is real, and hence

$$\int_a^b \operatorname{Re}\left(e^{-i\theta}\varphi(t)\right)\mathrm{d}\,t = \rho.$$

By making use of the triangle inequality for real-valued functions, we deduce

$$\left|\int_a^b \varphi(t)\,\mathrm{d}\,t\right| = \rho = \left|\int_a^b \operatorname{Re}\left(e^{-i\theta}\varphi(t)\right)\mathrm{d}\,t\right| \le \int_a^b \left|\operatorname{Re}\left(e^{-i\theta}\varphi(t)\right)\right|\mathrm{d}\,t \le \int_a^b |\varphi(t)|\,\mathrm{d}\,t. \qquad \square$$

4.2 Complex-valued functions of complex variable

Let γ be a smooth curve with initial point ξ_1 and terminal point ξ_2. For an integer $n \ge 1$ define $P_n = \{z_0, z_1, \ldots, z_n\}$ a set of distinct points lying on γ to be a partition of γ if $z_0 = \xi_1$

and $z_n = \xi_2$ and z_{k-1} precedes z_k for every $k = 1, \ldots, n$. Denote by $\mu(P_n)$ the mesh size of the partition P_n, defined by

$$\mu(P_n) = \max\left\{ \underset{\gamma}{\operatorname{length}}(z_{k-1}z_k) : k = 1, \ldots, n \right\},$$

where $\underset{\gamma}{\operatorname{length}}(z_{k-1}z_k)$ is the arc-length along γ between z_{k-1} and z_k. Clearly

$$\lim_{n \to +\infty} \mu(P_n) = 0.$$

Let $f : S \to \mathbb{C}$ be defined on an open subset $S \subset \mathbb{C}$. Let γ be a smooth curve in S. Then, a Riemann sum of f corresponding to a partition $P_n = \{z_0, z_1, \ldots, z_n\}$ is any sum given by

$$S(f, P_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}),$$

where c_k is any point lying on γ between z_{k-1} and z_k (may take one of the values z_{k-1}, z_k), for every $k = 1, \ldots, n$.

Definition 4.2. Let $f : S \to \mathbb{C}$ be defined on an open subset S, and let γ be a smooth curve in S. We say that f is integrable along γ if there exists $\ell \in \mathbb{C}$ such that for every $\varepsilon > 0$, there exits a partition $P_n = \{z_0, z_1, \ldots, z_n\}$ of γ, where $\lim_{n \to +\infty} \mu(P_n) = 0$, for which $|S(f, P_n) - \ell| < \varepsilon$. The number ℓ is called the integral (path-integral) of f along γ, and is denoted by

$$\ell = \int_\gamma f(z) \, \mathrm{d}\, z.$$

In this case (f is integrable along γ), we have

$$\int_\gamma f(z) \, \mathrm{d}\, z = \lim_{n \to +\infty} \sum_{k=1}^{n} f(z_{k-1})(z_k - z_{k-1}),$$

where the Riemann sum on the RHS is taken over all partitions P_n of γ, and the points c_k are replaced with z_{k-1}.

Now, let $z = \gamma(t)$, $t \in [a, b]$, be <u>any</u> parametrization of the curve γ, and let $\tilde{P}_n = \{t_0, t_1, \ldots, t_n\}$ be a subdivision of $[a, b]$ corresponding to $P_n = \{z_0, z_1, \ldots, z_n\}$, i.e., $\gamma(t_k) = z_k$ for every $k = 0, 1, \ldots, n$. The Riemann sum $S(f, P_n)$ can be "approximated" by

$$\widetilde{S}(\tilde{P}_n) = \sum_{k=1}^{n} f(\gamma(t_{k-1}))\gamma'(t_{k-1})(t_k - t_{k-1}),$$

for sufficiently large n. Notice that this is a Riemann sum for the function $f(\gamma(t))\gamma'(t)$ on the interval $[a, b]$. From this, we see analogously to te real case that the continuity is sufficient for the integrability.

Definition 4.3. Let $f : D \to \mathbb{C}$ be continuous on an open subset S. Let γ be a smooth curve in S, and let $z = \gamma(t)$, $t \in [a, b]$ be any parametrization of γ. Then the integral of f along γ is given by

$$\int_{\gamma} f(z)\,\mathrm{d}\,z = \int_{a}^{b} f(\gamma(t))\gamma'(t)\,\mathrm{d}\,t.$$

Remark 4.2. (1) Let $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_s$ be a contour in D, where γ_j, $j = 1, \ldots, s$, are the smooth components of Γ. Then we can define the integral along Γ by the relation

$$\int_{\Gamma} = \int_{\gamma_1} + \int_{\gamma_2} + \cdots + \int_{\gamma_s}.$$

(2) The integral along γ doesn't depend on the parametrization of γ. Indeed, if $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ are two parametrizations of γ, then there exists a strictly increasing bijection $\psi : [a, b] \to [c, d]$ such that $\gamma_1(t) = \gamma_2(\psi(t))$, $\forall t \in [a, b]$. Hence,

$$\int_{c}^{d} f(\gamma_2(s))\gamma_2'(s)\,\mathrm{d}\,s \overset{s=\psi(t)}{=} \int_{a=\psi^{-1}(c)}^{b=\psi^{-1}(d)} f(\gamma_1(t))\underbrace{\gamma_2'(\psi(t))\psi'(t)}_{=\gamma_1'(t)}\,\mathrm{d}\,t = \int_{a}^{b} f(\gamma_1(t))\gamma_1'(t)\,\mathrm{d}\,t.$$

(3) We can directly deduce that the integral along a contour Γ is a "linear operator". In addition, the integral along the opposite contour $-\Gamma$ is given by

$$\int_{-\Gamma} = -\int_{\Gamma}.$$

Example 4.1. According to the values of $n \in \mathbb{Z}$, Evaluate the integral $\int_{\gamma}(z-a)^n\,\mathrm{d}\,z$ along the positively oriented circle $\gamma : |z - a| = r > 0$.

Lec. 15 → **Notation.** If Γ is a positively oriented closed contour, then we often use the notation $\oint_{\Gamma}$ for the integral. In addition, when we write, e.g., $\oint_{|z-a|=r}$, we mean the integral along the circle of the equation $|z - a| = r$.

Recall. Let $z = \gamma(t)$, $t \in [a, b]$, be a parametrization of a smooth curve γ. Then the arclength of γ is given by

$$\ell(\gamma) = \int_{a}^{b} |\gamma'(t)|\,\mathrm{d}\,t = \int_{\gamma} |\,\mathrm{d}\,z|. \tag{4.1}$$

The second inequality follows from $\mathrm{d}\,z = \gamma'(t)\,\mathrm{d}\,t$, which yields $|\,\mathrm{d}\,z| = |\gamma'(t)|\,\mathrm{d}\,t$ (called: the arclength measure).

