Exercise 3.

(a) *Let f be a meromorphic function on $\mathbf{C}$. Then $f = g/h$ where g and h are entire functions.*

(b) Let f be a meromorphic function on the Riemann sphere. Then f is rational, that is $f = g/h$ where g and h are polynomials.

Sol.: (a) Let f be a meromorphic function on $\mathbf{C}$. The set of poles of f is a discrete subset of $\mathbf{C}$, possibly infinite. By Weierstrass' factorization theorem, there exists an entire function $h \in \mathcal{O}(\mathbf{C})$ with zeros on the poles of f, with the same order. Namely, if $\{a_j\}_j$ are the poles of f, each with order m_j, then there exist integers p_j so that the infinite product

$$h(z) = z^k \prod_j E_{p_j}(\frac{z}{a_j})^{m_j}$$

converges to an entire function with the assigned zeros. Now the function $g := h \cdot f$ is an entire function and $f = g/h$ is a quotient of entire functions.

Let's check that g is entire. Let a be a pole of f of order m. Locally around a

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \dots.$$

and $h(z) = (z-a)^m \chi(z)$, with $\chi(z) \neq 0$ holomorphic. Then locally around a one has $h(z)f(z) = \chi(z)(a_{-m} + a_{-m+1}z + \dots$ holomorphic.

(b) Let f be a meromorphic function on $S^2 = \mathbf{C} \cup \infty$. Since S^2 is compact, the poles of f are finitely many.

- If the point ∞ is not a pole of f, then there exists a polynomial g with zeros on the poles of f with the same order. It follows that $g \cdot f$ is holomorphic on S^2, and therefore constant. In this case $f = c/g$, with $c \in \mathbf{C}$.

- If the point ∞ is a pole of f, then there exists a polynomial h with zeros on the poles of f different from ∞, with the same order. Then $g = h \cdot f$ is a holomorpohic function on $\mathbf{C} \cong S^2 \setminus \{\infty\}$, with a pole at ∞. Hence g is a polynomial and $f = g/h$ is rational.

Exercise 4. *Show that* $\sum_{n=1}^{\infty} \frac{1}{z^2-n^2}$ *defines a meromorphic function on* $\mathbf{C}$*. Determine its poles, their orders and their residues.*

Sol.: Denote $f_n(z) = \frac{1}{z^2-n^2}$. Then for $n \geq 1$, the function f_n is meromorphic on $\mathbf{C}$ with poles of order 1 at $z = \pm n$.

Fix a $R > 0$. Then for all $n > R$, the functions f_n are holomorphic on $D(0,R)$. Moreover, the estimate

$$|z^2 - n^2| > n^2 - R^2, \quad \text{for all } |z| < R,$$

implies that the series of holomorphic functions

$$\sum_{n>R} |f_n(z)| \leq \sum_{n>R} \frac{1}{n^2 - R^2}$$

converges uniformly on the closed disk $\overline{D(0,R)}$ and

$$\sum_{n \leq R} |f_n(z)| + \sum_{n>R} |f_n(z)|$$

defines a meromorphic function therein. By taking R bigger and bigger, we can conclude that the series $\sum_{n=1}^{\infty} \frac{1}{z^2-n^2}$ defines a meromorphic function f on $\mathbf{C}$.

Since the poles of the functions f_n are pairwise disjoint, the poles of f are $\mathbf{Z} \setminus \{0\}$ and the residue of f at a pole is given by

$$Res_f(n) = Res_{f_n}(n) = \lim_{z \to n} (z-n) f_n(z) = \frac{1}{2n}, \quad n > 0$$

and

$$Res_f(-n) = Res_{f_n}(-n) = \lim_{z \to -n} (z+n) f_n(z) = -\frac{1}{2n}, \quad n > 0.$$