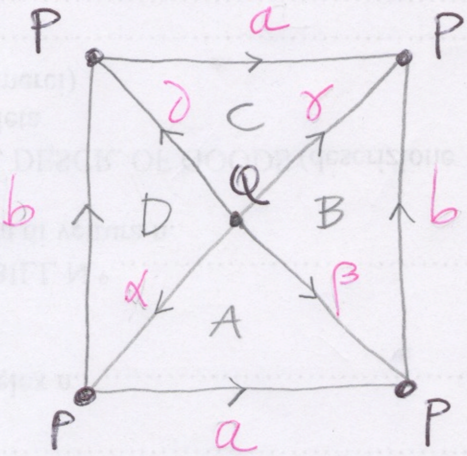
1 Omologia simpliciale del toro



$X =$ toro

$$C_0(X) = \mathbb{Z}P \oplus \mathbb{Z}Q$$

$$C_1(X) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma \oplus \mathbb{Z}\delta$$

$$C_2(X) = \mathbb{Z}A \oplus \mathbb{Z}B \oplus \mathbb{Z}C \oplus \mathbb{Z}D$$

$$\partial_0 \equiv 0$$

$$\partial_1(a) = \partial_1(b) = 0 \qquad \partial_1(\alpha) = \partial_1(\beta) = \partial_1(\gamma) = \partial_1(\delta) = P - Q$$

$$\partial_2(A) = \alpha + a - \beta$$

$$\partial_2(B) = \beta + b - \gamma$$

$$\partial_2(C) = \delta + a - \gamma$$

$$\partial_2(D) = \alpha + b - \delta$$

$$H_0(X) = \frac{\operatorname{Ker}\partial_0}{\operatorname{Im}\partial_1} = \frac{C_0(X)}{\operatorname{Im}\partial_1} = \frac{\mathbb{Z}P + \mathbb{Z}Q}{\mathbb{Z}(P-Q)} \cong \mathbb{Z}$$

$$H_2(X) = \frac{\operatorname{Ker}\partial_2}{\operatorname{Im}\partial_3} \cong \operatorname{Ker}\partial_2 = \mathbb{Z}(A+B-C-D) \cong \mathbb{Z}$$

$$\partial_3 = 0 \qquad \partial(A+B-C-D) = 0$$

$$\ker \partial_1 = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}(\alpha-\beta) \oplus \mathbb{Z}(\beta-\gamma) \oplus \mathbb{Z}(\gamma-\delta)$$

$$\operatorname{Im} \partial_2 = \mathbb{Z}\partial_2(A) \oplus \mathbb{Z}\partial_2(B) \oplus \mathbb{Z}\partial_2(C)$$

$$\cong \mathbb{Z}(\alpha+a-\beta) \oplus \mathbb{Z}(\beta+b-\gamma) \oplus \mathbb{Z}(\delta+a-\gamma)$$

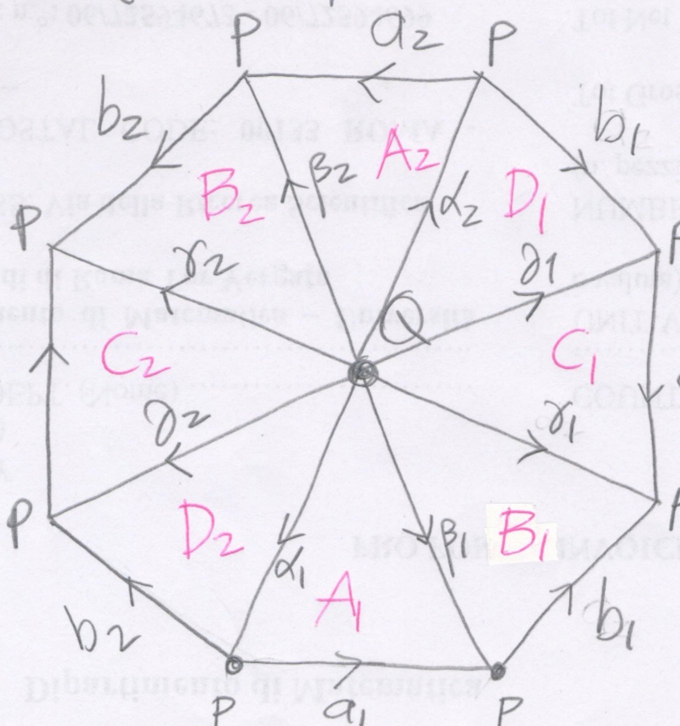
Reorganizzando i generatori di $\ker \partial_1$, troviamo