Let $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_s$ be a contour, where γ_j, $j = 1, \ldots, s$, are the smooth components of Γ. Then the arclength of Γ is given by

$$\ell(\Gamma) = \sum_{j=1}^{s} \ell(\gamma_j). \tag{4.2}$$

Theorem 4.1. *Let $f : S \to \mathbb{C}$ be continuous on an open subset S, and let γ be a smooth curve in S. Then*

$$\left| \int_\gamma f(z) \,\mathrm{d}\, z \right| \le \int_\gamma |f(z)| |\mathrm{d}\, z|.$$

Proof. Let $z = \gamma(t)$, $t \in [a, b]$, be any parametrization of γ. Then making use of Lemma 4.1 results in

$$\left| \int_\gamma f(z) \right| = \left| \int_a^b f(\gamma(t))\gamma'(t) \,\mathrm{d}\, t \right| \le \int_a^b |f(\gamma(t))||\gamma'(t)| \,\mathrm{d}\, t = \int_\gamma |f(z)| |\mathrm{d}\, z|.$$ □

Corollary 4.2. *If under the conditions of Theorem 4.1 we have in addition that $|f(z)| \le M$ for every $z \in \gamma$, then*

$$\left| \int_\gamma f(z) \,\mathrm{d}\, z \right| \le M\ell(\gamma).$$

Remark 4.3. Using (4.2), Corollary 4.2 holds true for γ as a contour.

Exercise 4.1. Find an upper bound for the modulus of the integral

$$\oint_{|z|=4} \frac{e^z}{z+1} \,\mathrm{d}\, z.$$

Solution. The length of the circle $\gamma = \{z : |z| = 4\}$ is $\ell(\gamma) = 8\pi$. We have

$$|z+1| \ge |z| - 1 = 4 - 1 = 3 \quad \text{and} \quad |e^z| = e^{\mathrm{Re}(z)} \le e^{|z|} = e^4.$$

Therefore,

$$\left| \oint_{|z|=4} \frac{e^z}{z+1} \,\mathrm{d}\, z \right| \le \oint_{|z|=4} \left| \frac{e^z}{z+1} \right| |\mathrm{d}\, z| \le \frac{e^4}{3} \oint_{|z|=4} |\mathrm{d}\, z| = \frac{8\pi e^4}{3}.$$ □

4.2.1 Fundamental Theorem of Integration

Definition 4.4 (Primitive)**.** Let $f : S \to \mathbb{C}$ be a continuous function on an open subset $S \subset \mathbb{C}$. A holomorphic function $F : S \to \mathbb{C}$ is said to be a *primitive* of f in S if $F'(z) = f(z)$ for every $z \in S$.

Theorem 4.3. *If f is continuous on an open subset $S \subset \mathbb{C}$, and if it has a primitive F in S, then*

$$\int_\Gamma f(z) \,\mathrm{d}\, z = F(z_1) - F(z_2), \qquad \text{F(z\_2)-F(z\_1)}$$

for every $z_1, z_2 \in S$ and for every contour Γ in S joining z_1 to z_2. In particular, if Γ is a closed contour in S, then

$$\int_\Gamma f(z) \,\mathrm{d}\, z = 0.$$

Proof. Let $z_1, z_2 \in S$, and let γ be any smooth contour in S joining z_1 to z_2. In case of γ being a contour is an obvious modifications. Consider any parametrization $z = \gamma(t)$, $t \in [a, b]$, of γ. Since F is a primitive of f in S, it follows that the function $F(\gamma)$ is a primitive of $F'(\gamma)\gamma'$ in $[a, b]$. Then, applying the fundamental theorem of integration for real variable, we deduce the conclusion of the theorem. $\square$

Remark 4.4. (1) For the real functions, every continuous function has a primitive. This is not true for complex functions. E.g., the function $f(z) = \overline{z}$ in continuous on $\mathbb{C}$, but

$$\oint_{|z|=1} \overline{z}\,\mathrm{d}\,z = \int_0^{2\pi} e^{-it}(ie^{it})\,\mathrm{d}\,t = 2\pi i \neq 0.$$

So, from Theorem 4.3, f doesn't have a primitive in $\mathbb{C}$. **Here, f is not holomorphic**
(2) One may expect that if f is holomorphic, then it has a primitive. Well, this is not true in general either. E.g., the function $f(z) = z^{-1}$ is holomorphic on $\mathbb{C} \setminus \{0\}$. However,

$$\oint_{|z|=1} \frac{\mathrm{d}\,z}{z} = 2\pi i \neq 0.$$

So, f doesn't have a primitive in $\mathbb{C} \setminus \{0\}$. **Here, D is not simply connected**

Definition 4.5 (Path-independence)**.** Let $f : S \to \mathbb{C}$ be continuous on an open subset $S \subset \mathbb{C}$. The integral of f is said to be *path-independent* in S if for any $z_1, z_2 \in S$ and for every γ_1, γ_2 two contours in S joining z_1 to z_2 we have

$$\int_{\gamma_1} f(z)\,\mathrm{d}\,z = \int_{\gamma_2} f(z)\,\mathrm{d}\,z.$$

In case of path-independence, the integral will be denoted by $\displaystyle\int_{z_1}^{z_2}$.

Remark 4.5. Not every continuous function has a path-independent integral. E.g., the function $f(z) = \overline{z}$ is continuous on $\mathbb{C}$. For -1 and 1, we have two smooth curves joining 1 to -1 defined by

$$\gamma_1(t) = e^{it}, \quad t \in [0, \pi] \quad \text{and} \quad \gamma_2(t) = e^{-it}, \quad t \in [0, \pi].$$

Then

$$\int_{\gamma_1} \overline{z}\,\mathrm{d}\,z = \pi i \quad \text{and} \quad \int_{\gamma_2} \overline{z}\,\mathrm{d}\,z = -\pi i.$$

Theorem 4.3 asserts that if f has a primitive in S, then its integral is path-independent in S. The following result shows that the converse is also true.

