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THE GLOBAL QUANTUM DUALITY PRINCIPLE: THEORY, EXAMPLES, AND APPLICATIONS

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Abstract. Let R be an integral domain, let $\hbar \in R \setminus \{0\}$ be such that $R \;\; \hbar R$ is a field, and $\mathcal{HA}$ the category of torsionless (or flat) Hopf algebras over R. We call $H \in \mathcal{HA}$ a *“quantized function algebra” (=QFA)*, resp. *“quantized restricted universal enveloping algebras” (=QrUEA)*, at $\hbar$ if $H \;\; \hbar H$ is the function algebra of a connected Poisson group, resp. the (restricted, if $R \;\; \hbar R$ has positive characteristic) universal enveloping algebra of a (restricted) Lie bialgebra.

We establish an “inner” Galois correspondence on $\mathcal{HA}$, via the definition of two endofunctors, $(\ )^\vee$ and $(\ )'$, of $\mathcal{HA}$ such that: *(a)* the image of $(\ )^\vee$, resp. of $(\ )'$, is the full subcategory of all QrUEAs, resp. QFAs, at $\hbar$; *(b)* if $p := \mathit{Char}\, R \;\; \hbar R \; = 0$, the restrictions $(\ )^\vee{}_{\text{QFAs}}$ and $(\ )'{}_{\text{QrUEAs}}$ yield equivalences inverse to each other; *(c)* if $p = 0$, starting from a QFA over a Poisson group G, resp. from a QrUEA over a Lie bialgebra $\mathfrak{g}$, the functor $(\ )^\vee$, resp. $(\ )'$, gives a QrUEA, resp. a QFA, over the dual Lie bialgebra, resp. a dual Poisson group. In particular, *(a)* yields a machine to produce quantum groups of both types (either QFAs or QrUEAs), *(b)* gives a characterization of them among objects of $\mathcal{HA}$, and *(c)* gives a “global” version of the so-called “quantum duality principle” (after Drinfeld’s, cf. [Dr]).

We then apply our result to Hopf algebras of the form $\Bbbk[\hbar] \otimes_{\Bbbk} H$ where H is a Hopf algebra over the field $\Bbbk$: this yields quantum groups, hence “classical” geometrical symmetries of Poisson type (Poisson groups or Lie bialgebras, via specialization) associated to the “generalized” symmetry encoded by H. Both our main result and the above mentioned application are illustrated by means of several examples, which are studied in some detail.

“Dualitas dualitatum
et omnia dualitas”

N. Barbecue, “Scholia”

INDEX

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Introduction

The most general notion of "symmetry" in mathematics is encoded in the notion of Hopf algebra. Among Hopf algebras H over a field, the commutative and the cocommutative ones encode "geometrical" symmetries, in that they correspond, under some technical conditions, to algebraic groups and to (restricted, if the ground field has positive characteristic) Lie algebras respectively: in the first case H is the algebra $F[G]$ of regular functions over an algebraic group G, whereas in the second case it is the (restricted) universal enveloping algebra $U(\mathfrak{g})$ ($\mathbf{u}(\mathfrak{g})$ in the restricted case) of a (restricted) Lie algebra $\mathfrak{g}$. A popular generalization of these two types of "geometrical symmetry" is given by quantum groups: roughly, these are Hopf algebras H depending on a parameter $\hbar$ such that setting $\hbar = 0$ the Hopf algebra one gets is either of the type $F[G]$ — hence H is a *quantized function algebra*, in short QFA — or of the type $U(\mathfrak{g})$ or $\mathbf{u}(\mathfrak{g})$ (according to the characteristic of the ground field) — hence H is a *quantized (restricted) universal enveloping algebra*, in short QrUEA. When a QFA exists whose specialization (i.e. its "value" at $\hbar = 0$) is $F[G]$, the group G inherits from this "quantization" a Poisson bracket, which makes it a Poisson (algebraic) group; similarly, if a QrUEA exists whose specialization is $U(\mathfrak{g})$ or $\mathbf{u}(\mathfrak{g})$, the (restricted) Lie algebra $\mathfrak{g}$ inherits a Lie cobracket which makes it a Lie bialgebra. Then by Poisson group theory one has Poisson groups G^* dual to G and a Lie bialgebra $\mathfrak{g}^*$ dual to $\mathfrak{g}$, so other geometrical symmetries are related to the initial ones.

The dependence of a Hopf algebra on $\hbar$ can be described as saying that it is defined over a ring R and $\hbar \in R$: so one is lead to dwell upon the category $\mathcal{HA}$ of Hopf R–algebras (maybe with some further conditions), and then raises three basic questions:

— **(1)** *How can we produce quantum groups?*

— **(2)** *How can we characterize quantum groups (of either kind) within $\mathcal{HA}$?*

— **(3)** *What kind of relationship, if any, does exist between quantum groups over mutually dual Poisson groups, or mutually dual Lie bialgebras?*

A first answer to question **(1)** and **(3)** together is given, in characteristic zero, by the so-called "quantum duality principle", known in literature with at least two formulations. One claims that quantum function algebras associated to dual Poisson groups can be taken to be dual — in the Hopf sense — to each other; and similarly for quantum enveloping algebras (cf. [FRT1] and [Se]). The second one, formulated by Drinfeld in local terms (i.e., using formal groups, rather than algebraic groups, and complete topological Hopf algebras; cf. [Dr], §7, and see [Ga4] for a proof), gives a recipe to get, out of a QFA over G, a QrUEA over $\mathfrak{g}^*$, and, conversely, to get a QFA over G^* out of a QrUEA over $\mathfrak{g}$. More precisely, Drinfeld defines two functors, inverse to each other, from the category of quantized universal enveloping algebras (in his sense) to the category of quantum formal series Hopf algebras (his analogue of QFAs) and viceversa, such that $U_\hbar(\mathfrak{g}) \mapsto F_\hbar[[G^*]]$ and $F_\hbar[[G]] \mapsto U_\hbar(\mathfrak{g}^*)$ (in his notation, where the subscript $\hbar$ stands as a reminder for "quantized" and the double square brackets stand for "formal series Hopf algebra").

In this paper we establish a *global* version of the quantum duality principle which gives a complete answer to questions **(1)** through **(3)**. The idea is to push as far as possible Drinfeld's original method so to apply it to the category $\mathcal{HA}$ of all Hopf algebras which are torsion-free — or flat, if one prefers this narrower setup — modules over some (integral) domain, say R, and to do it for each non-zero element $\hbar$ in R such that $R/\hbar R$ be a field.

To be precise, we extend Drinfeld's recipe so to define functors from $\mathcal{HA}$ to itself. We show that the image of these "generalized" Drinfeld's functors is contained in a category of quantum groups — one gives QFAs, the other QrUEAs — so we answer question **(1)**. Then, in the characteristic zero case, we prove that when restricted to quantum groups these functors yield equivalences inverse to each other. Moreover, we show that these equivalences exchange the types of quantum group (switching QFA with QrUEA) and the underlying Poisson symmetries (interchanging G or $\mathfrak{g}$ with G^* or $\mathfrak{g}^*$), thus solving **(3)**. Other details enter the picture to show that these functors endow $\mathcal{HA}$ with sort of a (inner) "Galois correspondence", in which QFAs on one side and QrUEAs on the other side are the subcategories (in $\mathcal{HA}$) of "fixed points" for the composition of both Drinfeld's functors (in the suitable order): in particular, this answers question **(2)**. It is worth stressing that, since our "Drinfeld's functors" are defined for each non-trivial point $(\hbar)$ of $\mathit{Spec}_{\mathit{max}}(R)$, for any such $(\hbar)$ and for any H in $\mathcal{HA}$ they yield two quantum groups, namely a QFA and a QrUEA, w.r.t. $\hbar$ itself. Thus we have a method to get, out of any single $H \in \mathcal{HA}$, several quantum groups.

Therefore the "global" in the title is meant in several respects: geometrically, we consider global objects (Poisson groups rather than Poisson *formal* groups, as in Drinfeld's approach); algebraically we consider quantum groups over any domain R, so there may be several different "semiclassical limits" (=specializations) to consider, one for each non-trivial point of type $(\hbar)$ in the maximal spectrum of R (while Drinfeld has $R = \Bbbk[[\hbar]]$ so one can specialise only at $\hbar = 0$); more generally, our recipe applies to *any* Hopf algebra, i.e. not only to quantum groups; finally, most of our results are characteristic-free, i.e. they hold not only in characteristic zero (as in Drinfeld's case) but also in positive characteristic. Furthermore, this "global version" of the quantum duality principle opens the way to formulate a "quantum duality principle for subgroups and homogeneous spaces", see [CG].

A key, long-ranging application of our *global quantum duality principle* (GQDP) is the following. Take as R the polynomial ring $R = \Bbbk[\hbar]$, where $\Bbbk$ is a field: then for any Hopf algebra over $\Bbbk$ we have that $H[\hbar] := R \otimes_{\Bbbk} H$ is a torsion-free Hopf R–algebra, hence we can apply Drinfeld's functors to it. The outcome of this procedure is the *crystal duality principle* (CDP), whose statement strictly resembles that of the GQDP: now Hopf $\Bbbk$–algebras are looked at instead of torsionless Hopf R–algebras, and quantum groups are replaced by Hopf algebras with canonical filtrations such that the associated graded Hopf algebra is either commutative or cocommutative. Correspondingly, we have a method to associate to H a Poisson group G and a Lie bialgebra $\mathfrak{k}$ such that G is an affine space (as an algebraic variety) and $\mathfrak{k}$ is graded (as a Lie algebra); in both cases, the "geometrical" Hopf algebra can be attained — roughly — through a continuous 1-parameter deformation process. This result can also be formulated in purely classical — i.e. "non-quantum" — terms and proved by purely classical means. However, the approach via the GQDP also yields further possibilities to deform H into other Hopf algebras of geometrical type, which is out of reach of any classical approach.

The paper is organized as follows. In §1 we fix notation and terminology, while §2 is devoted to define Drinfeld's functors and state our main result, the GQDP (Theorem 2.2). In §3 we extend Drinfeld's functors to a broader framework, that of *(co)augmented (co)algebras*, and study their properties in general. §4 instead is devoted to the analysis of the effect of such functors on quantum groups, and prove Theorem 2.2, i.e. the GQDP. In §5 we explain the CDP, which is deduced as an application of the CDP to trivial deformations of Hopf $\Bbbk$–algebras: in particular, we study in detail the case of group algebras. In the last part of the paper we illustrate our results by studying in full detail several relevant examples. First we dwell upon some well-known quantum groups: the standard quantization of the Kostant-Kirillov structure on a Lie algebra (§6), the standard Drinfeld-Jimbo quantization of semisimple groups (§7), the quantization of the Euclidean group (§8) and the quantization of the Heisenberg group (§9). Then we study a key example of non-commutative, non-cocommutative Hopf algebra — a non-commutative version of the Hopf algebra of formal diffeomorphisms — as a nice application of the CDP (§10).

Warning: this paper is not meant for publication! The results presented here will be published in separate articles; therefore, any reader willing to quote anything from the present preprint is kindly invited to ask the author for the precise reference(s).

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§ 1 Notation and terminology

1.1 The classical setting. Let $\Bbbk$ be a fixed field of any characteristic. We call "algebraic group" the maximal spectrum G associated to any commutative Hopf $\Bbbk$–algebra H (in particular, we deal with *proaffine* as well as *affine* algebraic groups); then H is called the algebra of regular function on G, denoted with $F[G]$. We say that G is connected if $F[G]$ has no non-trivial idempotents; this is equivalent to the classical topological notion when $F[G]$ is of finite type, i.e. $\dim(G)$ is finite.

If G is an algebraic group, we denote by $\mathfrak{m}_e$ the defining ideal of the unit element $e \in G$ (it is the augmentation ideal of $F[G]$); the cotangent space of G at e is $\mathfrak{g}^\times := \mathfrak{m}_e \big/ {\mathfrak{m}_e}^2$, which is naturally a Lie coalgebra. The tangent space of G at e is the dual space $\mathfrak{g} := \big(\mathfrak{g}^\times\big)^*$ to $\mathfrak{g}^\times$: this is a Lie algebra, which coincides with the set of all left-(or right-)invariant derivations of $F[G]$. By $U(\mathfrak{g})$ we mean the universal enveloping algebra of $\mathfrak{g}$: it is a connected cocommutative Hopf algebra, and there is a natural Hopf pairing (see §1.2*(a)*) between $F[G]$ and $U(\mathfrak{g})$. If $\mathit{Char}(\Bbbk) = p > 0$, then $\mathfrak{g}$ is a restricted Lie algebra, and $\mathbf{u}(\mathfrak{g}) := U(\mathfrak{g}) \Big/ \big(\big\{\, x^p - x^{[p]} \,\big|\, x \in \mathfrak{g} \,\big\}\big)$ is the restricted universal enveloping algebra of $\mathfrak{g}$. In the sequel, to unify notation and terminology, when $\mathit{Char}(\Bbbk) = 0$ we shall call any Lie algebra $\mathfrak{g}$ "restricted", and its "restricted universal enveloping algebra" will be $U(\mathfrak{g})$, and we shall write $\mathcal{U}(\mathfrak{g}) := U(\mathfrak{g})$ if $\mathit{Char}(\Bbbk) = 0$ and $\mathcal{U}(\mathfrak{g}) := \mathbf{u}(\mathfrak{g})$ if $\mathit{Char}(\Bbbk) > 0$.

We shall also consider $\mathit{Hyp}\,(G) := \big(F[G]^\circ\big)_\varepsilon = \big\{\, f \in F[G]^\circ \,\big|\, f({\mathfrak{m}_e}^n) = 0 \;\, \forall\, n \geqq 0 \,\big\}$, i.e. the connected component of the Hopf algebra $F[G]^\circ$ dual to $F[G]$, which is called the *hyperalgebra* of G. By construction $\mathit{Hyp}\,(G)$ is a connected Hopf algebra, containing $\mathfrak{g} = \mathit{Lie}(G)$; if $\mathit{Char}(\Bbbk) = 0$ one has $\mathit{Hyp}\,(G) = U(\mathfrak{g})$, whereas if $\mathit{Char}(\Bbbk) > 0$ one has a sequence of Hopf algebra morphisms $U(\mathfrak{g}) \relbar\joinrel\twoheadrightarrow \mathbf{u}(\mathfrak{g}) \lhook\joinrel\longrightarrow \mathit{Hyp}\,(G)$. In any case, there exists a natural perfect (= non-degenerate) Hopf pairing between $F[G]$ and $\mathit{Hyp}\,(G)$.

Now assume G is a Poisson group (for this and other notions hereafter see, e.g., [CP], but within an *algebraic geometry* setting): then $F[G]$ is a Poisson Hopf algebra, and its Poisson bracket induces on $\mathfrak{g}^\times$ a Lie bracket which makes it into a Lie bialgebra, and so $U(\mathfrak{g}^\times)$ and $\mathcal{U}(\mathfrak{g}^\times)$ are co-Poisson Hopf algebras too. On the other hand, $\mathfrak{g}$ turns into a Lie bialgebra — maybe in topological sense, if G is infinite dimensional — and $U(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$ are (maybe topological) co-Poisson Hopf algebras. The Hopf pairing above between $F[G]$ and $\mathcal{U}(\mathfrak{g})$ then is compatible with these additional co-Poisson and Poisson structures. Similarly, $\mathit{Hyp}\,(G)$ is a co-Poisson Hopf algebra as well and the Hopf pairing between $F[G]$ and $\mathit{Hyp}\,(G)$ is compatible with the additional structures. Moreover, the perfect (=non-degenerate) pairing $\mathfrak{g} \times \mathfrak{g}^\times \longrightarrow \Bbbk$ given by evaluation is compatible with the Lie bialgebra structure on either side (see §1.2*(b)*): so $\mathfrak{g}$ and $\mathfrak{g}^\times$ are Lie bialgebras *dual to each other*. In the sequel, we denote by $G^\star$ any connected algebraic Poisson group with $\mathfrak{g}$ as cotangent Lie bialgebra, and say it is *dual* to G.

Let H be a Hopf algebra over an integral domain D. We call H a *"function algebra"* (FA in short) if it is commutative, with no non-trivial idempotents, and such that, if $p := \mathit{Char}(\Bbbk) > 0$, then $\eta^p = 0$ for all η in the kernel of the counit of H. If D is a field, an FA is the algebra of regular functions of an algebraic group-scheme over D which is connected and, if $\mathit{Char}(\Bbbk) > 0$, is zero-dimensional of height 1; conversely, if G is such a group-scheme then $F[G]$ has these properties. Instead, we call H a "restricted universal enveloping algebra" (=rUEA) if it is cocommutative, connected, and generated by its primitive part. If D is a field, an rUEA is the restricted universal enveloping algebra of some (restricted) Lie algebra over D; conversely, if $\mathfrak{g}$ is such a Lie algebra, then $\mathcal{U}(\mathfrak{g})$ has these properties (see, e.g., [Mo], Theorem 5.6.5, and references therein).

For the Hopf operations in any Hopf algebra we shall use standard notation, as in [Ab].

Definition 1.2.

(a) Let H, K be Hopf algebras (in any category). A pairing $\langle\ ,\ \rangle : H \times K \longrightarrow R$ (where R is the ground ring) is a Hopf (algebra) pairing if $\big\langle x, y_1 \cdot y_2 \big\rangle = \big\langle \Delta(x), y_1 \otimes y_2 \big\rangle := \sum_{(x)} \big\langle x_{(1)}, y_1 \big\rangle \cdot \big\langle x_{(2)}, y_2 \big\rangle$, $\big\langle x_1 \cdot x_2, y \big\rangle = \big\langle x_1 \otimes x_2, \Delta(y) \big\rangle := \sum_{(y)} \big\langle x_1, y_{(1)} \big\rangle \cdot \big\langle x_2, y_{(2)} \big\rangle$, $\langle x, 1 \rangle = \epsilon(x)$, $\langle 1, y \rangle = \epsilon(y)$, $\big\langle S(x), y \big\rangle = \big\langle x, S(y) \big\rangle$, *for all* $x, x_1, x_2 \in H$, $y, y_1, y_2 \in K$.

(b) Let $\mathfrak{g}$, $\mathfrak{h}$ be Lie bialgebras (in any category). A pairing $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{h} \longrightarrow \Bbbk$ (where $\Bbbk$ is the ground ring) is called a Lie bialgebra pairing if $\big\langle x, [y_1, y_2] \big\rangle = \big\langle \delta(x), y_1 \otimes y_2 \big\rangle := \sum_{[x]} \big\langle x_{[1]}, y_1 \big\rangle \cdot \big\langle x_{[2]}, y_2 \big\rangle$, $\big\langle [x_1, x_2], y \big\rangle = \big\langle x_1 \otimes x_2, \delta(y) \big\rangle := \sum_{[y]} \big\langle x_1, y_{[1]} \big\rangle \cdot \big\langle x_2, y_{[2]} \big\rangle$, *for all* $x, x_1, x_2 \in \mathfrak{g}$ *and* $y, y_1, y_2 \in \mathfrak{h}$, *with* $\delta(x) = \sum_{[x]} x_{[1]} \otimes x_{[2]}$ *and* $\delta(y) = \sum_{[x]} y_{[1]} \otimes y_{[2]}$.

1.3 The quantum setting. Let R be a (integral) domain, and let $F = F(R)$ be its quotient field. Denote by $\mathcal{M}$ the category of torsion-free R–modules, and by $\mathcal{HA}$ the category of all Hopf algebras in $\mathcal{M}$; note that *flat* modules form a full subcategory of $\mathcal{M}$. Let $\mathcal{M}_F$ be the category of F–vector spaces, and $\mathcal{HA}_F$ be the category of all Hopf algebras in $\mathcal{M}_F$. For any $M \in \mathcal{M}$, set $M_F := F(R) \otimes_R M$. Scalar extension gives a functor $\mathcal{M} \longrightarrow \mathcal{M}_F$, $M \mapsto M_F$, which restricts to a functor $\mathcal{HA} \longrightarrow \mathcal{HA}_F$ as well.

Let $\hbar \in R$ be a non-zero element (which will be fixed throughout), and let $\Bbbk := R\big/(\hbar) = R\big/\hbar R$ be the quotient ring. For any R–module M, we set $M_\hbar\Big|_{\hbar=0} := M\big/\hbar M = \Bbbk \otimes_R M$: this is a $\Bbbk$–module (via scalar restriction $R \to R\big/\hbar R =: \Bbbk$), which we call the *specialization* of M at $\hbar = 0$; we use also notation $M \xrightarrow{\ \hbar \to 0\ } \overline{M}$ to mean shortly that $M_\hbar\Big|_{\hbar=0} \cong \overline{M}$. Moreover, set $M_\infty := \bigcap_{n=0}^{+\infty} \hbar^n M$ (this is the closure of $\{0\}$ in the $\hbar$–adic topology of M). For any $H \in \mathcal{HA}$, let $I_H := \mathit{Ker}\Big(H \overset{\epsilon}{\twoheadrightarrow} R \xrightarrow{\ \hbar \mapsto 0\ } \Bbbk\Big)$ and set $I_H^\infty := \bigcap_{n=0}^{+\infty} I_H^{\,n}$.

Finally, given $\mathbb{H}$ in $\mathcal{HA}_F$, a subset $\overline{H}$ of $\mathbb{H}$ is called *an R–integer form* (or simply *an R–form*) *of* $\mathbb{H}$ iff $\overline{H}$ is a Hopf R–subalgebra of $\mathbb{H}$ (so $\overline{H}$ is torsion-free as an R–module, hence $\overline{H} \in \mathcal{HA}$) and $H_F := F(R) \otimes_R \overline{H} = \mathbb{H}$.

We are now ready to introduce the notion of "quantum group".

Definition 1.4. *("Global quantum groups" [or "algebras"]) Let R, $\hbar \in R \setminus \{0\}$ be as in §1.3.*

(a) We call quantized restricted universal enveloping algebra *(in short, QrUEA) (at $\hbar$) any $\mathcal{U}_\hbar \in \mathcal{HA}$ such that $\mathcal{U}_\hbar\big|_{\hbar=0} := \mathcal{U}_\hbar \big/ \hbar\, \mathcal{U}_\hbar$ is (isomorphic to) an rUEA.*

We call $\mathcal{QrUEA}$ the full subcategory of $\mathcal{HA}$ whose objects are all the QrUEAs (at $\hbar$).

(b) We call quantized function algebra *(in short, QFA) (at $\hbar$) any $F_\hbar \in \mathcal{HA}$ such that $\left(F_\hbar\right)_\infty = {I_{F_\hbar}}^{\infty}$ (notation of §1.3)[1] and $F_\hbar\big|_{\hbar=0} := F_\hbar \big/ \hbar\, F_\hbar$ is (isomorphic to) an FA.*

We call $\mathcal{QFA}$ the full subcategory of $\mathcal{HA}$ whose objects are all the QFAs (at $\hbar$).

Remark 1.5: If $\mathcal{U}_\hbar$ is a QrUEA (at $\hbar$, that is w.r.t. to $\hbar$) then $\mathcal{U}_\hbar\big|_{\hbar=0}$ is a co-Poisson Hopf algebra, w.r.t. the Poisson cobracket δ defined as follows: if $x \in \mathcal{U}_\hbar\big|_{\hbar=0}$ and $x' \in \mathcal{U}_\hbar$ gives $x = x' \mod \hbar\,\mathcal{U}_\hbar$, then $\delta(x) := \big(\hbar^{-1}\left(\Delta(x') - \Delta^{\mathrm{op}}(x')\right)\big) \mod \hbar\left(\mathcal{U}_\hbar \otimes \mathcal{U}_\hbar\right)$. If $\Bbbk := R \big/ \hbar\, R$ is a field, then $\mathcal{U}_\hbar\big|_{\hbar=0} \cong \mathcal{U}(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$, and by [Dr], §3, the restriction of δ makes $\mathfrak{g}$ into a *Lie bialgebra* (the isomorphism $\mathcal{U}_\hbar\big|_{\hbar=0} \cong \mathcal{U}(\mathfrak{g})$ being one of *co-Poisson* Hopf algebras); in this case we write $\mathcal{U}_\hbar = \mathcal{U}_\hbar(\mathfrak{g})$.

Similarly, if $F_\hbar$ is a QFA at $\hbar$, then $F_\hbar\big|_{\hbar=0}$ is a *Poisson* Hopf algebra, w.r.t. the Poisson bracket $\{\ ,\ \}$ defined as follows: if $x, y \in F_\hbar\big|_{\hbar=0}$ and $x', y' \in F_\hbar$ give $x = x' \mod \hbar\, F_\hbar$, $y = y' \mod \hbar\, F_\hbar$, then $\{x, y\} := \big(\hbar^{-1}(x'\, y' - y'\, x')\big) \mod \hbar\, F_\hbar$. Therefore , if $\Bbbk := R \big/ \hbar\, R$ is a field, then $F_\hbar\big|_{\hbar=0} \cong F[G]$ for some connected *Poisson* algebraic group G (the isomorphism being one of *Poisson* Hopf algebras): in this case we write $F_\hbar = F_\hbar[G]$.

Definition 1.6.

(a) Let R be any (integral) domain, and let F be its field of fractions. Given two F–modules $\mathbb{A}$, $\mathbb{B}$, and an F–bilinear pairing $\mathbb{A} \times \mathbb{B} \longrightarrow F$, for any R–submodule $A \subseteq \mathbb{A}$ and $B \subseteq \mathbb{B}$ we set $A^\bullet := \Big\{ b \in \mathbb{B} \;\Big|\; \big\langle A, b \big\rangle \subseteq R \Big\}$ and $B^\bullet := \Big\{ a \in \mathbb{A} \;\Big|\; \big\langle a, B \big\rangle \subseteq R \Big\}$.

