1. *Evaluate* $\Gamma(-\frac{1}{2}), \quad \Gamma(-\frac{7}{2}), \quad \Gamma(-\frac{1}{3})$.

*Sol.*: From $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and the functional equation $\Gamma(x+1) = x\Gamma(x)$ we have

$$\Gamma(-\tfrac{1}{2}) = -2\Gamma(\tfrac{1}{2}) = -2\sqrt{\pi};$$

$$\Gamma(-\tfrac{7}{2}) = -\tfrac{2}{7}\Gamma(-\tfrac{5}{2}) = \quad \ldots \quad = \tfrac{2^4}{7\cdot 5\cdot 3\cdot 1}\sqrt{\pi};$$

$$\Gamma(-\tfrac{1}{3}) = -3\Gamma(\tfrac{2}{3}) = -3\int_0^\infty e^{-t}t^{-1/3}dt = -4.0623538182792012508358640844635413566\ldots.$$

2. *Let* $n \in \mathbf{N}$. *Show that*

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2^n}\sqrt{\pi}\,.$$

*Sol.*: One has $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. By the functional equation, one has

$$\Gamma(\tfrac{1}{2}+1) = \tfrac{1}{2}\sqrt{\pi}, \qquad \Gamma(\tfrac{1}{2}+2) = \tfrac{3}{2}\Gamma(\tfrac{1}{2}+1) = \tfrac{3}{2}\tfrac{1}{2}\sqrt{\pi} \quad \ldots$$

$$\Gamma(\frac{1}{2}+n) = \prod_{j=1}^{n}(\frac{1}{2}+j)\sqrt{\pi} = \prod_{j=1}^{n}(\frac{1+2j}{2})\sqrt{\pi} = \frac{1\cdot 3\cdot 5\cdot (2n-1)}{2^n}\sqrt{\pi}.$$

3. *Use Wielandt's theorem to prove that for* $Re(z) > 0$ *one has*

$$\Gamma(z) = \frac{1}{g(z)}, \quad \textit{where} \quad g(z) = ze^{\gamma z}\prod_{k=1}^{\infty}\left((1+\frac{z}{k})e^{-z/k}\right),$$

*where* $\gamma = \lim_{k\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\ldots-\log k)$ *is the Euler-Mascheroni constant.*

*Sol.*: Let's verify that $F(z) = \frac{1}{g(z)}$ satisfies the assumptions of Wielandt's theorem.
(a) $F(1) = 1$:
Recall that $\prod_{k=1}^{n}(1+\frac{1}{k}) = 2\cdot\frac{3}{2}\cdot\frac{4}{3}\ldots\cdot\frac{n+1}{n} = n+1$. We claim that $g(1) = 1$. Indeed

$$g(1) = \lim_{n\to\infty} 1\cdot e^{1+\frac{1}{2}+\frac{1}{3}+\ldots-\log(n)}\prod_{k=1}^{n}(1+\tfrac{1}{k})e^{\frac{-1}{k}}$$

$$= \lim_{n\to\infty} e^{1+\frac{1}{2}+\frac{1}{3}+\ldots-\log(n)-1-\frac{1}{2}-\frac{1}{3}-\ldots}\prod_{k=1}^{n}(1+\frac{1}{k})$$

$$= \lim_{n\to\infty}\tfrac{n+1}{n} = 1.$$

(b) $F(z+1) = zF(z)$.