Theorem 4.4. *If $f : D \to \mathbb{C}$ is continuous on a domain $D \subset \mathbb{C}$, and if its integral is path-independent in D, then f has a primitive in D.*

Lec. 16 $\longrightarrow$ *Proof.* Let $z_0 \in D$ be fixed, and for any $z \in D$, let γ_z be a curve in D joining z_0 to z (this is possible by the connectedness of D). Define the function

$$F(z) = \int_{\gamma_z} f(\xi)\, \mathrm{d}\,\xi.$$

As the integral of f is path-independent, it follows that $F : D \to \mathbb{C}$ is well-defined. Now, let $h \in \mathbb{C} \setminus \{0\}$ be sufficiently small such that $[z, z+h] \subset D$ (This is always possible as D is open) . Therefore, we have

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left(\int\limits_{\gamma_z + [z,z+h]} f(\xi)\, \mathrm{d}\,\xi - \int\limits_{\gamma_z} f(\xi)\, \mathrm{d}\,\xi \right) = \frac{1}{h} \int\limits_{[z,z+h]} f(\xi)\, \mathrm{d}\,\xi,$$

which implies that

$$\begin{aligned} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \left| \frac{F(z+h) - F(z)}{h} - \frac{f(z)}{h} \int_{[z,z+h]} \mathrm{d}\,\xi \right| \\ &\leq \frac{1}{|h|} \int_{[z,z+h]} |f(\xi) - f(z)||\, \mathrm{d}\,\xi|. \end{aligned} \tag{4.3}$$

Given $\varepsilon > 0$. Then by the continuity of f at z, there exists δ for which $|\xi - z| < \delta$ implies $|f(\xi) - f(z)| < \varepsilon$. Let h satisfy $|h| < \delta$. Then for every $\xi \in [z, z+h]$ we have $|\xi - z| \leq |(z+h) - z| = |h| < \delta$. Hence, from (4.3), we deduce

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \frac{\varepsilon}{|h|} \ell([z, z+h]) = \varepsilon.$$

This shows that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

Since z is arbitrary in D, it follows that F is holomorphic on D, and that $F'(z) = f(z)$ for every $z \in D$. That is, F is a primitive of f in D. This completes the proof. □

Corollary 4.5. *If $f : D \to \mathbb{C}$ is continuous on a domain D and $\oint_\gamma f(z)\, \mathrm{d}\, z = 0$ for every closed contour γ in D, then f has a primitive.*

Proof. Let $z_1, z_2 \in D$ be any arbitrary two distinct points in D, and let γ_1 and γ_2 be any two contours in D joining z_1 to z_2. Then the contour $\gamma = \gamma_1 + (-\gamma_2)$ is a closed contour in D. Therefore,

$$\int_{\gamma_1} f(z)\, \mathrm{d}\, z - \int_{\gamma_2} f(z)\, \mathrm{d}\, z = \int_{\gamma_1} f(z)\, \mathrm{d}\, z + \int_{-\gamma_2} f(z)\, \mathrm{d}\, z = \oint_\gamma f(z)\, \mathrm{d}\, z = 0.$$

This shows that the integral of f is path-independent in D, and by Theorem 4.4, we deduce that f has a primitive in D. □

4.3 Cauchy Integral Theorem and Consequences

Theorem 4.6 (Cauchy-Goursat)**.** *Let $f : D \to \mathbb{C}$ be holomorphic on a* ***simply connected*** *domain $D \subset \mathbb{C}$. Then for every Jordan contour γ in D we have*

$$\oint_\gamma f(z)\,\mathrm{d}\,z = 0.$$

Idea of the proof. we will divide the proof into three cases:

Case 1. Consider first that $\gamma = T$ is a positively oriented triangle. Divide T into four inscribed triangles S_1, S_2, S_3 and S_4 using the midpoints of its sides as in Figure 1. Notice that the integral along the four triangles reduces to the integral along the original triangle T, as the integral along the inscribed line segments will cancel out (They're traversed in opposite orientation). Therefore,

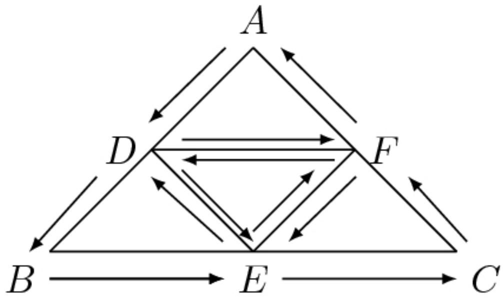


Figure 1: Triangles of Goursat

$$\oint_T f(z)\,\mathrm{d}\,z = \sum_{k=1}^{4} \oint_{S_k} f(z)\,\mathrm{d}\,z.$$

Notice that there must exist one triangle, denoted T_1, among S_1, S_2, S_3 and S_3 that satisfies

$$\left|\oint_{T_1} f(z)\,\mathrm{d}\,z\right| = \max_{1\le k\le 4}\left|\oint_{S_k} f(z)\,\mathrm{d}\,z\right|.$$

Then

$$\left|\oint_{T} f(z)\,\mathrm{d}\,z\right| \le 4\left|\oint_{T_1} f(z)\,\mathrm{d}\,z\right|.$$

Notice also that the perimeter of T_1 is $\ell(T_1) = \ell(T)/2$. By repeating the process, and continue to subdivide T_1 in the same way, we arrive at a sequence $T, T_1, T_2, \cdots, T_n, \cdots$ for which

$$\left|\oint_{T} f(z)dz\right| \le 4\left|\oint_{T_1} f(z)dz\right| \le \cdots \le 4^n\left|\oint_{T_n} f(z)dz\right| \le \cdots, \tag{4.4}$$

and $\ell(T_n) = \ell(T)/2^n$. Denote by $\Delta_n = T_n \cup \mathrm{int}(T_n)$. Then $\{\Delta_n\}_{n\ge1}$ is a sequence of nested compact sets, i.e.,

$$\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \cdots \supset \Delta_n \supset \cdots$$

and $\operatorname{diam}(\Delta_n) \leq \ell(T_n) = \ell(T)/2^n \longrightarrow 0$ as $n \to +\infty$. It follows from the Nested Sets Theorem that $\bigcap\limits_{n\geq 1} \Delta_n = \{z_0\}$ for some $z_0 \in D$ (D is simply connected). Since f is holomorphic at z_0, it follows that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z)(z - z_0), \tag{4.5}$$

where $R(z) \longrightarrow 0$ as $z \to z_0$. Now, for an arbitrary given $\varepsilon > 0$, there exists $\delta > 0$ such that $|R(z)| < \varepsilon$ whenever $|z - z_0| < \delta$. In addition, as $\operatorname{diam}(\Delta_n) \to 0$ as $n \to +\infty$, there exists $n_0 \geq 1$ such that $\Delta_n \subset D(z_0, \delta)$ for every $n \geq n_0$. Now, from (4.5), we have

$$\begin{aligned} \oint_{T_n} f(z)\,\mathrm{d}\,z &= f(z_0)\underbrace{\oint_{T_n}\mathrm{d}\,z}_{=0} + f'(z_0)\underbrace{\oint_{T_n}(z - z_0)\,\mathrm{d}\,z}_{=0} + \oint_{T_n} R(z)(z - z_0)\,\mathrm{d}\,z \\ &= \oint_{T_n} R(z)(z - z_0)\,\mathrm{d}\,z. \end{aligned} \tag{4.6}$$

Notice that both functions 1 and $(z - z_0)$ have primitives, and for that their integrals along closed contours vanish. In addition, notice that when $z \in T_n$, $|z - z_0| \leq \operatorname{diam}(T_n)$. Then, from (4.6) and for $n \geq n_0$ we obtain

$$\begin{aligned} \left|\oint_{T_n} f(z)\,\mathrm{d}\,z\right| &\leq \oint_{T_n} |R(z)||z - z_0||\,\mathrm{d}\,z| \\ &\leq \varepsilon\,\operatorname{diam}(T_n) \oint_{T_n} |\,\mathrm{d}\,z| \\ &= \varepsilon\,\operatorname{diam}(T_n)^2 \leq \varepsilon\frac{\ell(T)^2}{4^n}. \end{aligned}$$

Using this with (4.4) yield

$$\left|\oint_T f(z)dz\right| \leq \varepsilon \ell(T)^2.$$

As $\varepsilon > 0$ is arbitrary, we deduce that $\oint\limits_T f(z)dz = 0$.