(b) Let R be a domain. Given H, $K \in \mathcal{HA}$, we say that H and K are dual to each other *if there exists a perfect Hopf pairing between them for which $H = K^\bullet$ and $K = H^\bullet$.*

§ 2 The global quantum duality principle

2.1 Drinfeld's functors. (Cf. [Dr], §7) Let R, $\mathcal{HA}$ and $\hbar \in R$ be as in §1.3. For any $H \in \mathcal{HA}$, let $I = I_H := Ker\Big(H \overset{\epsilon}{\twoheadrightarrow} R \xrightarrow{\hbar \mapsto 0} R \big/ \hbar\, R = \Bbbk\Big) = Ker\Big(H \xrightarrow{\hbar \mapsto 0} H \big/ \hbar\, H \overset{\bar{\epsilon}}{\twoheadrightarrow} \Bbbk\Big)$ (as in §1.3), a maximal Hopf ideal of H (where $\bar{\epsilon}$ is the counit of $H\big|_{\hbar=0}$, and the two composed maps clearly coincide): we define

$$H^\vee := \textstyle\sum_{n \geq 0} \hbar^{-n} I^n = \sum_{n \geq 0} \left(\hbar^{-1} I\right)^n = \bigcup_{n \geq 0} \left(\hbar^{-1} I\right)^n \quad \left(\subseteq H_F\right).$$

If $J = J_H := Ker\left(\epsilon_H\right)$ then $I = J + \hbar \cdot 1_H$, so $H^\vee = \sum_{n \geq 0} \hbar^{-n} J^n = \sum_{n \geq 0} \left(\hbar^{-1} J\right)^n$ too.

[1]This requirement turns out to be a natural one, see Theorem 3.8.

Given any Hopf algebra H, for every $n \in \mathbb{N}$ define $\Delta^n \colon H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := \mathrm{id}_H$, and $\Delta^n := \big(\Delta \otimes \mathrm{id}_H^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ if $n > 2$. For any ordered subset $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, define the morphism $j_\Sigma : H^{\otimes k} \longrightarrow H^{\otimes n}$ by $j_\Sigma(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_{i_m} := a_m$ for $1 \leq m \leq k$; then set $\Delta_\Sigma := j_\Sigma \circ \Delta^k$, $\Delta_\emptyset := \Delta^0$, and $\delta_\Sigma := \sum_{\Sigma' \subset \Sigma} (-1)^{n - |\Sigma'|} \Delta_{\Sigma'}$, $\delta_\emptyset := \epsilon$. By the inclusion-exclusion principle, this definition admits the inverse formula $\Delta_\Sigma = \sum_{\Psi \subseteq \Sigma} \delta_\Psi$. We shall also use the notation $\delta_0 := \delta_\emptyset$, $\delta_n := \delta_{\{1,2,\dots,n\}}$, and the useful formula $\delta_n = (\mathrm{id}_H - \epsilon)^{\otimes n} \circ \Delta^n$, for all $n \in \mathbb{N}_+$.

Now consider any $H \in \mathcal{HA}$ and $\hbar \in R$ as in §1.3: we define

$$H' := \big\{ a \in H \,\big|\, \delta_n(a) \in \hbar^n H^{\otimes n}, \ \forall\, n \in \mathbb{N} \big\} \quad \big(\subseteq H\big).$$

Theorem 2.2. *("The Global Quantum Duality Principle") Assume $\Bbbk := R\big/\hbar R$ is a field.*

(a) The assignment $H \mapsto H^\vee$, resp. $H \mapsto H'$, defines a functor $(\ )^\vee \colon \mathcal{HA} \longrightarrow \mathcal{HA}$, resp. $(\ )' \colon \mathcal{HA} \longrightarrow \mathcal{HA}$, whose image lies in $\mathcal{QrUEA}$, resp. in $\mathcal{QFA}$. Moreover, for all $H \in \mathcal{HA}$ we have $H \subseteq \big(H^\vee\big)'$ and $H \supseteq \big(H'\big)^\vee$, hence also $H^\vee = \big(\big(H^\vee\big)'\big)^\vee$ and $H' = \big(\big(H'\big)^\vee\big)'$. Finally, if $H \in \mathcal{HA}$ is flat, then $H^\vee$ and H' are flat as well.

(b) Assume that $Char(\Bbbk) = 0$. Then for any $H \in \mathcal{HA}$

$$H = \big(H^\vee\big)' \iff H \in \mathcal{QFA} \qquad \text{and} \qquad H = \big(H'\big)^\vee \iff H \in \mathcal{QrUEA} \quad ;$$

thus we have two induced equivalences, namely $(\ )^\vee \colon \mathcal{QFA} \longrightarrow \mathcal{QrUEA}$, $H \mapsto H^\vee$, and $(\ )' \colon \mathcal{QrUEA} \longrightarrow \mathcal{QFA}$, $H \mapsto H'$, which are inverse to each other.

(c) ("Quantum Duality Principle") Assume that $Char(\Bbbk) = 0$. Then

$$F_\hbar[G]^\vee\Big|_{\hbar=0} := F_\hbar[G]^\vee \Big/ \hbar\, F_\hbar[G]^\vee = U(\mathfrak{g}^\times), \quad U_\hbar(\mathfrak{g})'\Big|_{\hbar=0} := U_\hbar(\mathfrak{g})' \Big/ \hbar\, U_\hbar(\mathfrak{g})' = F\big[G^\star\big]$$

(with G, $\mathfrak{g}$, $\mathfrak{g}^\times$, $\mathfrak{g}^\star$ and $G^\star$ as in §1.1, and $U_\hbar(\mathfrak{g})$ has the obvious meaning, cf. §1.5) where the choice of the group $G^\star$ — among all the connected Poisson algebraic groups with tangent Lie bialgebra $\mathfrak{g}^\star$ — depends on the choice of the QrUEA $U_\hbar(\mathfrak{g})$. In other words, $F_\hbar[G]^\vee$ is a QrUEA for the Lie bialgebra $\mathfrak{g}^\times$, and $U_\hbar(\mathfrak{g})'$ is a QFA for the Poisson group $G^\star$.

(d) Let $Char(\Bbbk) = 0$. Let $F_\hbar \in \mathcal{QFA}$, $U_\hbar \in \mathcal{QrUEA}$ be dual to each other (with respect to some pairing). Then ${F_\hbar}^\vee$ and ${U_\hbar}'$ are dual to each other (w.r.t. the same pairing).

(e) Let $Char(\Bbbk) = 0$. Then for all $\mathbb{H} \in \mathcal{HA}_F$ the following are equivalent:

$\mathbb{H}$ has an R–integer form $H_{(f)}$ which is a QFA at $\hbar$;

$\mathbb{H}$ has an R–integer form $H_{(u)}$ which is a QrUEA at $\hbar$.

Remarks 2.3: after stating our main theorem, some comments are in order.

(a) The Global Quantum Duality Principle as a *"Galois correspondence" type theorem.* Let $L \subseteq E$ be a Galois (not necessarily finite) field extension, and let $G := Gal\big(E/L\big)$ be its Galois group. Let $\mathcal{F}$ be the set of intermediate extensions (i.e. all fields F such that

$L \subseteq F \subseteq E$), let $\mathcal{S}$ be the set of all subgroups of G and let $\mathcal{S}^c$ be the set of all subgroups of G which are *closed* w.r.t. the Krull topology of G. Note that $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{S}^s$ can all be seen as lattices w.r.t. set-theoretical inclusion — $\mathcal{S}^c$ being a sublattice of $\mathcal{S}$ — hence as categories too. The celebrated Galois Theorem yields two maps, namely $\Phi \colon \mathcal{F} \longrightarrow \mathcal{S}$, $F \mapsto \mathit{Gal}\big(E/F\big) := \big\{ \gamma \in G \;\big|\; \gamma\big|_F = \mathrm{id}_F \big\}$, and $\Psi \colon \mathcal{S} \longrightarrow \mathcal{F}$, $H \mapsto E^H := \big\{ e \in E \;\big|\; \eta(e) = e \;\; \forall\, \eta \in H \big\}$, such that:

— *1)* Φ and Ψ are contravariant functors (that is, they are order-reversing maps of lattices, i.e. lattice antimorphisms); moreover, the image of Φ lies in the subcategory $\mathcal{S}^c$;

— *2)* for $H \in \mathcal{S}$ one has $\Phi\big(\Psi(H)\big) = \overline{H}$, the *closure* of H w.r.t. the Krull topology: thus $H \subseteq \Phi\big(\Psi(H)\big)$, and $\Phi \circ \Psi$ is a *closure operator*, so that $H \in \mathcal{S}^c$ iff $H = \Phi\big(\Psi(H)\big)$;

— *3)* for $F \in \mathcal{F}$ one has $\Psi\big(\Phi(F)\big) = F$;

— *4)* Φ and Ψ restrict to antiequivalences $\Phi : \mathcal{F} \to \mathcal{S}^c$ and $\Psi : \mathcal{S}^c \to \mathcal{F}$ which are inverse to each other.

Then one can see that Theorem 2.2 establishes a strikingly similar result, which in addition is much more symmetric: $\mathcal{HA}$ plays the role of both $\mathcal{F}$ and $\mathcal{S}$, whereas $(\ )'$ stands for Ψ and $(\ )^\vee$ stands for Φ. $\mathcal{QFA}$ plays the role of the distinguished subcategory $\mathcal{S}^c$, and symmetrically we have the distinguished subcategory $\mathcal{Q}r\mathcal{UEA}$. The composed operator $\big((\ )^\vee\big)' = (\ )' \circ (\ )^\vee$ plays the role of a "closure operator", and symmetrically $\big((\ )'\big)^\vee = (\ )^\vee \circ (\ )'$ plays the role of a "taking-the-interior operator": in other words, QFAs may be thought of as "closed sets" and QrUEAs as "open sets" in $\mathcal{HA}$.

(b) Duality between Drinfeld's functors. For any $n \in \mathbb{N}$ let $\mu_n \colon {J_H}^{\otimes n} \hookrightarrow H^{\otimes n} \xrightarrow{m^n} H$ be the composition of the natural embedding of ${J_H}^{\otimes n}$ into $H^{\otimes n}$ with the n–fold multiplication (in H): then μ_n is the "Hopf dual" to δ_n. By construction we have $H^\vee = \sum_{n \in \mathbb{N}} \mu_n\big(\hbar^{-n} {J_H}^{\otimes n}\big)$ and $H' = \bigcap_{n \in \mathbb{N}} {\delta_n}^{-1}\big(\hbar^{+n} {J_H}^{\otimes n}\big)$: this shows that the two functors are built up as "dual" to each other (cf. also part *(d)* of Theorem 2.2).

(c) Ambivalence QrUEA $\leftrightarrow$ QFA in $\mathcal{HA}_F$. Part (e) of Theorem 2.2 means that some Hopf algebras *over* $F(R)$ might be thought of *both* as "quantum function algebras" *and* as "quantum enveloping algebras": examples are U_F and F_F for $U \in \mathcal{Q}r\mathcal{UEA}$ and $F \in \mathcal{QFA}$.

(d) Drinfeld's functors for algebras, coalgebras and bialgebras. The definition of either of Drinfeld functors requires only "half of" the notion of Hopf algebra. In fact, one can define $(\ )^\vee$ for all "augmented algebras" (that is, roughly speaking, "algebras with a counit") and $(\ )'$ for all "coaugmented coalgebras" (roughly, "coalgebras with a unit"), and in particular for bialgebras: this yields again nice functors, and neat results extending the global quantum duality principle hold for them; we shall prove all this in the next section.

§ 3 General properties of Drinfeld's functors

3.1 Augmented algebras, coaugmented coalgebras and Drinfeld's functors for them. Let R be a commutative ring with 1, $\mathcal{M}$ the category of torsion-free R–modules.

We call *augmented algebra* the datum of a unital associative algebra $A \in \mathcal{M}$ with a distinguished unital algebra morphism $\underline{\epsilon} : A \longrightarrow R$ (so the unit map $u : R \longrightarrow A$ is a section of $\underline{\epsilon}$): these form a category in the obvious way. We call *indecomposable elements* of an augmented algebra A the elements of the set $Q(A) := J_{{}_A} \big/ J_{{}_A}^{\,2}$ with $J_{{}_A} := Ker\big(\underline{\epsilon} : A \longrightarrow R\big)$. We denote $\mathcal{A}^+$ the category of all augmented algebras in $\mathcal{M}$.

We call *coaugmented coalgebra* any counital coassociative coalgebra C with a distinguished counital coalgebra morphism $\underline{u} : R \longrightarrow C$ (so $\underline{u}$ is a section of the counit map $\epsilon : C \longrightarrow R$), and let $\underline{1} := \underline{u}(1)$, a group-like element in C: these form a category in the obvious way. For such a C we said *primitive* the elements of the set $P(C) := \{\, c \in C \,|\, \Delta(c) = c \otimes \underline{1} + \underline{1} \otimes c \,\}$. We denote $\mathcal{C}^+$ the category of all coaugmented coalgebras in $\mathcal{M}$.

We denote $\mathcal{B}$ the category of all bialgebras in $\mathcal{M}$; clearly each bialgebra B can be seen both as an augmented algebra, w.r.t. $\underline{\epsilon} = \epsilon \equiv \epsilon_{{}_B}$ (the counit of B) and as a coaugmented coalgebra, w.r.t. $\underline{u} = u \equiv u_{{}_B}$ (the unit map of B), so that $\underline{1} = 1 = 1_{{}_B}$: then $Q(B)$ is naturally a Lie coalgebra and $P(B)$ a Lie algebra over R. In the following we'll do such an interpretation throughout, looking at objects of $\mathcal{B}$ as objects of $\mathcal{A}^+$ and of $\mathcal{C}^+$.

Now let R be a domain, and fix $\hbar \in R \setminus \{0\}$ as in §1.3. Let $A \in \mathcal{A}^+$, and $I = I_{{}_A} := \\ := Ker\Big(A \overset{\underline{\epsilon}}{\twoheadrightarrow} R \xrightarrow{\hbar \mapsto 0} R\big/\hbar\, R = \Bbbk\Big) = Ker\Big(A \xrightarrow{\hbar \mapsto 0} A\big/\hbar\, A \xrightarrow{\underline{\epsilon}\,|_{\hbar=0}} \Bbbk\Big)$ as in §1.3, a maximal Hopf ideal of A (where $\underline{\epsilon}\,\big|_{\hbar=0}$ is the counit of $A\big|_{\hbar=0}$, and the two composed maps do coincide): like in §2.1, we define

$$A^\vee \; := \; \textstyle\sum_{n \geq 0} \hbar^{-n} I^n \; = \; \sum_{n \geq 0} \big(\hbar^{-1} I\big)^n \; = \; \bigcup_{n \geq 0} \big(\hbar^{-1} I\big)^n \quad \big(\subseteq A_F\big)\,.$$

If $J = J_{{}_A} := Ker(\epsilon_{{}_A})$ then $I = J + \hbar \cdot 1_{{}_A}$, thus $A^\vee = \sum_{n \geq 0} \hbar^{-n} J^n = \sum_{n \geq 0} \big(\hbar^{-1} J\big)^n$.

Given any coalgebra C, for every $n \in \mathbb{N}$ define $\Delta^n : C \longrightarrow C^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := \mathrm{id}_{{}_C}$, and $\Delta^n := \big(\Delta \otimes \mathrm{id}_{{}_C}^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ if $n > 2$. If C is *coaugmented*, for any ordered subset $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, define the morphism $j_{{}_\Sigma} : C^{\otimes k} \longrightarrow C^{\otimes n}$ by $j_{{}_\Sigma}(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := \underline{1}$ if $i \notin \Sigma$ and $b_{i_m} := a_m$ for $1 \leq m \leq k$; then set $\Delta_\Sigma := j_{{}_\Sigma} \circ \Delta^k$, $\Delta_\emptyset := \Delta^0$, and $\delta_\Sigma := \sum_{\Sigma' \subset \Sigma} (-1)^{n - |\Sigma'|} \Delta_{\Sigma'}$, $\delta_\emptyset := \epsilon$. Like in §2.1, the inverse formula $\Delta_\Sigma = \sum_{\Psi \subseteq \Sigma} \delta_\Psi$ holds. We'll also use notation $\delta_0 := \delta_\emptyset$, $\delta_n := \delta_{\{1,2,\dots,n\}}$, and the useful formula $\delta_n = \big(\mathrm{id}_{{}_C} - \epsilon \cdot \underline{1}\,\big)^{\otimes n} \circ \Delta^n$, for all $n \in \mathbb{N}_+$.

Now consider any $C \in \mathcal{C}^+$ and $\hbar \in R$ as in §1.3. We define

$$C' \; := \; \big\{\, c \in C \,\big|\, \delta_n(c) \in \hbar^n C^{\otimes n}, \; \forall \; n \in \mathbb{N} \,\big\} \quad \big(\subseteq C\big)\,.$$

In particular, according to our general remark above for any $B \in \mathcal{B}$ (and any prime element $\hbar \in R$ as above) $B^\vee$ is defined w.r.t. $\underline{\epsilon} = \epsilon_{{}_B}$ and B' is defined w.r.t. $\underline{1} = 1_{{}_B}$.

Lemma 3.2. *Let $H \in \mathcal{HA}$, and set $\overline{H} := H \big/ H_\infty$ (notation of §1.3). Then:*

(a) $H_\infty = (H')_\infty$, $H_\infty \subseteq \big(H^\vee\big)_\infty$, H_∞ is a Hopf ideal of H, and $\big(\overline{H}\,\big)_\infty = \{0\}$. Moreover, there are natural isomorphisms $\big(\overline{H}\,\big)^\vee = H^\vee \big/ H_\infty$, $\big(\overline{H}\,\big)' = H' \big/ H_\infty$.

(b) $\overline{H} \in \mathcal{HA}$, and $\overline{H}\big|_{\hbar=0} = H\big|_{\hbar=0}$. In particular, if $H\big|_{\hbar=0}$ has no zero-divisors the same holds for H, and if H is a QFA, resp. a QrUEA, then $\overline{H}$ is a QFA, resp. a QrUEA.

(c) Analogous statements hold for any $A \in \mathcal{A}^+$, any $C \in \mathcal{C}^+$ and any $B \in \mathcal{B}$.

Proof. Trivial from definitions. $\square$

Proposition 3.3. *Let $A \in \mathcal{A}^+$, $B \in \mathcal{B}$, $H \in \mathcal{HA}$. Then $A^\vee \in \mathcal{A}^+$, $B^\vee \in \mathcal{B}$, and $H^\vee \in \mathcal{HA}$. If in addition A, resp. B, is flat, then $A^\vee$, resp. $B^\vee$, is flat as well.*

Proof. First, we have $A^\vee$, $B^\vee$, $H^\vee \in \mathcal{M}$, for they are clearly torsion-free. In addition, $A^\vee$ is obtained from A in two steps: localisation — namely, $A \rightarrowtail\!\!\rightsquigarrow A\big[\hbar^{-1}\big]$ — and restriction, i.e. taking a submodule — namely, $A\big[\hbar^{-1}\big] \rightarrowtail\!\!\rightsquigarrow A^\vee \big(\subseteq A\big[\hbar^{-1}\big] \big)$. Both these steps preserve flatness, hence if A is flat then $A^\vee$ is flat too, and the same for B and $B^\vee$.

Second, $A^\vee := \sum_{n=0}^{\infty} \hbar^n J^n$ where $J := Ker\,(\epsilon_A)$, so $A^\vee$ is clearly an R–subalgebra of A_F, hence $A^\vee \in \mathcal{A}^+$; similarly holds for B and H of course. Moreover, J_B is bi-ideal of B, so $\Delta(J_B) \subseteq B \otimes J_B + J_B \otimes B$, hence $\Delta\big({J_B}^n\big) \subseteq \sum\limits_{r+s=n} {J_B}^r \otimes {J_B}^s$ for all $n \in \mathbb{N}$, thus $\Delta\big(\hbar^{-n} {J_B}^n\big) \subseteq \hbar^{-n} \sum\limits_{r+s=n} {J_B}^r \otimes {J_B}^s = \sum\limits_{r+s=n} \big(\hbar^{-r} {J_B}^r\big) \otimes \big(\hbar^{-s} {J_B}^s\big) \subseteq B^\vee \otimes B^\vee$ for all $n \in \mathbb{N}$, whence $\Delta\big(B^\vee\big) \subseteq B^\vee \otimes B^\vee$ which means $B^\vee \in \mathcal{B}$. Finally, for H we have in addition $S\big({J_H}^n\big) = {J_H}^n$ (for all $n \in \mathbb{N}$) because J_H is a Hopf ideal, therefore $S\big(\hbar^{-n} {J_H}^n\big) = \hbar^{-n} {J_H}^n$ (for all $n \in \mathbb{N}$), thus $S\big(H^\vee\big) = H^\vee$ and so $H^\vee \in \mathcal{HA}$. $\square$

Lemma 3.4. *Let B be any bialgebra. Let a, $b \in B$, and let $\Phi \subseteq \mathbb{N}$, with Φ finite. Then*

(a) $\quad \delta_\Phi(ab) = \sum\limits_{\Lambda \cup Y = \Phi} \delta_\Lambda(a)\, \delta_Y(b)\,;$

(b) $\quad$ *if $\Phi \neq \emptyset$, then* $\quad \delta_\Phi(ab - ba) = \sum\limits_{\substack{\Lambda \cup Y = \Phi \\ \Lambda \cap Y \neq \emptyset}} \big(\delta_\Lambda(a)\, \delta_Y(b) - \delta_Y(b)\, \delta_\Lambda(a) \big)\,;$

(c) $\quad$ *if the ground ring of B is a field, and if $D_n := Ker\,(\delta_{n+1})$ (for all $n \in \mathbb{N}$), then*

$$\underline{D} \ : \quad \{0\} =: D_{-1} \subseteq D_0 \subseteq D_1 \subseteq \cdots D_n \subseteq \cdots \quad (\subseteq B\,)$$

is a bialgebra filtration of B with $\big[D_m, D_n\big] \subseteq D_{m+n-1}$ ($\forall\, m, n \in \mathbb{N}$), hence the associated graded bialgebra is commutative. If $B = H$ is a Hopf algebra, then $\underline{D}$ is a Hopf algebra filtration, so the associated graded bialgebra is a commutative graded Hopf algebra.

Proof. (a) (cf. [KT], Lemma 3.2) First, notice that the inversion formula $\Delta_\Phi = \sum_{\Psi \subseteq \Phi} \delta_\Psi$ (see §2.1) gives $\sum_{\Psi \subseteq \Phi} \delta_\Psi(ab) = \Delta_\Phi(ab) = \Delta_\Phi(a)\, \Delta_\Phi(b) = \sum_{\Lambda, Y \subseteq \Phi} \delta_\Lambda(a)\, \delta_Y(b)$; this can be rewritten as

$$\textstyle\sum_{\Psi \subseteq \Phi} \delta_\Psi(ab) \ = \ \sum_{\Psi \subseteq \Phi} \sum_{\Lambda \cup Y = \Psi} \delta_\Lambda(a)\, \delta_Y(b)\,. \tag{3.1}$$

We prove the claim by induction on the cardinality $|\Phi|$ of Φ. If $\Phi = \emptyset$ then $\delta_\Phi = j_\emptyset \circ \epsilon$, which is a morphism of algebras, so the claim does hold. Now assume it holds for all sets of cardinality less than $|\Phi|$, hence also for all proper subsets of Φ: then the right-hand-side of (3.1) equals $\sum\limits_{\Psi \subsetneqq \Phi} \delta_\Psi(ab) + \sum\limits_{\Lambda \cup Y = \Phi} \delta_\Lambda(a)\, \delta_Y(b)$. Then the claim follows by subtracting from both sides of (3.1) the summands corresponding to the proper subsets Ψ of Φ.

(b) (cf. [KT], Lemma 3.2) The very definitions give $\delta_\Lambda(a)\,\delta_Y(b) = \delta_Y(b)\,\delta_\Lambda(a)$ when $\Lambda \cap Y = \emptyset$, so the claim follows from this and from *(a)*.

(c) Let $a \in D_m$, $b \in D_n$: then $a\,b \in D_{m+n}$ because part *(a)* gives $\delta_{m+n+1}(ab) = \sum_{\Lambda \cup Y = \{1,\dots,m+n+1\}} \delta_\Lambda(a)\,\delta_Y(b) = 0$ since in the sum one has $|\Lambda| > m$ or $|Y| > n$ which forces $\delta_\Lambda(a) = 0$ or $\delta_Y(b) = 0$. Similarly, $[a,b] \in D_{m+n-1} \leq m+n-1$ because part *(b)* yields $\delta_{m+n}\big([a,b]\big) = \sum_{\substack{\Lambda \cup Y = \{1,\dots,m+n\} \\ \Lambda \cap Y \neq \emptyset}} \delta_\Lambda(a)\,\delta_Y(b) = 0$.

Second, we prove that $\Delta\big(D_n\big) \subseteq \sum_{r+s=n} D_r \otimes D_s$, for all $n \in \mathbb{N}$. Let $\eta \in D_n \setminus D_{n-1}$. Then $\Delta(\eta) = \epsilon(\eta) \cdot 1 \otimes 1 + \eta \otimes 1 + 1 \otimes \eta + \delta_2(\eta)$; since $D_0 := Ker(\delta_1) = \langle 1 \rangle = \Bbbk \cdot 1$ we need only to show that $\delta_2(\eta) \in \sum_{r+s=n} D_r \otimes D_s$. We can write $\delta_2(\eta) = \sum_j u_j \otimes v_j$ with $u_j, v_j \in J := Ker(\epsilon)$ — so that $\delta_1(u_j) = u_j$ for all j — and the u_j's linearly independent among themselves. By coassociativity of Δ one has $(\delta_r \otimes \delta_s) \circ \delta_2 = \delta_{r+s}$ (for all $r, s \in \mathbb{N}$); therefore, $0 = \delta_{n+1}(\eta) = \sum_j \delta_1(u_j) \otimes \delta_n(v_j) = \sum_j u_j \otimes \delta_n(v_j)$: since the u_j's are linearly independent, this yields $\delta_n(v_j) = 0$, that is $v_j \in Ker(\delta_n) =: D_{n-1}$, for all j.