$$\ker \partial_1 = \mathbb{Z}a \oplus \mathbb{Z}(a+(\alpha-\beta)) \oplus \mathbb{Z}b \oplus \mathbb{Z}(b+(\beta-\gamma)) + \oplus \mathbb{Z}(a+(\gamma-\delta))$$

da cui

$$H_1(X) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} \cong \mathbb{Z}a \oplus \mathbb{Z}b \cong \mathbb{Z} \oplus \mathbb{Z}.$$

2 omologia simpliciale di $X = T \# T$



$$C_0(X) = \mathbb{Z}_p \oplus \mathbb{Z}_q$$

$$C_1(X) = \underbrace{\mathbb{Z}a_i \oplus \mathbb{Z}b_i \oplus \mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i \oplus \mathbb{Z}\gamma_i \oplus \mathbb{Z}\delta_i}_{12 \text{ generatori}} \qquad i=1,2$$

$$C_2(X) = \underbrace{\mathbb{Z}A_i \oplus \mathbb{Z}B_i \oplus \mathbb{Z}C_i \oplus \mathbb{Z}D_i}_{8 \text{ generatori}}$$

$$\partial_0 = 0$$

$$\partial_1(a_i) = \partial_1(b_i) = 0$$

$$\partial_1(\alpha_i) = \partial_1(\beta_i) = \partial_1(\gamma_i) = \partial_1(\delta_i) = P - Q$$

$$\alpha_1 \quad \alpha_2 \quad \beta_1 \quad \beta_2 \quad \gamma_1 \quad \gamma_2 \quad \delta_1 \quad \delta_2$$

$$A \quad \underbrace{\alpha_1-\alpha_2 \quad \alpha_2-\beta_1 \quad \beta_1-\beta_2 \quad \beta_2-\gamma_1 \quad \gamma_1-\gamma_2 \quad \gamma_2-\delta_1 \quad \delta_1-\delta_2}_{7 \text{ elem. indip. di } \ker \partial_1}$$

$$\ker \partial_1 = \mathbb{Z}a_i \oplus \mathbb{Z}b_i \oplus A$$

gruppo libero abel. di rango $4+7=11$

$$\partial_2(A_1) = \alpha_1 + a_1 - \beta_1$$
$$\partial_2(B_1) = \beta_1 + b_1 - \gamma_1$$
$$\partial_2(C_1) = \delta_1 + a_1 - \gamma_1$$
$$\partial_2(D_1) = \alpha_2 + b_1 - \delta_1$$
$$\partial_2(A_2) = \alpha_2 + a_2 - \beta_2$$
$$\partial_2(B_2) = \beta_2 + b_2 - \gamma_2$$
$$\partial_2(C_2) = \delta_2 + a_2 - \gamma_2$$
$$\partial_2(D_2) = \alpha_1 + b_2 - \delta_2$$

$$\ker \partial_2 = \big(A_1 + B_1 - (C_1 + D_1) + (A_2 + B_2) - (C_2 + D_2) \big)$$

$$H_0(X) = \frac{\ker\partial_0}{\operatorname{Im}\partial_1} = \frac{\mathbb{Z}_P \oplus \mathbb{Z}_Q}{\mathbb{Z}(P-Q)} \cong \mathbb{Z}$$

$$= \frac{C_0(X)}{\operatorname{Im}\partial_1}$$

$$H_2(X) = \frac{\ker\partial_2}{\operatorname{Im}\partial_3} = \ker\partial_2 = \mathbb{Z}\left(\sum (A_i+B_i) - (C_i+D_i)\right)$$

$$\cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker\partial_1}{\operatorname{Im}\partial_2} = \frac{\text{gruppo di rk} = 11}{\text{gruppo di rk} = 7}$$

$$\cong \underbrace{\mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}b_1 \oplus \mathbb{Z}b_2}_{rk = 4}$$

dim:

$\partial_2(A_1) = a_1 + (\alpha_1-\beta_1)$ $\qquad \alpha_1-\beta_1 = (\alpha_1-\alpha_2)+(\alpha_2-\beta_1)$

$\partial_2(B_1) = b_1 + (\beta_1-\gamma_1)$ $\qquad \beta_1-\gamma_1 = (\beta_1-\beta_2)+(\beta_2-\gamma_1)$

$\partial_2(C_1) = a_1 - (\gamma_1-\delta_1)$ $\qquad \gamma_1-\delta_1 = (\gamma_1-\gamma_2)+(\gamma_2-\delta_1)$

$\partial_2(D_1) = b_1 - (\alpha_2-\delta_1)$ $\qquad \alpha_2-\delta_1 = (\alpha_2-\beta_1)+(\beta_1-\beta_2)+(\beta_2-\gamma_1)+(\gamma_1-\gamma_2)+(\gamma_2-\delta_1)$

$\partial_2(A_2) = a_2 + (\alpha_2-\beta_2)$ $\qquad \alpha_2-\beta_2 = (\alpha_2-\beta_1)+(\beta_1-\beta_2)$

$\partial_2(B_2) = b_2 + (\beta_2-\gamma_2)$ $\qquad \beta_2-\gamma_2 = (\beta_2-\gamma_1)+(\gamma_1-\gamma_2)$

$\partial_2(C_2) = a_2 - (\gamma_2-\delta_2)$ $\qquad \gamma_2-\delta_2 = (\gamma_2-\delta_1)+(\delta_1-\delta_2)$

Riorganizziamo i generatori di $\ker \partial_1$:

tramite opportune combinazioni lineari a coefficienti interi di

$$a_1, a_2, b_1, b_2, \alpha_1-\alpha_2, \alpha_2-\beta_1, \beta_1-\beta_2, \beta_2-\gamma_1, \gamma_1-\gamma_2, \gamma_2-\delta_1, \delta_1-\delta_2$$

troviamo che un altro insieme di generatori indipendenti di $\ker \partial_1$ è dato da:

$$a_1, a_2, b_1, b_2,$$

$$a_1 + (\alpha_1-\alpha_2) + (\alpha_2-\beta_1) = a_1 + (\alpha_1-\beta_1) = \partial_2(A_1)$$

$$a_1 - (\gamma_1-\gamma_2) - (\gamma_2-\delta_1) = a_1 - (\gamma_1-\delta_1) = \partial_2(C_1)$$

$$b_1 + (\beta_1-\beta_2) + (\beta_2-\gamma_1) = b_1 + (\beta_1-\gamma_1) = \partial_2(B_1)$$

$$b_1 + (\alpha_2-\beta_1) + (\beta_1-\beta_2) + (\beta_2-\gamma_1) + (\gamma_2-\delta_1) = b_1 + (\alpha_2-\delta_1) = \partial_2(D_1)$$

$$a_2 + (\alpha_1-\beta_1) + (\beta_1-\beta_2) = a_2 + (\alpha_1-\beta_2) = \partial_2(A_2)$$

$$a_2 - (\gamma_2-\delta_1) - (\delta_1-\delta_2) = a_2 - (\gamma_2-\delta_2) = \partial_2(C_2)$$

$$b_2 + (\beta_2-\gamma_1) + (\gamma_1-\gamma_2) = b_2 + (\beta_2-\gamma_2) = \partial_2(B_2)$$

