

1. (a) Show that \mathbf{C}^* and Δ^* are not biholomorphic;
- (b) Show that Δ^* is not biholomorphic to an annulus $A = \{z \in \mathbf{C} \mid 0 < r < |z| < 1\}$.

If you are curious, a proof that two annuli $A_{r,1} = \{0 < r < |z| < 1\}$ and $A_{s,1} = \{0 < s < |z| < 1\}$ are biholomorphic if and only if $r = s$ can be found in Greene-Krantz, Thm.7.9.1, p.240.

Sol.: (a) Suppose that there exists a biholomorphism $f: \mathbf{C}^* \rightarrow \Delta^*$. Then $f \circ \exp: \mathbf{C} \rightarrow \Delta^*$ would define a bounded entire holomorphic map. This is impossible by Liouville's theorem.

(b) Suppose that there exists a biholomorphism $f: \Delta^* \rightarrow A$. By the same arguments used in the previous case, $z = 0$ is necessarily a removable singularity for f and f extends to a holomorphic map $\tilde{f}: \Delta \rightarrow A$. On the other hand, by topological reasons (Δ is simply connected, while A is not), the image $\tilde{f}(\Delta)$ is properly contained in A . Contradiction.

Alternatively one could use the fact that the automorphism groups of $Aut(\mathbf{C}^*)$, $Aut(\Delta^*)$ and $Aut(A)$ are non-isomorphic.

- $Aut(\mathbf{C}^*) = \{z \mapsto az, a \in \mathbf{C}^*\} \cup \{z \mapsto a/z, a \in \mathbf{C}^*\}$.

Let $f: \mathbf{C}^* \rightarrow \mathbf{C}^*$ be a biholomorphism. Then neither $z = 0$ nor ∞ can be essential singularities of f , otherwise injectivity would fail by Casorati-Weierstrass' theorem.

If $z = 0$ is a removable singularity, then $0 \mapsto 0$ and f extends to an automorphism of \mathbf{C} . Namely, it is a degree 1 polynomial of the form $f(z) = az$, with $a \in \mathbf{C}^*$.

If $z = 0$ is a pole, then $\lim_{z \rightarrow 0} |f(z)| = \infty$. In this case $1/f$ extends to an automorphism on \mathbf{C} mapping 0 to 0. Namely $\frac{1}{f(z)} = az$, for $a \in \mathbf{C}^*$. Equivalently $f(z) = a/z$, for $a \in \mathbf{C}^*$.

Alternatively: Since f has no essential singularities, then either 0 and ∞ are both removable singularities or they are both poles. In the first case, f extends to an automorphism of the Riemann sphere $\tilde{f}: S^2 \rightarrow S^2$ mapping $0 \mapsto 0$ and $\infty \mapsto \infty$. Namely $f(z) = az$, with $a \in \mathbf{C}^*$. In the second case, f extends to an automorphism of the Riemann sphere mapping $0 \mapsto \infty$ and $\infty \mapsto 0$. Namely, $f(z) = a/z$, with $a \in \mathbf{C}^*$.

- $Aut(\Delta^*) = \{z \mapsto e^{i\theta}z, \theta \in \mathbf{R}\}$.

Let $f: \Delta^* \rightarrow \Delta^*$ be a biholomorphism. Then $z = 0$ is necessarily a removable singularity of f and f extends to an automorphism of Δ , mapping $0 \mapsto 0$. Hence $f(z) = e^{i\theta}z$, with $\theta \in \mathbf{R}$.

- $Aut(A)$ contains $\{z \mapsto e^{i\theta}z, \theta \in \mathbf{R}\} \cup \{z \mapsto e^{i\theta}r/z\}$.

Infinite series of holomorphic and meromorphic functions.

2. Let D be a domain in \mathbf{C} . Prove that if the series of holomorphic functions $\sum_k g_k(z)$ converges normally on $A \subset D$, then it converges uniformly on $A \subset D$.