Case 2. If γ is a positively oriented closed polygonal line in D, then by triangulation of the polygonal region enclosed by γ into finitely many triangles, we can use the previous case to deduce that $\oint\limits_\gamma f(z)\,\mathrm{d}\,z = 0$. (check it!)

Case 3. Consider now that γ is a general Jordan contour in D. We know that the integral $\oint_\gamma f(z)\,\mathrm{d}\,z$ is the limit of the sums

$$S(f, P_n) = \sum_{k=1}^{n} f(z_{k-1})(z_k - z_{k-1})$$

over all the partitions $z_0, z_1, \ldots, z_n = z_0$ of γ with $\mu(P_n) \longrightarrow 0$ as $n \to +\infty$. Then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and P_n a partition of γ with $\mu(P_n) < \delta_1$ such that

$$\left|\oint_\gamma f(z)\,\mathrm{d}\,z - S(f, P_n)\right| < \frac{\varepsilon}{2}. \tag{4.7}$$

Let L_n be the closed polygonal line whose vertex set is P_n. Choose n sufficiently large so that $\mu(P_n)$ is sufficiently small in order to ensure that $L_n \subset D$. Recall that f is continuous on the compact $K_n = Ł_n \cup \text{int}(L_n)$, and hence it is uniformly continuous there. Therefore, there exists $\delta_2 > 0$ such that $|f(z_1) - f(z_2)| < \dfrac{\varepsilon}{2\ell(L_n)}$ whenever $|z_1 - z_2| < \delta_2$ and $z_1, z_2 \in K_n$. Taking $\delta < \min\{\delta_1, \delta_2\}$ we obtain

$$\left| S(f, P_n) - \oint_{L_n} f(z)\,\mathrm{d}\,z \right| \leq \sum_{k=1}^{n} \int_{[z_{k-1}, z_k]} |f(z_{k-1}) - f(z)|\,|dz| < \frac{\varepsilon}{2}.$$

We know from Case 2 that

$$\oint_{L_n} f(z)\,\mathrm{d}\,z = 0.$$

Thus $|S(f, P_n)| < \varepsilon/2$. Using this with (4.7) we deduce that

$$\left| \oint_{\gamma} f(z)\,\mathrm{d}\,z \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\oint\limits_{\gamma} f(z)\,\mathrm{d}\,z = 0$. This completes the proof. □

Lec. 17 → The more practical version of Cauchy-Goursat theorem is as follows: *If a function f is holomorphic on $\overline{\text{int}(\gamma)}$, where γ is a Jordan contour, then* $\oint\limits_{\gamma} f(z)\,\mathrm{d}\,z = 0$.

Example 4.2. Since any polynomial P is an entire function, then for any closed contour in $\mathbb{C}$ we have $\oint\limits_{\gamma} P(z)\,\mathrm{d}\,z = 0$.

Remark 4.6. (1) From Corollary 4.5, every holomorphic function f in a simply connected domain D has a primitive in D. Consequently, the integral of f is path-independent.
(2) Cauchy-Goursat theorem is not valid in a domain that is not simply connected. E.g., $f(z) = 1/z$ is holomorphic in $\mathbb{C} \setminus \{0\}$, and $\oint\limits_{|z|=1} 1/z\,\mathrm{d}\,z = 2\pi i$.

Procedure in case of multiply-connected domains. Let $f : D \to \mathbb{C}$ is holomorphic on D, which has two holes (multiply-connected). Let γ be a Jordan contour enclosing both holes, and let γ_1 and γ_2 be two Jordan contours, each enclosing one of the two holes individually.

Let $z_1 \in \gamma$, $z_2, z_3 \in \gamma_1$ and $z_4 \in \gamma_3$. Then make a cut along some curves γ_{z_1,z_2} joining z_1 and z_2 and γ_{z_3,z_4} joining z_3 and z_4, and both lie within $\text{int}(\gamma)$. Then f is holomorphic on the simply connected region

$$\overline{\text{int}\,(\gamma)} \setminus \Big(\text{int}(\gamma_1) \cup \text{int}(\gamma_2) \cup \gamma_{z_1,z_2} \cup \gamma_{z_3,z_4} \Big).$$

Let's denote by $-\gamma_1(z_2, z_3)$ the arc of γ_1 joining z_2 to z_3 in the negative orientation of γ_1. Therefore, by Cauchy-Goursat theorem, we obtain

$$\oint_{\gamma} + \int_{\gamma_{z_1,z_2}} + \int_{-\gamma_1(z_2,z_3)} + \int_{\gamma_{z_3,z_4}} + \oint_{-\gamma_2} + \int_{-\gamma_{z_3,z_4}} + \int_{-\gamma_1(z_3,z_2)} + \int_{-\gamma_{z_1,z_2}} = 0$$

Therefore,

$$\oint_\gamma = \oint_{\gamma_1} + \oint_{\gamma_2}.$$

We can easily generalize this into a domain with n holes as follows.

Theorem 4.7 (Cauchy theorem for multiply-connected domains)**.** *Let γ be a Jordan contour, and let $\gamma_1, \ldots, \gamma_n$ be Jordan contours such that $\operatorname{int}(\gamma_j)$, $j = 1, \ldots, n$, are pairwise disjoint and all are inside $\operatorname{int}(\gamma)$. If f is holomorphic on*

$$\overline{\operatorname{int}(\gamma)} \setminus \bigcup_{j=1}^{n} \operatorname{int}(\gamma_j),$$

then

$$\oint_\gamma f(z)\,\mathrm{d}\,z = \sum_{j=1}^{n} \oint_{\gamma_j} f(z)\,\mathrm{d}\,z.$$