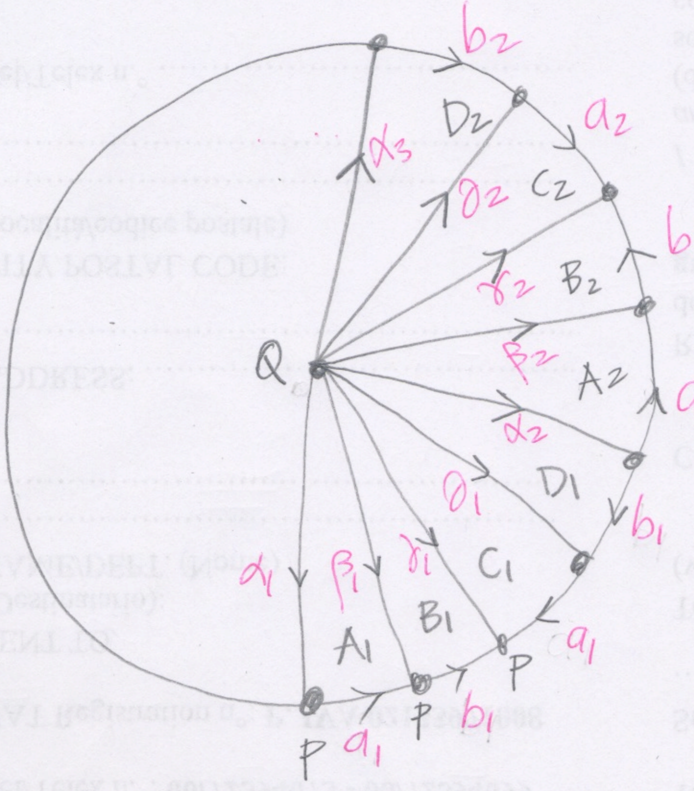
Adesso è chiaro che

$$H_1(X) \cong \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}b_1 \oplus \mathbb{Z}b_2.$$

Ⓐ

Generalizziamo il calcolo precedente e calcoliamo l'omologia simpliciale della somma connessa di n tori

$$X = \underbrace{T \# \cdots \# T}_{n}$$



$$C_0(X) = \mathbb{Z}P \oplus \mathbb{Z}Q$$

$$C_1(X) = \mathbb{Z}a_i \oplus \mathbb{Z}b_i \oplus \mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i \oplus \mathbb{Z}\gamma_i \oplus \mathbb{Z}\delta_i$$

$$C_2(X) = \mathbb{Z}A_i \oplus \mathbb{Z}B_i \oplus \mathbb{Z}C_i \oplus \mathbb{Z}D_i$$

$$\partial_0 = \partial_3 = 0$$

$$\partial_1(a_i) = \partial_1(b_i) = 0$$

$$\partial_1(\alpha_i) = \partial_1(\beta_i) = \partial_1(\gamma_i) = \partial_1(\delta_i) = P - Q$$

(B)

Calcoliamo $\ker \partial_1$:

$\ker \partial_1$ contiene $\mathbb{Z}a_i \oplus \mathbb{Z}b_i$

e contiene tutti gli elementi della forma

$$(\star)\ \begin{cases} \alpha_i - \alpha_j,\ \beta_i - \beta_j,\ \gamma_i - \gamma_j,\ \delta_i - \delta_j \quad i<j \\ \&\ \lambda_i - \mu_j,\ \text{con } \lambda_i, \mu_j \in \{\alpha_i, \beta_i, \gamma_i, \delta_i\} \\ \lambda \neq \mu. \end{cases}$$

Ordiniamo

$$\alpha_1 \dots \alpha_n\ \beta_1 \dots \beta_n\ \gamma_1 \dots \gamma_n\ \delta_1 \dots \delta_n.$$

Allora gli elementi di $(\star)$ sono generati da

$$(\star\star)\ \begin{cases} \alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_{n-1}\alpha_n, \alpha_n - \beta_1, \beta_1 \beta_2, \dots \\ \beta_{n-1} - \beta_n, \beta_n - \gamma_1, \gamma_1 - \gamma_2, \dots \gamma_{n-1} - \gamma_n, \\ \gamma_n - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n. \end{cases}$$

cioè da $4n-1$ generatori indipendenti.

