Cauchy-Riemann equations.

1. Determine all points where the following functions satisfy the Cauchy-Riemann equations:

$$f(x,y) = xy^2, \quad f(z) = |z|^2(|z|^2-1), \quad f(z) = \sin(|z|^2), \quad f(z) = z(z+\bar{z})^2.$$

2. Verify that $f(z) = \bar{z}$ is not $\mathbf{C}$-differentiable.

3. Let $f\colon D \to \mathbf{C}$ be a function on an open set $D \subset \mathbf{C}$. Verify that $\frac{\partial f}{\partial \bar{z}} = \overline{\frac{\partial \bar{f}}{\partial z}}$.

4. Let $f\colon D \to \mathbf{C}$ be a holomorphic function on an open set $D \subset \mathbf{C}$. Verify that the function $g(z) := \overline{f(\bar{z})}$ is holomorphic on $D^* = \{\bar{z} \ : z \in D\}$.

5. Let $f\colon D \to \mathbf{C}$ be a holomorphic function on an open connected set $D \subset \mathbf{C}$:
   (a) if $f'(z) \equiv 0$, then f is constant.
   (b) if $|f(z)|^2$ is constant, then f is constant.
   (c) if $\arg(f(z))$ is constant, then f is constant.

Elementary functions; complex power series.

6. Compute the values of the following functions:
   (a) $e^{i\pi/4}, \quad e^{i3\pi/4}, \quad e^{1+i2\pi/3}$.
   (b) $2^i, \quad i^i$.
   (c) $\log 3i, \quad \log(-4), \quad \log(e+i), \quad \mathrm{Log}(\mathrm{Log}(i))$
   (here log denotes the multivalued complex logarithm and Log denotes its principal value).

7. Consider the function $f(z) = z^2,\ z \in \mathbf{C}$.
   (a) Determine the image of the set $Q = \{z = x+iy \in \mathbf{C} \mid x,y > 0\}$ under f.
   (b) Determine the largest disc $B(1,R) = \{z \in \mathbf{C} \mid |z-1| < R\}$ on which f is injective.
   (c) Answer questions (a) and (b) when $f(z) = z^4$.

8. Determine the preimage of $Q = \{z = x+iy \in \mathbf{C} \mid x,y > 0\}$ under $f(z) = \sqrt{z}$.

9. Determine the image of the following sets of the plane under the exponential map $z \mapsto e^z$:
   (a) the line $x = 2$;
   (b) the line $y = x$;
   (c) the square of vertices $0,\ i,\ 1,\ 1+i$.
   (d) the strip $x - \frac{\pi}{2} < y < x + \frac{\pi}{2}$.

10. Let $f(z) = e^{z^2}$. Draw the curves $|f|$=constant and $\arg(f)$=constant.

11. Let $\cos z := \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z := \frac{e^{iz}-e^{-iz}}{2i}$.
    (a) Verify that $(\sin z)' = \cos z$ e $(\cos z)' = -\sin z$.
    (b) Verify that $\cos^2 z + \sin^2 z = 1$.

Complex integration.

12. Compute

$$\int_\gamma \frac{e^z}{z} dz, \qquad \int_\gamma \frac{e^z}{(z-1/2)^2} dz,$$

where $\gamma = \{e^{i\theta},\ \theta \in [0, 2\pi]\}$.

13. Compute

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{(z-1)(z-2i)} dz,$$

where $\gamma = \{4e^{i\theta},\ \theta \in [0, 2\pi]\}$.

14. Compute

$$\int_\gamma \frac{\cos z}{(z-\pi/2)^{10}} dz,$$

where $\gamma = \{2e^{i\theta},\ \theta \in [0, 2\pi]\}$.

15. Let $|a| < 1 < |b|$. For $n, m \in \mathbf{Z}$, compute

$$\frac{1}{2\pi i}\int_\gamma \frac{(z-b)^m}{(z-a)^n} dz,$$

where $\gamma = \{e^{i\theta},\ \theta \in [0, 2\pi]\}$.

Identity principle, Liouville theorem.

16. Let $\Omega \subset \mathbf{C}$ be an open connected set. Let $f, g\colon \Omega \longrightarrow \mathbf{C}$ be holomorphic functions such that $f \cdot g \equiv 0$. Then either $f \equiv 0$ or $g \equiv 0$.

17. Let $f\colon \mathbf{C} \longrightarrow \mathbf{C}$ be a holomorphic function, such that $|f(z)| \le |e^z|$, for all $z \in \mathbf{C}$. Prove that $f(z) = ce^z$, for some constant $c \in \mathbf{C}$.

18. Let $f\colon \mathbf{C} \longrightarrow \mathbf{C}$ be a holomorphic function. Assume that there exist positive constants M and c for which $|f(z)| \le M(c + |z|^n)$, for all $z \in \mathbf{C} \setminus B(0, R)$, with $R > 0$.
    (a) Show that f is a polynomial of degree $\le n$.
    (b) Assume $|f(z)| \le M|z|^2$. Determine $f(0)$ and $f'(0)$.

19. Determine all the zeros of $\sin z$ and $\cos z$. Let $U = \mathbf{C} \setminus \{\pi/2 \pm k\pi,\ k \in \mathbf{Z}\}$. Let f be a holomorphic function on U, such that $f(\pi/n) = \tan(\pi/n)$, $n \ge 3$. Deduce that f is not holomorphic on all $\mathbf{C}$ and that it does not assume the value i.