Exercise 4.2. Find the value of the integral

$$I = \oint_{|z-2|=2} \frac{5z+7}{z^2+2z-3}\,\mathrm{d}\,z.$$

Solution. We have $z^2 + 2z - 3 = (z-1)(z+3)$. So, the function

$$f(z) = \frac{5z+7}{z^2+2z-3}$$

is holomorphic on the disc $D(2,2)\setminus\{1\}$. By Cauchy theorem for multiply-connected domains we have

$$\oint_{|z-2|=2} f(z)\,\mathrm{d}\,z = \oint_{|z-1|=1} f(z)\,\mathrm{d}\,z.$$

By decomposition, we have

$$f(z) = \frac{3}{z-1} + \frac{2}{z+3}.$$

Therefore,

$$I = \oint_{|z-1|=1} \frac{3}{z-1}\,\mathrm{d}\,z + \oint_{|z-1|=1} \frac{2}{z+3}\,\mathrm{d}\,z = 3(2\pi i) + 0 = 6\pi i.$$

For the second integral, we know that $z \mapsto 2/(z+3)$ is holomorphic on $\{|z-1| \le 1\}$, and hence by Cauchy-Goursat theorem, its integral along $|z-1| = 1$ is 0. □

Theorem 4.8 (Cauchy's integral formula)**.** *Let γ be a Jordan contour, and let f be holomorphic on $\overline{\operatorname{int}(\gamma)}$. Then for every $z \in \operatorname{int}(\gamma)$, we have*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi - z}\,\mathrm{d}\,\xi.$$

Remark 4.7. This formula is valid for Jordan contours only (i.e., simple closed contours). There is a general formula for any closed contours, which depends on the Winding numbers. This will be discussed in the upcoming sections.

Example 4.3. we have

$$\oint_{|z-2|=2} \frac{5z+7}{z^2+2z-3}\,\mathrm{d}\,z = \oint_{|z-2|=2} \frac{(5z+7)/(z+3)}{(z-1)}\,\mathrm{d}\,z = 2\pi i\left(\frac{5z+7}{z+3}\right)_{z=1} = 2\pi i\left(\frac{12}{4}\right) = 6\pi i.$$

Exercise 4.3. Find the value of

$$I = \oint_{|z|=2} \frac{e^z}{z(z-1)}\,\mathrm{d}\,z.$$

Solution. The function $f(z) := \dfrac{e^z}{z(z-1)}$ is holomorphic on $\{z : |z| \leq 2\} \setminus \{0,1\}$. By Cauchy's theorem for multiply-connected domains, we have

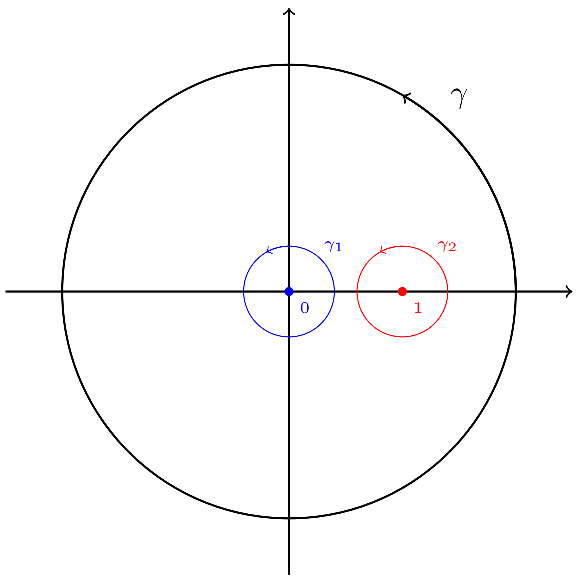
$$I = \oint_{|z|=2} f(z)\,\mathrm{d}\,z = \oint_{|z|=\varepsilon_1} f(z)\,\mathrm{d}\,z + \oint_{|z-1|=\varepsilon_2} f(z)\,\mathrm{d}\,z,$$

where ε_1 and ε_2 are too small positive numbers. Moreover, by Cauchy's integral formula we have

$$\oint_{|z|=\varepsilon_1} f(z)\,\mathrm{d}\,z = \oint_{|z|=\varepsilon_1} \frac{e^z/(z-1)}{z}\,\mathrm{d}\,z = 2\pi i\left(\frac{e^z}{(z-1)}\right)_{z=0} = -2\pi i,$$

$$\oint_{|z-1|=\varepsilon_2} f(z)\,\mathrm{d}\,z = \oint_{|z-1|=\varepsilon_2} \frac{e^z/z}{z-1}\,\mathrm{d}\,z = 2\pi i\left(\frac{e^z}{z}\right)_{z=1} = 2e\pi i.$$

Thus $I = 2(e-1)\pi i$. □



Lec. 18 $\longrightarrow$ *Proof of Theorem 4.8.* Let $z \in \operatorname{int}(\gamma)$, and let $r > 0$ be sufficiently small so that $D(z,r) \subset \operatorname{int}(\gamma)$. The function $\xi \mapsto f(\xi)/(\xi - z)$ is holomorphic on $\overline{\operatorname{int}(\gamma)} \setminus \{z\}$. Then by Cauchy theorem for multiply connected domains, we have

$$\oint\limits_{\gamma} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\,\xi = \oint\limits_{|\xi-z|=r} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\,\xi = \oint\limits_{|\xi-z|=r} \frac{f(\xi)-f(z)}{\xi - z}\,\mathrm{d}\,\xi + f(z) \oint\limits_{|\xi-z|=r} \frac{\mathrm{d}\,\xi}{\xi - z}. \tag{4.8}$$

Let us denote by $I(r)$ the first integral on the right-hand side, while the second integral equals $2\pi i f(z)$.

Claim. $I(r) \longrightarrow 0$ as $r \to 0^+$.

Proof of Claim. By continuity of f at z, we have for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(\xi) - f(z)| < \frac{\varepsilon}{2\pi}$ whenever $|\xi - z| < \delta$. Taking $r < \delta$ yields

$$|I(r)| \le \oint\limits_{|\xi-z|=r} \frac{|f(\xi)-f(z)|}{|\xi - z|}|\,\mathrm{d}\,\xi| \le \frac{\varepsilon}{2\pi r} \oint\limits_{|\xi-z|=r} |\,\mathrm{d}\,\xi| = \frac{\varepsilon}{2\pi r}(2\pi r) = \varepsilon.$$

This proves the Claim. □

Now, from (4.8) we deduce that

$$\oint\limits_{\gamma} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\,\xi = 2\pi i f(z),$$

which proves the required formula. □

Corollary 4.9 (Gauss Mean Value Property)**.** *Let $f : D \to \mathbb{C}$ be holomorphic on a domain $D \subset \mathbb{C}$. Let $z \in D$ and $r > 0$ such that $D(z,r) \subset D$. Then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\,\mathrm{d}\,\theta.$$

Proof. Since f is holomorphic on $\{\xi : |\xi - z| \le r\}$, it follows by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|\xi-z|=r} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\,\xi.$$

Using the parametrization $\xi = z + r^{i\theta}$, $\theta \in [0, 2\pi]$, we obtain

$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + re^{i\theta})}{re^{i\theta}}(ire^{i\theta})\,\mathrm{d}\,\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\,\mathrm{d}\,\theta.$$ □

Next, we proceed to establish Cauchy's integral formula for the derivatives.