Conclusione

$\ker \partial_1$ è un gruppo libero con

$$2n + 4n - 1 = 6n - 1$$

generatori indipendenti.

$$\partial_2(A_1)=\alpha_1+a_1-\beta_1$$
$$\partial_2(B_1)=\beta_1+b_1-\gamma_1$$
$$\partial_2(C_1)=\delta_1+a_1-\gamma_1$$
$$\partial_2(D_1)=\alpha_2+b_1-\delta_1$$

$$\partial_2(A_1+B_1-(C_1+D_1))=\alpha_1-\alpha_2$$

$$\partial_2(A_i)=\alpha_i+a_i-\beta_i$$
$$\partial_2(B_i)=\beta_i+b_i-\gamma_i$$
$$\partial_2(C_i)=\delta_i+a_i-\gamma_i$$
$$\partial_2(D_i)=\alpha_{i+1}+b_i-\delta_i$$

$$\partial_2(A_i+B_i-(C_i+D_i))=\alpha_i-\alpha_{i+1}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\partial_2(A_n)=\alpha_n+a_n-\beta_n$$
$$\partial_2(B_n)=\beta_n+b_n-\gamma_n$$
$$\partial_2(C_n)=\delta_n+a_n-\gamma_n$$
$$\partial_2(D_n)=\alpha_1+b_n-\delta_n$$

$$\partial_2((A_n+B_n)-(C_n+D_n))=\alpha_n-\alpha_1$$

$$\Longrightarrow \partial_2\Big(\underbrace{\sum_{i=1}^{n}(A_i+B_i)-(C_i+D_i)}\Big)=0.$$

<u>in effetti</u> $\quad \ker\partial_2=\mathbb{Z}\left(\sum_{i=1}^{n}(A_i+B_i)-(C_i+D_i)\right)$

(D)

Dai calcoli fatti, segue immediatamente che

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{C_0(X)}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}P \oplus \mathbb{Z}Q}{\mathbb{Z}(P-Q)} \cong \mathbb{Z}$$

$$H_2(X) = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \ker \partial_2 = \mathbb{Z}\left(\sum_{i=1}^{n} (A_i + B_i) - (C_i + D_i)\right) \cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\langle a_i, b_i, \text{generat. } \overset{4n-1}{\#?} \rangle}{\langle 4n-1 \text{ generat.: cf pag } ⑤ \rangle}$$

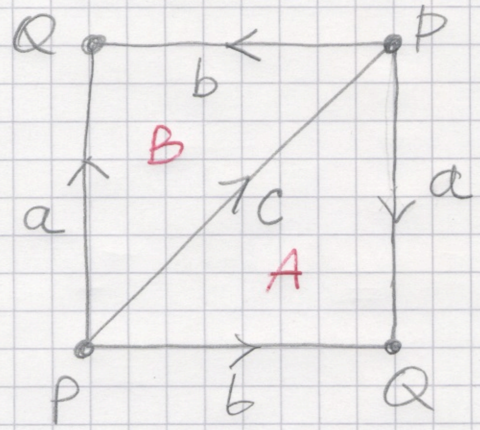
$$\cong \bigoplus_{i=1}^{n} \mathbb{Z}a_i \oplus \mathbb{Z}b_i$$

per vedere questo fatto è necessario riarrangiare i generatori di $\ker \partial_1$, come abbiamo fatto nei casi $n=1$ e $n=2$, ossia verificare che

$$\ker \partial_1 = \langle \underset{i=1,\dots,n}{a_i, b_i}, \underbrace{\partial_2(A_i), \partial_2(B_i), \partial_2(C_i), \partial_2(D_i)}_{i=1,\dots,n-1} \rangle$$