Sol.: By assumption the series converges normally, i.e. $\sum_k \|g_k\|_A < \infty$, where $\|g_k\|_A = \sup_{z \in A} |g_k(z)|$. Then for all $z \in A$ and $N > M$ one has

$$\left| \sum_{k=1}^N g_k(z) - \sum_{k=1}^M g_k(z) \right| \leq \sum_{k=M}^N |g_k(z)| \leq \sum_{k=M}^N \sup_{z \in A} |g_k(z)| \leq \sum_{k=M}^N \|g_k\|_A \rightarrow 0, \quad \text{for } N, M \rightarrow \infty.$$

Hence the sequence of the partial sums of the series is uniformly Cauchy and the series converges uniformly on $A \subset D$.

3. Show that the series $\sum_{n \geq 1} \frac{z^n}{n^2}$ converges uniformly on the unit disc.

Sol.: For all $z \in \Delta$ one has

$$\left| \frac{z^n}{n^2} \right| \leq \frac{1}{n^2}.$$

Hence

$$\sum_{n \geq 1} \left| \frac{z^n}{n^2} \right| \leq \sum_{n \geq 1} \frac{1}{n^2} < \infty,$$

which implies that the series $\sum_{n \geq 1} \frac{z^n}{n^2}$ converges absolutely and uniformly on Δ .

4. (a) Show that $\frac{1}{(1-z)^2} = \sum_{k=1}^{\infty} k z^{k-1}$, for $z \in \Delta$, and that $4 = \sum_k k \frac{1}{2^{k-1}}$.

(b) Show that $\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$, for $z \in \Delta$.

Sol.: (a) We know that the geometric series $\sum_{k \geq 0} z^k$ converges uniformly on compact sets in Δ to the sum $\frac{1}{1-z}$. Then, by Weierstarss theorem, the uniform holds true also for the derivatives:

$$\left(\sum_{k \geq 0} z^k \right)' = \sum_{k \geq 1} k z^{k-1} \longrightarrow \left(\frac{1}{1-z} \right)' = \frac{1}{(1-z)^2}.$$

In particular, for $z = \frac{1}{2}$, we obtain the identity

$$\sum_{k \geq 1} k \frac{1}{2^{k-1}} = \frac{1}{(1-\frac{1}{2})^2} = 4.$$

(b) By Exercises 5 and 6 in Sheet 3, the uniform convergence holds true for the primitives with value 0 at $z = 0$:

$$\int \frac{1}{1-z} = -\log(1-z) = \int \sum_{k \geq 0} z^k = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

and

$$\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}.$$

5. Show that $\sum_{n=1}^{\infty} \frac{1}{(z+n)^2}$ defines a meromorphic function on \mathbf{C} . Determine its poles and their orders.

Sol.: The n^{th} term of the series $f_n(z) = \frac{1}{(z+n)^2}$ on \mathbf{C} is a meromorphic function on \mathbf{C} with a pole of order 2 at $z = -n$.

Fix $R > 0$. Then for $n > R$ the functions f_n are holomorphic on $\{z \in \mathbf{C} : |z| \leq R\}$. Since

$$\left| \frac{1}{(z+n)^2} \right| \leq \frac{1}{|n-|z||^2} \leq \frac{1}{|n-R|^2},$$

the series

$$\left| \sum_{n=R}^{\infty} \frac{1}{(z+n)^2} \right| \leq \sum_{n=R}^{\infty} \frac{1}{|n-R|^2}$$

converges uniformly to a holomorphic function on $D(0, R)$ and $\sum_{n \geq 1}^{\infty} f_n(z)$ defines a meromorphic function therein. By taking R larger and larger, we conclude that the original series converges locally uniformly and defines a meromorphic function on \mathbf{C} with poles of order 2 at all integers $n \geq 1$. This last statement follows from the fact that the poles of the functions f_n are pairwise disjoint.

6. Show that $\sum_{n=1}^{\infty} \frac{1}{z^2+n^2}$ defines a meromorphic function on \mathbf{C} . Determine its poles and their orders.