Now, set $D_n^J := D_n \cap J$ for $n \in \mathbb{N}$. Splitting J as $J = D_1^J \oplus W_1$ — for some subspace W_1 of J — we can rewrite $\delta_2(\eta)$ as $\delta_2(\eta) = \sum_i u_i^{(1)} \otimes v_i^{(n-1)} + \sum_h u_h^+ \otimes v_h^+$, where $u_i^{(1)} \in D_1^J$, $u_h^+ \in W_1$, $v_i^{(n-1)}, v_h^+ \in D_{n-1}^J$ (for all i, h) and the u_h^+'s are linearly independent. Then also the $\delta_2\big(u_h^+\big)$'s are linearly independent: indeed, if $\sum_h c_h\, \delta_2\big(u_h^+\big) = 0$ for some scalars c_h then $\sum_h c_h\, u_h^+ \in Ker(\delta_2) =: D_1$, forcing $c_h = 0$ for all h. Then again by coassociativity $0 = \delta_{n+1}(\eta) = (\delta_2 \otimes \delta_{n-1})\big(\delta_2(\eta)\big) = \sum_i \delta_2\big(u_h^+\big) \otimes \delta_{n-1}\big(v_h^+\big)$, which — as the $\delta_2\big(u_h^+\big)$'s are linearly independent — yields $\delta_{n-1}\big(v_h^+\big) = 0$, i.e. $v_h^+ \in D_{n-2}$, for all h.

Now we repeat the argument. Splitting J as $J = D_2 \oplus W_2$ — for some subspace W_2 of J — we can rewrite $\delta_2(\eta)$ as $\delta_2(\eta) = \sum_i u_i^{(1)} \otimes v_i^{(n-1)} + \sum_j u_j^{(2)} \otimes v_j^{(n-2)} + \sum_k u_k^* \otimes v_k^*$, where $u_j^{(2)} \in D_2 \bigcap J$, $v_j^{(n-2)}, v_k^* \in D_{n-2} \bigcap J$, $u_k^* \in W_2$ (for all j, k) and the u_k^*'s are linearly independent. Then also the $\delta_3\big(u_k^*\big)$'s are linearly independent (as above), and by coassociativity we get $0 = \delta_{n+1}(\eta) = (\delta_3 \otimes \delta_{n-2})\big(\delta_2(\eta)\big) = \sum_k \delta_3\big(u_k^*\big) \otimes \delta_{n-2}\big(v_k^*\big)$, which gives $\delta_{n-2}\big(v_k^*\big) = 0$, i.e. $v_k^* \in D_{n-3}$, for all k. Iterating this argument, we eventually stop getting $\delta_2(\eta) = \sum_i u_i^{(1)} \otimes v_i^{(n-1)} + \sum_j u_j^{(2)} \otimes v_j^{(n-2)} + \cdots + \sum_\ell u_\ell^{(n-1)} \otimes v_\ell^{(1)} = \sum_{s=1}^{n-1} \sum_t u_{s,t}^{(s)} \otimes v_{s,t}^{(n-s)}$ with $u_{s,t}^{(s)} \in D_s$, $v_{s,t}^{(n-s)} \in D_{n-s}$ for all s, t, so $\delta_2(\eta) \in \sum_{a+b=n} D_a \otimes D_b$, q.e.d.

Finally, if $B = H$ is a Hopf algebra then $\Delta \circ S = S^{\otimes 2} \circ \Delta$, hence $\Delta^n \circ S = S^{\otimes n} \circ \Delta^n$ ($n \in \mathbb{N}$), and $\epsilon \circ S = S \circ \epsilon$, thus $\delta^n \circ S = S^{\otimes n} \circ \delta^n$ (for all $n \in \mathbb{N}$) follows, which yields $S\big(D_n\big) \subseteq D_n$ for all $n \in \mathbb{N}$. Thus $\underline{D}$ is a Hopf algebra filtration, and the rest follows. $\square$

Proposition 3.5. *Assume that* $\Bbbk := R\big/\hbar R$ *is a field. Let* $C \in \mathcal{C}^+$, $B \in \mathcal{B}$, $H \in \mathcal{HA}$. *Then* $C' \in \mathcal{C}^+$, $B' \in \mathcal{B}$, *and* $H' \in \mathcal{HA}$. *Moreover, if* C, B, *is flat, then* C', B', *is flat too.*

Proof. First, by definition C' is an R–submodule of C, because the maps δ_n $(n \in \mathbb{N})$ are R–linear; since C is torsion-free, its submodule C' is torsion-free too, i.e. $C' \in \mathcal{M}$. In addition, C' is an R–submodule of C, and taking a submodule preserve flatness: hence if C is flat then C' is flat too. The same holds for B and B' as well.

We must show that C' is a subcoalgebra. Due to Lemma 3.2*(c)*, we can reduce to prove it for $\big(\overline{C}\,\big)'$, that is we can assume from scratch that $C_\infty = \{0\}$.

Let $\widehat{R}$ be the $\hbar$–adic completion of R. Let also $\widehat{C}$ be the $\hbar$–adic completion of C: this is a separated complete topological $\widehat{R}$–module, hence it is topologically free (i.e. of type $\widehat{R}^{\,Y}$ for some set Y); moreover, it is a topological Hopf algebra, whose coproduct takes values into the $\hbar$–adic completion $C \,\widehat{\otimes}\, C$ of $C \otimes C$. Since $C_\infty = \{0\}$, the natural map $C \longrightarrow \widehat{C}$ is a *monomorphism* of (topological) Hopf R–algebras, so C identifies with a Hopf R–subalgebra of $\widehat{C}$. Further, we have $\widehat{C}\Big/\hbar^n\widehat{C} = C\Big/\hbar^n C$ for all $n \in \mathbb{N}$. Finally, we set $\widehat{C}^* := Hom_{\widehat{R}}\Big(\widehat{C}\,,\,\widehat{R}\Big)$ for the dual of $\widehat{C}$.

Pick $a \in C'$; first we prove that $\Delta(a) \in C' \otimes C$: to this end, since $\widehat{C}$ is topologically free it is enough to show that $(\mathrm{id} \otimes f)\big(\Delta(a)\big) \in C' \otimes_R \widehat{R} \cdot \underline{1}$ for all $f \in \widehat{C}^*$, which amounts to show that $\big((\delta_n \otimes f) \circ \Delta\big)(a) \in \hbar^n C^{\otimes n} \otimes_R \widehat{R} \cdot \underline{1}$ for all $n \in \mathbb{N}_+$, $f \in \widehat{C}^*$. Now, we can rewrite the latter term as

$$\big((\delta_n \!\otimes\! f) \!\circ\! \Delta\big)(a) = \Big(\big((\mathrm{id} - \epsilon \cdot \underline{1}\,\big)^{\otimes n} \!\otimes\! f\,\big) \circ \Delta^{n+1}\Big)(a) = \delta_n(a) \!\otimes\! f(\underline{1}\,) \!\cdot\! \underline{1} + \big(\mathrm{id}^{\otimes n} \!\otimes\! f\,\big)\big(\delta_{n+1}(a)\big)$$

and the right-hand-side term does lie in $\hbar^n C^{\otimes n} \otimes_R \widehat{R} \cdot \underline{1}$, for $a \in C'$, q.e.d.

Definitions imply $\Delta(x) = -\epsilon(x) \cdot \underline{1} \otimes \underline{1} + x \otimes \underline{1} + \underline{1} \otimes x + \delta_2(x)$ for all $x \in C$. Due to the previous analysis, we argue that $\delta_2(a) \in C' \otimes C$ for all $a \in C'$, and we only need to show that $\delta_2(a) \in C' \otimes C'$: this will imply $\Delta(a) \in C' \otimes C'$ since $\underline{1} \in C'$ (as it is group-like).

Let $\widehat{C'}$ be the $\hbar$–adic completion of C': again, this is a topologically free $\widehat{R}$–module, and since $(C')_\infty = C_\infty = \{0\}$ (by Lemma 3.2*(a)* and our assumptions) the natural map $C' \longrightarrow \widehat{C'}$ is in fact an embedding, so C' identifies with an R–submodule of $\widehat{C'}$. If $\big\{\, \beta_j \,\big|\, j \in \mathcal{J} \,\big\}$ is a subset of C' whose image in $C'\big|_{\hbar=0}$ is a basis of the latter $\Bbbk$–vector space, then it is easy to see that $C' = \bigoplus_{j \in \mathcal{J}} \widehat{R}\, \beta_j$: fixing a section $\nu \colon \Bbbk \lhook\joinrel\longrightarrow R$ of the projection map $R \relbar\joinrel\twoheadrightarrow R\big/\hbar\, R =: \Bbbk$, this implies that each element $a \in C'$ has a unique expansion as a series $a = \sum_{n \in \mathbb{N}} \sum_{j \in \mathcal{J}} \nu(\kappa_{j,n})\, \hbar^n\, \beta_j$ for some $\kappa_{j,n} \in \Bbbk$ which, for fixed n, are almost all zero. Finally $\widehat{C'}\Big/\hbar\,\widehat{C'} = C'\Big/\hbar\, C' = B\Big/\hbar\, B$, with $B := \bigoplus_{j \in \mathcal{J}} R\, \beta_j$.

We shall also consider $\Big(\widehat{C'}\,\Big)^* := Hom_{\widehat{R}}\Big(\widehat{C'}\,,\,\widehat{R}\Big)$, the dual of $\widehat{C'}$.

Now, we have $\delta_2(a) \in C' \otimes C \subseteq \widehat{C'} \otimes C$, so we can expand $\delta_2(a)$ inside $\widehat{C'} \otimes C$ as $\delta_2(a) = \sum_{i \in \mathcal{I}} \Big(\sum_{n \in \mathbb{N}} \sum_{j \in \mathcal{J}} \nu\big(\kappa^i_{j,n}\big)\, \hbar^n\, \beta_j \Big) \otimes c_i$ for some $\kappa^i_{j,n} \in \Bbbk$ as above and $c_i \in C$

($\mathcal{I}$ being some finite set). Then we can rewrite $\delta_2(a)$ as

$$\delta_2(a) \,=\, \sum_{j \in \mathcal{J}} \beta_j \otimes \bigg(\sum_{n \in \mathbb{N}} \sum_{i \in \mathcal{I}} \nu\big(\kappa^i_{j,n}\big)\, \hbar^n \cdot c_i \bigg) \,=\, \sum_{j \in \mathcal{J}} \beta_j \otimes \gamma_j \;\in\; C' \mathbin{\widetilde{\otimes}} \widehat{C}$$

where $\gamma_j := \sum_{n \in \mathbb{N}} \sum_{i \in \mathcal{I}} \nu\big(\kappa^i_{j,n}\big)\, \hbar^n \cdot c_i \in \widehat{C}$ for all j, and $C' \mathbin{\widetilde{\otimes}} \widehat{C}$ is the completion of $C' \otimes \widehat{C}$ w.r.t. the weak topology. We contend that all the γ_j's belong to $\big(\widehat{C}\,\big)'$.

In fact, assume this is false: then there is $s \in \mathbb{N}_+$ such that $\delta_s(\gamma_i) \notin \hbar^s \widehat{C}^{\,\widehat{\otimes}\, s}$ for some $i \in \mathcal{J}$; we can choose such an i so that s be minimal, thus $\delta_{s'}(\gamma_j) \in \hbar^{s'} \widehat{C}^{\,\widehat{\otimes}\, s'}$ for all $j \in \mathcal{J}$ and $s' < s$; since $\delta_s = (\delta_2 \otimes \mathrm{id}) \circ \delta_{s-1}$ — by coassociativity — we have also $\delta_s(\gamma_j) \in \hbar^{s-1} \widehat{C}^{\,\widehat{\otimes}\, s}$ for all $j \in \mathcal{J}$. Now consider the element

$$A \,:=\, \textstyle\sum_{j \in \mathcal{J}} \overline{\beta_j} \otimes \overline{\delta_s(\gamma_j)} \;\in\; \Big(\widehat{C'} \Big/ \hbar\, \widehat{C'} \Big) \mathbin{\widetilde{\otimes}}_{\Bbbk} \Big(\hbar^{s-1} \widehat{C}^{\,\widehat{\otimes}\, s} \Big/ \hbar^s \widehat{C}^{\,\widehat{\otimes}\, s} \Big)$$

the right-hand-side space being equal to $\big(C' \big/ \hbar\, C'\big) \mathbin{\widetilde{\otimes}}_{\Bbbk} \big(\hbar^{s-1} C^{\otimes s} \big/ \hbar^s C^{\otimes s}\big)$; hereafter, such notation as $\overline{x}$ will always denote the coset of x in the proper quotient space. By construction, the $\overline{\beta_j}$'s are linearly independent and some of the $\overline{\delta_s(\gamma_j)}$'s are non zero: therefore A is non zero, and we can write it as $A = \sum_{\ell \in \mathcal{L}} \overline{\lambda_\ell} \otimes \overline{\chi_\ell}$ $(\neq 0)$ where $\mathcal{L}$ is a suitable non-empty index set, λ_ℓ (for all ℓ) belongs to the completion $\widetilde{C'}$ of $\widehat{C'}$ w.r.t. the weak topology, $\chi_\ell \in \hbar^{s-1} \widehat{C}^{\,\widehat{\otimes}\, s}$, the $\overline{\lambda_\ell}$'s are linearly independent in the $\Bbbk$–vector space $\widetilde{C'} \big/ \hbar\, \widetilde{C'}$ (which is just the completion of $C'\big|_{\hbar=0} := C' \big/ \hbar\, C'$ w.r.t. the weak topology), and the $\overline{\chi_\ell}$'s are linearly independent in the $\Bbbk$–vector space $\hbar^{s-1} \widehat{C}^{\,\widehat{\otimes}\, s} \Big/ \hbar^s \widehat{C}^{\,\widehat{\otimes}\, s}$. In particular $\lambda_\ell \notin \hbar\, \widetilde{C'}$ for all ℓ: so there is $r \in \mathbb{N}_+$ such that $\delta_r(\lambda_\ell) \in \hbar^r \widetilde{C}^{\,\widetilde{\otimes}\, r} \setminus \hbar^{r+1} \widetilde{C}^{\,\widetilde{\otimes}\, r}$ for all $\ell \in \mathcal{L}$ (hereafter, $K^{\widetilde{\otimes}\, m}$ denotes the completion of $K^{\otimes m}$ w.r.t. the weak topology), hence $\overline{\delta_r(\lambda_\ell)} \neq 0 \in \hbar^r \widetilde{C}^{\,\widetilde{\otimes}\, r} \Big/ \hbar^{r+1} \widetilde{C}^{\,\widetilde{\otimes}\, r}$. Now write $\overline{\delta_n}$ for the composition of δ_n with a projection map (such as $X \twoheadrightarrow X \big/ \hbar\, X$, say): then the outcome of this analysis is that

$$\big(\overline{\delta_r} \otimes \overline{\delta_s}\,\big)\big(\delta_2(a)\big) \,=\, \big(\overline{\delta_r} \otimes \mathrm{id}\big)\Big(\textstyle\sum_{j \in J} \overline{\beta_j} \otimes \overline{\delta_s(\gamma_j)}\Big) \,=\, \sum_{\ell \in \mathcal{L}} \overline{\delta_r(\lambda_\ell)} \otimes \overline{\chi_\ell} \,\neq\, 0$$

in the $\Bbbk$–vector space $\Big(\hbar^r C^{\otimes r} \Big/ \hbar^{r+1} C^{\otimes r}\Big) \mathbin{\widetilde{\otimes}}_{\Bbbk} \Big(\hbar^{s-1} C^{\otimes s} \Big/ \hbar^s C^{\otimes s}\Big)$.

On the other hand, coassociativity yields $(\delta_r \otimes \delta_s)\big(\delta_2(a)\big) = \delta_{r+s}(a)$. Therefore, since $a \in C'$ we have $\delta_{r+s}(a) \in \hbar^{r+s} C^{\otimes (r+s)}$, hence $\overline{\delta_{r+s}(a)} = 0$ in the $\Bbbk$–vector space $\hbar^{r+s-1} C^{\otimes (r+s)} \Big/ \hbar^{r+s} C^{\otimes (r+s)}$. Now, there are standard isomorphisms

$$\hbar^\ell C^{\otimes \ell} \Big/ \hbar^{\ell+1} C^{\otimes r} \;\cong\; \big(\hbar^\ell C^{\otimes \ell}\,\big) \otimes_R \Bbbk \qquad \text{for} \quad \ell \in \{r, s-1, r+s-1\}$$

$$\big(\hbar^{r+s-1} C^{\otimes s}\,\big) \otimes_R \Bbbk \,\cong\, \Big(\big(\hbar^r C^{\otimes r}\,\big) \otimes_R \Bbbk\Big) \otimes_{\Bbbk} \Big(\big(\hbar^{s-1} C^{\otimes s}\,\big) \otimes_R \Bbbk\Big)$$

$$\hbar^{r+s-1} C^{\otimes (r+s)} \Big/ \hbar^{(r+s)} C^{\otimes (r+s)} \;\cong\; \Big(\hbar^r C^{\otimes r} \Big/ \hbar^{r+1} C^{\otimes r}\Big) \otimes_{\Bbbk} \Big(\hbar^{s-1} C^{\otimes s} \Big/ \hbar^s C^{\otimes s}\Big).$$

Moreover, $\Big(\hbar^r C^{\otimes r} \Big/ \hbar^{r+1} C^{\otimes r}\Big) \otimes_{\Bbbk} \Big(\hbar^{s-1} C^{\otimes s} \Big/ \hbar^s C^{\otimes s}\Big)$ naturally embeds, as a dense subset, into $\Big(\hbar^r C^{\otimes r} \Big/ \hbar^{r+1} C^{\otimes r}\Big) \mathbin{\widetilde{\otimes}}_{\Bbbk} \Big(\hbar^{s-1} C^{\otimes s} \Big/ \hbar^s C^{\otimes s}\Big)$, so via the last isomorphism above we get

$$\hbar^{r+s-1} C^{\otimes (r+s)} \Big/ \hbar^{(r+s)} C^{\otimes (r+s)} \lhook\joinrel\longrightarrow \Big(\hbar^r C^{\otimes r} \Big/ \hbar^{r+1} C^{\otimes r} \Big) \,\widetilde{\otimes}_{\Bbbk} \Big(\hbar^{s-1} C^{\otimes s} \Big/ \hbar^s C^{\otimes s} \Big) .$$

This last monomorphism maps $\overline{(\delta_r \otimes \delta_s)\big(\delta_2(a)\big)} = \overline{\delta_{r+s}(a)} = 0$ onto $\big(\overline{\delta_r} \otimes \overline{\delta_s}\big)\big(\delta_2(a)\big) \neq 0$, a contradiction! Therefore we must have $\gamma_j \in \big(\widehat{C}\,\big)'$ for all $j \in \mathcal{J}$, as contended.

The outcome is $\delta_2(a) = \sum_{j \in \mathcal{J}} \beta_j \otimes \gamma_j \in C' \widetilde{\otimes} \big(\widehat{C}\,\big)'$, so $\delta_2(a) \in \big(C' \otimes C\big) \bigcap \big(C' \widetilde{\otimes} \big(\widehat{C}\,\big)'\big)$.

For all $n \in \mathbb{N}$, the result above yields $\big(\mathrm{id} \otimes \delta_n\big)\big(\delta_2(a)\big) \in \big(C' \otimes C^{\otimes n}\big) \bigcap \Big(C' \widetilde{\otimes}\, \delta_n\big(\big(\widehat{C}\,\big)'\big)\Big) \subseteq$ $\subseteq \big(C' \otimes C^{\otimes n}\big) \bigcap \Big(C' \,\widetilde{\otimes}\, \hbar^n \widehat{C}^{\,\widehat{\otimes}\, n}\Big) = C' \otimes \hbar^n C^{\otimes n}$, because $\widehat{C} \Big/ \hbar^n \widehat{C} = C \Big/ \hbar^n C$ (see above) implies $C \bigcap \hbar^n \widehat{C} = \hbar^n C$. So we found $\big(\mathrm{id} \otimes \delta_n\big)\big(\delta_2(a)\big) \in C' \otimes \hbar^n C^{\otimes n}$ for all $n \in \mathbb{N}$. Acting like in the first part of the proof, we'll show that this implies $\delta_2(a) \in C' \otimes C'$. To this end, it is enough to show that $(f \otimes \mathrm{id})\big(\delta_2(a)\big) \in \widehat{R} \cdot \underline{1} \otimes_R C'$ for all $f \in \big(\widehat{C'}\,\big)^*$, which amounts to show that $(f \otimes \delta_n)\big(\delta_2(a)\big) \in \widehat{R} \cdot \underline{1} \otimes_R \hbar^n C^{\otimes n}$ for all $n \in \mathbb{N}_+$, $f \in \big(\widehat{C'}\,\big)^*$. But this is true because $(f \otimes \delta_n)\big(\delta_2(a)\big) = (f \otimes \mathrm{id})\big((\mathrm{id} \otimes \delta_n)\big(\delta_2(a)\big)\big) \in (f \otimes \mathrm{id})\big(C' \otimes \hbar^n C^{\otimes n}\big) \subseteq$ $\widehat{R} \cdot \underline{1} \otimes_R \hbar^n C^{\otimes n}$. We conclude that $C' \in \mathcal{C}^+$, q.e.d.

Now look at $B \in \mathcal{B}$. By the previous part we have $B' \in \mathcal{C}^+$. Moreover, B' is multiplicatively closed, thanks to Lemma 3.4*(a)*, and $1 \in B'$ by the very definitions. Thus B' is an R–sub-bialgebra of B, so $B' \in \mathcal{B}$.

Finally, for $H \in \mathcal{HA}$ one has in addition $\Delta \circ S = S^{\otimes 2} \circ \Delta$, which implies $\Delta^n \circ S = S^{\otimes n} \circ \Delta^n$ hence $\delta_n \circ S = S^{\otimes n} \circ \delta_n$, for all $n \in \mathbb{N}$. This clearly yields $S(H') = H'$, whence H' is a Hopf subalgebra of H, thus $H' \in \mathcal{HA}$, q.e.d. $\square$

Remark: The "hard step" in the previous proof — i.e. proving that $\Delta(C') \subseteq C' \otimes C'$ — is much simpler when, after the reduction step to $C_\infty = \{0\}$, one has that C is free, as an R–module (note also that for C free one has automatically $C_\infty = \{0\}$). In fact, in this case — i.e. if C is free — we don't need to use completions. The argument to prove that $\delta_2(a) \in C' \otimes C$ goes through untouched, just using C instead of $\widehat{C}$, the freeness of C taking the role of the topological freeness of $\widehat{C}$; similarly, later on if C' also is free (for instance, when R is a PID, for C' is an R–submodule of the free R–module C) we can directly use it instead of the topologically free module $\widehat{C'}$, just taking $\big\{ \beta_j \,\big|\, j \in \mathcal{J} \big\}$ to be an R–basis of C': then we can write $\delta_2(a) = \sum_{j \in \mathcal{J}} \beta_j \otimes \gamma_j \in C' \otimes C$ for some $\gamma_j \in C$, and the argument we used applies again to show that now $\gamma_j \in C'$ for all j, so that $\delta_2(a) \in C' \otimes C'$, q.e.d.

Theorem 3.6. *Assume that* $\Bbbk := R \big/ \hbar R$ *is a field.*

(a) $X \mapsto X^\vee$ *gives well-defined functors from* $\mathcal{A}^+$ *to* $\mathcal{A}^+$*, from* $\mathcal{B}$ *to* $\mathcal{B}$*, from* $\mathcal{HA}$ *to* $\mathcal{HA}$.

(b) $X \mapsto X'$ *gives well-defined functors from* $\mathcal{C}^+$ *to* $\mathcal{C}^+$*, from* $\mathcal{B}$ *to* $\mathcal{B}$*, from* $\mathcal{HA}$ *to* $\mathcal{HA}$.

(c) For any $B \in \mathcal{B}$ *we have* $B \subseteq \big(B^\vee\big)'$, $B \supseteq \big(B'\big)^\vee$*, hence* $B^\vee = \Big(\big(B^\vee\big)'\Big)^\vee$, $B' = \Big(\big(B'\big)^\vee\Big)'$.

Proof. In force of Propositions 3.3–5, to define the functors we only have to set them on morphisms. So let $\varphi \in Mor_{\mathcal{A}^+}(A, E)$ be a morphism in $\mathcal{A}^+$: by scalar extension it gives a morphism $A_F \longrightarrow E_F$ of Hopf $F(R)$–algebras, which maps $\hbar^{-1} J_{{}_A}$ into $\hbar^{-1} J_{{}_E}$, hence $A^\vee$ into $E^\vee$: this yields the morphism $\varphi^\vee \in Mor_{\mathcal{A}^+}\big(A^\vee, E^\vee\big)$ we were looking for. On the other hand, if $\psi \in Mor_{\mathcal{C}^+}(C,\Gamma)$ is a morphism in $\mathcal{C}^+$ then $\delta_n \circ \psi = \psi^{\otimes n} \circ \delta_n$ for all n, so $\psi\big(C'\big) \subseteq \Gamma'$: thus as $\psi' \in Mor_{\mathcal{C}^+}\big(C',\Gamma'\big)$ we take the restriction $\psi\big|_{C'}$ of ψ to C'.

Now consider $B \in \mathcal{B}$. For any $n \in \mathbb{N}$ we have $\delta_n(B) \subseteq {J_{{}_B}}^{\otimes n}$ (see §2.1); this can be read as $\delta_n(B) \subseteq {J_{{}_B}}^{\otimes n} = \hbar^n \big(\hbar^{-1} J_{{}_B}\big)^{\otimes n} \subseteq \hbar^n \big(B^\vee\big)^{\otimes n}$, which gives $B \subseteq \big(B^\vee\big)'$, q.e.d. On the other hand, let $I' := Ker\Big(B' \overset{\epsilon}{\twoheadrightarrow} R \xrightarrow{\hbar \mapsto 0} \Bbbk\Big)$; since $\big(B'\big)^\vee := \bigcup_{n=0}^{\infty} \big(\hbar^{-1} I'\big)^n$, in order to show that $B \supseteq \big(B'\big)^\vee$ it is enough to check that $B \supseteq \hbar^{-1} I'$. So let $x' \in I'$: then $\delta_1(x') \in \hbar B$, hence $x' = \delta_1(x') + \epsilon(x') \in \hbar B$. Therefore $\hbar^{-1} x' \in B$, q.e.d. Finally, the last two identities follow easily from the two inclusions we've just proved. □

Theorem 3.7. *Assume that* $\Bbbk := R\big/\hbar R$ *is a field.*

Let $B \in \mathcal{B}$. *Then* $B^\vee\Big|_{\hbar=0}$ *is an rUEA (see §1.1), generated by* $\hbar^{-1} J \mod \hbar B^\vee$.