⓪

OMOLOGIA SIMPLICIALE di $X = \mathbb{P}^2$



$C_0(X) = \mathbb{Z}P \oplus \mathbb{Z}Q$

$C_1(X) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$

$C_2(X) = \mathbb{Z}A \oplus \mathbb{Z}B$

$\partial_0 \equiv \partial_3 \equiv 0$

$\partial_1(b) = Q - P$

$\partial_1(a) = Q - P$

$\partial_1(c) = 0$

$\left.\begin{array}{l}\partial_2(A) = c + a - b \\ \partial_2(B) = c + b - a\end{array}\right\}$ 2 generat. indip.

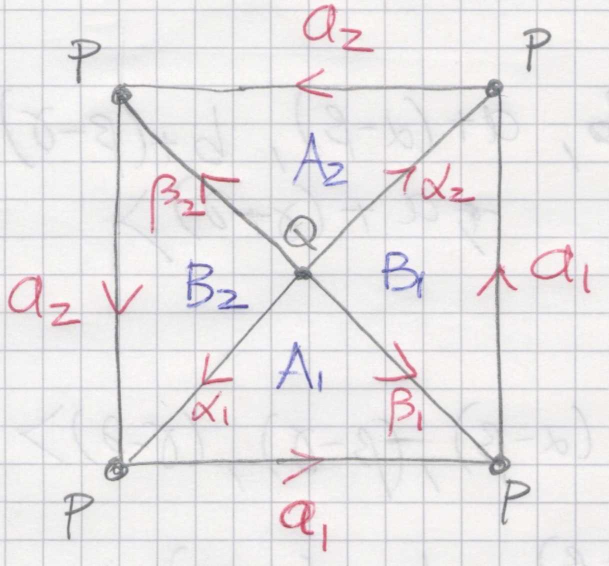
Alternativamente, una base di $\operatorname{Im}\partial_2$ è $\{2(a-b),\ c+(a-b)\}$.

$$H_0(X) = \frac{\ker\partial_0}{\operatorname{Im}\partial_1} = \frac{C_0(X)}{\operatorname{Im}\partial_1} \cong \frac{\mathbb{Z}P \oplus \mathbb{Z}Q}{\mathbb{Z}(Q-P)} \cong \mathbb{Z}$$

$$H_2(X) = \frac{\ker\partial_2}{\operatorname{Im}\partial_3} = \ker\partial_2 = \{0\}$$

$$H_1(X) = \frac{\ker\partial_1}{\operatorname{Im}\partial_2} = \frac{\mathbb{Z}(a-b) \oplus \mathbb{Z}(c+(a-b))}{\mathbb{Z}2(a-b) \oplus \mathbb{Z}(c+(a-b))} \cong \mathbb{Z}/_2$$

2 OMOLOGIA SIMPLICIALE di $X = \mathbb{P}^2 \# \mathbb{P}^2$



$C_0(X) = \mathbb{Z}P \oplus \mathbb{Z}Q$ 2 generatori

$C_1(X) = \mathbb{Z}a_i \oplus \mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i$ 6 generatori
$i=1,2$

$C_2(X) = \mathbb{Z}A_i \oplus \mathbb{Z}B_i$ 4 generatori
$i=1,2$

$\partial_0 \equiv 0$

$\partial_1(a_i) \equiv 0 \quad \partial_1(\alpha_i) = \partial_1(\beta_i) = P - Q \quad i=1,2$

$\ker \partial_1 = \langle a_1, a_2, \alpha_1 - \alpha_2, \alpha_2 - \beta_1, \beta_1 - \beta_2 \rangle$ 5 gener. indip.