Theorem 4.10. *Let* γ *be a Jordan contour, and let* f *be continuous on* $\overline{\operatorname{int}(\gamma)}$*. Then the function* $g : \operatorname{int}(\gamma) \to \mathbb{C}$*, defined by*

$$g(z) = \oint_\gamma \frac{f(\xi)}{\xi - z} \,\mathrm{d}\,\xi,$$

is holomorphic on $\operatorname{int}(\gamma)$ *and*

$$g'(z) = \oint_\gamma \frac{f(\xi)}{(\xi - z)^2} \,\mathrm{d}\,\xi, \quad \forall z \in \operatorname{int}(\gamma).$$

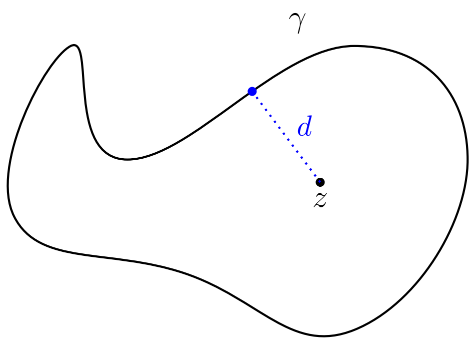
Proof of Theorem 4.10. Let $z \in \operatorname{int}(\gamma)$, and let $h \in \mathbb{C}$ such that $z + h \in \operatorname{int}(\gamma)$. Then

$$\frac{g(z+h) - g(z)}{h} = \oint_\gamma \frac{f(\xi)}{(\xi - z)(\xi - z - h)} \,\mathrm{d}\,\xi.$$

Therefore,

$$\left| \frac{g(z+h) - g(z)}{h} - \oint_\gamma \frac{f(\xi)}{(\xi - z)^2} \,\mathrm{d}\,\xi \right| \leq |h| \oint_\gamma \frac{|f(\xi)|}{|(\xi - z)^2(\xi - z - h)|} |\,\mathrm{d}\,\xi|. \tag{4.9}$$

Now, let $d := \min_{\xi \in \gamma} |\xi - z|$. Clearly, $|\xi - z| \geq d > 0$, for every $\xi \in \gamma$. By taking $|h| < \frac{d}{2}$, we obtain $|\xi - z - h| \geq |\xi - z| - |h| \geq \frac{d}{2}$, for every $\xi \in \gamma$.



Since f is continuous on the compact $\overline{\operatorname{int}(\gamma)}$, it is then bounded. In particular, there exists $M > 0$ such that $|f(\xi)| \leq M$ for every $\xi \in \gamma$. It follows that

$$\oint_\gamma \frac{|f(\xi)|}{|(\xi - z)^2(\xi - z - h)|} |\,\mathrm{d}\,\xi| \leq \frac{M}{d^2 \times \frac{d}{2}} \oint_\gamma |\,\mathrm{d}\,\xi| = \frac{2M\ell(\gamma)}{d^3}.$$

Plugging this into (4.9) yields

$$\left| \frac{g(z+h) - g(z)}{h} - \oint_\gamma \frac{f(\xi)}{(\xi - z)^2} \,\mathrm{d}\,\xi \right| \leq \frac{2M\ell(\gamma)}{d^3} \cdot |h| \longrightarrow 0, \quad \text{as } h \to 0.$$

That implies that f is $\mathbb{C}$-differentiable at $z \in \operatorname{int}(\gamma)$. Since z is arbitrary in $\operatorname{int}(\gamma)$, it follows that f is holomorphic on $\operatorname{int}(\gamma)$, and we have

$$g'(z) = \oint_\gamma \frac{f(\xi)}{(\xi - z)^2} \,\mathrm{d}\,\xi, \quad \forall z \in \operatorname{int}(\gamma).$$ □

One can follow the same proof of Theorem 4.10 to prove the following result.

Theorem 4.11. *Let $n \geq 1$ be an integer, and let γ be a Jordan contour. If f is continuous on $\overline{\operatorname{int}(\gamma)}$, then the function $g_n : \operatorname{int}(\gamma) \to \mathbb{C}$, defined by*

$$g_n(z) = \oint_\gamma \frac{f(\xi)}{(\xi - z)^n} \,\mathrm{d}\,\xi,$$

is holomorphic on $\operatorname{int}(\gamma)$ and

$$g_n(z) = n \oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} \,\mathrm{d}\,\xi, \quad \forall z \in \operatorname{int}(\gamma).$$

Proof. Left as an exercise ! □

Note that when f is holomorphic on $\overline{\operatorname{int}(\gamma)}$, then by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi - z} \,\mathrm{d}\,\xi, \quad \forall z \in \operatorname{int}(\gamma).$$

In addition, Theorem 4.11 asserts that

$$f'(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^2} \,\mathrm{d}\,\xi, \quad \forall z \in \operatorname{int}(\gamma).$$

Again, Theorem 4.11 asserts that even f' is holomorphic on $\operatorname{int}(\gamma)$, and

$$f''(z) = \frac{1}{\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^3} \,\mathrm{d}\,\xi, \quad \forall z \in \operatorname{int}(\gamma).$$

Hence, by induction, we clearly have the following result.

Theorem 4.12. *If $f : D \to \mathbb{C}$ is holomorphic on a domain $D \subset \mathbb{C}$, then all its derivatives $f^{(k)}$, $k \in \mathbb{N}$, exist and are holomorphic on D. Moreover, $\forall k \in \mathbb{N}$, we have*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{k+1}} \,\mathrm{d}\,\xi, \quad \forall z \in \operatorname{int}(\gamma), \tag{4.10}$$

where γ is any Jordan contour in D such that $\operatorname{int}(\gamma) \subset D$.

Remark 4.8. (1) The formula (4.10) is called the Cauchy's integral formula for the derivatives.

(2) Let $f(z) = u(x, y) + iv(x, y)$, for $z = x + iy \in D$. If f is holomorphic on D, then both u and v are differentiable on D. In addition, from Theorem 4.12, we deduce that both u and v are $\mathcal{C}^\infty(D)$.