The above relation follows from $g(z) = zg(z+1)$.

$$\frac{g(z)}{g(z+1)} = \lim_{n\to\infty} \frac{ze^{(1+\frac{1}{2}+\ldots-\log(n))z}\prod_{k=1}^{n}(1+\frac{z}{k})e^{-z/k}}{(z+1)e^{(1+\frac{1}{2}+\ldots-\log(n))(z+1)}\prod_{k=1}^{n}(1+\frac{(z+1)}{k})e^{-(z+1)/k}}$$

$$= \lim_{n\to\infty} \frac{z}{z+1} \frac{e^{(1+\frac{1}{2}+\ldots-\log(n)-1-\frac{1}{2}-\ldots)z}\prod_{k=1}^{n}(1+\frac{z}{k})}{e^{(1+\frac{1}{2}+\ldots-\log(n)-1-\frac{1}{2}-\ldots)(z+1)}\prod_{k=1}^{n}(1+\frac{(z+1)}{k})}$$

$$= \lim_{n\to\infty} \frac{z}{z+1} \frac{e^{-\log(n)z}}{e^{-\log(n)(z+1)}} \prod_{k=1}^{n}(1+\frac{z}{k})\left(1+\frac{(z+1)}{k}\right)^{-1}$$

$$= \lim_{n\to\infty} \frac{z}{z+1}\, n \prod_{k=1}^{n} \frac{k+z}{k} \frac{k}{k+1+z} = \lim_{n\to\infty} n \frac{z}{z+1} \frac{1+z}{n+1+z}$$

$$= \lim_{n\to\infty} \frac{zn}{n+1+z} = z.$$

(c) $F$ is bounded on the strip $S_{1,2} = \{1 \le Re(z) \le 2\}$.
We show that $g$ is bounded by below on $S_{1,2}$.

4. *Use Wielandt's theorem to prove the Legendre duplication formula*

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}).$$

*(hint: let $w = 2z$ and consider the function $f(w) = 2^w\Gamma(\frac{w}{2})\Gamma(\frac{w}{2}+\frac{1}{2})$).*

*Sol.*: Let's verify that $f(w)$ satisfies the assumptions of Wielandt's theorem:
(a) $f$ is clearly holomorphic on $\{w \in \mathbf{C} \mid Re(w) > 0\}$, since $\Gamma$ and $2^w$ are;
(b) $f(w+1) = wf(w)$:

$$f(w+1) = 2\cdot 2^w\Gamma(\frac{w}{2}+\frac{1}{2})\Gamma(\frac{w}{2}+1) = 2\cdot 2^w\Gamma(\frac{w}{2}+\frac{1}{2})\frac{w}{2}\Gamma(\frac{w}{2})$$

$$= w2^w\Gamma(\frac{w}{2})\Gamma(\frac{w}{2}+\frac{1}{2}) = wf(w).$$

(c) $f$ is bounded on the vertical strip $S_{1,2} = \{w \in \mathbf{C} \ : \ 1 \le Re(W) \le 2\}$:
The functions $\Gamma(\frac{w}{2})$ and $\Gamma(\frac{w}{2}+\frac{1}{2})$ are bounded on $S_{1,2}$. Since also $|2^w| = |e^{w\log(2)}| = e^{Re(w)\log(2)}$ is bounded on $S_{1,2}$, the same is true for $f$.

By Wielandt's theorem $f(w) = f(1)\Gamma(w)$. Since $f(1) = 2\Gamma(\frac{1}{2})\Gamma(\frac{w}{2}+\frac{1}{2}) = 2\sqrt{\pi}$, one obtains

$$2^w\Gamma(\frac{w}{2})\Gamma(\frac{w}{2}+\frac{1}{2}) = 2\sqrt{\pi}\Gamma(w) \quad \Leftrightarrow \quad \Gamma(2z) = \frac{1}{\sqrt{\pi}}2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}).$$

5. *Prove that if* $y > 0$*, then*

$$|\Gamma(iy)| = \sqrt{\frac{\pi}{y \sinh \pi y}}.$$

*Sol.*: Directly from the definition of $\Gamma$ we see that, for $y > 0$,

$$\Gamma(-iy) = \int_0^\infty e^{-t} e^{-\log(t)} \left(\cos(y \log(t)) - i \sin(y \log(t))\right) dt$$

and

$$\Gamma(iy) = \int_0^\infty e^{-t} e^{-\log(t)} \left(\cos(y \log(t)) + i \sin(y \log(t))\right) dt.$$