$$\left.\begin{aligned}
\partial_2(A_1) &= \alpha_1 + a_1 - \beta_1 \\
\partial_2(B_1) &= \beta_1 + a_1 - \alpha_2 \\
\partial_2(A_2) &= \alpha_2 + a_2 - \beta_2 \\
\partial_2(B_2) &= \beta_2 + a_2 - \alpha_1
\end{aligned}\right\}$$ 4 generat. indip. di $\operatorname{Im}\partial_2$.

$$H_0(X) \cong \frac{\mathbb{Z}P \oplus \mathbb{Z}Q}{\mathbb{Z}(P-Q)} \cong \frac{\mathbb{Z}(P+Q) \oplus \mathbb{Z}(P-Q)}{\mathbb{Z}(P-Q)} \cong \mathbb{Z}$$

$$H_2(X) = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \ker \partial_2 = \{0\}$$

$$\partial_3 \equiv 0$$

$$H_1(X) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} \cong \mathbb{Z}_2 \oplus \mathbb{Z}$$

dim: Come base di $\ker \partial_1$ si può anche prendere

$$\begin{aligned}
&a_1 \\
&a_1 + a_2 \\
&a_1 + (\alpha_1 - \alpha_2) + (\alpha_2 - \beta_1) &&= \partial_2(A_1) \\
&a_1 - (\alpha_2 - \beta_1) &&= \partial_2(B_1) \\
&a_2 - (\alpha_1 - \beta_2) &&= \partial_2(B_2)
\end{aligned}$$

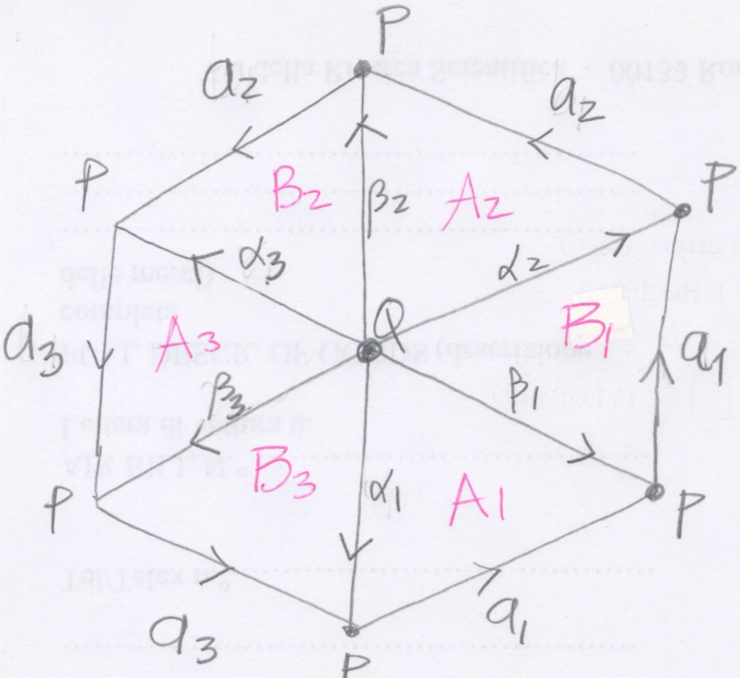
& come base di $\operatorname{Im} \partial_2$

$$\partial_2(A_1),\ \partial_2(B_1),\ \partial_2(B_2),\ \partial_2(A_1 + B_1 + A_2 + B_2) = 2(a_1 + a_2)$$

Adesso è chiaro che

$$\frac{\ker \partial_1}{\operatorname{Im} \partial_2} \cong \mathbb{Z}(a_1) \oplus \frac{\mathbb{Z}(a_1 + a_2)}{\mathbb{Z}\, 2(a_1 + a_2)} \cong$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}_2 . \qquad \boxtimes$$



$$X = \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$$

$$C_0(X) = \mathbb{Z}P \oplus \mathbb{Z}Q$$

$$C_1(X) = \mathbb{Z}a_i \oplus \mathbb{Z}b_i \qquad i=1,2,3$$

$$C_2(X) = \mathbb{Z}A_i \oplus \mathbb{Z}B_i \qquad i=1,2,3$$

$$\partial_0 \equiv 0 \equiv \partial_3$$

$$\partial_1(a_i) = 0 \qquad \partial_1(\alpha_i) = \partial_1(\beta_i) = P - Q \qquad i=1,2,3$$

$$\ker \partial_1 = \langle a_i, \alpha_1-\alpha_2, \alpha_2-\alpha_3, \alpha_3-\beta_1, \beta_1-\beta_2, \beta_2-\beta_3 \rangle$$

