**Exercise 1.** Compute the following integrals (giving details on the steps)

(i)
$$\int_{|z|=2} \frac{1}{1+z^2} dz;$$

(ii)
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx.$$

**Solution 1.i.** By the residue theorem
$$\int_{|z|=2} \frac{1}{1+z^2} dz = 2\pi i (Res_i f + Res_{-i} f)\,,$$
where $f(z) = \frac{1}{1+z^2}$. As $Res_i f = \lim_{z\to i}(z-i)f(z) = \frac{1}{2i}$ and $Res_i f = \lim_{z\to -i}(z+i)f(z) = -\frac{1}{2i}$, one has
$$\int_{|z|=2} \frac{1}{1+z^2} dz = 0.$$
As the ball $\{|z| < 2\}$ contains all poles of the rational function $\frac{1}{1+z^2}$ and the degree of the denumerator is greater than the degree of the numerator plus one, the result can also be obtained by a general argument discussed during the lectures.

Moreover, it can be obtain by applying Cauchy's integral formula twice, as
$$\frac{1}{1+z^2} = \frac{1}{2i}\Big(\frac{1}{z-i} - \frac{1}{z+i}\Big)\,.$$

**Solution 1.i.** A real analyst would notice that
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx = \lim_{r\to\infty} \int_{-r}^{r} \frac{1}{1+x^2} dx = \lim_{r\to\infty} \arctan|_{-r}^{r} = \pi.$$
A compex analyst would apply the sector method, as discussed during the lectures.

**Exercise 2.** Prove two of the following statements.

(i) Let $f\colon D \to \mathbb{C}$ be a holomorphic function. Let $z_0$ be an essential singularity of $f$. Then there exists a sequence $\{z_n\} \subset D$, with $z_n \to z_0$, such that $\lim_n f(z_n) = \infty$.

(ii) Let $f\colon D \to \mathbb{C}$ be a holomorphic function. Let $z_0$ be an essential singularity of $f$. Then there exists a sequence $\{z_n\} \subset D$, with $z_n \to z_0$, such that $\lim_n f(z_n)$ does not exist.

(iii) Let $g : \mathbb{C} \to \mathbb{C}$ be a non-polynomial holomorphic function. Then there exists a sequence $\{z_n\} \subset D$, with $z_n \to \infty$, such that $\lim_n g(z_n)$ does not exist.

**Solution 1.i.** Casorati-Weierstrass theorem may be useful. Picard's theorem more. Indeed it says that for every $\varepsilon$ small enough $\mathbb{C} \setminus \{w_0\} \subseteq f(D^*(z_0, \varepsilon))$, for some $w_0 \in \mathbb{C}$. Hence, by choosing $\varepsilon = 1/n$, for $n$ large enough, we obtain a sequence $z_n \to z_0$ such that $f(z_n) = w_0 + n$. This proves (i).

**Solution 1.ii.** Arguing as in (i), we can choose a sequence $z_n \to z_0$ such that $f(z_{2n}) = w_0 + 1$ and $f(z_{2n+1}) = w_0 + 2$. This proves (ii).

**Solution 1.iii.** Zero is an essential singularity of $f(\zeta) := g(\frac{1}{\zeta})$ (just look at the Laurent series in 0). By (ii) it follows that there exists $\zeta_n \to 0$, such that $\lim_n f(\zeta_n)$ does not exist. Let $z_n := \frac{1}{\zeta_n}$. Then $z_n \to \infty$ and $\lim_n g(z_n)$ does not exists. This proves (iii).