Sol.: The n^{th} term of the series $f_n(z) = \frac{1}{z^2+n^2}$ is a meromorphic function on \mathbf{C} with poles of order 1 at $\pm in$, with residue $\frac{1}{2in}$ and $\frac{-1}{2in}$, respectively.

Fix $R > 0$. Then for all $n > R$, the functions f_n are holomorphic on $D(0, R)$. In addition, one has

$$\left| \frac{1}{z^2+n^2} \right| \leq \frac{1}{|R^2-n^2|},$$

which implies that the series $\sum_{n>R} \frac{1}{z^2+n^2}$ converges uniformly to a holomorphic function on $D(0, R)$. It follows that the series $\sum_{n \geq 1} \frac{1}{z^2+n^2}$ converges uniformly to a meromorphic function on $D(0, R)$. By taking R larger and larger, we conclude that the original series converges locally uniformly and defines a meromorphic function on \mathbf{C} with poles of order 1 at $\pm in$, for all $n \geq 1$. This last statement follows from the fact that the poles of the functions f_n are pairwise disjoint.

7. Show that $g(z) = \left(\frac{\pi}{\sin \pi z}\right)^2$ converges uniformly to 0 for $|Im(z)| \rightarrow \infty$.

Sol.: For $z = x + iy$, one has the following estimate, independent of x :

$$\left| \frac{\pi}{\sin \pi z} \right|^2 = \frac{\pi^2}{\sin^2 \pi x \cosh^2 \pi y + \cos^2 \pi x \sinh^2 \pi y} = \frac{\pi^2}{\sin^2 \pi x + \sinh^2 \pi y} \leq \frac{\pi^2}{\sinh^2 \pi y}.$$

Hence

$$\lim_{|y| \rightarrow \infty} \left| \frac{\pi}{\sin \pi z} \right|^2 = 0,$$

uniformly with respect to x .

8. Compute $\lim_{z \rightarrow 0} \left(\frac{\pi}{\sin \pi z}\right)^2 - \frac{1}{z^2}$ and $\lim_{z \rightarrow 0} \sum_{n \in \mathbf{Z}} \frac{1}{(z-n)^2} - \frac{1}{z^2}$. Deduce that $\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol.: From the Taylor expansion $\sin \pi z = \pi z - \frac{1}{6}\pi^3 z^3 + \dots$ we obtain

$$\left(\frac{\pi}{\sin \pi z}\right)^2 = \left(\frac{\pi}{\pi z - \frac{1}{6}\pi^3 z^3 + \dots}\right)^2 = \frac{1}{z^2} \left(\frac{1}{1 - \frac{1}{6}\pi^2 z^2 + \dots}\right)^2 = \frac{1}{z^2} + \frac{\pi^2}{3} + z^2(\dots)$$

and

$$\lim_{z \rightarrow 0} \left(\frac{\pi}{\sin \pi z}\right)^2 - \frac{1}{z^2} = \frac{\pi^2}{3}.$$

On the other hand, from the identity $\left(\frac{\pi}{\sin \pi z}\right)^2 = \sum_{n \in \mathbf{Z}} \frac{1}{(z-n)^2}$ and the fact that the function $\sum_{n \in \mathbf{Z}} \frac{1}{(z-n)^2} - \frac{1}{z^2}$ is holomorphic around 0, we deduce

$$\lim_{z \rightarrow 0} \sum_{n \in \mathbf{Z}} \frac{1}{(z-n)^2} - \frac{1}{z^2} = 2 \lim_{z \rightarrow 0} \sum_{n \geq 1} \frac{1}{(z-n)^2} = \sum_{n \geq 1} \frac{1}{n^2}.$$

Conclusion:

$$\frac{\pi^2}{3} = 2 \sum_{n \geq 1} \frac{1}{n^2} \quad \text{and} \quad \sum_{n \geq 1} \frac{1}{n^2} = \pi^2/6.$$

9. Show that $\sum_{n \in \mathbf{Z}} \frac{1}{(z^3 - n^3)}$ converges normally to a meromorphic function. Locate the poles and find the corresponding principal parts of the function.