$\Rightarrow$ ha rango $3+5=8$

$$(A) \begin{cases} \partial_2(A_1) = \alpha_1 + a_1 - \beta_1 \\ \partial_2(B_1) = \beta_1 + a_1 - \alpha_2 \\ \partial_2(A_2) = \alpha_2 + a_2 - \beta_2 \\ \partial_2(B_2) = \beta_2 + a_2 - \alpha_3 \\ \partial_2(A_3) = \alpha_3 + a_3 - \beta_3 \\ \partial_2(B_3) = \beta_3 + a_3 - \alpha_1 \end{cases}$$

▶ si può verificare che:

$$\ker \partial_2 = \{0\}$$

$$H_0(X) = \frac{\ker\partial_0}{\mathrm{Im}\,\partial_1} = \frac{C_0(X)}{\mathrm{Im}\,\partial_1} = \frac{\mathbb{Z}P \oplus \mathbb{Z}Q}{\mathbb{Z}(P-Q)} = \mathbb{Z}$$

$$H_2(X) = \frac{\ker\partial_2}{\mathrm{Im}\,\partial_3} = \ker\partial_2 = \{0\}$$

$$H_1(X) = \frac{\ker\partial_1}{\mathrm{Im}\,\partial_2} = \frac{\langle a_i, \alpha_1-\alpha_2, \alpha_2-\alpha_3, \alpha_3-\beta_1, \beta_1-\beta_2, \beta_2-\beta_3 \rangle}{\langle \quad \rangle_{rk=6}}$$

generato da $\{\partial_2(A_i), \partial_2(B_i)\}_{i=1,2,3}$

riarrangiamo i generatori di $\mathrm{Im}\,\partial_2$

$$\partial_2\left(\sum_{i=1}^{3} A_i+B_i\right) = 2(a_1+a_2+a_3)$$

$$\partial_2(A_1) = a_1 + (\alpha_1-\beta_1) = a_1 + (\alpha_1-\alpha_2) + (\alpha_2-\alpha_3) + (\alpha_3-\beta_1)$$

$$\partial_2(B_1) = a_1 - (\alpha_2-\beta_1) = a_1 - (\alpha_2-\alpha_3) - (\alpha_3-\beta_1)$$

$$\partial_2(A_2) = a_2 + (\alpha_2-\beta_2) = a_2 + (\alpha_2-\alpha_3) + (\alpha_3-\beta_1) + (\beta_1-\beta_2)$$

$$\partial_2(B_2) = a_2 - (\alpha_3-\beta_2) = a_2 - (\alpha_3-\beta_1) - (\beta_1-\beta_2)$$

$$\partial_2(B_3) = a_3 + (\alpha_3-\beta_3) = a_3 + (\alpha_3-\beta_1) + (\beta_1-\beta_2) + (\beta_2-\beta_3)$$

riarrangiamo i generatori di $\ker\partial_1$

$$a_1,\ a_2,\ a_1+a_2+a_3,\ \partial_2(A_1),\ \partial_2(B_1),\ \partial_2(A_2),\ \partial_2(B_2),\ \partial_2(B_3).$$

Da cui

$$\Longrightarrow H_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$$

$\boxtimes$

Nel caso generale si procede in modo analogo . . . . .

e si trova

$$X = \underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{n\text{-volte}}$$

$$H_1(X) \cong \mathbb{Z}_2 \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{(n-1)\text{ volte}}$$