Exercise 4.4. Let f be a holomorphic function on the disc $D(0,\rho)$, where $\rho > 1$. Evaluate

$$\oint_{|z|=1} \left(2 + z + \frac{1}{z}\right) \frac{f(z)}{z} \, \mathrm{d}\, z.$$

Then deduce the value of

$$\int_0^{2\pi} f(e^{it}) \cos^2\left(\frac{t}{2}\right) \mathrm{d}\, t$$

Proof. (a) We have

$$\begin{aligned}\oint_{|z|=1} \left(2 + z + \frac{1}{z}\right) \frac{f(z)}{z} \, \mathrm{d}\, z &= \oint_{|z|=1} \frac{(2+z)\, f(z)}{z} \, \mathrm{d}\, z + \oint_{|z|=1} \frac{f(z)}{z^2} \, \mathrm{d}\, z \\ &= 2\pi i (2+z) f(z)\Big|_{z=0} + 2\pi i f'(z)\Big|_{z=0} \\ &= 2\pi i \Big(2 f(0) + f'(0)\Big).\end{aligned}$$

(b) By a suitable parameterization of the positively oriented unit circle, we obtain

$$\begin{aligned}\oint_{|z|=1} \left(2 + z + \frac{1}{z}\right) \frac{f(z)}{z} \, \mathrm{d}\, z &= \int_0^{2\pi} \left(2 + e^{it} + e^{-it}\right) \frac{f(e^{it})}{e^{it}} (ie^{it}) \, \mathrm{d}\, t \\ &= i \int_0^{2\pi} \underbrace{(2 + 2\cos t)}_{=4\cos^2(t/2)} f(e^{it}) \, \mathrm{d}\, t \\ &= 4i \int_0^{2\pi} f(e^{it}) \cos^2\left(\frac{t}{2}\right) \mathrm{d}\, t.\end{aligned}$$

From (a), we deduce that

$$\int_0^{2\pi} f(e^{it}) \cos^2\left(\frac{t}{2}\right) \mathrm{d}\, t = \pi f(0) + \frac{\pi}{2} f'(0).$$

□

Exercise 4.5. Let $\rho > 2$. Evaluate the integral

$$I = \oint_{|z|=\rho} \frac{z-1}{z^3 - 2z^2} \, \mathrm{d}\, z.$$

Solution. The function $z \mapsto \dfrac{z-1}{z^3 - 2z^2}$ is holomorphic on $\{z : |z| \leq \rho\} \setminus \{0, 2\}$. By Cauchy theorem for multiply connected domains, we have

$$I = \oint_{|z|=\varepsilon_1} \frac{z-1}{z^3 - 2z^2} \, \mathrm{d}\, z + \oint_{|z-2|=\varepsilon_2} \frac{z-1}{z^3 - 2z^2} \, \mathrm{d}\, z,$$

where $\varepsilon_1, \varepsilon_2 > 0$ are sufficiently small. In addition, we have

$$\oint_{|z|=\varepsilon_1} \frac{z-1}{z^3 - 2z^2} \, \mathrm{d}\, z = \oint_{|z|=\varepsilon_1} \frac{(z-1)/(z-2)}{z^2} \, \mathrm{d}\, z = 2\pi i \left(\frac{z-1}{z-2}\right)'_{z=0} = 2\pi i \left(\frac{-1}{(z-2)^2}\right)_{z=0} = \frac{-\pi i}{2},$$

$$\oint_{|z-2|=\varepsilon_2} \frac{z-1}{z^3 - 2z^2} \, \mathrm{d}\, z = \oint_{|z-2|=\varepsilon_2} \frac{(z-1)/z^2}{(z-2)} \, \mathrm{d}\, z = 2\pi i \left(\frac{z-1}{z^2}\right)_{z=2} = \frac{\pi i}{2}.$$

Thus $\boxed{\text{I=0}}$.

□

Lec. 19

Corollary 4.13 (Cauchy's inequality). *Let f be holomorphic on a domain D, and let $z \in D$, and $r > 0$ such that $\overline{D(z,r)} \subset D$. If $\exists M > 0, |f(\xi)| \leq M, \forall |\xi| = r$, then*

$$|f^{(n)}(z)| \leq \frac{n!M}{r^n}, \quad \forall n \geq 1.$$

Proof. By Cauchy's formula for the derivatives, we have, for $n \geq 1$,

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \oint_{|\xi - z| = r} \frac{|f(\xi)|}{|\xi - z|^{n+1}} |\mathrm{d}\,\xi| \leq \frac{n!M}{2\pi r^{n+1}}(2\pi r) = \frac{n!M}{r^n}.$$

□

Theorem 4.14 (Liouville). *Let f be entire function (Holomorphic on $\mathbb{C}$). If $\exists M > 0$ such that $|f(z)| \leq M$ for every $z \in \mathbb{C}$, then f is constant in $\mathbb{C}$.*

Proof. Let $z \in \mathbb{C}$. Then for every $r > 0$, Cauchy's inequality yields $|f'(z)| \leq \dfrac{M}{r}$. By letting $r \to +\infty$, we get $f'(z) = 0$. Since $z \in \mathbb{C}$ is arbitrary, we deduce that $f' \equiv 0$ and hence f is constant in $\mathbb{C}$. □

Exercise 4.6. Let f be an entire function satisfying $\mathrm{Re}(f(z)) \geq 0$ for every $z \in \mathbb{C}$. Show that f is constant in $\mathbb{C}$.

Solution. Let $g(z) = e^{-f(z)}$ for all $z \in \mathbb{C}$. Since f and e^{-z} are both entire functions, it follows that g is also an entire function. We have, by using the assumption in the statement,

$$|g(z)| = \left|e^{-f(z)}\right| = e^{-\mathrm{Re}(f(z))} \leq 1, \quad \forall z \in \mathbb{C}.$$

By Liouville's theorem, we get that g must be a constant, say $g(z) = K \neq 0$ for every $z \in \mathbb{C}$. Since we have $|K| = |g(z)| = e^{-\mathrm{Re}(f(z))}$ for every $z \in \mathbb{C}$, it follows that $\mathrm{Re}(f(z)) = -\ln|K|$ for every $z \in \mathbb{C}$, i.e., $\mathrm{Re}(f(z))$ is constant, and as f is entire (Holomorphic on $\mathbb{C}$), we deduce that f is also constant in $\mathbb{C}$. □

We established in Theorem 2.8 that

Analyticity $\Longrightarrow$ Holomorphicity.

In the following result, we will prove that

Holomorphicity $\Longrightarrow$ Analyticity.

Theorem 4.15 (Taylor). *Let $f : D \to \mathbb{C}$ be holomorphic on a domain $D \subset \mathbb{C}$. Then f is analytic on D, and for each $z_0 \in D$, we have*

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad \forall z \in D(z_0, R),$$

where $R = \sup\left\{r > 0 : \overline{D(z_0, r)} \subset D\right\}$.