In other words, $\Gamma(-iy) = \overline{\Gamma(iy)}$. Now

$$\Gamma(iy)\Gamma(1 - iy) = \Gamma(iy)(-iy)\Gamma(-iy) = (-iy)|\Gamma(iy)|^2 = \frac{\pi}{\sin(\pi i y)} = \frac{\pi}{i \sinh(\pi y)},$$

from which we deduce the statement

$$|\Gamma(iy)| = \sqrt{\frac{\pi}{y \sinh \pi y}}.$$

6. *Recall that* $\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$*, for* $s \in \mathbf{C}$ *with* $Re(s) > 1$*. Compute the logarithmic derivative of* $\zeta$ *(justify the steps).*

*Sol.*: Let $D$ be a domain in $\mathbf{C}$. If an infinite product $\prod_n f_n(z)$ converges normally on compact sets of $D$ to a holomorphic function $f(z)$, then the series $\sum_n \frac{f_n'(z)}{f_n(z)}$ converges normally on compact sets of $D$ to the logarithmic derivative $\frac{f'(z)}{f(z)}$ of $f$ (see Cartan, V.3.2, Thm.2 ).

The infinite product $\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$, converges normally for $s \in \mathbf{C}$ with $Re(s) > 1$. Hence the logarithmic derivative of $\zeta$ is given by

$$\log(\zeta(s))' = \frac{\zeta(s)'}{\zeta(s)} = \sum_p \frac{f_p(s)'}{f_p(s)},$$

where $f_p(s) = \frac{1}{1 - p^{-s}}$ and $f_p(s)' = -\frac{p^{-s}\log(p)}{(1 - p^{-s})^2}$. Hence

$$\log(\zeta(s))' = -\sum_p \frac{p^{-s}\log(p)}{1 - p^{-s}}.$$

7. *Let* $2, 3, 5, 7, \ldots$ *be the series of prime numbers. Prove that*

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{5^2})(1 - \frac{1}{7^2}) \ldots = \frac{6}{\pi^2}.$$

*Sol.*: One has

$$\prod_{p\,\mathrm{prime}} 1-\frac{1}{p^2} = \prod_p \frac{1}{\frac{1}{1-\frac{1}{p^2}}} = \frac{1}{\prod_p \frac{1}{1-\frac{1}{p^2}}} = \frac{1}{\zeta(2)} = \frac{1}{\sum_{n\geq 1}\frac{1}{n^2}} = \frac{6}{\pi^2},$$

since $\sum_{n\geq 1}\frac{1}{n^2} = \pi^2/6$.

8. *Verify that* $\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos(\frac{\pi s}{2})\zeta(s)$.

*Sol.*: The above relation is also equivalent to

$$\zeta(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$$

(just substitute $s$ with $1-s$). Recall that the function

$$Z(s) := \Gamma(s/2)\,\pi^{-s/2}\,\zeta(s)$$

satisfies the functional equation $Z(s) = Z(1-s)$. This implies

$$\Gamma(s/2)\,\pi^{-s/2}\,\zeta(s) = \Gamma(\frac{1-s}{2})\,\pi^{-\frac{1-s}{2}}\,\zeta(1-s)$$

and

$$\zeta(s) = \frac{\Gamma(\frac{1-s}{2})}{\Gamma(s/2)}\pi^{s-\frac{1}{2}}\zeta(1-s). \qquad (*)$$

From Legendre's duplication formula (cf. Exercise 4) applied to $\frac{1-s}{2}$, namely

$$\Gamma(1-s) = \frac{1}{\sqrt{\pi}}\,2^{-s}\,\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2}),$$

we obtain

$$\Gamma(\frac{1-s}{2}) = \frac{\Gamma(1-s)}{\Gamma(1-\frac{s}{2})}\sqrt{\pi}\,2^s.$$