In particular, if $H \in \mathcal{HA}$ *then* $H^\vee \in \mathcal{Q}r\mathcal{UEA}$.

Proof. A famous characterization theorem in Hopf algebra theory claims the following (cf. for instance [Ab], Theorem 2.5.3, or [Mo], Theorem 5.6.5, and references therein, noting also that in the cocommutative case *connectedness* and *irreducibility* coincide):

A Hopf algebra H over a ground field $\Bbbk$ is the restricted universal enveloping algebra of a restricted Lie algebra $\mathfrak{g}$ if and only if H is generated by $P(H)$ (the set of primitive elements of H) and it is cocommutative and connected. In that case, $\mathfrak{g} = P(H)$.

Thus we must prove that the bialgebra $B^\vee\big|_{\hbar=0}$ is in fact a Hopf algebra, it is generated by its primitive part $P\big(B^\vee\big|_{\hbar=0}\big)$ and it is cocommutative and connected, for then $B^\vee\big|_{\hbar=0} = \mathcal{U}(\mathfrak{g})$ with $\mathfrak{g} = P\big(B^\vee\big|_{\hbar=0}\big)$ being a restricted Lie *bialgebra* (by Remark 1.5).

Since $B^\vee = \sum_{n \geq 0} \big(\hbar^{-1} J\big)^n$, it is generated, as a unital algebra, by $J^\vee := \hbar^{-1} J$. Consider $j^\vee \in J^\vee$, and $j := \hbar\, j^\vee \in J$; then

$$\Delta(j) = \delta_2(j) + j \otimes 1 + 1 \otimes j - \epsilon(j) \cdot 1 \otimes 1 \in j \otimes 1 + 1 \otimes j + J \otimes J$$

for $\Delta = \delta_2 + \mathrm{id} \otimes 1 + 1 \otimes \mathrm{id} - \epsilon \cdot 1 \otimes 1$ and $Im\,(\delta_2) \subseteq J \otimes J$ by construction. Therefore

$$\begin{aligned} \Delta\big(j^\vee\big) = \delta_2\big(j^\vee\big) + j^\vee \otimes 1 + 1 \otimes j^\vee - \epsilon\big(j^\vee\big) \cdot 1 \otimes 1 = \delta_2\big(j^\vee\big) + j^\vee \otimes 1 + 1 \otimes j^\vee \in \\ \in\; j^\vee \otimes 1 + 1 \otimes j^\vee + \hbar^{-1} J \otimes J = j^\vee \otimes 1 + 1 \otimes j^\vee + \hbar^{+1} J^\vee \otimes J^\vee \end{aligned}$$

whence

$$\Delta\big(j^\vee\big) \equiv j^\vee \otimes 1 + 1 \otimes j^\vee \mod \hbar B^\vee \qquad \big(\forall\, j^\vee \in J^\vee\big)\,. \tag{3.2}$$

This proves that $J^\vee\big|_{\hbar=0} \subseteq P\big(B^\vee\big|_{\hbar=0}\big)$, and since $J^\vee\big|_{\hbar=0}$ generates $B^\vee\big|_{\hbar=0}$ (for $J^\vee$ generates $B^\vee$), *a fortiori* $B^\vee\big|_{\hbar=0}$ is generated by $P\big(B^\vee\big|_{\hbar=0}\big)$, hence $B^\vee\big|_{\hbar=0}$ is cocommutative too. In addition, (3.2) enables us to apply Lemma 5.5.1 in [Mo] — which is stated there for Hopf algebras, but *holds indeed for bialgebras as well* — to the bialgebra $B^\vee\big|_{\hbar=0}$, with $A_0 = \Bbbk\cdot 1$ and $A_1 = J^\vee \Big/ \big(J^\vee \cap \hbar\, B^\vee\big)$: then that lemma proves that $B^\vee\big|_{\hbar=0}$ is connected. Another classical result (cf. [Ab], Theorem 2.4.24) then ensures that $B^\vee\big|_{\hbar=0}$ is indeed a Hopf algebra; as it is also connected, cocommutative and generated by its primitive part, we can apply the characterization theorem and get the claim. $\square$

Theorem 3.8. *Assume that $\Bbbk := R\big/\hbar R$ is a field.*

Let $B \in \mathcal{B}$. Then $(B')_\infty = {I_{B'}}^\infty$ and $B'\Big|_{\hbar=0}$ is commutative and has no non-trivial idempotents. In addition, when $p := Char(\Bbbk) > 0$ each non-zero element of $J_{B'|_{\hbar=0}}$ has nilpotency order p, that is $\overline{\eta}^{\,p} = 0$ for all $\overline{\eta} \in J_{B'|_{\hbar=0}}$.

In particular, if $H \in \mathcal{HA}$ then $H' \in \mathcal{QFA}$.

Proof. The second part of the claim (about $H \in \mathcal{HA}$) is simply a straightforward reformulation of the first part (about $B \in \mathcal{B}$), so in fact it is enough to prove the latter.

First we must show that $B'\big|_{\hbar=0}$ is commutative, $(B')_\infty = {I_{B'}}^\infty$ and $B'\big|_{\hbar=0}$ has no non-trivial idempotents (cf. §1.3–4). For later use, set $I := I_B$, $J := J_B$, $J' := J_{B'}$, $I' := I_{B'}$.

As for commutativity, we have to show that $ab - ba \in \hbar\, B'$ for all $a, b \in B'$. First, by the inverse formula for Δ^n (see §2.1) we have $\mathrm{id}_B = \Delta^1 = \delta_1 + \delta_0 = \delta_1 + \epsilon$; so $x = \delta_1(x) + \epsilon(x)$ for all $x \in B$. If $x \in B'$ we have $\delta_1(x) \in \hbar\, B$, hence there exists $x_1 \in B$ such that $\delta_1(x) = \hbar\, x_1$. Now take $a, b \in B'$: then $a = \hbar\, a_1 + \epsilon(a)$, $b = \hbar\, b_1 + \epsilon(b)$, whence $ab - ba = \hbar\, c$ with $c = \hbar\,(a_1 b_1 - b_1 a_1)$; therefore we are left to show that $c \in B'$. To this end, we have to check that $\delta_\Phi(c)$ is divisible by $\hbar^{|\Phi|}$ for any nonempty finite subset Φ of $\mathbb{N}_+$: as multiplication by $\hbar$ is injective (for B is torsion-free!), it is enough to show that $\delta_\Phi(ab - ba)$ is divisible by $\hbar^{|\Phi|+1}$.

Let Λ and Y be subsets of Φ such that $\Lambda \cup Y = \Phi$ and $\Lambda \cap Y \neq \emptyset$: then $|\Lambda| + |Y| \geq |\Phi| + 1$. Now, $\delta_\Lambda(a)$ is divisible by $\hbar^{|\Lambda|}$ and $\delta_Y(b)$ is divisible by $\hbar^{|Y|}$. From this and from Lemma 3.4*(b)* it follows that $\delta_\Phi(ab - ba)$ is divisible by $\hbar^{|\Phi|+1}$, q.e.d.

Second, we show that $(B')_\infty = (I')^\infty$. By definition $\hbar\, B' \subseteq I'$, whence $B'_\infty := \bigcap_{n=0}^{+\infty} \hbar^n B' \subseteq \bigcap_{n=0}^{+\infty} \big(I'\big)^n =: \big(I'\big)^\infty$, i.e. $\big(B'\big)_\infty \subseteq \big(I'\big)^\infty$. Conversely, $I' = \hbar\, B' + J'$ with $\hbar\, B' \subseteq \hbar\, B$ and $J' = \delta_1(J') \subseteq \hbar\, B$: thus $I' \subseteq \hbar\, B$, hence $\big(I'\big)^\infty \subseteq \bigcap_{n=0}^{+\infty} \hbar^n B =: B_\infty$. Now definitions give $B_\infty \subseteq B'$ and $\hbar^\ell B_\infty = B_\infty$ for all $\ell \in \mathbb{Z}$, so $\hbar^{-n}\big(I'\big)^\infty \subseteq \hbar^{-n} B_\infty = B_\infty \subseteq B'$ hence $\big(I'\big)^\infty \subseteq \hbar^n B'$ for all $n \in \mathbb{N}$, thus finally $\big(I'\big)^\infty \subseteq \big(B'\big)_\infty$.

Third, we prove that $B'\big|_{\hbar=0}$ has no non-trivial idempotents.

Let $a \in B'$, and suppose that $\overline{a} := a \mod \hbar\, B' \in B'\big|_{\hbar=0}$ is idempotent, i.e. $\overline{a}^{\,2} = \overline{a}$. Then $a^2 = a + \hbar\, c$ for some $c \in \hbar\, B'$. Set $a_0 := \epsilon(a)$, $a_1 := \delta_1(a)$, and $c_0 := \epsilon(c)$, $c_1 := \delta_1(c)$; since $a, c \in B'$ we have $a_1, c_1 \in \hbar\, B \cap J = \hbar\, J$.

First, applying δ_n to the identity $a^2 = a + \hbar\, c$ and using Lemma 3.4*(a)* we get

$$\sum_{\Lambda \cup Y = \{1,\dots,n\}} \delta_\Lambda(a)\, \delta_Y(a) \;=\; \delta_n\big(a^2\big) \;=\; \delta_n(a) + \hbar\, \delta_n(c) \qquad \forall \;\; n \in \mathbb{N}_+ \;. \tag{3.3}$$

Since $a,\, c \in B'$ we have $\delta_n(a),\, \delta_n(c) \in \hbar^n B^{\otimes n}$ for all $n \in \mathbb{N}$. Therefore (3.3) yields

$$\delta_n(a) \;\equiv \sum_{\Lambda \cup Y = \{1,\dots,n\}} \delta_\Lambda(a)\, \delta_Y(a) \;=\; 2\,\delta_0(a)\,\delta_n(a) + \sum_{\substack{\Lambda \cup Y = \{1,\dots,n\} \\ \Lambda, Y \neq \emptyset}} \delta_\Lambda(a)\, \delta_Y(a) \mod \hbar^{n+1} B^{\otimes n}$$

for all $n \in \mathbb{N}_+$, which, recalling that $a_0 := \delta_0(a)$, gives (for all $n \in \mathbb{N}_+$)

$$\big(1 - 2\,a_0\big)\, \delta_n(a) \;\equiv \sum_{\substack{\Lambda \cup Y = \{1,\dots,n\} \\ \Lambda, Y \neq \emptyset}} \delta_\Lambda(a)\, \delta_Y(a) \mod \hbar^{n+1} B^{\otimes n} \quad . \tag{3.4}$$

Now, applying ϵ to the identity $a^2 = a + \hbar\, c$ gives ${a_0}^2 = a_0 + \hbar\, c_0$. This implies $\big(1 - 2\,a_0\big) \notin \hbar\, B$: this is trivial if $\mathit{Char}(R) = 2$ or $a_0 = 0$; otherwise, if $\big(1 - 2\,a_0\big) \in \hbar\, B$ then $a_0 = 1/2 + \hbar\,\alpha$ for some $\alpha \in B$, and so ${a_0}^2 = \big(1/2 + \hbar\,\alpha\big)^2 = 1/4 + \hbar\,\alpha + \hbar^2\,\alpha^2 \neq 1/2 + \hbar\,\alpha + \hbar\, c_0 = a_0 + \hbar\, c_0$, thus contradicting the identity ${a_0}^2 = a_0 + \hbar\, c_0$. Now using $\big(1 - 2\,a_0\big) \notin \hbar\, B$ and formulas (3.4) — for all $n \in \mathbb{N}_+$ — an easy induction argument gives $\delta_n(a) \in \hbar^{n+1} B$, for all $n \in \mathbb{N}_+$. Now consider $a_1 = a - a_0 = \hbar\,\alpha$ for some $\alpha \in \hbar\, J$: we have $\delta_0(\alpha) = \epsilon(\alpha) = 0$ and $\delta_n(\alpha) = \hbar^{-1} \delta_n(a) \in \hbar^n B$, for all $n \in \mathbb{N}_+$, which mean $\alpha \in B'$. Thus $a = a_0 + \hbar\,\alpha \equiv a_0 \mod \hbar\, B'$, whence $\overline{a} = \overline{a_0} \in B'\big|_{\hbar=0}$; then ${\overline{a_0}}^{\,2} = \overline{a_0} \in \Bbbk$ gives us $\overline{a_0} \in \{0,1\}$, hence $\overline{a} = \overline{a_0} \in \{0,1\}$, q.e.d.

Finally, assume that $p := \mathit{Char}(\Bbbk) > 0$; then we have to show that $\overline{\eta}^{\,p} = 0$ for each $\overline{\eta} \in J_{B'|_{\hbar=0}}$, or simply $\eta^p \in \hbar\, J_{B'}$ for each $\eta \in J_{B'}$. Indeed, for any $n \in \mathbb{N}$ from the multiplicativity of Δ^n and from $\Delta^n(\eta) = \sum_{\Lambda \subseteq \{1,\dots,n\}} \delta_\Lambda(\eta)$ (cf. §2.1) we have

$$\Delta^n(\eta^p) = \big(\Delta^n(\eta)\big)^p = \Big(\textstyle\sum_{\Lambda \subseteq \{1,\dots,n\}} \delta_\Lambda(\eta) \Big)^p \;\in\; \textstyle\sum_{\Lambda \subseteq \{1,\dots,n\}} \delta_\Lambda(\eta)^p \; +$$

$$+ \sum_{\substack{e_1,\dots,e_p < p \\ e_1 + \cdots e_p = p}} \binom{p}{e_1,\dots,e_p} \sum_{\Lambda_1,\dots,\Lambda_p \subseteq \{1,\dots,n\}} \textstyle\prod_{k=1}^p \delta_{\Lambda_k}(\eta)^{e_k} \;+\; \hbar \cdot \displaystyle\sum_{k=0}^{n-1} \sum_{\substack{\Psi \subseteq \{1,\dots,n\} \\ |\Psi| = k}} j_\Psi\big({J_{B'}}^{\otimes k}\big) \;+\; \hbar \cdot {J_{B'}}^{\otimes n}$$

because $\delta_\Lambda(\eta) \in j_\Lambda\Big({J_{B'}}^{\otimes |\Lambda|}\Big)$ (for all $\Lambda \subseteq \{1,\dots,n\}$) and $\big[J_{B'}, J_{B'}\big] \subseteq \hbar\, J_{B'}$. Then

$$\delta^n(\eta^p) = \big(\mathrm{id}_B - \epsilon\big)^{\otimes n}\big(\Delta^n(\eta^p)\big) \;\in\; \delta_n(\eta)^p + \sum_{\substack{e_1,\dots,e_p < p \\ e_1 + \cdots e_p = p}} \binom{p}{e_1,\dots,e_p} \sum_{\cup_k \Lambda_k = \{1,\dots,n\}} \textstyle\prod_{k=1}^p \delta_{\Lambda_k}(\eta)^{e_k} + \hbar\, {J_{B'}}^{\otimes n}.$$

Now, $\delta^n(\eta)^p \in \big(\hbar^n B^{\otimes n}\big)^p \subseteq \hbar^{n+1} B^{\otimes n}$ because $\eta \in B'$, and similarly we have also $\prod_{k=1}^p \delta_{\Lambda_k}(\eta)^{e_k} \in \hbar^{\sum_k |\Lambda_k|\, e_k} B^{\otimes n} \subseteq \hbar^n B^{\otimes n}$ whenever $\bigcup_{k=1}^n \Lambda_k = \{1,\dots,n\}$; in addition, the multinomial coefficient $\binom{p}{e_1,\dots,e_p}$ (with $e_1, \dots, e_p < p$) is a multiple of p, hence it is zero in $R \big/ \hbar\, R = \Bbbk$, that is $\binom{p}{e_1,\dots,e_p} \in \hbar\, R$: therefore

$$\sum_{\substack{e_1,\dots,e_p < p \\ e_1 + \cdots + e_p = p}} \binom{p}{e_1,\dots,e_p} \sum_{\cup_{k=1}^p \Lambda_k = \{1,\dots,n\}} \prod_{k=1}^p \delta_{\Lambda_k}(\eta)^{e_k} \;\in\; \hbar^{n+1} B^{\otimes n} \quad .$$

Finally, since $J_{B'} \subseteq \hbar\, J_B$ we have also $\hbar\, {J_{B'}}^{\otimes n} \subseteq \hbar^{n+1} B^{\otimes n}$. The outcome then is that $\delta_n(\eta^p) \in \hbar^{n+1} B^{\otimes n}$ for all $n \in \mathbb{N}$, thus $\eta \in \hbar\, B'$ as expected. $\square$

§ 4 Drinfeld's functors on quantum groups

From now on, we **assume that** $\Bbbk := R\big/\hbar R$ **is a field**.

Lemma 4.1. *Let $F_\hbar \in \mathcal{QFA}$, and assume that $F_\hbar\big|_{\hbar=0}$ is reduced. Let $I := I_{F_\hbar}$, let $\widehat{F_\hbar}$ be the I–adic completion of $F_\hbar$, and $\widehat{I^{\,n}}$ the I–adic closure of I^n in $\widehat{F_\hbar}$, for all $n \in \mathbb{N}$.*

(a) $\widehat{F_\hbar}$ is isomorphic as an $\widehat{R}$–module (where $\widehat{R}$ is the $\hbar$–adic completion of R) to a formal power series algebra of type $\widehat{R}\big[\big[\{Y_b\}_{b\in\mathcal{S}}\big]\big]$ (where $\mathcal{S}$ stands for some index set).

(b) Letting $\nu : \Bbbk \lhook\joinrel\longrightarrow R$ be a section of the quotient map $R \relbar\joinrel\twoheadrightarrow R\big/\hbar\, R =: \Bbbk$, use it to identify (set-theoretically) $\widehat{F_\hbar} \cong \widehat{R}\big[\big[\{Y_b\}_{b\in\mathcal{S}}\big]\big]$ with $\nu(\Bbbk)\big[\big[\{Y_0\} \cup \{Y_b\}_{b\in\mathcal{S}}\big]\big]$ (with $\hbar \cong Y_0$). Then via such an identification both $\widehat{\big(\widehat{I}\,\big)^n}$ and $\widehat{I^{\,n}}$ coincide with the set of all formal series of (least) degree n (in the Y_i's, with $i \in \{0\} \cup \mathcal{S}$), for all $n \in \mathbb{N}$.

(c) There exist $\Bbbk$–module isomorphisms $G_I\big(F_\hbar\big) \cong \Bbbk\big[Y_0, \{Y_b\}_{b\in\mathcal{S}}\big] \cong G_{\widehat{I}}\big(\widehat{F}\,\big)$ for the graded rings associated to $F_\hbar$ and $\widehat{F_\hbar}$ with the I–adic and the $\widehat{I}$–adic filtration.

(d) Let $\mu : F_\hbar \longrightarrow \widehat{F_\hbar}$ be the natural map. Then $\mu(F_\hbar) \bigcap \widehat{I^{\,n}} = \mu(I^n)$ for all $n \in \mathbb{N}$.

Proof. Let $F[G] \equiv F_\hbar\big|_{\hbar=0} := F_\hbar\big/\hbar\, F_\hbar$, and let $\widehat{F[G]} = F[[G]]$ be the $\mathfrak{m}$–adic completion of $F[G]$, where $\mathfrak{m} = \mathit{Ker}\big(\epsilon_{F[G]}\big)$ is the maximal ideal of $F[G]$ at the unit element of G. Then $I = \pi^{-1}(\mathfrak{m})$, the preimage of $\mathfrak{m}$ under the specialization map $\pi : F_\hbar \relbar\joinrel\twoheadrightarrow F_\hbar\big/\hbar\, F_\hbar = F[G]$. Therefore π induces a continuous epimorphism $\widehat{\pi} : \widehat{F_\hbar} \relbar\joinrel\twoheadrightarrow \widehat{F[G]} = F[[G]]$, which again is nothing but specialization at $\hbar = 0$. Note also that the ground ring of $\widehat{F_\hbar}$ is $\widehat{R}$, because the ground ring of the I–adic completion of a unital R–algebra is the $(R \cap I)$–adic completion of R, and the $R \cap I = \hbar\, R$. Then of course $\widehat{F_\hbar}$ is also a topological $\widehat{R}$–module. Moreover, by construction we have $\big(\widehat{F_\hbar}\,\big)_\infty = \{0\}$.

Now, let $\{y_b\}_{b\in\mathcal{S}}$ be a $\Bbbk$–basis of $\mathfrak{m}\big/\mathfrak{m}^2 = Q\big(F[G]\big)$; by hypothesis $F[G]$ is reduced, thus $F[[G]]$ is just the formal power series algebra in the y_b's, i.e. $F[[G]] \cong \Bbbk\big[\big[\{Y_b\}_{b\in\mathcal{S}}\big]\big]$. For any $b \in \mathcal{S}$, pick a $j_b \in \pi^{-1}(y_b) \bigcap J$ (with $J := \mathit{Ker}\big(\epsilon_{F_\hbar}\big)$), and fix also a section $\nu : \Bbbk \lhook\joinrel\longrightarrow R$ of the quotient map $R \relbar\joinrel\twoheadrightarrow R\big/\hbar\, R = \Bbbk$ as in *(b)*. Using these, we can define a continuous morphism of $\widehat{R}$–modules $\Psi : \widehat{R}\big[\big[\{Y_b\}_{b\in\mathcal{S}}\big]\big] \relbar\joinrel\relbar\joinrel\longrightarrow \widehat{F_\hbar}$ mapping $Y^{\,\underline{e}} := \prod_{b\in\mathcal{S}} Y_b^{\underline{e}(b)}$ to $j^{\,\underline{e}} := \prod_{b\in\mathcal{S}} j_b^{\underline{e}(b)}$ for all $\underline{e} \in \mathbb{N}_f^{\mathcal{S}} := \big\{\, \sigma \in \mathbb{N}^{\mathcal{S}} \,\big|\, \sigma(b) = 0$ *for almost all* $b \in \mathcal{S} \,\big\}$ (hereafter, monomials like the previous ones are *ordered* w.r.t. any fixed order of the index set $\mathcal{S}$). In addition, using ν one can identify (set-theoretically) $\widehat{R} \cong \nu(\Bbbk)[[Y_0]]$ (with $\hbar \cong Y_0$), whence a bijection $\nu(\Bbbk)\big[\big[Y_0 \cup \{Y_b\}_{b\in\mathcal{S}}\big]\big] \cong \widehat{R}\big[\big[\{Y_b\}_{b\in\mathcal{S}}\big]\big]$ arises.

We claim that Ψ is surjective. Indeed, since $\big(\widehat{F_\hbar}\,\big)_\infty = \{0\}$, for any $f \in \widehat{F_\hbar}$ there is a unique $v_\hbar(f) \in \mathbb{N}$ such that $f \in \hbar^{v_\hbar(f)} \widehat{F_\hbar} \setminus \hbar^{v_\hbar(f)+1} \widehat{F_\hbar}$, so $\widehat{\pi}\Big(\hbar^{-v_\hbar(f)} f\Big) = \sum_{\underline{e}\in\mathbb{N}_f^{\mathcal{S}}} c_{\underline{e}} \cdot y^{\underline{e}}$ for some $c_{\underline{e}} \in \Bbbk$ not all zero. Then for $f_1 := f - \hbar^{v_\hbar(f)} \cdot \sum_{\underline{e}\in\mathbb{N}_f^{\mathcal{S}}} \nu(c_{\underline{e}}) \cdot j^{\,\underline{e}}$ we have $v_\hbar(f_1) > v_\hbar(f)$. Iterating, we eventually find for f a formal power series expression of the type $f = \sum_{n\in\mathbb{N}} \hbar^n \cdot \sum_{\underline{e}\in\mathbb{N}_f^{\mathcal{S}}} \nu(c_{\underline{e},n}) \cdot j^{\,\underline{e}} = \sum_{(e_0,\underline{e})\in\mathbb{N}\times\mathbb{N}_f^{\mathcal{S}}} \nu(\kappa_{\underline{e}}) \cdot \hbar^{e_0} j^{\,\underline{e}}$, so $f \in \mathit{Im}(\Psi)$, q.e.d. Thus in order to prove *(a)* we are left to show that Ψ is injective too.

Consider the graded ring associated to the $\hbar$–adic filtration of $\widehat{F_\hbar}$, that is $G_\hbar\big(\widehat{F_\hbar}\big) := \oplus_{n=0}^{+\infty}\Big(\hbar^n \widehat{F_\hbar}/\hbar^{n+1}\widehat{F_\hbar}\Big)$: this is *commutative*, because $\widehat{F_\hbar}\Big/\hbar\,\widehat{F_\hbar} = \widehat{\pi}\big(\widehat{F_\hbar}\big) = F[[G]]$ is commutative, and more precisely $G_\hbar\big(\widehat{F_\hbar}\big) \cong \Bbbk\big[\,\bar{\hbar}\,\big] \otimes_\Bbbk \Big(\widehat{F_\hbar}\Big/\hbar\,\widehat{F_\hbar}\Big) \cong \Bbbk[Y_0] \otimes_\Bbbk F[[G]] \cong$ $\cong \big(F[[G]]\big)[Y_0]$ as graded $\Bbbk$–algebras. In addition, the epimorphism (of $\widehat{R}$*–modules*) $\Psi : \widehat{R} \otimes_\Bbbk F[[G]] \relbar\joinrel\twoheadrightarrow \widehat{F_\hbar}$ induces an epimorphism $G_{Y_0}(\Psi) : G_{Y_0}\big(\widehat{R} \otimes_\Bbbk F[[G]]\big) \relbar\joinrel\twoheadrightarrow G_\hbar\big(\widehat{F_\hbar}\big)$ of *graded* $\Bbbk$*–algebras*, and by the very construction $G_{Y_0}(\Psi)$ is clearly an isomorphism yielding $\big(F[[G]]\big)[Y_0] \cong G_{Y_0}\big(\widehat{R} \otimes_\Bbbk F[[G]]\big) \cong G_\hbar\big(\widehat{F_\hbar}\big)$: then by a standard argument (cf. [Bo], Ch. III, §2.8, Corollary 1) we conclude that Ψ is an isomorphism as well, q.e.d.

