Liouville's theorem, identity principle, maximum modulus principle, harmonic functions, Schwarz's lemma, automorphisms of the unit disk.

1. Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic doubly periodic function (it means that $f(z+\omega_1) = f(z+\omega_2) = f(z)$ for all $z \in \mathbb{C}$, where $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent vectors). Then f is constant.

2. Let $U \subset \mathbb{C}$ be an open neighbourhood of 0. Show that there are no holomorphic functions $f\colon U \to \mathbb{C}$, such that:
   (a) $f(\frac{1}{n}) = (-1)^n \frac{1}{n^2}$,
   (b) $f(\frac{1}{n}) = \frac{1}{2^n}$,
   (c) $|f^{(n)}(0)| > n!\, n^n$.

3. Let $D \subset \mathbb{C}$ be a domain and let $f\colon D \to \mathbb{C}$ be a holomorphic function, not identically zero. Prove that the set of zeros of f in D is at most countable (use: D is a countable union of compact sets).

4. Let $f\colon U \to \mathbb{C}$ be a holomorphic function, where U is an open neighbourhood of the closed unit disc $\overline{\Delta}$. Assume that f is not identically zero.
   (a) Show that f has at most finitely many zeros in Δ.
   (b) Determine the zeros of $f(z) = \sin(\frac{1}{1-z})$ on the disc Δ and compare the result with (a).

5. Let $S = \{z \in \mathbb{C} \ : \ 0 < \mathrm{Re} z < \frac{\pi}{2}\}$. Determine whether there exists a holomorphic function $f\colon S \to \mathbb{C}$ such that
   (a) $\mathrm{Re} f(z) = x^2 y + y^2 x + \sin x \sinh y$;
   (b) $\mathrm{Re} f(z) = y^3 - yx^2 - 2x^2 y + \cos x \cosh y + \sin x \sinh y$.
   Justify your answer: either exhibit one such function or explain why it cannot exist.

6. (*Liouville theorem for harmonic functions*). Let $u\colon \mathbb{C} \to \mathbb{R}$ be harmonic and bounded either from above or from below.
   (a) Show that u is constant.
   (b) Verify that the real and the imaginary parts of the following holomorphic functions are not bounded:
   $$e^z, \qquad \sin z, \qquad \cos z, \qquad z^2.$$

7. Set $D = D(z_0, r)$. Let $f\colon D \to \mathbb{C}$ be a holomorphic function. Show that
   $$f(z_0) = \frac{1}{Area(D)} \int_D f(z)\,dxdy\,.$$

8. Let D be a domain in $\mathbb{C}$ and let $f\colon D \to \mathbb{C}$ be a nonconstant holomorphic function. Show that the local minima of $|f|$ coincide with the zeros of f.

9. Let $f\colon U \to \mathbb{C}$ be a nonconstant holomorphic function defined on a neighbourhood of the unit disc Δ. Show that if $|f|$ is constant on the boundary of Δ, then f admits at least one zero in Δ.

10. *Automorphisms of* Δ.
    (a) Show that every automorphism of the unit disc Δ extends injectively to a neighbourhood of its closure $\overline{\Delta}$ and admits at least a fixed point in $\overline{\Delta}$.
    (b) Show that if f has a fixed point in Δ, then it is necessarily unique.

11. Define $D(0,r) = \{z \in \mathbb{C} \ : \ |z| < r\}$. Let $f : D(0,1) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \partial D(0,1)$. Show that if f is nonconstant, then there exists an automorphism g of $D(0,1)$ such that $f \circ g(0) = 0$.

12. Let $f : \Delta \to \Delta$ be a holomorphic function with a zero of order m at 0. Show that for all $z \in \Delta$ one has $|f(z)| \leq |z|^m$.

13. Verify that the Cayley transform

$$C(z) := i\,\frac{1+z}{1-z}$$

is a biholomorphism between the unit disc Δ and the upper half plane $\mathbb{H}^+$.

14. Show that the strip $S_r := \{\, z \in \mathbb{C} \ : \ 0 < \operatorname{Im} z < r \,\}$, with $r > 0$, is biholomorphic to $\mathbb{H}^+$ (and therefore to Δ). Determine an explicit biholomorphism $f\colon S_1 \to \Delta$.

15. Determine whether there exists a nonconstant holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the border $\partial\Delta$ of the unit disc Δ.

16. Determine whether there exists a nonconstant holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the real line $\mathbb{R}$.