Uniform convergence on compact sets

1. *Show that the sequence* $f_n(z) = \frac{1-z^{n+1}}{1-z}$ *converges pointwise to* $\frac{1}{1-z}$, *for* $z \in \Delta$. *Show that the convergence is locally uniform but not uniform on* Δ.

Sol.: For every $z_0 \in \Delta$, the sequence converges pointwise: $f_n(z_0) \to \frac{1}{1-z_0}$.
The convergence is locally uniform: for all z with $|z| \leq r < 1$

$$\left|\frac{1-z^{n+1}}{1-z} - \frac{1}{1-z}\right| = \frac{|z|^{n+1}}{|1-z|} \leq \frac{|z|^{n+1}}{|1-|z||} \leq \frac{r^{n+1}}{r-1} \to 0.$$

The convergence is not uniform on Δ: for n_0 fixed,

$$\lim_{\substack{x\to 1\\ x\in\mathbf{R}}} \frac{x^{n_0+1}}{|1-x|} = \infty.$$

Therefore, in order to make $\frac{|z|^{n+1}}{|1-z|}$ small one needs to choose n_0 bigger and bigger as $|z| \to 1$.

2. *Consider the following sequences of functions* $\mathbf{C} \to \mathbf{C}$

$$f_n(z) = \frac{z}{n}, \qquad g_n(z) = z^n, \qquad h_n(z) = \frac{1}{n^z}.$$

For each sequence:
(a) *Find the pointwise limit;*
(b) *Find a set* A *where the convergence is uniform;*
(c) *Find a set* U *where the convergence is locally uniform;*
(d) *Determine whether the convergence is uniform on* U *or not.*

Sol.: (a) The sequence converges pointwise to the function identically zero: in fact for all $z_0 \in \mathbf{C}$, one has $\lim_{n\to\infty} \frac{z_0}{n} = 0$.
The convergence is locally uniform: for $|z| \leq R$ one has

$$\frac{|z|}{n} \leq \frac{R}{n} \to 0.$$

The convergence is not uniform on $\mathbf{C}$: the above estimate cannot be extended uniformly to all $\mathbf{C}$.
(b) Pointwise, the sequence g_n behaves as follows

$$g_n(z_0) \to \begin{cases} 0, & |z_0| < 1; \\ \infty, & |z_0| > 1; \\ \text{may have no limit}, & |z_0| = 1. \end{cases}$$

On the unit disk Δ, the convergence is locally uniform, but not uniform : for $|z| \leq r < 1$

$$|z|^n \leq r^n \to 0.$$

(c) One has $\left|\frac{1}{n^z}\right| = \frac{1}{n^x}$, where $z = x + iy$. Pointwise the sequence h_n behaves as follows

$$h_n(z_0) \to \begin{cases} 0, & x_0 > 0; \\ 1, & x_0 = 0; \\ +\infty, & x_0 < 0. \end{cases}$$

Hence h_n converges pointwise to 0 on $\{Re(z) > 0\}$. There the convergence is locally uniform but not uniform. On the set $\{Re(z) > r > 0\}$, one has

$$\left|\frac{1}{n^z}\right| = \frac{1}{n^x} \leq \frac{1}{n^r} \to 0.$$

3. *Let $\{p_n\}_{n\in\mathbf{N}}$ be a sequence of polynomials of degree $\leq N$, where N is a fixed positive integer. Show that if $p_n \to p$ uniformly on compact sets, then the limit function p is a polynomial of degree $\leq N$.*

Sol.: A function $p(z)$ is a polynomial of degree $\leq N$ if and only if the k^{th} derivative $p^{(k)}(z)$ is identically 0, for all $k > N$. If the polynomials p_n converge to p uniformly on compact sets, then they converge together with all the derivatives. Let K be a compact set. Then for every $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that for all $n > n_0$

$$|p_n^{(k)}(z) - p^{(k)}(z)| < \epsilon, \qquad \forall n > n_0, \quad \forall z \in K.$$

It follows that $p^{(k)}(z) \equiv 0$, for all $k > N$ and $p(z)$ is a polynomial of degree $\leq N$.

4. *Let D be a domain and let $f_n: D \to \mathbf{C}$ be a sequence of holomorphic functions converging to a non-constant function f uniformly on compact sets. Show that if f has m zeros in D, then all but finitely many f_n have at least m zeros in D.*

Sol.: Let $z_1, \ldots, z_l$ be the zeros of $f \not\equiv 0$. By a reformulation of Hurwitz theorem, for all $j = 1, \ldots, l$ there exists n_j such that $f_n(z_j) = 0$, for all $n > n_j$. Then for all $n > m = max\{n_1, \ldots, n_l\}$ one has $f_n(z_j) = 0$. Hence the number of zeros of f_n is $\geq m$, for all $n > m$.
Remark. Let z_0 be a zero of a non-constant function f. Then there exists a disk $D(z_0, r)$ such that $f \neq 0$ on $\overline{D(z_0, r)} \setminus \{z_0\}$. If a sequence f_n converges locally uniformly to f, then it converges pointwise to f and for n sufficiently large, $f_n \neq 0$ on $\overline{D(z_0, r)} \setminus \{z_0\}$. For a contour $\gamma \subset \overline{D(z_0, r)} \setminus \{z_0\}$ one has

$$\frac{1}{2\pi i}\int_\gamma \frac{f_n'(\zeta)}{f_n(\zeta)} d\zeta \to \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta = p \in \mathbf{N},$$

where p is the multiplicity of z_0. Therefore also $\frac{1}{2\pi i}\int_\gamma \frac{f_n'(\zeta)}{f_n(\zeta)} d\zeta = p$.