As for part *(b)*, we start by noting that $\widehat{I} = \widehat{\pi}^{-1}\big(\mathit{Ker}\,(\epsilon_{F[[G]]})\big) = \mathit{Ker}\,(\epsilon_{\widehat{F_\hbar}}) + \hbar\,\widehat{F_\hbar}$, so each element of $\widehat{I}$ is expressed — via the isomorphism Ψ — by a series of degree at least 1; moreover, for all $b, d \in \mathcal{S}$ we have $j_b\, j_d - j_d\, j_b = \hbar\, j_+$ for some $j_+ \in \mathit{Ker}\,(\epsilon_{\widehat{F_\hbar}})$. This implies that when multiplying n factors from $\widehat{I}$ expressed by n series of positive degree, we can reorder the unordered monomials in the y_b's occurring in the multiplication process and eventually get a formal series — with *ordered* monomials — of degree at least n. This proves the claim for both $\widehat{I^n}$ and $\widehat{\big(\widehat{I}\,\big)^n}$.

For part *(c)*, the analysis above shows that the natural map $\mu : F_\hbar \longrightarrow \widehat{F_\hbar}$ induces $\Bbbk$–module isomorphisms $\big(\widehat{I}\,\big)^n\Big/\big(\widehat{I}\,\big)^{n+1} \cong \widehat{\big(\widehat{I}\,\big)^n}\Big/\widehat{\big(\widehat{I}\,\big)^{n+1}} \cong \widehat{I^n}\Big/\widehat{I^{n+1}}$ (for all $n \in \mathbb{N}$), so $G_I\big(F_\hbar\big) := \bigoplus_{n=0}^{+\infty} I^n\Big/I^{n+1} \cong \bigoplus_{n=0}^{+\infty}\big(\widehat{I}\,\big)^n\Big/\big(\widehat{I}\,\big)^{n+1} =: G_{\widehat{I}}\big(\widehat{F_\hbar}\big) \cong \bigoplus_{n=0}^{+\infty}\widehat{I^n}\Big/\widehat{I^{n+1}}$; moreover, the given description of the $\widehat{I^n}$'s implies $G_{\widehat{I}}\big(\widehat{F_\hbar}\big) := \bigoplus_{n=0}^{+\infty}\widehat{I^n}\Big/\widehat{I^{n+1}} \cong \Bbbk\big[Y_0, \{Y_b\}_{b\in\mathcal{S}}\big]$ as $\Bbbk$–modules, and the like for $G_I\big(F_\hbar\big)$, thus *(c)* is proved.

Finally, *(d)* is a direct consequence of *(c)*: for the latter yields $\Bbbk$–module isomorphisms $F_\hbar/I^n \cong G_I\big(F_\hbar\big)\Big/G_I\big(I^n\big) \cong G_I\big(\widehat{F_\hbar}\big)\Big/G_I\big(\widehat{I^n}\big) \cong \widehat{F_\hbar}\Big/\widehat{I^n}$, thus $\mu(I^n) = \widehat{I^n} \bigcap \mu(F_\hbar)$. $\square$

Remark: the previous description of the "formal quantum group" $\widehat{F_\hbar}$ shows that the latter looks exactly like expected. In particular, in the finite dimensional case we can say it is a local ring which is also "regular", in the sense that the four numbers

— dimension of the "cotangent space" $I_{F_\hbar}\Big/{I_{F_\hbar}}^2$,

— least number of generators of the maximal ideal $I_{F_\hbar}$,

— Hilbert dimension (= degree of the Hilbert polynomial of the graded ring $G_{\widehat{I}}\big(\widehat{F_\hbar}\big)$),

— Krull dimension of the associated graded ring $G_{\widehat{I}}\big(\widehat{F_\hbar}\big)$,

are *all equal*. Another way to say it is to note that, if $\big\{j_1, \dots, j_d\big\}$ is a lift in $J_{F_\hbar}$ of any system of parameters of $G = \mathit{Spec}\,\big(F_\hbar\big|_{\hbar=0}\big)$ around the identity (with $d = \dim(G)$), then the set $\big\{j_0 := \hbar, j_1, \dots, j_d\big\}$ is a "system of parameters" for $F_\hbar$ (or, more precisely, for the local ring $\widehat{F_\hbar}$). A suggestive way to interpret all this is to think at quantization as "adding one dimension (or deforming) in the direction of the quantization parameter $\hbar$":

and here we stress the fact that this is to be done "in a *regular* way".

Lemma 4.2. *Let $F_\hbar \in \mathcal{QFA}$, and assume that $F_\hbar\big|_{\hbar=0}$ is reduced. Then:*

(a) if $\varphi \in F_\hbar$ and $\hbar^s \varphi \in {I_{F_\hbar}}^n$ $(s, n \in \mathbb{N})$, then $\varphi \in {I_{F_\hbar}}^{n-s}$;

(b) if $y \in I_{F_\hbar} \setminus {I_{F_\hbar}}^2$, then $\hbar^{-1} y \notin \hbar {F_\hbar}^\vee$;

(c) $\big({F_\hbar}^\vee\big)_\infty = \big(F_\hbar\big)_\infty \; \big(= {I_{F_\hbar}}^\infty \big)$.

(d) Let $Char(\mathbb{k}) = 0$, and let $U_\hbar \in \mathcal{QrUEA}$. Let $x' \in {U_\hbar}'$, and let $x \in U_\hbar \setminus \hbar U_\hbar$, $n \in \mathbb{N}$, be such that $x' = \hbar^n x$. Set $\bar{x} := x \mod \hbar U_\hbar$. Then $\partial(\bar{x}) \leq n$ (hereafter $\partial(\bar{x})$ is the degree of $\bar{x}$ w.r.t. the standard filtration of the universal enveloping algebra $U_\hbar\big|_{\hbar=0}$).

Proof. *(a)* Set $I := I_{F_\hbar}$. Consider I^∞ (cf. Definition 1.4*(b)*) and the quotient Hopf algebra $\overline{F}_\hbar := F_\hbar \big/ I^\infty$: then $\bar{I} := I_{\overline{F}_\hbar} = I \big/ I^\infty$. By Lemma 3.2*(a)*, $\overline{F}_\hbar$ is again a QFA, having the same specialization at $\hbar = 0$ than $F_\hbar$, and such that $\bar{I}^\infty := {I_{\overline{F}_\hbar}}^\infty = \{0\}$. Now, $\phi \in I^\ell \iff \overline{\phi} \in \bar{I}^\ell$ for all $\phi \in F_\hbar$, $\ell \in \mathbb{N}$, with $\overline{\phi} := \phi + I^\infty \in \overline{F}_\hbar$: thus it is enough to make the proof for $\overline{F}_\hbar$, i. e. we can assume from scratch that $I^\infty = \{0\}$. In particular the natural map from $F_\hbar$ to its I–adic completion $\widehat{F_\hbar}$ is injective, as its kernel is I^∞.

Consider the embedding $F_\hbar \hookrightarrow \widehat{F_\hbar}$: from the proof of Lemma 4.1 one easily sees that $\widehat{I^\ell} \bigcap F_\hbar = I^\ell$, for all ℓ (because $F_\hbar \big/ I^\ell \cong \widehat{F_\hbar} \big/ \widehat{I^\ell}$): then, using the description of $\widehat{I^\ell}$ in Lemma 4.1, $\varphi \in F_\hbar$ and $\hbar^s \varphi \in I^n$ give at once $\varphi \in \widehat{I^{n-s}} \bigcap F_\hbar = I^{n-s}$, q.e.d.

(b) Let $y \in I_{F_\hbar} \setminus {I_{F_\hbar}}^2$. Assume $\hbar^{-1} y = \hbar \eta$ for some $\eta \in {F_\hbar}^\vee \setminus \{0\}$. Since ${F_\hbar}^\vee := \bigcup_{N \geq 0} \hbar^{-N} {I_{F_\hbar}}^n$ we have $\eta = \hbar^{-N} i_N$ for some $N \in \mathbb{N}_+$, $i_N \in {I_{F_\hbar}}^N$. Then we have $\hbar^{-1} y = \hbar \eta = \hbar^{1-N} i_N$, whence $\hbar^{N-1} y = \hbar \, i_N$: but the right-hand-side belongs to ${I_{F_\hbar}}^{N+1}$, whilst the left-hand-side cannot belong to ${I_{F_\hbar}}^{N+1}$, due to *(a)*, because $y \notin {I_{F_\hbar}}^2$, a contradiction.

(c) Clearly $F_\hbar \subset {F_\hbar}^\vee$ implies $\big(F_\hbar\big)_\infty := \bigcap_{n=0}^{+\infty} \hbar^n F_\hbar \subseteq \bigcap_{n=0}^{+\infty} \hbar^n {F_\hbar}^\vee =: \big({F_\hbar}^\vee\big)_\infty$. For the converse inclusion, note that by definitions $\big(F_\hbar\big)_\infty$ is a two-sided ideal both inside $F_\hbar$ and inside ${F_\hbar}^\vee$, and ${\overline{F}_\hbar}^\vee \equiv \Big(F_\hbar \big/ \big(F_\hbar\big)_\infty\Big)^\vee = {F_\hbar}^\vee \big/ \big(F_\hbar\big)_\infty$, so we have also $\big({F_\hbar}^\vee\big)_\infty \mod \big(F_\hbar\big)_\infty \subseteq \Big({F_\hbar}^\vee \big/ \big(F_\hbar\big)_\infty\Big)_\infty = \Big({\overline{F}_\hbar}^\vee\Big)_\infty$, with $\overline{F}_\hbar := F_\hbar \big/ {I_{F_\hbar}}^\infty = F_\hbar \big/ \big(F_\hbar\big)_\infty$ (a QFA, by Lemma 3.2*(a)*). So, we prove that $\Big({\overline{F}_\hbar}^\vee\Big)_\infty = \{0\}$ for then $\big({F_\hbar}^\vee\big)_\infty \subseteq \big(F_\hbar\big)_\infty$ will follow.

Let $\mu \colon F_\hbar \longrightarrow \widehat{F_\hbar}$ be the natural map from $F_\hbar$ to its $I_{F_\hbar}$–adic completion, whose kernel is ${I_{F_\hbar}}^\infty = \big(F_\hbar\big)_\infty$: this makes $\overline{F}_\hbar$ embed into $\widehat{F_\hbar}$, and gives ${\overline{F}_\hbar}^\vee \subseteq {\widehat{F_\hbar}}^\vee := \bigcup_{n \geq 0} \hbar^{-n} \widehat{I^n}$ (notation of Lemma 4.1), whence $\Big({\overline{F}_\hbar}^\vee\Big)_\infty \subseteq \Big({\widehat{F_\hbar}}^\vee\Big)_\infty$. Now, the description of $\widehat{F_\hbar}$ and $\widehat{I^n}$ in Lemma 4.1 yields that ${\widehat{F_\hbar}}^\vee$ is contained in the $\hbar$–adic completion of the R–subalgebra of $F(R) \otimes_R \widehat{F_\hbar}$ generated by $\big\{\hbar^{-1} j_b\big\}_{b \in \mathcal{S}}$ (as in the proof of Lemma 4.1), which is a *polynomial* algebra. But then ${\widehat{F_\hbar}}^\vee$ is separated in the $\hbar$–adic topology, i.e. $\Big({\widehat{F_\hbar}}^\vee\Big)_\infty = \{0\}$.

(d) (cf. [EK], Lemma 4.12) By hypothesis $\delta_{n+1}(x') \in \hbar^{n+1} {U_\hbar}^{\otimes (n+1)}$, whence $\delta_{n+1}(x) \in \hbar {U_\hbar}^{\otimes (n+1)}$, so $\delta_{n+1}(\bar{x}) = 0$, i.e. $\bar{x} \in Ker\Big(\delta_{n+1} \colon U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\otimes (n+1)}\Big)$, where $\mathfrak{g}$ is the

Lie bialgebra such that $U_\hbar\big|_{\hbar=0} := U_\hbar \Big/ \hbar\, U_\hbar = U(\mathfrak{g})$. But since $Char(\Bbbk) = 0$, the latter kernel equals $U(\mathfrak{g})_n := \big\{\, \bar{y} \in U(\mathfrak{g}) \,\big|\, \partial(\bar{y}) \leq n \,\big\}$ (cf. [KT], §3.8), whence the claim. $\square$

Proposition 4.3. *Let* $Char(\Bbbk) = 0$. *Let* $F_\hbar \in \mathcal{QFA}$. *Then* $\big({F_\hbar}^\vee\big)' = F_\hbar$.

Proof. Theorem 3.6 gives $F_\hbar \subseteq \big({F_\hbar}^\vee\big)'$, so we have to prove only the converse. Let $\overline{F}_\hbar := F_\hbar \Big/ (F_\hbar)_\infty$; by Lemma 4.2*(c)* we have $\big({F_\hbar}^\vee\big)_\infty = (F_\hbar)_\infty$; by Lemma 3.2*(a)* we have $\big(\overline{F}_\hbar\big)^\vee = {F_\hbar}^\vee \Big/ (F_\hbar)_\infty = {F_\hbar}^\vee \Big/ \big({F_\hbar}^\vee\big)_\infty$, whence again by Lemma 3.2*(a)* we get $\Big(\big(\overline{F}_\hbar\big)^\vee\Big)' = \Big({F_\hbar}^\vee \Big/ \big({F_\hbar}^\vee\big)_\infty\Big)' = \big({F_\hbar}^\vee\big)' \Big/ \big({F_\hbar}^\vee\big)_\infty = \big({F_\hbar}^\vee\big)' \Big/ (F_\hbar)_\infty$. Thus, if the claim is true for $\overline{F}_\hbar$ then $F_\hbar \Big/ (F_\hbar)_\infty =: \overline{F}_\hbar = \Big(\big(\overline{F}_\hbar\big)^\vee\Big)' = \big({F_\hbar}^\vee\big)' \Big/ (F_\hbar)_\infty$, whence clearly $\big({F_\hbar}^\vee\big)' = F_\hbar$. Therefore it is enough to prove the claim for $\overline{F}_\hbar$: in other words, we can assume ${I_{F_\hbar}}^\infty = (F_\hbar)_\infty = \big({F_\hbar}^\vee\big)_\infty = \{0\}$ (see Lemma 4.2*(c)*). In the sequel, set $I := I_{F_\hbar}$.

Let $x' \in \big({F_\hbar}^\vee\big)'$ be given; since $(F_\hbar)_\infty = \{0\}$ there are $n \in \mathbb{N}$ and $x^\vee \in {F_\hbar}^\vee \setminus \hbar\, {F_\hbar}^\vee$ such that $x' = \hbar^n x^\vee$. By Theorem 3.7, ${F_\hbar}^\vee$ is a QrUEA, with semiclassical limit $U(\mathfrak{g})$ where the Lie bialgebra $\mathfrak{g}$ is $\mathfrak{g} = I^\vee \Big/ \big(\hbar\, {F_\hbar}^\vee \bigcap I^\vee\big)$, with $I^\vee := \hbar^{-1} I$.

Fix an ordered basis $\{b_\lambda\}_{\lambda \in \Lambda}$ of $\mathfrak{g}$ over $\Bbbk$, and fix also a subset $\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}$ of $I_{F_\hbar}^\vee$ such that $x_\lambda^\vee \mod \hbar\, {F_\hbar}^\vee = b_\lambda$ for all $\lambda \in \Lambda$: so $x_\lambda^\vee = \hbar^{-1} x_\lambda$ for some $x_\lambda \in J$, for all λ.

Lemma 4.2*(d)* gives $d := \partial(\bar{x}) \leq n$, so we can write $\overline{x^\vee}$ as a polynomial $P\big(\{b_\lambda\}_{\lambda \in \Lambda}\big)$ in the b_λ's of degree $d \leq n$; hence $x^\vee \equiv P\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big) \mod \hbar\, {F_\hbar}^\vee$, so $x^\vee = P\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big) + \hbar\, x_{[1]}^\vee$ for some $x_{[1]}^\vee \in {F_\hbar}^\vee$. Now $x' = \hbar^n x^\vee = \hbar^n P\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big) + \hbar^{n+1} x_{[1]}^\vee$ with

$$\hbar^n P\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big) = \hbar^n P\Big(\big\{\hbar^{-1}\, x_\lambda\big\}_{\lambda \in \Lambda}\Big) \in F_\hbar$$

because P has degree $d \leq n$; thus since $F_\hbar \subseteq \big({F_\hbar}^\vee\big)'$ (by Theorem 3.6) we get

$$x_1' := x' - \hbar^n P\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big) \in \big({F_\hbar}^\vee\big)' \qquad \text{and} \qquad x_1' = \hbar^{n+1} x_{[1]}^\vee = \hbar^{n_1} x_1^\vee$$

for some $n_1 \in \mathbb{N}$, $n_1 > n$, and some $x_1^\vee \in {F_\hbar}^\vee \setminus \hbar\, {F_\hbar}^\vee$. Therefore, we can repeat this construction with x_1' instead of x', n_1 instead of n, and $x_1^\vee$ instead of $x^\vee$, and so on. Iterating, we eventually get an increasing sequence $\big\{n_s\big\}_{s \in \mathbb{N}}$ of natural numbers and a sequence $\Big\{P_s\big(\{X_\lambda\}_{\lambda \in \Lambda}\big)\Big\}_{s \in \mathbb{N}}$ of polynomials such that the degree of $P_s\big(\{X_\lambda\}_{\lambda \in \Lambda}\big)$ is at most n_s, for all $s \in \mathbb{N}$, and $x' = \sum_{s \in \mathbb{N}} \hbar^{n_s} P_s\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big)$.

How should we look at the latter formal series? By construction, each one of the summands $\hbar^{n_s} P_s\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big)$ belongs to $F_\hbar$: more precisely, $\hbar^{n_s} P_s\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big) \in {I_{F_\hbar}}^{n_s}$ for all $s \in \mathbb{N}$; this means that $\sum_{s \in \mathbb{N}} \hbar^{n_s} P_s\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big)$ is a well-defined element of $\widehat{F_\hbar}$, the $I_{F_\hbar}$–adic completion of $F_\hbar$, and the formal expression $x' = \sum_{s \in \mathbb{N}} \hbar^{n_s} P_s\big(\big\{x_\lambda^\vee\big\}_{\lambda \in \Lambda}\big)$ is an identity in $\widehat{F_\hbar}$. So we find $x' \in \big({F_\hbar}^\vee\big)' \bigcap \widehat{F_\hbar}$. Now, consider the embedding $\mu \colon F_\hbar \hookrightarrow \widehat{F_\hbar}$

and the specialization map $\pi \colon F_\hbar \twoheadrightarrow F_\hbar\big|_{\hbar=0} = F[G]$: like in the proof of Lemma 4.1, π extends by continuity to $\widehat{\pi} \colon \widehat{F_\hbar} \twoheadrightarrow \widehat{F[G]} = F[[G]]$: then one easily checks that the map $\mu\big|_{\hbar=0} \colon F[G] = F_\hbar\big|_{\hbar=0} \longrightarrow \widehat{F_\hbar}\big|_{\hbar=0} = F[[G]]$ is injective too. Since $Ker(\pi) = \hbar\, F_\hbar$ and $Ker(\widehat{\pi}) = \hbar\, \widehat{F_\hbar}$, this implies $F_\hbar \bigcap \hbar\, \widehat{F_\hbar} = \hbar\, F_\hbar$, whence $F_\hbar \bigcap \hbar^\ell\, \widehat{F_\hbar} = \hbar^\ell F_\hbar$ for all $\ell \in \mathbb{N}$. Getting back to our $x' \in \big({F_\hbar}^\vee\big)' \bigcap \widehat{F_\hbar}$, we have $x' = \hbar^{-n} y$ for some $n \in \mathbb{N}$ and $y \in F_\hbar$; thus, we conclude that $y = \hbar^n x' \in F_\hbar \bigcap \hbar^n\, \widehat{F_\hbar} = \hbar^n F_\hbar$, so that $x' \in F_\hbar$, q.e.d. $\square$

Proposition 4.4. *Let $H, K \in \mathcal{HA}$, and $\langle\ ,\ \rangle \colon H \times K \longrightarrow R$ a Hopf pairing. Then*

(a) $H^\vee \subseteq \big(K'\big)^\bullet$ and $K' \subseteq \big(H^\vee\big)^\bullet$ (and viceversa). Therefore, the above pairing induces a Hopf pairing $\langle\ ,\ \rangle \colon H^\vee \times K' \longrightarrow R$.

(b) If in addition the pairing $H \times K \longrightarrow R$ and its specialization $H\big|_{\hbar=0} \times K\big|_{\hbar=0} \longrightarrow \Bbbk$ at $\hbar = 0$ are both perfect, and $K = H^\bullet$, then we have also $K' = \big(H^\vee\big)^\bullet$.

(c) Similar results hold for $B, \Omega \in \mathcal{B}$ and $\langle\ ,\ \rangle \colon B \times \Omega \longrightarrow R$ a bialgebra pairing (i.e. a pairing with the properties of Definition 1.2(a) but the one about the antipode).

Proof. *(a)* The Hopf pairing $H_F \times K_F \longrightarrow F(R)$ given by scalar extension clearly restricts to a similar Hopf pairing $H^\vee \times K' \longrightarrow F(R)$: we must prove this takes values in R .

Let $I = I_{{}_H}$, so $H^\vee = \bigcup_{n=0}^\infty \hbar^{-n} I^n$ (cf. §2.1). Pick $c_1, \dots, c_n \in I$, $y \in K'$: then

$$\Big\langle \prod_{i=1}^n c_i \,,\, y \Big\rangle = \Big\langle \otimes_{i=1}^n c_i \,,\, \Delta^n(y) \Big\rangle = \Big\langle \otimes_{i=1}^n c_i \,,\, \sum_{\Psi \subseteq \{1,\dots,n\}} \delta_\Psi(y) \Big\rangle =$$
$$= \sum_{\Psi \subseteq \{1,\dots,n\}} \Big\langle \otimes_{i=1}^n c_i \,,\, \delta_\Psi(y) \Big\rangle = \sum_{\Psi \subseteq \{1,\dots,n\}} \Big\langle \otimes_{i \in \Psi} c_i \,,\, \delta_{|\Psi|}(y) \Big\rangle \cdot \prod_{j \not\in \Psi} \langle c_j \,, 1 \rangle \in$$
$$\in \sum_{\Psi \subseteq \{1,\dots,n\}} \hbar^{n-|\Psi|} R \cdot \hbar^{|\Psi|} R = \hbar^n R \,.$$

The outcome is $\big\langle I^n, K' \big\rangle \subseteq \hbar^n R$, whence $\big\langle \hbar^n I^n, K' \big\rangle \subseteq R$, for all $n \in \mathbb{N}$; since $H^\vee = \bigcup_{n=0}^\infty \hbar^{-n} I^n$, we get $H^\vee \subseteq \big(K'\big)^\bullet$ and $K' \subseteq \big(H^\vee\big)^\bullet$: then it follows also that the restricted pairing $H^\vee \times K' \longrightarrow F(R)$ does take values in R, as claimed.

(b) We revert the previous argument to show that $\big(H^\vee\big)^\bullet \subseteq K'$.

Let $\psi \in \big(H^\vee\big)^\bullet$: then $\big\langle \hbar^{-s} I^s, \psi \big\rangle \in R$ so $\big\langle I^s, \psi \big\rangle \in \hbar^s R$, for all s. For $s = 0$ we get $\big\langle H, \psi \big\rangle \in R$, thus $\psi \in H^\bullet = K$ and so $\delta_n(\psi) \in K^{\otimes n}$ for all n . If $n \in \mathbb{N}$, $i_1, \dots, i_n \in I$,

$$\Big\langle \otimes_{k=1}^n i_k \,,\, \delta_n(\psi) \Big\rangle \;=\; \sum_{\Psi \subseteq \{1,\dots,n\}} (-1)^{n-|\Psi|} \cdot \big\langle \prod_{k \in \Psi} i_k \,, \psi \big\rangle \cdot \prod_{k \not\in \Psi} \epsilon(i_k) \;\in$$
$$\in \sum_{\Psi \subseteq \{1,\dots,n\}} \big\langle I^{|\Psi|}, \psi \big\rangle \cdot \hbar^{n-|\Psi|} R \;\subseteq\; \sum_{s=0}^n \hbar^s \cdot \hbar^{n-s} R = \hbar^n R$$

therefore $\big\langle I^{\otimes n}, \delta_n(\psi) \big\rangle \subseteq \hbar^n R$. Now, H splits as $H = R \cdot 1_{{}_H} \oplus J_{{}_H}$, with $J_{{}_H} := Ker(\epsilon_{{}_H})$; then $H^{\otimes n}$ splits into direct sum of ${J_{{}_H}}^{\otimes n}$ plus other direct summands which are tensor products with at least one tensor factor $R \cdot 1_{{}_H}$. As $J_{{}_K} := Ker(\epsilon_{{}_K}) = \big\{\, y \in K \,\big|\, \langle 1_{{}_H}, y \rangle = 0 \,\big\}$, we have $\big\langle H^{\otimes n}, j^\otimes \big\rangle = \big\langle {J_{{}_H}}^{\otimes n}, j^\otimes \big\rangle$ for $j^\otimes \in {J_{{}_K}}^{\otimes n}$. Now $\delta_n(\psi) \in {J_{{}_K}}^{\otimes n}$: this and the previous analysis together give $\big\langle H^{\otimes n}, \delta_n(\psi) \big\rangle \subseteq \big\langle {I_{{}_H}}^{\otimes n}, \delta_n(\psi) \big\rangle \subseteq \hbar^n R$, for all $n \in \mathbb{N}$.