Sol.: For $n \in \mathbf{Z}_{\neq 0}$, the n^{th} term of the series $f_n(z) = \frac{1}{z^3 - n^3}$ is a meromorphic function on \mathbf{C} with poles of order 1 at $n, n\alpha, n\bar{\alpha}$, where α is a third root of unity, and principal parts given by

$$\frac{a}{z - n}, \quad \frac{b}{z - n\alpha}, \quad \frac{c}{z - n\bar{\alpha}}, \quad \text{for some } a, b, c \in \mathbf{C},$$

respectively. For $n = 0$, the function $f_0(z) = \frac{1}{z^3}$ has a pole of order 3 at $z = 0$, with principal part equal to $\frac{1}{z^3}$. Now the arguments are similar to the ones used in Exercises 5 and 6. Fix $R > 0$. Then for all $|n| > R$, the functions f_n are holomorphic on $D(0, R)$ and on $D(0, R)$ the following estimate holds

$$\left| \frac{1}{z^3 - n^3} \right| \leq \frac{1}{||n|^3 - R^3|}.$$

Hence the series $\sum_{|n| > R} \frac{1}{z^3 - n^3}$ converges uniformly to a holomorphic function on $D(0, R)$ and the series $\sum_{n \in \mathbf{Z}} \frac{1}{z^3 - n^3}$ converges uniformly to a meromorphic function on $D(0, R)$. We conclude that the original series converges locally uniformly and defines a meromorphic function f on \mathbf{C} with poles of order 1 at $n, n\alpha, n\bar{\alpha}$, for all $n \neq 0$ and a pole of order 3 at $z = 0$. The principal parts of f at the poles coincide with the principal parts of the functions f_n .

10. Find a meromorphic function $f: \mathbf{C} \rightarrow \mathbf{C}$ with simple poles at the positive integers with residue 1.

Sol.: We construct f as the sum of a converging series of meromorphic functions f_n , for $n \geq 1$, each with a simple pole of residue 1 at n . Since the series $\sum_{n \geq 1} \frac{1}{z - n}$ does not converge, we take

$$\sum_{n \geq 1} f_n(z), \quad \text{with } f_n(z) = \frac{1}{z - n} + \frac{1}{n}.$$

Next we show that the above series converges locally uniformly on \mathbf{C} to a meromorphic function with the required properties.

Fix $R > 0$. For all $n > R$, the functions f_n are holomorphic on the disk $D(0, R)$. Moreover, on $D(0, R)$ one has

$$\left| \frac{z}{n(z - n)} \right| \leq \left| \frac{R}{n^2 - nR} \right|,$$

which implies that $\sum_{n > R} f_n(z)$ converges uniformly to a holomorphic function on $D(0, R)$ and $\sum_{n \geq 1} f_n(z)$ converges uniformly to a meromorphic function on $D(0, R)$. By taking R larger and larger, we conclude that the original series defines a meromorphic function on \mathbf{C} with poles of order 1 and residue 1 at all integers $n \geq 1$.

Infinite products of holomorphic functions.

11. Prove that the infinite product $\prod_j a_j$, where $a_j = \frac{1}{2}$ for every j , does not converge.

Sol.: The partial products $\prod_{j=1}^n \frac{1}{2} = \frac{1}{2^n}$ tend to 0, for $n \rightarrow \infty$. This contradicts the definition of a converging infinite product, for which the above limit of the partial products has to be $\neq 0$.