5. *Let $g: D(0, \rho) \to \mathbf{C}$ be holomorphic. Let $G: D(0, \rho) \to \mathbf{C}$ be such that $G' = g$ and $G(0) = 0$. Show that for all $z \in D(0, \rho)$ one has*

$$|G(z)| \leq |z| \sup\{|g(w)| \,:\, |w| \leq |z|\}.$$

Deduce that $\|G\|_{D(0,\rho)} \leq \rho \|G\|_{D(0,\rho)}$ (suggestion: take $G(z) := \int_{\gamma_z} g(w)dw$, where $\gamma_z(t) = tz$, for $t \in [0, 1]$).

Sol.: As $D(0,r)$ is a convex set, every holomorphic function g on $D(0,r)$ admits a holomorphic primitive (determined up to a constant)

$$G: D(0,r) \to \mathbf{C}, \quad G'(z) = g(z), \quad G(0) = 0.$$

For example one can take

$$G(z) = \int_{\gamma_z} g(w)dw,$$

where $\gamma_z: [0,1] \to D(0,r)$ is the segment $\gamma_z(t) = tz$ joining 0 and z, namely

$$G(z) = \int_0^1 g(tz)zdt.$$

Now from the estimate

$$|G(z)| \leq |z| \sup_{\gamma_z}\{|g(tz)|\} \leq |z| \sup\{|g(w)| \ : \ |w| \leq z\}, \qquad \forall z \in D(0,r),$$

we obtain

$$\|G\|_{D(0,r)} \leq r\|g\|_{D(0,r)}.$$

6. *Let f_n be a sequence of holomorphic functions converging locally uniformly to f on $D(0,R)$. Show that $F_n \to F$ locally uniformly on $D(0,R)$, where $F_n' = f_n$, $F_n(0) = 0$ for all n, and $F' = f$, $F(0) = 0$.*

Sol.: From Exercise 5, for all $r_0 < R$ we have the following estimate on $D(0,r_0)$:

$$\|F\|_{D(0,r_0)} \leq r_0\|f\|_{D(0,r_0)}.$$

In particular

$$\|F_n - F\|_{D(0,r_0)} \leq r_0\|f_n - f\|_{D(0,r_0)}.$$

Hence $\|f_n - f\|_{D(0,r_0)} \to 0$ implies $\|F_n - F\|_{D(0,r_0)} \to 0$, as claimed.

7. *Discuss the convergence and uniform convergence of the sequence $f_n(z) = nz^n$, for $n \in \mathbf{N}$.*

Sol.: Pointwise, the sequence behaves as follows

$$n|z_0|^n \to \infty, \quad \forall\ z_0\ |z_0| \geq 1, \qquad n|z_0|^n \to 0, \quad \forall\ z_0\ |z_0| < 1.$$

On the unit disk Δ, the convergence is locally uniform, but not uniform. In fact, for $|z| \leq r < 1$.

$$n|z|^n \leq nr^n = \frac{n}{\frac{1}{r^n}} \to 0.$$

8. *Prove that the sequence $f_n(z) = \frac{1}{1+nz}$ is uniformly convergent to the function identically 0, for all z such that $|z| > 2$. Can the region of uniform convergence be extended?*

Sol.: For every fixed $n \in \mathbf{N}$, the function f_n is a meromorphic function with a simple pole at $z = -\frac{1}{n}$. Hence all the poles of the sequence are contained in the closed disk $\overline{D(-\frac{1}{2}, \frac{1}{2})}$. Then all the functions f_n are holomorphic on the set $\{z \in \mathbf{C} \ : \ |z| > 2\}$ and on that set the sequence converges pointwise to the function identically 0. Fix $R > \frac{1}{2}$. The estimate

$$|nz + 1| \geq |nR - 1| \to \infty, \quad \text{for } n \to \infty,$$

which is independent of z, shows that the convergence is locally uniform, and therefore uniform on compact sets.

Actually, the sequence converges locally uniformly on $\mathbf{C}^*$. Given an arbitrary fixed compact set K, it is contained in some set of the form $\{z \in \mathbf{C} \ : \ |z| > r\}$, for some $r > 0$. After removing finitely many functions (depending on r), we are left with a family of holomorphic functions on $\{z \in \mathbf{C} \ : \ |z| > r\}$ and estimates similar to the above ones show the local uniform convergence.