Now, $H^\bullet = K$ implies $\big(H^{\otimes n}\big)^\bullet = K^{\otimes n}$ for the induced pairing $H^{\otimes n} \times K^{\otimes n} \longrightarrow R$. On the other hand, $\Big\langle H^{\otimes n}, \delta_n(\psi) \Big\rangle \subseteq \hbar^n R$ (for all n) gives $\hbar^{-n} \delta_n(\psi) \in \big(H^{\otimes n}\big)^\bullet = K^{\otimes n}$, that is $\delta_n(\psi) \in \hbar^n K^{\otimes n}$ for all $n \in \mathbb{N}$, whence finally $\psi \in K'$, q.e.d.

(c) We don't need antipode to prove *(a)* and *(b)*: the like arguments prove *(c)* too. $\square$

Proposition 4.5. *Let $Char(\Bbbk) = 0$. Let $U_\hbar \in \mathcal{Q}r\mathcal{UEA}$. Then $\big({U_\hbar}'\big)^\vee = U_\hbar$.*

Proof. First, let $\overline{U}_\hbar := U_\hbar \Big/ \big(U_\hbar\big)_\infty$, and assume the claim holds for $\overline{U}_\hbar$: then repeated applications of Lemma 3.2*(a)* give $U_\hbar \Big/ \big(U_\hbar\big)_\infty = \overline{U}_\hbar = \Big(\big(\overline{U}_\hbar\big)'\Big)^\vee = \big({U_\hbar}'\big)^\vee \Big/ \big(U_\hbar\big)_\infty$, whence $\big({U_\hbar}'\big)^\vee = U_\hbar$ follows at once; therefore we are left to prove the claim for $\overline{U}_\hbar$, which means we may assume $\big(U_\hbar\big)_\infty = \{0\}$. In order to simplify notation, we set $H := U_\hbar$.

Our purpose now is essentially to resort to a similar result which holds for quantum groups "à la Drinfeld": so we mimic the procedure followed in [Ga4] (in particular Proposition 3.7 therein), noting in addition that in the present case we can get rid of the hypotheses $\dim(\mathfrak{g}) < +\infty$ (with $U(\mathfrak{g}) = U_\hbar / \hbar U_\hbar$), as one can check getting through the entire procedure developed in [Ga4] in light of [*loc. cit.*], §3.9.

Let $\widehat{H}$ be the $\hbar$–adic completion of H: this is a separated complete topological $\widehat{R}$–module, $\widehat{R}$ being the $\hbar$–adic completion of R, and a topological Hopf algebra, whose coproduct takes values into $\widehat{H} \,\widehat{\otimes}\, \widehat{H} := H \,\widehat{\otimes}\, H$, the $\hbar$–adic completion of $H \otimes H$ (indeed, $\widehat{H}$ is a *quantized universal enveloping algebra* in the sense of Drinfeld). As $H_\infty = \{0\}$, the natural map $H \longrightarrow \widehat{H}$ embeds H as a (topological) Hopf R–subalgebra of $\widehat{H}$. Then we set also $\widehat{H}' := \big\{\, \eta \in \widehat{H} \,\big|\, \delta_n(\eta) \in \hbar^n \widehat{H}^{\,\widehat{\otimes}\, n} \,\big\}$ and $\big(\widehat{H}'\big)^\times := \bigcup_{n \geq 0} \hbar^{-n} I_{\widehat{H}'}^{\,n} \, \Big(\subseteq Q\big(\widehat{R}\,\big) \otimes_{\widehat{R}} \widehat{H}\Big)$, where $I_{\widehat{H}'} := Ker\big(\epsilon_{\widehat{H}'}\big) + \hbar \cdot \widehat{H}'$ (as in §1.3), and we let $\big(\widehat{H}'\big)^\vee$ be the $\hbar$–adic completion of $\big(\widehat{H}'\big)^\times$.

Now consider $\widehat{K} := \widehat{H}^* \equiv Hom_{\widehat{R}}\Big(\widehat{H}, \widehat{R}\Big)$, the dual of $\widehat{H}$: this is a topological Hopf $\widehat{R}$–algebra, w.r.t. the weak topology, in natural perfect Hopf pairing with $\widehat{H}$: in Drinfeld's terminology, it is a *quantized formal series Hopf algebra*. We define $\widehat{K}^\times := \sum_{n \geq 0} \hbar^{-n} J_{\widehat{K}}^{\,n} \, \Big(\subseteq Q\big(\widehat{R}\,\big) \otimes_{\widehat{R}} \widehat{K}\Big)$, where $J_{\widehat{K}} := Ker\big(\epsilon_{\widehat{K}}\big)$ (as in §1.3) and we let $\widehat{K}^\vee$ be the $\hbar$–adic completion of $\widehat{K}^\times$, and we define $\big(\widehat{K}^\vee\big)'$ in the obvious way. With much the same arguments used for Proposition 4.3, one proves that $\big(\widehat{K}^\vee\big)' = \widehat{K}$. Like in [Ga4], one proves — with much the same arguments as for Proposition 4.4 — that $\widehat{H}' = \big(\widehat{K}^\vee\big)^\bullet$ and $\widehat{K}^\vee \subseteq \big(\widehat{H}'\big)^\bullet$; moreover, one has also $\widehat{H} = \widehat{K}^*$, whence one argues $\widehat{K}^\vee = \big(\widehat{H}'\big)^\bullet$. Using this and the equality $\big(\widehat{K}^\vee\big)' = \widehat{K}$ one proves $\big(\widehat{H}'\big)^\vee = \widehat{H}$ as well (see [Ga4] for details).

Now, definitions imply $\widehat{H} \Big/ \hbar^n \widehat{H} = H \Big/ \hbar^n H$ for all $n \in \mathbb{N}$: thus $\hbar^n \widehat{H} \bigcap H = \hbar^n H$, and similarly $\hbar^n \widehat{H}^{\widehat{\otimes} \ell} \bigcap H^{\otimes \ell} = \hbar^n H^{\otimes \ell}$, for all $n, \ell \in \mathbb{N}$, whence $\widehat{H}' \bigcap H = H'$ follows at once; this easily implies $I_{\widehat{H}'} \bigcap H = I_{H'}$ as well. By construction H is dense inside $\widehat{H}$ w.r.t. the $\hbar$–adic topology; then H' is dense inside $\widehat{H}'$ w.r.t. the topology induced on the latter by the $\hbar$–adic topology of $\widehat{H}$. Now, the description of $\widehat{H}'$ in [Ga4], §3.5 (which can be given also when $\dim(\mathfrak{g}) = +\infty$), tells us that $\widehat{H}' \bigcap \hbar^n H = I_{\widehat{H}'}^n$; then we argue that H' is dense within $\widehat{H}'$ w.r.t. the $I_{\widehat{H}'}$–adic topology of $\widehat{H}'$. This together with $I_{\widehat{H}'} \bigcap H = I_{H'}$ implies, by a standard argument, that $I_{\widehat{H}'}^n \bigcap H = I_{H'}^n$ for all $n \in \mathbb{N}$.

Finally, take $\eta \in H \setminus \hbar H$. We can show that there exists an $\eta' \in I_{\widehat{H}'}^{\partial(\overline{\eta})}$ (notation of Lemma 4.2*(d)*) such that $\eta' = \hbar^{\partial(\overline{\eta})}\eta + \eta'_+$ for some $\eta'_+ \in I_{\widehat{H}'}^{\partial(\overline{\eta})+1}$, just proceeding like in [Ga4], §3.5 (noting again that we can drop the assumption $\dim(\mathfrak{g}) < +\infty$): roughly, we consider any basis of $H\big|_{\hbar=0} = \widehat{H}\big|_{\hbar=0}$ containing $\overline{\eta}$, we look at the dual basis inside $\widehat{K}\big|_{\hbar=0}$ and lift it to a topological basis of K, then rescale the latter — dividing out each element by the proper power of $\hbar$ — to sort a topological basis of $K^\vee$: the dual basis of $\widehat{H}'$ will contain an element η' as required. But then $\hbar^{\partial(\overline{\eta})}\eta = \eta' - \eta'_+ \in I_{\widehat{H}'}^{\partial(\overline{\eta})} \bigcap H = I^{\,\partial(\overline{\eta})}$, thanks to the previous analysis: therefore $\eta = \hbar^{-\partial(\overline{\eta})} \cdot \hbar^{\partial(\overline{\eta})}\eta \in \hbar^{-\partial(\overline{\eta})} I^{\,\partial(\overline{\eta})} \subseteq \big(H'\big)^\vee$. The outcome is $H \subseteq \big(H'\big)^\vee$, whilst the reverse inclusion follows from Theorem 3.6. $\square$

Corollary 4.6. *Let $Char(\Bbbk) = 0$. Let $U_\hbar \in \mathcal{Q}r\mathcal{UEA}$. Then $\big({U_\hbar}'\big)_F = (U_\hbar)_F$.*

Proof. Definitions give ${H^\vee}_F = H_F$ for all $H \in \mathcal{HA}$. Therefore, since $U_\hbar = \big({U_\hbar}'\big)^\vee$ by Proposition 4.5, we have $\big({U_\hbar}'\big)_F = \Big(\big({U_\hbar}'\big)^\vee\Big)_F = (U_\hbar)_F$, q.e.d. $\square$

<u>*Remark*</u>: it is worth noticing that, while ${H^\vee}_F = H_F$ for all $H \in \mathcal{HA}$, *we have not in general* ${H'}_F = H_F$; in particular, example exist of non-trivial $H \in \mathcal{HA}$ such that $H' = R \cdot 1_{{}_H}$, so that ${H'}_F = F(R) \cdot 1_{{}_H} \subsetneqq H$. These cases also yield counterexamples to Proposition 4.5, namely some $H \in \mathcal{HA}$ for which $\big(H'\big)^\vee \subsetneqq H$.

Theorem 4.7. *Let $F_\hbar[G] \in \mathcal{QFA}$ (notation of Remark 1.5) such that $F_\hbar[G]\big|_{\hbar=0}$ is reduced. Then $F_\hbar[G]^\vee\big|_{\hbar=0}$ is a universal enveloping algebra, namely*

$$F_\hbar[G]^\vee\Big|_{\hbar=0} := F_\hbar[G]^\vee \Big/ \hbar\, F_\hbar[G]^\vee = U(\mathfrak{g}^\times)$$

where $\mathfrak{g}^\times$ is the cotangent Lie bialgebra of G (cf. §1.1).

Proof. Set for simplicity $F_\hbar := F_\hbar[G]$, $F_0 := F_\hbar \big/ \hbar\, F_\hbar = F[G]$, and ${F_\hbar}^\vee := F_\hbar[G]^\vee$, ${F_0}^\vee := {F_\hbar}^\vee \big/ \hbar\, {F_\hbar}^\vee$. By Theorem 3.7, ${F_\hbar}^\vee$ is a QrUEA, so ${F_0}^\vee = {F_\hbar}^\vee\big|_{\hbar=0}$ is the restricted universal enveloping algebra $\mathbf{u}(\mathfrak{k})$ of some restricted Lie bialgebra $\mathfrak{k}$. Our purpose is to prove that: first, ${F_0}^\vee = \mathbf{u}(\mathfrak{k}) = U(\mathfrak{h})$ for some Lie bialgebra $\mathfrak{h}$; second, $\mathfrak{h} \cong \mathfrak{g}^\times$.

Once again we can reduce to the case when $F_\hbar$ is separated w.r.t. the $\hbar$–adic topology. Indeed, we have $\big(F_\hbar\big)_\infty = \big({F_\hbar}^\vee\big)_\infty$ by Lemma 4.2*(c)*; then $\overline{{F_\hbar}^\vee} := {F_\hbar}^\vee \Big/ \big({F_\hbar}^\vee\big)_\infty = {F_\hbar}^\vee \big/ \big(F_\hbar\big)_\infty = \big(\overline{F_\hbar}\big)^\vee$ by Lemma 3.2*(a)* (taking notation from there), and so ${F_\hbar}^\vee\Big|_{\hbar=0} = \overline{{F_\hbar}^\vee}\Big|_{\hbar=0} = \big(\overline{F_\hbar}\big)^\vee\Big|_{\hbar=0}$, where the first identity follows from Lemma 3.2*(b)*. Therefore it is enough to prove the claim for $\overline{F_\hbar}$, which means that we can assume that $\big(F_\hbar\big)_\infty = \{0\}$.

Like in Lemma 4.1, let $I := I_{F_\hbar}$, and let $\widehat{F_\hbar}$ be the I–adic completion of $F_\hbar$. By assumption $I^\infty = \big(F_\hbar\big)_\infty = \{0\}$, hence the natural map $F_\hbar \longrightarrow \widehat{F}_\hbar$ is a monomorphism.

Consider $J := Ker\big(\epsilon \colon F_\hbar \longrightarrow R\big)$, and let $J^\vee := \hbar^{-1} J \subset {F_\hbar}^\vee$. As in the proof of Lemma 4.1, let $\{y_b\}_{b \in \mathcal{S}}$ be a $\Bbbk$–basis of $J_0 \Big/ {J_0}^2 = Q\big(F[G]\big)$, where $J_0 := Ker\big(\epsilon_{F[G]}\big) = \mathfrak{m}$, and pull it back to a subset $\{j_b\}_{b \in \mathcal{S}}$ of J. Using notation of Lemma 4.1, we have $I^n \big/ I^{n+1} \cong \widehat{I^n} \big/ \widehat{I^{n+1}}$ for all $n \in \mathbb{N}$; then from the description of the various $\widehat{I}^\ell$ ($\ell \in \mathbb{N}$) given there we see that $I^n \big/ I^{n+1}$ is a $\Bbbk$–vector space with basis the set of (cosets of) ordered monomials $\big\{ \hbar^{e_0} j^{\,\underline{e}} \mod I^{n+1} \,\big|\, e_0 \in \mathbb{N}, \underline{e} \in \mathbb{N}_f^{\mathcal{S}},\ e_0 + |\underline{e}| = n \big\}$ where $|\underline{e}| := \sum_{b \in \mathcal{S}} \underline{e}(b)$. As a consequence, and noting that $\hbar^{-n} I^{n+1} = \hbar \cdot \hbar^{-(n+1)} I^{n+1} \equiv 0 \mod \hbar\, {F_\hbar}^\vee$, we argue that $\hbar^{-n} I^n \mod \hbar\, {F_\hbar}^\vee$ is spanned over $\Bbbk$ by $\big\{ \hbar^{-|\underline{e}|} j^{\,\underline{e}} \mod \hbar\, {F_\hbar}^\vee \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}},\ |\underline{e}| \leq n \big\}$: we claim this set is in fact a basis of $\hbar^{-n} I^n \mod \hbar\, {F_\hbar}^\vee$. Indeed, if not we find a non-trivial linear combination of the elements of this set which is zero: multiplying by $\hbar^n$ this gives an element $\gamma_n \in I^n \setminus I^{n+1}$ such that $\hbar^{-n} \gamma_n \equiv 0 \mod \hbar\, {F_\hbar}^\vee$; then there is $\ell \in \mathbb{N}$ such that $\hbar^{-n} \gamma_n \in \hbar \cdot \hbar^{-\ell} I^\ell$, so $\hbar^\ell \gamma_n = \hbar^{1+n} I^\ell \subseteq I^{1+n+\ell}$: but then Lemma 4.2*(a)* yields $\gamma_n \in I^{n+1}$, a contradiction! The outcome is that $\big\{ \hbar^{-|\underline{e}|} j^{\,\underline{e}} \mod \hbar\, {F_\hbar}^\vee \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}} \big\}$ is a $\Bbbk$–basis of ${F_0}^\vee$.

Now let $j_\beta^\vee := \hbar^{-1} j_\beta$ for all $\beta \in \mathcal{S}$. Since $j_\mu\, j_\nu - j_\nu\, j_\mu \in \hbar\, J$, for any $\mu, \nu \in \mathcal{S}$, we can write $j_\mu\, j_\nu - j_\nu\, j_\mu = \hbar \sum_{\beta \in \mathcal{S}} c_\beta\, j_\beta + \hbar^2 \gamma_1 + \hbar\, \gamma_2$ for some $c_\beta \in R$, $\gamma_1 \in J$ and $\gamma_2 \in J^2$, whence $\big[j_\mu^\vee, j_\nu^\vee\big] := j_\mu^\vee\, j_\nu^\vee - j_\nu^\vee\, \gamma_\mu^\vee = \sum_{\beta \in \mathcal{S}} c_\beta\, j_\beta^\vee + \gamma_1 + \hbar^{-1} \gamma_2 \equiv \sum_{\beta \in \mathcal{S}} c_\beta\, j_\beta^\vee \mod J + J^\vee J$: but $J + J^\vee J = \hbar \big(J^\vee + J^\vee J^\vee\big) \subseteq \hbar\, {F_\hbar}^\vee$, so $\big[j_\mu^\vee, j_\nu^\vee\big] \equiv \sum_{\beta \in \mathcal{S}} c_\beta\, j_\beta^\vee \mod \hbar\, {F_\hbar}^\vee$ which shows that $\mathfrak{h} := J^\vee \mod \hbar\, {F_\hbar}^\vee$ is a Lie subalgebra of ${F_0}^\vee$. By the previous analysis ${F_0}^\vee$ has the $\Bbbk$–basis $\big\{ \big(j^\vee\big)^{\underline{e}} \mod \hbar\, {F_\hbar}^\vee \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}} \big\}$, hence the Poincaré-Birkhoff-Witt theorem tells us that ${F_0}^\vee = U(\mathfrak{h})$ as associative algebras. On the other hand, we saw in the proof of Theorem 3.7 that $\Delta\big(j^\vee\big) \equiv j^\vee \otimes 1 + 1 \otimes j^\vee \mod \hbar\, \big({F_\hbar}^\vee\big)^{\otimes 2}$ for all $j^\vee \in J^\vee$, which gives $\Delta(\mathrm{j}) = \mathrm{j} \otimes 1 + 1 \otimes \mathrm{j}$ for all $\mathrm{j} \in \mathfrak{h}$, whence ${F_0}^\vee = U(\mathfrak{h})$ as *Hopf* algebras too.

Now for the second step. The specialization map $\pi^\vee \colon {F_\hbar}^\vee \twoheadrightarrow {F_0}^\vee = U(\mathfrak{h})$ restricts to $\eta \colon J^\vee \twoheadrightarrow \mathfrak{h} := J^\vee \bmod \hbar\, {F_\hbar}^\vee = J^\vee \Big/ J^\vee \cap \big(\hbar\, {F_\hbar}^\vee\big) = J^\vee \Big/ \big(J + J^\vee J_\hbar\big)$, for $J^\vee \cap \big(\hbar\, {F_\hbar}^\vee\big) = J^\vee \cap \hbar^{-1} {I_{F_\hbar}}^2 = J_\hbar + J^\vee J_\hbar$ by Lemma 4.2*(b)*. Moreover, multiplication by $\hbar^{-1}$ yields an R–module isomorphism $\mu \colon J \overset{\cong}{\hookrightarrow\!\!\!\rightarrow} J^\vee$. Let $\rho \colon J_0 \twoheadrightarrow J_0 \big/ {J_0}^2 =: \mathfrak{g}^\times$ be the natural projection map, and $\nu \colon \mathfrak{g}^\times \hookrightarrow J_0$ a section of ρ. The specialization map $\pi \colon F_\hbar \twoheadrightarrow F_0$ restricts to $\pi' \colon J \twoheadrightarrow J \big/ (J \cap \hbar\, F_\hbar) = J_\hbar \big/ \hbar\, J_\hbar = J_0$: we fix a section $\gamma \colon J_0 \hookrightarrow J_\hbar$ of π'.

Consider the composition map $\sigma := \eta \circ \mu \circ \gamma \circ \nu \colon \mathfrak{g}^\times \longrightarrow \mathfrak{h}$. This is well-defined, i.e. it is independent of the choice of ν and γ. Indeed, if $\nu, \nu' \colon \mathfrak{g}^\times \hookrightarrow J_0$ are two sections of ρ, and σ, σ' are defined correspondingly (with the same fixed γ for both), then $Im\,(\nu - \nu') \subseteq Ker\,(\rho) = {J_0}^2 \subseteq Ker\,(\eta \circ \mu \circ \gamma)$, so that $\sigma = \eta \circ \mu \circ \gamma \circ \nu = \eta \circ \mu \circ \gamma \circ \nu' = \sigma'$. Similarly, if $\gamma, \gamma' \colon J_0 \hookrightarrow J_\hbar$ are two sections of π', and σ, σ' are defined correspondingly (with the same ν for both), we have $Im\,(\gamma - \gamma') \subseteq Ker\,(\pi') = \hbar\, J = Ker\,(\eta \circ \mu)$, thus

$\sigma = \eta \circ \mu \circ \gamma \circ \nu = \eta \circ \mu \circ \gamma' \circ \nu = \sigma'$, q.e.d. In a nutshell, σ is the composition map

$$\mathfrak{g}^\times \xhookrightarrow{\;\bar{\nu}\;} J_0 \big/ {J_0}^2 \xhookrightarrow{\;\bar{\gamma}\;} J \Big/ \big(J^2 + \hbar\, J\,\big) \xhookrightarrow{\;\bar{\mu}\;} J^\vee \Big/ \big(J + J^\vee J\,\big) \xhookrightarrow{\;\bar{\eta}\;} \mathfrak{h}$$

where the maps $\bar{\nu}$, $\bar{\gamma}$, $\bar{\mu}$, $\bar{\eta}$, are the ones canonically induced by ν, γ, μ, η, and $\bar{\nu}$, *resp.* $\bar{\gamma}$, *does not depend on the choice of* ν, *resp.* γ, as it is the inverse of the isomorphism $\bar{\rho}\colon J_0 \big/ {J_0}^2 \xhookrightarrow{\;\cong\;} \mathfrak{g}^\times$, resp. $\overline{\pi'}\colon J \Big/ \big(J^2 + \hbar\, J\big) \xhookrightarrow{\;\cong\;} J_0 \big/ {J_0}^2$, induced by ρ, resp. by π'. We use this remark to show that σ is also an isomorphism of the Lie bialgebra structure.

Using the vector space isomorphism $\sigma : \mathfrak{g}^\times \xrightarrow{\;\cong\;} \mathfrak{h}$ we pull-back the Lie bialgebra structure of $\mathfrak{h}$ onto $\mathfrak{g}^\times$, and denote it by $\big(\mathfrak{g}^\times, [\ ,\ ]_\bullet, \delta_\bullet\big)$; on the other hand, $\mathfrak{g}^\times$ also carries its natural structure of Lie bialgebra, dual to that of $\mathfrak{g}$ (e.g., the Lie bracket is induced by restriction of the Poisson bracket of $F[G]$), denoted by $\big(\mathfrak{g}^\times, [\ ,\ ]_\times, \delta_\times\big)$: we must prove that these two structures coincide.

First, *for all* x_1, $x_2 \in \mathfrak{g}^\times$ *we have* $\big[x_1, x_2\big]_\bullet = \big[x_1, x_2\big]_\times$.

Indeed, let $f_i := \nu(x_i)$, $\varphi_i := \gamma(f_i)$, $\varphi_i^\vee := \mu(\varphi_i)$, $y_i := \eta\big(\varphi_i^\vee\big)$ $(\, i = 1, 2)$. Then

$$\begin{aligned}\big[x_1, x_2\big]_\bullet := \sigma^{-1}\Big(\big[\sigma(x_1), \sigma(x_2)\big]_\mathfrak{h}\Big) = \sigma^{-1}\big([y_1, y_2]\big) = \big(\rho \circ \pi' \circ \mu^{-1}\big)\Big(\big[\varphi_1^\vee, \varphi_2^\vee\big]\Big) = \\ = \big(\rho \circ \pi'\big)\big(\hbar^{-1}[\varphi_1, \varphi_2]\big) = \rho\big(\{f_1, f_2\}\big) =: \big[x_1, x_2\big]_\times \ , \qquad \text{q.e.d.}\end{aligned}$$

The case of Lie cobrackets can be treated similarly; but since they take values in tensor squares, we make use of suitable maps $\nu_\otimes := \nu^{\otimes 2}$, $\gamma_\otimes := \gamma^{\otimes 2}$, etc.; we also make use of notation $\chi_\otimes := \eta_\otimes \circ \mu_\otimes = (\eta \circ \mu)^{\otimes 2}$ and $\nabla := \Delta - \Delta^{\text{op}}$.

Now, *for all* $x \in \mathfrak{g}^\times$ *we have* $\delta_\bullet(x) = \delta_\times(x)$.

Indeed, let $f := \nu(x)$, $\varphi := \gamma(f)$, $\varphi^\vee := \mu(\varphi)$, $y := \eta\big(\varphi^\vee\big)$. Then we have

$$\begin{aligned}&\delta_\bullet(x) := {\sigma_\otimes}^{-1}\big(\delta_\mathfrak{h}(\sigma(x))\big) = {\sigma_\otimes}^{-1}\big(\delta_\mathfrak{h}\big(\eta(\varphi^\vee)\big)\big) = {\sigma_\otimes}^{-1}\Big(\eta_\otimes\big(\hbar^{-1}\nabla(\varphi^\vee)\big)\Big) = \\ &= {\sigma_\otimes}^{-1}\Big(\big(\eta \circ \mu\big)_\otimes\big(\nabla(\varphi)\big)\Big) = {\sigma_\otimes}^{-1}\Big(\big(\eta \circ \mu \circ \gamma\big)_\otimes\big(\nabla(f)\big)\Big) = \rho_\otimes\big(\nabla(f)\big) = \rho_\otimes\big(\nabla(\nu(x))\big) = \delta_\times(x)\end{aligned}$$

where the last equality holds because $\delta_\times(x)$ is uniquely defined as the unique element in $\mathfrak{g}^\times \otimes \mathfrak{g}^\times$ such that $\big\langle u_1 \otimes u_2\, , \delta_\times(x)\big\rangle = \big\langle [u_1, u_2]\, , x\big\rangle$ for all $u_1, u_2 \in \mathfrak{g}$, and we have

$$\big\langle [u_1, u_2]\, , x\big\rangle = \big\langle [u_1, u_2]\, , \rho(f)\big\rangle = \big\langle u_1 \otimes u_2\, , \nabla(f)\big\rangle = \big\langle u_1 \otimes u_2\, , \rho_\otimes\big(\nabla(\nu(x))\big)\big\rangle\ . \qquad \square$$

4.8 Interlude: quantizations of pointed Poisson manifolds and of their linear approximation. The proof of Theorem 4.7 in fact leads to a more general result; to mention it, we need some more terminology.