12. Prove that $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.

Sol.: One has

$$\begin{aligned} \prod_{n=2}^N \left(1 - \frac{1}{n^2}\right) &= \prod_{n=2}^N \frac{n+1}{n} \cdot \prod_{n=2}^N \frac{n-1}{n} = \frac{3}{2} \frac{4}{3} \cdots \frac{N}{N-1} \frac{N+1}{N} \cdot \frac{1}{2} \frac{2}{3} \cdots \frac{N-1}{N} \\ &= \frac{1}{2} \frac{N+1}{N} \rightarrow \frac{1}{2}, \quad \text{for } N \rightarrow \infty. \end{aligned}$$

13. Prove that for $|z| < 1$ one has

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\dots = \frac{1}{1-z}.$$

Sol.: One has

$$(1+z)(1+z^2) = 1+z+z^2+z^3, \quad (1+z)(1+z^2)(1+z^4) = 1+z+z^2+z^3+z^4+z^5+z^6+z^7, \dots$$

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\dots(1+z^{2^k}) = 1+z+z^2+z^3+z^4+\dots+z^{1+2+\dots+2^k}.$$

Hence for $k \rightarrow \infty$ we get the sum of the geometric series

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\dots(1+z^{2^k}) \longrightarrow \frac{1}{1-z}.$$

14. Prove that

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges absolutely and uniformly on every compact set.

Sol.: Let $a_n = -n$ and $p_n \equiv 1$. Since

$$\sum_{n=1}^{\infty} \left(\frac{R}{n}\right)^2 < \infty \quad \forall R > 0,$$

by the convergence criterion of Thm.8.2.2 in Greene-Krantz, the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{z/n}$$

converges absolutely and uniformly on all \mathbf{C} . It defines an entire holomorphic function with simple zeros at negative integers $n \in \mathbf{Z}_{<0}$, and no other zeros.

15. *Construct an entire function on \mathbf{C} with simple zeros at n^2 , for $n \geq 0$, and no other zeros.*

Sol.: Let $a_n = n^2$ and $p_n \equiv 0$. Since

$$\sum_{n=1}^{\infty} \frac{R}{n^2} < \infty, \quad \forall R > 0,$$

by the convergence criterion of Thm.8.2.2 in Greene-Krantz, the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right)$$

converges absolutely and uniformly on all \mathbf{C} . Then

$$z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right)$$

defines an entire holomorphic function with simple zeros at n^2 , for $n \geq 0$, and no other zeros.

16. *Construct an entire function that has simple zeros on the positive real axis at the points \sqrt{n} , for $n \geq 1$, double zeros on the imaginary axis at the points $\pm i\sqrt{n}$, for $n \geq 1$, and no other zeros.*

Sol.: An entire function that has simple zeros on the positive real axis at the points \sqrt{n} , for $n \geq 1$ is given by

$$f(z) = \prod_{n \geq 1} \left(1 - \frac{z}{\sqrt{n}}\right) e^{\frac{z}{\sqrt{n}} + \frac{1}{2}\left(\frac{z}{\sqrt{n}}\right)^2 + \frac{1}{3}\left(\frac{z}{\sqrt{n}}\right)^3}.$$

This can be seen by applying the convergence criterion of Thm.8.2.2 in Greene-Krantz with $a_n = \sqrt{n}$ and $p_n = 3$ for all n .

An entire function that has double zeros on the imaginary axis at the points $\pm i\sqrt{n}$, for $n \geq 1$ is given by

$$\begin{aligned} g(z) &= \prod_{n \geq 1} \left(\left(1 - \frac{z}{i\sqrt{n}}\right) e^{\frac{z}{i\sqrt{n}} + \frac{1}{2}\left(\frac{z}{i\sqrt{n}}\right)^2 + \frac{1}{3}\left(\frac{z}{i\sqrt{n}}\right)^3} \right)^2 \left(\left(1 + \frac{z}{i\sqrt{n}}\right) e^{-\left(\frac{z}{i\sqrt{n}} + \frac{1}{2}\left(\frac{z}{i\sqrt{n}}\right)^2 + \frac{1}{3}\left(\frac{z}{i\sqrt{n}}\right)^3}\right) \right)^2 \\ &= \prod_{n \geq 1} \left(1 + \frac{z^2}{n}\right)^2. \end{aligned}$$

This can be seen by applying the convergence criterion of Thm.8.2.2 in Greene-Krantz with $a_n = \pm i\sqrt{n}$ and $p_n = 3$ for all n .

Now the function

$$F(z) := f(z)g(z)$$

is an entire function with the required zeros, and no other zeros.