The point $z = 0$ cannot be included in the convergence set (not even pointwise) since all functions have value 1 at $z = 0$.

9. *Let $\mathcal{F}$ be the family of all analytic functions on the open unit disc Δ whose coefficients in the Taylor expansion*
$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots,$$
satisfy $|a_n| \leq n$, for each n. Show that $\mathcal{F}$ is relatively compact (with respect to the topology of locally uniform convergence).

Sol.: We need to show that every family of functions f_n in $\mathcal{F}$ admits a subsequence f_{n_j} converging to an element $f_0 \in \mathcal{O}(\Delta)$. By Montel's theorem, it is sufficient to show that $\mathcal{F}$ is equibounded on compact sets (equivalently, locally equibounded). Fix $D(0,r) \subset \Delta$, for $0 < r < 1$. Let $f \in \mathcal{F}$. Then

$$|f(z)| \leq 1 + \sum_{n \geq 1} |a_n| r^n \leq 1 + \sum_{n \geq 1} n r^n, \qquad \forall z, \ |z| \leq r.$$

Since the series $\sum_{n \geq 1} n r^n$ converges, then

$$|f(z)| \leq S, \qquad \forall z, \ |z| \leq r.$$

This shows that the family $\mathcal{F}$ is locally equibounded and therefore equibounded on compact sets.

10. *Let f_n be a sequence of analytic functions on Δ, uniformly bounded. Assume that for each $z \in \Delta$ the sequence $f_n(z)$ converges. Show that f_n converges uniformly on compact subsets of Δ.*

Sol.: We are going to show that the sequence f_n is uniformly Cauchy on discs $D(z_0, r)$ in Δ. This is sufficient since any compact set in Δ can be covered by finitely many disks of a fixed radius.
Let $z \in D(z_0, r)$. Write

$$\begin{aligned} |f_n(z) - f_m(z)| &= |f_n(z) - f_n(z_0) + f_n(z_0) - f_m(z_0) - (f_m(z) - f_m(z_0))| \\ &\leq |f_n(z) - f_n(z_0)| + |f_n(z_0) - f_m(z_0)| + |f_m(z) - f_m(z_0)|. \qquad (*) \end{aligned}$$

Since f_n is uniformly bounded, then it is uniformly continuous (cf. proof of Montel's theorem). Namely if $z, z_0 \in \Delta$, then

$$|f(z) - f(z_0| \leq M|z - z_0|, \qquad \text{for some } M \in \mathbf{R}. \qquad (**)$$

By (**), we may choose r such that the first two summands of (*) are $< \epsilon/3$, for all n; since the sequence is pointwise convergent, we may choose N such that for all $n, m > N$, the third summand is $< \epsilon/3$. As a result, for r sufficiently small and n sufficiently large (depending on ϵ)

$$|f_n(z) - f_m(z)| \leq M|z - z_0| + M|z - z_0| + |f_n(z_0) - f_m(z_0)| < \epsilon.$$

11. *Consider the family* $\mathcal{F} = \{f_n(z) = \frac{z}{n}\}_{n \in \mathbf{N}}$, *defined on* $\mathbf{C}$. *Verify that* $\mathcal{F}$ *is not equibounded on* $\mathbf{C}$, *but it is equibounded on compact sets. Indeed it converges uniformly on compact sets to* $f \equiv 0$.

Sol.: For all $|z| \leq R$ and for all $n \geq 1$ one has

$$\left|\frac{z}{n}\right| \leq R.$$

Since every compact set K is contained in some disk $D(0, R)$, for some $R > 0$, then the family $\mathcal{F}$ is equibounded on compact sets. But it is not equibounded on all $\mathbf{C}$.

By Montel's theorem, every sequence in $\mathcal{F}$ admits a subsequence converging to a function $f \colon \mathbf{C} \to \mathbf{C}$. In this case $\mathcal{F}$ itself converges uniformly on compact sets to $f \equiv 0$.

Some examples for comparing the real and the complex cases.

a. (*Weierstrass' convergence theorem does not hold in the real case*) Consider the sequence of functions $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, defined on $[-1, 1]$. Verify that the functions f_n are smooth, and converge uniformly to a non-smooth function f.

b. (*Hurwitz's theorem does not hold in the real case*) Consider the sequence $f_n : \mathbf{R} \to \mathbf{R}$, defined by $f_n(x) = x^2 + \frac{1}{n}$. Verify that the functions f_n are never 0 on $\mathbf{R}$. Nevertheless they converge uniformly on compact sets to $f(x) = x^2$, which takes the value $f(0) = 0$.

c. (*Montel's theorem does not hold in the real case*) Consider the family $\mathcal{F} = \{f_n(x) = \sin(nx)\}_n$, defined on $[0, 2\pi]$. Verify that $\mathcal{F}$ is equibounded. Nevertheless it does not admit any converging subsequence (not even pointwise).