Namely, among algebraic $\Bbbk$–varieties let us consider the *pointed Poisson varieties*, defined to be pairs $(M, \bar{m})$ where M is a Poisson variety and $\bar{m} \in M$ is a point of M where the rank of the Poisson bivector is zero: in other words, $\{\bar{m}\}$ is a symplectic leaf of M. A *morphism of pointed Poisson varieties* $(M, \bar{m})$ *and* $(N, \bar{n})$ is any Poisson map $\varphi : M \longrightarrow N$ such that $\varphi(\bar{m}) = \bar{n}$. Clearly this defines a subcategory of the category of all Poisson varieties, whose morphisms are those morphisms of Poisson varieties which map distinguished points into distinguished points. In terms of their function algebras, any pointed Poisson variety M is given by the datum $\big(F[M], \mathfrak{m}_{\bar{m}}\big)$ where $\mathfrak{m}_{\bar{m}}$ is the defining ideal of $\bar{m} \in M$.

By assumptions, the Poisson bracket of $F[M]$ restricts to a Lie bracket onto $\mathfrak{m}_{\bar{m}}$: from this the quotient space $\mathcal{L}_M := \mathfrak{m}_{\bar{m}} \big/ \mathfrak{m}_{\bar{m}}^2$ (the cotangent space to M at $\bar{m}$) inherits a Lie algebra structure too, the so-called "linear approximation of M at $\bar{m}$" (see e.g. [We], §4). In the following we also call it the cotangent Lie algebra of $(M, \bar{m})$, or simply of M.

Natural examples of pointed Poisson varieties are the coisotropic Poisson homogeneous spaces, also called *Poisson quotients*, i.e. those Poisson homogeneous spaces of the form G/H, where G is a Poisson group and H is a (closed) coisotropic subgroup, where coisotropic means that the ideal $I(H)$ in $F[G]$ of all functions vanishing on H is a Poisson subalgebra of $F[G]$. The distinguished point is the coset eH of the unit element $e \in G$.

Another special class is given by the category of Poisson monoids (= unital Poisson semigroups): each one of them is naturally pointed by its unit element. If $(M, \bar{m}) = (\mathit{\Lambda}, e)$ is any Poisson monoid, then $F[\mathit{\Lambda}]$ is a bialgebra (and conversely), and $\mathcal{L}_{\mathit{\Lambda}}$ has a natural structure of *Lie bialgebra* — the *cotangent Lie bialgebra* of $\mathit{\Lambda}$ — the Lie cobracket being induced by the coproduct of $F[\mathit{\Lambda}]$, hence (dually) by the multiplication in $\mathit{\Lambda}$. It follows then that $U(\mathcal{L}_{\mathit{\Lambda}})$ is a co-Poisson Hopf algebra. When in particular the monoid $\mathit{\Lambda}$ is a Poisson group G we have $\mathit{\Lambda} = \mathfrak{g}^\times$.

We call *quantization of a pointed Poisson variety* $(M, \bar{m})$ any $A \in \mathcal{A}^+$ such that $A\big|_{\hbar=0} \cong F[M]$ as Poisson $\Bbbk$–algebras, and if $\pi : A \twoheadrightarrow A\big|_{\hbar=0} \cong F[M]$ is the specialisation map ($\hbar \mapsto 0$), then $Ker(\pi \circ \underline{\epsilon}_M) = \mathfrak{m}_{\bar{m}}$; in this case we write $A = F_\hbar[M]$. For any such object we set $J_M := Ker(\underline{\epsilon}_M)$ and $I_M := J_M + \hbar A$. A *morphism of quantizations of Poisson varieties* is any morphism $\phi : F_\hbar[M] \longrightarrow F_\hbar[N]$ in $\mathcal{A}^+$ such that $\phi(J_M) \subseteq J_N$. Quantizations of pointed Poisson varieties and their morphisms form a subcategory of $\mathcal{A}^+$.

A quick check throughout the proof of Theorem 4.7 (and of Theorem 3.7 for the last part of the claim) then shows that the same arguments also prove the following:

Theorem 4.9. *Let $F_\hbar[M] \in \mathcal{A}^+$ be a quantization of a pointed Poisson manifold $(M, \bar{m})$ (as above) such that $F_\hbar[M]\big|_{\hbar=0}$ is reduced. Then $F_\hbar[M]^\vee\big|_{\hbar=0}$ is a universal enveloping algebra, namely*

$$F_\hbar[M]^\vee\Big|_{\hbar=0} := F_\hbar[M]^\vee \Big/ \hbar\, F_\hbar[M]^\vee = U(\mathcal{L}_M)$$

(notation of §4.8). If in addition M is a Poisson monoid and $F_\hbar[M]$ is a quantization of $F[M]$ in $\mathcal{B}$, then the last identification above is one of Hopf algebras. $\square$

Theorem 4.10. *Let* $Char(\Bbbk) = 0$ *. Let* $U_\hbar(\mathfrak{g}) \in \mathcal{QrUEA}$ *(notation of Remark 1.5). Then*
$$U_\hbar(\mathfrak{g})'\Big|_{\hbar=0} := U_q(\mathfrak{g})' \Big/ \hbar\, U_\hbar(\mathfrak{g})' = F\big[G^\star\big]$$
where $G^\star$ *is a connected algebraic Poisson group dual to* G *(as in §1.1).*

Proof. Due to Theorem 3.8, $U_\hbar(\mathfrak{g})'$ is a QFA, with $U_\hbar(\mathfrak{g})' \xrightarrow{\ \hbar\to 0\ } F[H]$ for some connected algebraic Poisson group H ; in addition we know by assumption that $F[H]$ is reduced: we have to show that H is a group of type $G^\star$ as in the claim.

Applying Theorem 4.7 to the QFA $U_\hbar(\mathfrak{g})'$ yields $\big(U_\hbar(\mathfrak{g})'\big)^\vee \xrightarrow{\ \hbar\to 0\ } U(\mathfrak{h}^\times)$. Since Proposition 4.5 gives $U_\hbar(\mathfrak{g}) = \big(U_\hbar(\mathfrak{g})'\big)^\vee$, we have then
$$U(\mathfrak{g}) \xleftarrow{\ 0\leftarrow\hbar\ } U_\hbar(\mathfrak{g}) = \big(U_\hbar(\mathfrak{g})'\big)^\vee \xrightarrow{\ \hbar\to 0\ } U(\mathfrak{h}^\times)$$
so that $\mathfrak{h}^\times = \mathfrak{g}$: thus $\mathfrak{h} := \big(\mathfrak{h}^\times\big)^\star = \mathfrak{g}^\star$, whence $H = G^\star$, q.e.d. $\square$

Theorem 4.11. *Let* $Char(\Bbbk) = 0$ *. Consider* $F_\hbar \in \mathcal{QFA}$ *,* $U_\hbar \in \mathcal{QrUEA}$ *, and a perfect Hopf pairing* $\langle\ ,\ \rangle : F_\hbar \times U_\hbar \longrightarrow R$ *such that* $F_\hbar = {U_\hbar}^\bullet$ *and* $U_\hbar = {F_\hbar}^\bullet$*. Then*
$${U_\hbar}' = \big({F_\hbar}^\vee\big)^\bullet \qquad \text{and} \qquad {F_\hbar}^\vee = \big({U_\hbar}'\big)^\bullet .$$

Proof. First of all notice that the assumptions imply that the specialized Hopf pairing $F_\hbar\big|_{\hbar=0} \times U_\hbar\big|_{\hbar=0} \longrightarrow \Bbbk$ is perfect as well: then Proposition 4.4*(b)* gives ${U_\hbar}' = \big({F_\hbar}^\vee\big)^\bullet$. In addition ${F_\hbar}^\vee \subseteq \big({U_\hbar}'\big)^\bullet$ by Proposition 4.4*(a)*, and we have to prove the reverse inclusion.

Let $\varphi \in \big({U_\hbar}'\big)^\bullet$; in particular, we can choose φ such that $\big\langle \varphi\,, {U_\hbar}' \big\rangle = R$. Since $\big({U_\hbar}'\big)^\bullet \subseteq F(R) \otimes_R F_\hbar = F(R) \otimes_R {F_\hbar}^\vee$, there exists $c \in R \setminus \{0\}$ such that $\varphi_+ := c\,\varphi \in {F_\hbar}^\vee \setminus \hbar\, {F_\hbar}^\vee$: it follows that $\big\langle \varphi_+\,, {U_\hbar}' \big\rangle = c\,R$. If $F_\hbar = F_\hbar[G]$, $U_\hbar = U_\hbar(\mathfrak{g})$, then Theorems 4.7–8 give ${F_\hbar}^\vee\big|_{\hbar=0} = U(\mathfrak{g}^\times)$ and ${U_\hbar}'\big|_{\hbar=0} = F[G^\star]$. Thus there is $\overline{\eta} \in F[G^\star]$ such that $\big\langle \varphi_+\big|_{\hbar=0}\,, \overline{\eta}\,\big\rangle = 1$, hence there is $\eta \in {U_\hbar}'$ (a lift of $\overline{\eta}$) such that $\big\langle \varphi_+, \eta \big\rangle = 1 + \hbar\,\kappa$ for some $\kappa \in R$; but $\big\langle \varphi_+, \eta \big\rangle \in c\,R$ by construction, hence c divides $(1 + \hbar\,\kappa)$ in R.

As $\varphi_+ \in {F_\hbar}^\vee := \bigcup_{n \in \mathbb{N}} \hbar^{-n} I_{F_\hbar}^{\,n}$ we have $\varphi_+ = \hbar^{-n} \varphi_0$ for some $n \in \mathbb{N}$ and $\varphi_0 \in I_{F_\hbar}^{\,n}$; therefore $\big\langle \varphi_0, {U_\hbar}' \big\rangle = c\,\hbar^n R$. On the other hand, since $U_\hbar = \big({U_\hbar}'\big)^\vee$ (by Proposition 4.5) each $y \in U_\hbar$ can be written as $y = \hbar^{-\ell} y'$ for some $\ell \in \mathbb{N}$ and $y' \in {U_\hbar}'$; then $\big\langle \varphi_0, y \big\rangle = c\,\hbar^{n-\ell} \big\langle \varphi, y' \big\rangle \in R \bigcap c\,\hbar^{n-\ell} R$ because $\big\langle \varphi_0, y \big\rangle \in R$ and $\big\langle \varphi, y' \big\rangle \in R$. Now, if $\hbar^{n-\ell} \big\langle \varphi, y' \big\rangle \notin R$ then $n - \ell < 0$ and so $\hbar$ divides c. Since c divides $(1 + \hbar\,\kappa)$ we get an absurd, unless c is invertible in R: hence $\varphi = c^{-1} \varphi_+ \in {F_\hbar}^\vee$, q.e.d. Otherwise, we have always $\hbar^{n-\ell} \big\langle \varphi, y' \big\rangle \in R$, which means $\big\langle \varphi_0, y \big\rangle \in c\,R$ for all $y \in U_\hbar$; thus $c^{-1} \varphi_0 \in {U_\hbar}^\bullet = F_\hbar$. Now consider the $I_{F_\hbar}$–adic completion $\widehat{F_\hbar}$ of $F_\hbar$: the kernel of the natural map $\mu : F_\hbar \longrightarrow \widehat{F_\hbar}$ is $I_{F_\hbar}^{\,\infty} = (F_\hbar)_\infty$ (because $F_\hbar \in \mathcal{QFA}$), and the latter is zero because it is contained in the trivial left radical of the perfect pairing between $F_\hbar$ and $U_\hbar$; therefore $F_\hbar$ embeds into $\widehat{F_\hbar}$ via μ. We have $c^{-1} \varphi_0 \in F_\hbar \subseteq \widehat{F_\hbar}$ and $\varphi_0 \in I_{F_\hbar}^{\,n} \subseteq \widehat{I_{F_\hbar}^{\,n}}$: then from Lemma 4.1*(a)–(b)* we argue that $c^{-1} \varphi_0 \in \widehat{I_{F_\hbar}^{\,n}}$, hence finally $c^{-1} \varphi_0 \in \widehat{I_{F_\hbar}^{\,n}} \bigcap F_\hbar = I_{F_\hbar}^{\,n}$, thanks to Lemma 4.1*(d)*. The outcome is $\varphi = c^{-1} \hbar^{-n} \varphi_0 \in h^{-n} I_{F_\hbar}^{\,n} \subseteq {F_\hbar}^\vee$, q.e.d. $\square$

At last, we can gather our partial results to prove the main Theorem:

Proof of Theorem 2.2. Part *(a)* is proved by patching together Proposition 3.3, Proposition 3.5 and Theorems 3.6–8. Now recall that if $Char(\Bbbk) = 0$ every commutative Hopf $\Bbbk$–algebra is reduced; then part *(b)* follows from Theorem 3.6 and Propositions 4.3 and 4.5. Part *(c)* is proved by Theorems 4.7–8, whereas Theorem 4.11 proves part *(d)*. Finally, assume $Char(\Bbbk) = 0$ and consider $\mathbb{H} \in \mathcal{HA}_F$: if $H_{(f)} \in \mathcal{QFA}$ (w.r.t. a fixed prime $\hbar \in R$) is an R–integer form of $\mathbb{H}$, then $H_{(f)}^\vee$ is an integer form too — by the very definitions — and it is a QrUEA (at $\hbar$), by Proposition 3.3; conversely, if $H_{(u)} \in \mathcal{Q}r\mathcal{UEA}$ (w.r.t. a fixed prime $\hbar \in R$) is an R–integer form of $\mathbb{H}$, then also $H'_{(u)}$ is an integer form — by Corollary 4.6 — and it is a QFA (again at $\hbar$), by Proposition 3.5; this proves part *(e)*. $\square$

§ 5 Application to trivial deformations: the Crystal Duality Principle

5.1 Trivial deformations and GQDP. In this section, we apply the GQDP to the framework of trivial deformations of Hopf algebras over a field. In particular, we consider more closely some key examples: function algebras over algebraic groups, universal enveloping algebras of Lie algebras, and group algebras of abstract groups. The outcome seems to be of special interest in its own, as a chapter of classical — rather than "quantum" — Hopf algebra theory, and we propose it as a new tool for specialists in that matter.

To be short we perform our analysis for Hopf algebras only: however, as Drinfeld's functors are defined not only for Hopf algebras but for augmented algebras and coaugmented coalgebras too, we might do the same study for them as well (indeed, we do it in [Ga5]).

Let us now be more precise. Let $\mathcal{HA}_\Bbbk$ be the category of all Hopf algebras over the field $\Bbbk$. For all $n \in \mathbb{N}$, let $J^n := \big(Ker(\epsilon\colon H \longrightarrow \Bbbk)\big)^n$ and $D_n := Ker\big(\delta_{n+1}\colon H \longrightarrow H^{\otimes n}\big)$, and set $\underline{J} := \big\{J^n\big\}_{n\in\mathbb{N}}$, $\underline{D} := \big\{D_n\big\}_{n\in\mathbb{N}}$. Of course $\underline{J}$ is a decreasing filtration of H (maybe with $\bigcap_{n\geq 0} J^n \supsetneqq \{0\}$), and $\underline{D}$ is an increasing filtration of H (maybe with $\bigcup_{n\geq 0} D_n \subsetneqq H$), by coassociativity of the δ_n's.

Let $R := \Bbbk[\hbar]$ be the polynomial ring in the indeterminate $\hbar$: then R is a PID (= principal ideal domain), and $\hbar$ is a non-zero prime in R such that $R\big/\hbar\, R$ is the field $\Bbbk$. Let $H_\hbar := H[\hbar] = R \otimes_\Bbbk H$, the scalar extension of H: this is the *trivial deformation* of H. Clearly $H_\hbar$ is a torsion free Hopf algebra over R, hence one can apply Drinfeld's functors to it; in this section we do it with respect to the element $\hbar$ itself. We shall see that the outcome is quite neat, and can be expressed purely in terms of Hopf algebras in $\mathcal{HA}_\Bbbk$: because of the special relation between some features of H — namely, the filtrations $\underline{J}$ and $\underline{D}$ — and some properties of Drinfeld's functors, we call this result "Crystal Duality Principle", in that it is obtained through sort of a "crystallization" process. Here we bear in mind, in a sense, Kashiwara's motivation for the terminology "crystal bases" in the context of quantum groups: see [CP], §14.1, and references therein. Indeed, this theorem can also be proved almost entirely using only classical Hopf algebraic methods within $\mathcal{HA}_\Bbbk$, i.e. without resorting to deformations: this is accomplished in [Ga5].

Note that the same analysis and results (with only a bit more annoying details to take care of) still hold if we take as R any domain which is also a $\Bbbk$–algebra and as $\hbar$ any element in $R \setminus \{0\}$ such that $R/\hbar R = \Bbbk$; for instance, one can take $R = \Bbbk[[h]]$ and $\hbar := h$, or $R = \Bbbk\big[q, q^{-1}\big]$ and $\hbar := q-1$. Finally, in the sequel to be short we perform our analysis for Hopf algebras only: however, as Drinfeld's functors are defined not only for Hopf algebras but for augmented algebras and coaugmented coalgebras too, we might do the same study for them as well. In particular, the final result (the Crystal Duality Principle) has a stronger version which concerns these more general objects too (see [Ga5]).

We first discuss the general situation (§§5.2–4), second we look at the case of function algebras and enveloping algebras (§§5.6–7), then we state and prove the theorem of Crystal Duality Principle and eventually (§§5.12–13) we dwell upon two other interesting applications: hyperalgebras, and group algebras and their duals.

Lemma 5.2.

$$H_\hbar^\vee \; = \; \sum\nolimits_{n \geq 0} R \cdot \hbar^{-n} J^n \; = \; R \cdot J^0 + R \cdot \hbar^{-1} J^1 + \cdots + R \cdot \hbar^{-n} J^n + \cdots \tag{5.1}$$

$$H_\hbar{}' \; = \; \sum\nolimits_{n \geq 0} R \cdot \hbar^{n} D_n \; = \; R \cdot D_0 + R \cdot \hbar\, D_1 + \cdots + R \cdot \hbar^{n} D_n + \cdots \tag{5.2}$$

Proof. As for (5.1), we have $J_{H_\hbar} = R \cdot J$, whence $H_\hbar^\vee := \sum_{n \geq 0} \hbar^{-n} J_\hbar^{\,n} = \sum_{n \geq 0} \hbar^{-n} J^n$.

On the other hand, one has trivially $H_\hbar{}' \supseteq \sum_{n=0}^{+\infty} R \cdot \hbar^n D_n$. Conversely, let $\eta \in H_\hbar{}'$: then $\eta = \sum_k c_k \, \eta_k$ for some $c_k \in R$ and $\eta_k \in H$; in addition, we can assume the η_k's enjoy the following: $\eta_1, \dots, \eta_{k_1} \in D_{\ell_1} \setminus D_{\ell_1 - 1}$, $\eta_{k_1+1}, \dots, \eta_{k_2} \in D_{\ell_2} \setminus D_{\ell_2 - 1}$, $\dots$, η_1, $\dots$, $\eta_{k_t} \in D_{\ell_t} \setminus D_{\ell_t - 1}$ for some $k_i, \ell_j, t \in \mathbb{N}$ with $k_1 < k_2 < \cdots < k_t$, they are linearly independent over $\Bbbk$, and moreover $\eta_{k_i+1}, \dots, \eta_{k_{i+1}}$ belong to a vector subspace W_i of H such that $W_i \bigcap D_{\ell_i} = \{0\}$, for all i. By the assumptions on R, for any k there is a unique $v_k \in \mathbb{N}$ such that $c_k \in \hbar^{v_k} R \setminus \hbar^{v_k+1} R$; then for all $n \in \mathbb{N}$ we have $c_k \equiv 0 \mod \hbar^n R$ when $v_k \geq n$ and $c_k \equiv \sum_{s=v_k}^{n-1} a_s^{(k)} \, \hbar^s \mod \hbar^n R$, for some $a_s^{(k)} \in \Bbbk$ with $a_{v_k}^{(k)} \neq 0$, when $v_k < n$. Then $\sum_{v_k < n} \sum_{s=v_k}^{n-1} a_s^{(k)} \hbar^s \, \delta_n(\eta_k) \equiv \delta_n(\eta) \equiv 0 \mod \hbar^n$ and $\delta_n(\eta_k) \in H^{\otimes n} \subset H_\hbar^{\otimes n} \setminus \hbar \, H_\hbar^{\otimes n}$ imply — since $H_\hbar^{\otimes n} \Big/ \hbar^n H_\hbar^{\otimes n} \cong \Big(R/(\hbar^n R) \Big) \otimes_\Bbbk H^{\otimes n} \cong \Big(\Bbbk[x]/(x^n) \Big) \otimes_\Bbbk H^{\otimes n}$ — that $\sum_{n > v_k = v_-} a_{v_-}^{(k)} \delta_n(\eta_k) = 0$, where $v_- := \min\limits_k \{v_k\}$, hence $\sum_{n > v_k = v_-} a_{v_-}^{(k)} \eta_k \in \mathit{Ker}\,(\delta_n) =: D_{n-1}$: since all coefficients $a_{v_-}^{(k)}$ in this sum are non-zero, by our assumptions on the η_k's this forces $\eta_k \in \mathit{Ker}\,(\delta_n) =: D_{n-1}$, for all k such that $v_k = v_-$. The outcome is: $v_k < n \implies \eta_k \in D_{n-1}$ (for all k, n), whence we get $\eta_k \in D_{v_k}$ for all k, so that $\eta = \sum_k c_k \, \eta_k \in \sum_{s=0}^{+\infty} R \cdot \hbar^s D_s$, q.e.d. $\square$

5.3 Rees Hopf algebras and their specializations. We need some more terminology. Let M be a module over a commutative unitary ring R, and let

$$\underline{M} \;:=\; \{M_z\}_{z \in \mathbb{Z}} \;=\; \Big(\cdots \subseteq M_{-m} \subseteq \cdots \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots \Big)$$

be a bi-infinite filtration of M by submodules M_z ($z \in \mathbb{Z}$). In particular, we consider increasing filtrations (i.e., those with $M_z = \{0\}$ for all $z < 0$) and decreasing filtrations (those with $M_z = \{0\}$ for all $z > 0$) as special cases of bi-infinite filtrations. First we define the associated *blowing module* to be the R–submodule $\mathcal{B}_{\underline{M}}(M)$ of $M\big[t, t^{-1}\big]$ (where t is any indeterminate) given by $\mathcal{B}_{\underline{M}}(M) := \sum_{z \in \mathbb{Z}} t^z M_z$; this is isomorphic to the *first graded module*[2] *associated to* M, namely $\bigoplus_{z \in \mathbb{Z}} M_z$. Second, we define the associated *Rees module* to be the $R[t]$–submodule $\mathcal{R}^t_{\underline{M}}(M)$ of $M\big[t, t^{-1}\big]$ generated by $\mathcal{B}_{\underline{M}}(M)$; straightforward computations then give R–module isomorphisms

$$\mathcal{R}^t_{\underline{M}}(M) \Big/ (t-1)\, \mathcal{R}^t_{\underline{M}}(M) \;\cong\; \bigcup_{z \in \mathbb{Z}} M_z \; , \qquad \mathcal{R}^t_{\underline{M}}(M) \Big/ t\, \mathcal{R}^t_{\underline{M}}(M) \;\cong\; G_{\underline{M}}(M)$$

where $G_{\underline{M}}(M) := \bigoplus_{z \in Z} M_z \big/ M_{z-1}$ is the *second graded module associated to* M . In other words, $\mathcal{R}^t_{\underline{M}}(M)$ is an $R[t]$–module which specializes to $\bigcup_{z \in \mathbb{Z}} M_z$ for $t = 1$ and specializes to $G_{\underline{M}}(M)$ for $t = 0$; therefore the R–modules $\bigcup_{z \in \mathbb{Z}} M_z$ and $G_{\underline{M}}(M)$ can be seen as 1-parameter (polynomial) deformations of each other via the 1-parameter family of R–modules given by $\mathcal{R}^t_{\underline{M}}(M)$.

We can repeat this construction within the category of algebras, coalgebras, bialgebras or Hopf algebras over R with a filtration in the proper sense (by subalgebras, subcoalgebras, etc.): then we'll end up with corresponding objects $\mathcal{B}_{\underline{M}}(M)$, $\mathcal{R}^t_{\underline{M}}(M)$, etc. of the like type (algebras, coalgebras, etc.). In particular we'll cope with Rees Hopf algebras.

5.4 Drinfeld's functors on $\boldsymbol{H_\hbar}$ and filtrations on $\boldsymbol{H}$. Lemma 5.2 sets a link between properties of ${H_\hbar}'$, resp. of ${H_\hbar}^{\vee}$, and properties of the filtration $\underline{D}$, resp. $\underline{J}$, of H .