Proof. Let $z_0 \in D$ be arbitrary, and let $R = \sup\left\{r > 0 : \overline{D(z_0, r)} \subset D\right\}$. Let $z \in D(z_0, R)$, and let $\rho > 0$ satisfy $|z - z_0| < \rho < R$. It follows from Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|\xi - z_0| = \rho} \frac{f(\xi)}{\xi - z} \,\mathrm{d}\,\xi.$$

Recall that for any $|w| < 1$ we have

$$\frac{1}{1-w} = \frac{1-w^n}{1-w} + \frac{w^n}{1-w} = \sum_{j=0}^{n-1} w^j + \frac{w^n}{1-w}, \quad \forall n \geq 1.$$

Since $\left|\frac{z - z_0}{\xi - z_0}\right| = \frac{|z - z_0|}{\rho} < 1$, it follows that

$$\begin{aligned}
\frac{1}{\xi - z} &= \frac{1}{(\xi - z_0) - (z - z_0)} \\
&= \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \dfrac{z - z_0}{\xi - z_0}} \\
&= \frac{1}{\xi - z_0} \left(\sum_{j=0}^{n-1} \left(\frac{z - z_0}{\xi - z_0}\right)^j + \left(\frac{z - z_0}{\xi - z_0}\right)^n \frac{1}{1 - \dfrac{z - z_0}{\xi - z_0}} \right) \\
&= \sum_{j=0}^{n-1} \frac{(z - z_0)^j}{(\xi - z_0)^{j+1}} + \left(\frac{z - z_0}{\xi - z_0}\right)^n \frac{1}{\xi - z}.
\end{aligned}$$

From this we obtain that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \sum_{j=0}^{n-1} (z - z_0)^j \oint\limits_{|\xi - z_0| = \rho} \frac{f(\xi)}{(\xi - z_0)^{j+1}} \,\mathrm{d}\,\xi + \frac{1}{2\pi i} \oint\limits_{|\xi - z_0| = \rho} \left(\frac{z - z_0}{\xi - z_0}\right)^n \frac{f(\xi)}{\xi - z} \,\mathrm{d}\,\xi \\
&= \sum_{j=1}^{n-1} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j + \frac{1}{2\pi i} \oint\limits_{|\xi - z_0| = \rho} \left(\frac{z - z_0}{\xi - z_0}\right)^n \frac{f(\xi)}{\xi - z} \,\mathrm{d}\,\xi.
\end{aligned} \tag{4.11}$$

Set

$$R_n(z) := \oint\limits_{|\xi - z_0| = \rho} \left(\frac{z - z_0}{\xi - z_0}\right)^n \frac{f(\xi)}{\xi - z} \,\mathrm{d}\,\xi.$$

It is left as an exercise that $R_n(z) \to 0$ as $n \to +\infty$. as This with (4.11) show that $f(z)$ is written as a power series about z_0. Since z is arbitrary on $D(z_0, R)$, it follows that f is analytic at z_0, with the obtained Taylor's expansion in $D(z_0, R)$. Therefore, as z_0 is arbitrary in D, f is analytic on D, with the obtained Taylor's expansion. □

A $\mathbb{C}$-differentiability and $\mathbb{R}^2$-differentiability

Complex linear mapping. A complex function $f : \mathbb{C} \to \mathbb{C}$ that satisfies $f(z + z') = f(z) + f(z')$ and $f(az) = af(z)$ for every $z, z', a \in \mathbb{C}$, is called a complex linear mapping (or, $\mathbb{C}$-linear mapping[1]). Notice, in particular, that constant functions are not linear mappings. One can also see that if f is a linear mapping, then $f(0) = 0$ and $f(z) \neq 0$ for every $z \in \mathbb{C}^*$.

The functions $f(z) = cz$, where $c \in \mathbb{C}^*$, are $\mathbb{C}$-linear mappings. Conversely, every $\mathbb{C}$-linear mapping $f : \mathbb{C} \to \mathbb{C}$ must satisfy, for every $z \in \mathbb{C}$, $f(z) = f(1 \cdot z) = f(1) \cdot z$, which is of the form $f(z) = cz$ with $c = f(1) \in \mathbb{C}^*$. Thus, the only $\mathbb{C}$-linear mappings are the complex functions of the form $f(z) = cz$, where $c \in \mathbb{C}$.

Analogously to the real linear mappings (or, $\mathbb{R}^2$-linear mapping), $\mathbb{C}$-linear mappings can be represented by matrices. In fact, by letting $c = a + ib \in \mathbb{C}$ and $z = x + iy \in \mathbb{C}$, we obtain

$$cz = (a + ib)(x + iy) = (ax - by) + i(bx + ay).$$

Now, using the cartesian representation of complex numbers, and then using the matrix form, we can write

$$cz = \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

From this formula, we can deduce that every $\mathbb{C}$-linear mapping is also $\mathbb{R}^2$-linear mapping. However, not every $\mathbb{R}^2$-linear mapping is $\mathbb{C}$-linear, as show by the following example.

Example A.1. Consider the $\mathbb{R}^2$-linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ given in the matrix form

$$f \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x + y \end{bmatrix}.$$

By setting $z = x + iy$ and using the relations $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/(2i)$, we may write f is its complex form

$$f(z) = (x + y) + i(x + y) = (1 + i)(x + y) = z + i\overline{z}.$$

This shows that f cannot be $\mathbb{C}$-linear. Simply note that $f(i) = i + 1 \neq i - 1 = if(1)$.

Summarizing the discussion above we can state that the $\mathbb{R}^2$-linear mapping represented by the matrix

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

is $\mathbb{C}$-linear if and only if $\alpha = \delta$ and $\beta = -\gamma$.

[1] If there is no ambiguity, we may simply refer to a complex linear mapping as a linear mapping.

$\mathbb{R}^2$-differentiable functions. Let $U \subset \mathbb{R}^2$ be an open subset, and let $\mathrm{x}_0 \in U$. Recall that a function $f : U \to \mathbb{R}^2$ is $\mathbb{R}^2$-differentiable at x_0 if there exists an $\mathbb{R}^2$-linear mapping T such that

$$\lim_{\substack{\mathrm{h}\to(0,0)\\ \mathrm{h}\neq(0,0)}} \frac{\|f(\mathrm{x}_0+\mathrm{h}) - f(\mathrm{x}_0) - T(\mathrm{h})\|_2}{\|\mathrm{h}\|_2} \quad \text{exists.}$$

Equivalently, we say that f is $\mathbb{R}^2$-differentiable at x_0 if and only if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ for which

$$f(\mathrm{x}_0+\mathrm{h}) = f(\mathrm{x}_0) + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mathrm{h} + o(\|\mathrm{h}\|_2), \quad \text{as } \|\mathrm{h}\|_2 \to 0.$$

By setting $\mathrm{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $f(\mathrm{x}) = \begin{bmatrix} u(\mathrm{x}) \\ v(\mathrm{x}) \end{bmatrix}$, we obtain

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x}(\mathrm{x}_0) & \dfrac{\partial v}{\partial x}(\mathrm{x}_0) \\ \\ \dfrac{\partial u}{\partial y}(\mathrm{x}_0) & \dfrac{\partial v}{\partial y}(\mathrm{x}_0) \end{bmatrix},$$

which is the Jacobean matrix of f at the point x_0.
TO BE CONTINUED!