Recall that $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$. Then

$$\zeta(s) = \frac{\Gamma(1-s)}{\Gamma(1-\frac{s}{2})}\frac{1}{\Gamma(s/2)}\sqrt{\pi}\,2^s\,\pi^{s-\frac{1}{2}}\,\zeta(1-s)$$

$$= \Gamma(1-s)\frac{\sin(\frac{\pi s}{2})}{\pi}\,2^s\,\pi^s\,\zeta(1-s) = \Gamma(1-s)\sin(\frac{\pi s}{2})\,2^s\,\pi^{s-1}\,\zeta(1-s).$$

9. *Using the analytic continuation given by the formula of the previous exercise, prove that* $\zeta(-1) = -\frac{1}{12}$ *and that* $\zeta(-3) = \frac{1}{120}$.

*Sol.*: By applying the formula

$$\zeta(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$$

for $s = -1$, we get

$$\zeta(-1) = \frac{1}{2}\frac{1}{\pi^2}\sin(-\frac{\pi}{2})\Gamma(2)\zeta(2) = -\frac{1}{2}\frac{1}{\pi^2}\frac{\pi^2}{6} = -\frac{1}{12}.$$

For $s = -3$, we get

$$\zeta(-3) = \frac{1}{2^3}\frac{1}{\pi^4}\sin(-\frac{3\pi}{2})\Gamma(4)\zeta(4) = -\frac{1}{8}\frac{1}{\pi^4}\,3!\,\frac{16\pi^4}{2\cdot 24}B_4 = \frac{-1}{24}\cdot 6\cdot\frac{-1}{30} = \frac{1}{120}.$$

We used that on positive even integers $m$

$$\zeta(m) = -\frac{(2\pi i)^m}{2\cdot m!}B_m,$$

where $B_m$ is the $m^{th}$ Bernoulli number.

10. *Compute $\zeta(m)$, where $m$ is a negative integer.*

*Sol.*: Let $m = -2k+1$, with $k \geq 1$, be a negative odd integer. Then $1-m = 2k$ is a positive even integer. We recover the formula $\zeta(-2k+1) = -\frac{B_{2k}}{2k}$ using the functional equation of Exercise 8:

$$\zeta(-2k+1) = 2^{-2k+1}\pi^{-2k}\sin(\frac{\pi}{2}(-2k+1))\Gamma(2k)\zeta(2k)$$

$$= 2^{-2k+1}\pi^{-2k}\,\cos(-k\pi)\,(2k-1)!\,\zeta(2k)$$

$$= 2^{-2k+1}\pi^{-2k}\,\cos(-k\pi)\,(2k-1)!\,\cdot -\frac{2\pi i)^{2k}}{2\cdot(2k)!}B_{2k}$$

$$= \cos(-k\pi)\,i^{2k}\,(2k-1)!\,\cdot -\frac{B_{2k}}{(2k)!} = -\frac{B_{2k}}{2k}.$$

Let $m$ be a negative even integer. We are going to show that $\zeta(m) = 0$. These are the so called "trivial zeros" of the $\zeta$ function. Recall that
- $\zeta(s) \neq 0$ for $Re(s) > 1$, by Euler's product formula;
- $\Gamma(s) \neq 0$ for all $s \in \mathbf{C}$ and $\Gamma$ has poles at the negative integers;
- the function $Z(s) := \Gamma(s/2)\,\pi^{-s/2}\,\zeta(s)$ satisfies the functional equation $Z(s) = Z(1-s)$.

It follows that $Z(s) \neq 0$ for $Re(s) > 1$ and, by the functional equation, $Z(s) \neq 0$ for $Re(s) < -1$. Then, for $Re(s) < -1$, one has that $\zeta(s) = 0$ if and only if $s/2$ is a pole of $\Gamma$, namely if and only if $s$ is an even negative integer.