First, formula (5.1) together with the fact that ${H_\hbar}^{\vee} \in \mathcal{HA}$ implies that $\underline{J}$ is a Hopf algebra filtration of H ; conversely, if one proves that $\underline{J}$ is a Hopf algebra filtration of H (which is straightforward) then from (5.1) we get a one-line direct proof that ${H_\hbar}^{\vee} \in \mathcal{HA}$. Second, we can look at $\underline{J}$ as a bi-infinite filtration by reversing the index notation and then extending it trivially on the positive indices, namely

$$\underline{J} \;=\; \Big(\cdots \subseteq J^n \subseteq \cdots J^2 \subseteq J \subseteq J^0 \big(=H\big) \subseteq H \subseteq \cdots \subseteq H \subseteq \cdots \Big) \, ;$$

then the Rees Hopf algebra $\mathcal{R}^\hbar_{\underline{J}}(H)$ is defined (see §5.3). Now (5.1) give ${H_\hbar}^{\vee} = \mathcal{R}^\hbar_{\underline{J}}(H)$, so ${H_\hbar}^{\vee} \Big/ \hbar\, {H_\hbar}^{\vee} \cong \mathcal{R}^\hbar_{\underline{J}}(H) \Big/ \hbar \mathcal{R}^\hbar_{\underline{J}}(H) \cong G_{\underline{J}}(H)$. Thus $G_{\underline{J}}(H)$ is cocommutative because ${H_\hbar}^{\vee} \Big/ \hbar\, {H_\hbar}^{\vee}$ is; conversely, we get an easy proof of the cocommutativity of ${H_\hbar}^{\vee} \Big/ \hbar\, {H_\hbar}^{\vee}$

[2]Hereafter, I pick such terminology from Serge Lang's textbook *"Algebra"*.

once we prove that $G_{\underline{J}}(H)$ is cocommutative, which is straightforward. Finally, $G_{\underline{J}}(H)$ is generated by $Q(H) = J\big/J^2$ whose elements are primitive, so *a fortiori* $G_{\underline{J}}(H)$ is generated by its primitive elements; then the latter holds for ${H_\hbar}^{\vee}\Big/\hbar\,{H_\hbar}^{\vee}$ as well. To sum up, as ${H_\hbar}^{\vee} \in \mathcal{Q}r\mathcal{UEA}$ we argue that $G_{\underline{J}}(H) = \mathcal{U}(\mathfrak{g})$ for some restricted Lie bialgebra $\mathfrak{g}$; conversely, we can get ${H_\hbar}^{\vee} \in \mathcal{Q}r\mathcal{UEA}$ directly from the properties of the filtration $\underline{J}$ of H. Moreover, since $G_{\underline{J}}(H) = \mathcal{U}(\mathfrak{g})$ is graded, $\mathfrak{g}$ *as a restricted Lie algebra is graded too.*

On the other hand, it is straightforward to see that (5.2) together with the fact that ${H_\hbar}' \in \mathcal{HA}$ implies that $\underline{D}$ is a Hopf algebra filtration of H; conversely, if one proves that $\underline{D}$ is a Hopf algebra filtration of H (as we did in Lemma 3.4*(c)*) then from (5.2) we get an easy direct proof that ${H_\hbar}' \in \mathcal{HA}$. Second, we can look at $\underline{D}$ as a bi-infinite filtration by extending it trivially on the negative indices, namely

$$\underline{D} \;=\; \Big(\cdots \subseteq \{0\} \subseteq \cdots \{0\} \subseteq \big(\{0\} = \big) D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n \subseteq \cdots \Big) \quad ;$$

then the Rees Hopf algebra $\mathcal{R}^\hbar_{\underline{D}}(H)$ is defined (see §5.3). Now (5.2) gives ${H_\hbar}' = \mathcal{R}^\hbar_{\underline{D}}(H)$; but then ${H_\hbar}'\Big/\hbar\,{H_\hbar}' \cong \mathcal{R}^\hbar_{\underline{D}}(H)\Big/\hbar\,\mathcal{R}^\hbar_{\underline{D}}(H) \cong G_{\underline{D}}(H)$. Thus $G_{\underline{D}}(H)$ is commutative because ${H_\hbar}'\Big/\hbar\,{H_\hbar}'$ is; or, conversely, we get an easy proof of the commutativity of ${H_\hbar}'\Big/\hbar\,{H_\hbar}'$ once we prove that $G_{\underline{D}}(H)$ is commutative, as we did in Lemma 3.4*(c)*. Finally, $G_{\underline{D}}(H)$ is graded with 1-dimensional 0-component — by construction — hence it has no non-trivial idempotents; therefore the latter is true for ${H_\hbar}'\Big/\hbar\,{H_\hbar}'$ as well. Note also that $I_{{H_\hbar}'}^{\infty} = \{0\}$ by construction (because $H_\hbar$ is free over R). To sum up, since ${H_\hbar}' \in \mathcal{QFA}$ we get that $G_{\underline{D}}(H) = F[G]$ for some connected algebraic Poisson group G; conversely, we can argue that ${H_\hbar}' \in \mathcal{QFA}$ directly from the properties of the filtration $\underline{D}$.

In addition, since $G_{\underline{D}}(H) = F[G]$ is graded, when $Char(\Bbbk) = 0$ the (pro)affine variety $G_{(cl)}$ of closed points of G is a (pro)affine space[3], that is $G_{(cl)} \cong \mathbb{A}_{\Bbbk}^{\times\mathcal{I}} = \Bbbk^{\mathcal{I}}$ for some index set $\mathcal{I}$, and so $F[G] = \Bbbk\big[\{x_i\}_{i\in\mathcal{I}}\big]$ is a polynomial algebra.

Finally, when $p := Char(\Bbbk) > 0$ the group G has dimension 0 and height 1: indeed, we can see this as a consequence of the last part of Theorem 3.8 via the identity $H'_\hbar\Big/\hbar\,H'_\hbar = G_{\underline{D}}(H)$, or conversely we can prove that part of Theorem 3.8 via this identity by observing that G has those properties. In fact, we must show that $\bar{\eta}^p = 0$ for each $\eta \in \widetilde{H} := G_{\underline{D}}(G)$: letting $\eta \in H'_\hbar$ be any lift of $\bar{\eta}$ in $H'_\hbar$, we have $\eta \in D_\ell$ for some $\ell \in \mathbb{N}$, hence $\delta_{\ell+1}(x) = 0$. From $\Delta^{\ell+1}(\eta) = \sum_{\Lambda\subseteq\{1,\dots,\ell+1\}} \delta_\Lambda(\eta)$ (cf. §2.1) and the multiplicativity of $\Delta^{\ell+1}$ we have

[3]For it is a *cone* — since H is graded — without vertex — since $G_{(cl)}$, being a group, is smooth.

$$\Delta^{\ell+1}(\eta^p) = \big(\Delta^{\ell+1}(\eta)\big)^p = \Big(\sum_{\Lambda\subseteq\{1,\dots,\ell+1\}} \delta_\Lambda(\eta)\Big)^p \in \sum_{\Lambda\subseteq\{1,\dots,\ell+1\}} \delta_\Lambda(\eta)^p + \\ + \sum_{\substack{e_1,\dots,e_p<p \\ e_1+\cdots e_p=p}} \binom{p}{e_1,\dots,e_p} \sum_{\Lambda_1,\dots,\Lambda_p\subseteq\{1,\dots,\ell+1\}} \prod_{k=1}^p \delta_{\Lambda_k}(\eta)^{e_k} + \\ + \sum_{k=0}^{\ell} \sum_{\substack{\Psi\subseteq\{1,\dots,\ell+1\} \\ |\Psi|=k}} j_\Psi\big(J_{H'}^{\otimes k}\big) + \big(\mathrm{ad}_{[\ ,\ ]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big)$$

(since $\delta_\Lambda(\eta) \in j_\Lambda\Big(J_{H'}^{\otimes|\Lambda|}\Big)$ for all $\Lambda \subseteq \{1,\dots,\ell+1\}$) where $D_{(\ell)} := \sum_{\sum_k s_k=\ell} \otimes_{k=1}^{\ell+1} D_{s_k}$ and $\big(\mathrm{ad}_{[\ ,\ ]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big) := \big[\underbrace{D_{(\ell)}, \big[D_{(\ell)}, \dots, \big[D_{(\ell)}, \big[D_{(\ell)}, D_{(\ell)}}_{p}\big]\big]\cdots\big]\big]$. Then

$$\delta^{\ell+1}(\eta^p) = (\mathrm{id}_H - \epsilon)^{\otimes(\ell+1)}\big(\Delta^{\ell+1}(\eta^p)\big) \in \delta_{\ell+1}(\eta)^p + \sum_{\substack{e_1,\dots,e_p<p \\ e_1+\cdots e_p=p}} \binom{p}{e_1,\dots,e_p} \times \\ \times \sum_{\cup_{k=1}^p \Lambda_k=\{1,\dots,\ell+1\}} \prod_{k=1}^p \delta_{\Lambda_k}(\eta)^{e_k} + (\mathrm{id}_H - \epsilon)^{\otimes(\ell+1)}\Big(\big(\mathrm{ad}_{[\ ,\ ]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big)\Big).$$

Now, $\delta^{\ell+1}(\eta)^p = 0$ by construction, and $\binom{p}{e_1,\dots,e_p}$ (with $e_1, \dots, e_p < p$) is a multiple of p, hence it is zero because $p = \mathit{Char}(\Bbbk)$; therefore we end up with

$$\delta_{\ell+1}(\eta) \in (\mathrm{id}_H - \epsilon)^{\otimes(\ell+1)}\Big(\big(\mathrm{ad}_{[\ ,\ ]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big)\Big).$$

Now, by Lemma 3.4 we have $D_{s_i} \cdot D_{s_j} \subseteq D_{s_i+s_j}$ and $\big[D_{s_i}, D_{s_j}\big] \subseteq D_{(s_i+s_j)-1}$; these together with Leibniz' rule imply that $\big(\mathrm{ad}_{[\ ,\ ]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big) \subseteq \sum_{\sum_t r_t=p\,\ell+1-p} \otimes_{t=1}^{\ell+1} D_{r_t}$; moreover, since $D_0 = \mathit{Ker}(\delta_1) = \mathit{Ker}(\mathrm{id}-\epsilon)$ we have

$$(\mathrm{id}_H - \epsilon)^{\otimes(\ell+1)}\Big(\big(\mathrm{ad}_{[\ ,\ ]}(D_{(\ell)})\big)^{p-1}\big(D_{(\ell)}\big)\Big) \subseteq \sum_{\substack{\sum_t r_t=p\,\ell+1-p \\ r_1,\dots,r_{\ell+1}>0}} \otimes_{t=1}^{\ell+1} D_{r_t} .$$

In particular, in the last term above we have $D_{r_1} \subseteq D_{(p-1)\ell+1-p} := \mathit{Ker}\big(\delta_{(p-1)\ell+2-p}\big) \subseteq \mathit{Ker}\big(\delta_{(p-1)\ell}\big)$: therefore, using the coassociativity of the maps δ_n's, we get

$$\delta_{p\,\ell}(\eta) = \big(\big(\delta_{(p-1)\ell} \otimes \mathrm{id}^\ell\big) \circ \delta_{\ell+1}\big)(\eta) \subseteq \sum_{\substack{\sum_t r_t=p\,\ell-1 \\ r_1,\dots,r_{\ell+1}>0}} \delta_{(p-1)\ell}(D_{r_1}) \otimes D_{r_2} \otimes \cdots \otimes D_{r_{\ell+1}} = 0$$

i.e. $\delta_{p\,\ell}(\eta) = 0$. This means $\eta \in D_{p\,\ell-1}$, whereas, on the other hand, $\eta^p \in D_\ell^{\,p} \subseteq D_{p\,\ell}$: then $\bar{\eta}^p := \overline{\eta^p} = \bar{0} \in D_{p\,\ell}\Big/D_{p\,\ell-1} \subseteq G_{\underline{D}}(H)$, by the definition of the product in $G_{\underline{D}}(H)$. Finally, by general theory since G has dimension 0 and height 1 the function algebra $F[G] = G_{\underline{D}}(H) = H'_\hbar\Big/\hbar\, H'_\hbar$ is *truncated polynomial*, namely $F[G] = \Bbbk\big[\{x_i\}_{i\in\mathcal{I}}\big]\Big/\big(\{x_i^p\}_{i\in\mathcal{I}}\big)$.

5.5 Special fibers of ${H_\hbar}^\vee$ and ${H_\hbar}'$ and deformations. Given $H \in \mathcal{HA}_{\Bbbk}$, consider $H_\hbar$: our goal is to study ${H_\hbar}^\vee$ and ${H_\hbar}'$.

As for ${H_\hbar}^\vee$, the natural map from H to $\widehat{H} := G_{\underline{J}}(H) = {H_\hbar}^\vee \big/ \hbar\, {H_\hbar}^\vee =: {H_\hbar}^\vee \Big|_{\hbar=0}$ sends $J^\infty := \bigcap_{n \geq 0} J^n$ to zero, by definition; also, letting $H^\vee := H \big/ J^\infty$ (which is a Hopf algebra quotient of H because $\underline{J}$ is a Hopf algebra filtration), we have $\widehat{H} = \widehat{H^\vee}$. Thus $(H^\vee)_\hbar^\vee \Big|_{\hbar=0} = \widehat{H^\vee} = \widehat{H} = \mathcal{U}(\mathfrak{g}_-)$ for some graded restricted Lie bialgebra $\mathfrak{g}_-$. On the other hand, $(H^\vee)_\hbar^\vee \Big|_{\hbar=1} := (H^\vee)_\hbar^\vee \big/ (\hbar - 1)\, (H^\vee)_\hbar^\vee = \sum_{n \geq 0} \overline{J}^{\,n} = H^\vee$ (see §5.3). Thus we can see $(H^\vee)_\hbar^\vee = \mathcal{R}_{\underline{J}}^\hbar(H^\vee)$ as a 1-parameter family inside $\mathcal{HA}_{\Bbbk}$ with *regular* fibers — that is, they are isomorphic to each other as $\Bbbk$–vector spaces (indeed, we switch from H to $H^\vee$ just in order to achieve this regularity) — which links $\widehat{H^\vee}$ and $H^\vee$ as (polynomial) deformations of each other, namely

$$\mathcal{U}(\mathfrak{g}_-) = \widehat{H^\vee} = (H^\vee)_\hbar^\vee \Big|_{\hbar=0} \xleftrightarrow[(H^\vee)_\hbar^\vee]{0 \leftarrow \hbar \rightarrow 1} (H^\vee)_\hbar^\vee \Big|_{\hbar=1} = H^\vee .$$

Now look at $\big((H^\vee)_\hbar^\vee\big)'$: by construction, we have $\big((H^\vee)_\hbar^\vee\big)' \Big|_{\hbar=1} = (H^\vee)_\hbar^\vee \Big|_{\hbar=1} = H^\vee$, whereas $\big((H^\vee)_\hbar^\vee\big)' \Big|_{\hbar=1} = F[K_-]$ for some connected algebraic Poisson group K_-: in addition, if $Char(\Bbbk) = 0$ then $K_- = G_-^\star$ by Theorem 2.2*(c)*. So $\big((H^\vee)_\hbar^\vee\big)'$ can be thought of as a 1-parameter family inside $\mathcal{HA}_{\Bbbk}$, with regular fibers, linking $H^\vee$ and $F[G_-^\star]$ as (polynomial) deformations of each other, namely

$$H^\vee = \big((H^\vee)_\hbar^\vee\big)' \Big|_{\hbar=1} \xleftrightarrow[((H^\vee)_\hbar^\vee)']{1 \leftarrow \hbar \rightarrow 0} \big((H^\vee)_\hbar^\vee\big)' \Big|_{\hbar=0} = F[K_-] \; \Big(= F[G_-^\star] \text{ if } Char(\Bbbk) = 0 \Big) .$$

Therefore $H^\vee$ is both a deformation of an enveloping algebra and a deformation of a function algebra, via two different 1-parameter families (with regular fibers) in $\mathcal{HA}_{\Bbbk}$ which match at the value $\hbar = 1$, corresponding to the common element $H^\vee$. At a glance,

$$\mathcal{U}(\mathfrak{g}_-) \xleftrightarrow[(H^\vee)_\hbar^\vee]{0 \leftarrow \hbar \rightarrow 1} H^\vee \xleftrightarrow[((H^\vee)_\hbar^\vee)']{1 \leftarrow \hbar \rightarrow 0} F[K_-] \; \Big(= F[G_-^\star] \text{ if } Char(\Bbbk) = 0 \Big) . \tag{5.3}$$

Now consider ${H_\hbar}'$. We have ${H_\hbar}' \Big|_{\hbar=0} := {H_\hbar}' \big/ \hbar\, {H_\hbar}' = G_{\underline{D}}(H) =: \widetilde{H}$, and $\widetilde{H} = F[G_+]$ for some connected algebraic Poisson group G_+. On the other hand, we have also ${H_\hbar}' \Big|_{\hbar=1} := {H_\hbar}' \big/ (\hbar - 1)\, {H_\hbar}' = \sum_{n \geq 0} D_n =: H'$; note that the latter is a Hopf subalgebra of H, because $\underline{D}$ is a Hopf algebra filtration; moreover we have $\widetilde{H} = \widetilde{H'}$, by the very definitions. Therefore we can think at ${H_\hbar}' = \mathcal{R}_{\underline{D}}^\hbar(H')$ as a 1-parameter family inside $\mathcal{HA}_{\Bbbk}$ with regular fibers which links $\widetilde{H}$ and H' as (polynomial) deformations of each other, namely

$$F[G_+] = \widetilde{H} = {H_\hbar}' \Big|_{\hbar=0} \xleftrightarrow[{H_\hbar}']{0 \leftarrow \hbar \rightarrow 1} {H_\hbar}' \Big|_{\hbar=1} = H' .$$

Consider also $\big({H_\hbar}'\big)^\vee$: by construction, we have $\big({H_\hbar}'\big)^\vee\Big|_{\hbar=1} = {H_\hbar}'\Big|_{\hbar=1} = H'$, whereas $\big({H_\hbar}'\big)^\vee\Big|_{\hbar=0} = \mathcal{U}(\mathfrak{k}_+)$ for some restricted Lie bialgebra $\mathfrak{k}_+$: in addition, if $Char(\Bbbk) = 0$ then $\mathfrak{k}_+ = \mathfrak{g}_+^\times$ thanks to Theorem 2.2*(c)*. Thus $\big({H_\hbar}'\big)^\vee$ can be thought of as a 1-parameter family inside $\mathcal{HA}_\Bbbk$ with regular fibers which links $\mathcal{U}(\mathfrak{k}_+)$ and H' as (polynomial) deformations of each other, namely

$$H' = \big({H_\hbar}'\big)^\vee\Big|_{\hbar=1} \xleftarrow[({H_\hbar}')^\vee]{1 \leftarrow \hbar \rightarrow 0} \big({H_\hbar}'\big)^\vee\Big|_{\hbar=0} = \mathcal{U}(\mathfrak{k}_+) \quad \Big(= U(\mathfrak{g}_+^\times) \ \text{ if } \ Char(\Bbbk) = 0 \Big) .$$

Therefore, H' is at the same time a deformation of a function algebra and a deformation of an enveloping algebra, via two different 1-parameter families inside $\mathcal{HA}_\Bbbk$ (with regular fibers) which match at the value $\hbar = 1$, corresponding (in both families) to H'. In short,

$$F[G_+] \xleftrightarrow[{H_\hbar}']{0 \leftarrow \hbar \rightarrow 1} H' \xleftrightarrow[({H_\hbar}')^\vee]{1 \leftarrow \hbar \rightarrow 0} \mathcal{U}(\mathfrak{k}_+) \quad \Big(= U(\mathfrak{g}_+^\times) \ \text{ if } \ Char(\Bbbk) = 0 \Big) . \tag{5.4}$$

Finally, it is worth noticing that in the special case $H' = H = H^\vee$ we can splice together (5.3) and (5.4) to get

$$\begin{array}{ccccc} F[G_+] & \xleftrightarrow[{H_\hbar}']{0 \leftarrow \hbar \rightarrow 1} & H' & \xleftrightarrow[({H_\hbar}')^\vee]{1 \leftarrow \hbar \rightarrow 0} & \mathcal{U}(\mathfrak{k}_+) \ \Big(= U(\mathfrak{g}_+^\times) \ \text{ if } \ Char(\Bbbk) = 0 \Big) \\ & & \Vert & & \\ & & H & & \\ & & \Vert & & \\ \mathcal{U}(\mathfrak{g}_-) & \xleftrightarrow[(H^\vee)_\hbar^\vee]{0 \leftarrow \hbar \rightarrow 1} & H^\vee & \xleftrightarrow[((H^\vee)_\hbar^\vee)']{1 \leftarrow \hbar \rightarrow 0} & F[K_-] \ \Big(= F[G_-^\star] \ \text{ if } \ Char(\Bbbk) = 0 \Big) \end{array} \tag{5.5}$$

which gives *four* different regular 1-parameter deformations from H to Hopf algebras encoding geometrical objects of Poisson type (i.e. Lie bialgebras or Poisson algebraic groups).

5.6 The function algebra case. Let G be any algebraic group over the field $\Bbbk$. Let $R := \Bbbk[\hbar]$ be as in §5.1, and set $F_\hbar[G] := \big(F[G]\big)_\hbar = R \otimes_\Bbbk F[G]$: this is trivially a QFA at $\hbar$, for $F_\hbar[G] \big/ \hbar F_\hbar[G] = F[G]$, inducing on G the trivial Poisson structure, so that its cotangent Lie bialgebra is simply $\mathfrak{g}^\times$ with trivial Lie bracket and Lie cobracket dual to the Lie bracket of $\mathfrak{g}$. In the sequel we identify $F[G]$ with $1 \otimes F[G] \subset F_\hbar[G]$.

We begin by computing $F_\hbar[G]^\vee$ (w.r.t. the non-zero element $\hbar$).

Let $J := J_{F[G]} \equiv Ker\big(\epsilon_{F[G]}\big)$, let $\{y_b\}_{b \in \mathcal{S}}$ be a $\Bbbk$–basis of $Q\big(F[G]\big) = J \Big/ J^2 = \mathfrak{g}^\times$, and pull it back to a subset $\{j_b\}_{b \in \mathcal{S}}$ of J. Then we see that $J^n \big/ J^{n+1}$ is a $\Bbbk$–vector space spanned by the set of (cosets of) ordered monomials (using multiindices and all the notation introduced in the proof of Theorem 4.7) $\big\{\, j^{\underline{e}} \mod J^{n+1} \;\big|\; \underline{e} \in \mathbb{N}_f^{\mathcal{S}} ,\ |\underline{e}| = n \,\big\}$ where $|\underline{e}| := \sum_{b \in \mathcal{S}} \underline{e}(b)$; therefore $I^n \big/ I^{n+1}$ as a $\Bbbk$–vector space is spanned by $\big\{\, \hbar^{e_0} j^{\underline{e}}$

$\mod I^{n+1} \,\big|\, e_0 \in \mathbb{N}, \underline{e} \in \mathbb{N}_f^{\mathcal{S}}\,,\; e_0 + |\underline{e}| = n \,\big\}$. Noting that $\hbar^{-n} I^{n+1} = \hbar \cdot \hbar^{-(n+1)} I^{n+1} \equiv 0 \mod \hbar \, F_\hbar[G]^\vee$, we can argue that $\hbar^{-n} I^n \mod \hbar \, F_\hbar[G]^\vee$ is spanned over $\Bbbk$ by $\big\{ \hbar^{-|\underline{e}|} j^{\,\underline{e}} \mod \hbar \, F_\hbar[G]^\vee \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}}\,,\; |\underline{e}| \leq n \,\big\}$. Now we have to distinguish the various cases.

First of all, let $y \in F[G]$ be idempotent: switching if necessary to $y_+ := y - \epsilon(y)$ we can assume that $y \in J$. Then $y = y^2 = \cdots = y^n \in J^n \subset I^n$ for all $n \in \mathbb{N}$, so that $y \equiv 0 \mod \hbar \, F_\hbar[G]$. Thus in order to compute $F_\hbar[G] \Big/ \hbar \, F_\hbar[G]$ the idempotents of $F[G]$ are irrelevant: this means $F_\hbar[G] \Big/ \hbar \, F_\hbar[G] = F_\hbar\big[G^0\big] \Big/ \hbar \, F_\hbar\big[G^0\big]$, where G^0 is the connected component of G; thus we can assume from scratch that G be connected.

First assume G is *smooth*, i.e. $F[G]$ is *reduced*, which is always the case if $\mathit{Char}\,(\Bbbk) = 0$. Then the set above is a basis of $\hbar^{-n} I^n \mod \hbar \, F_\hbar[G]^\vee$: for if we have a non-trivial linear combination of the elements of this set which is zero, multiplying by $\hbar^n$ gives an element $\gamma_n \in I^n \setminus I^{n+1}$ such that $\hbar^{-n} \gamma_n \equiv 0 \mod \hbar \, F_\hbar[G]^\vee$; then there is $\ell \in \mathbb{N}$ such that $\hbar^{-n} \gamma_n \in \hbar \cdot \hbar^{-\ell} I^\ell$, so $\hbar^\ell \gamma_n = \hbar^{1+n} I^\ell \subseteq I^{1+n+\ell}$, whence clearly $\gamma_n \in I^{n+1}$, a contradiction! The outcome is that $\big\{ \hbar^{-|\underline{e}|} j^{\,\underline{e}} \mod \hbar \, F_\hbar[G]^\vee \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}} \big\}$ is a $\Bbbk$–basis of $F_\hbar[G]^\vee\Big|_{\hbar=0} := F_\hbar[G]^\vee \big/ \hbar \, F_\hbar[G]^\vee$. Now let $j_\beta^\vee := \hbar^{-1} j_\beta$ for all $\beta \in \mathcal{S}$. By the previous analysis $F_\hbar[G]^\vee\Big|_{\hbar=0}$ has $\Bbbk$–basis $\big\{ \big(j^\vee\big)^{\underline{e}} \mod \hbar \, F_\hbar[G]^\vee \,\big|\, \underline{e} \in \mathbb{N}_f^{\mathcal{S}} \big\}$, hence the Poincaré-Birkhoff-Witt theorem tells us that $F_\hbar[G]^\vee\Big|_{\hbar=0} = U(\mathfrak{h}) \equiv S(\mathfrak{h})$ as associative algebras, where $\mathfrak{h}$ is the Lie algebra spanned by $\big\{ j_b^\vee \mod \hbar \, F_\hbar[G]^\vee \big\}_{b \in \mathcal{S}}$ (as in the proof of Theorem 4.7), whose Lie bracket is trivial for it is given by $\big[j_b^\vee, j_\beta^\vee \big] := \hbar^{-2} (j_b \, j_\beta - j_\beta \, j_b) \mod \hbar \, F_\hbar[G]^\vee \equiv 0$. Further, we have also $\Delta\big(j^\vee\big) \equiv j^\vee \otimes 1 + 1 \otimes j^\vee \mod \hbar \, \big(F_\hbar[G]^\vee\big)^{\otimes 2}$ for all $j^\vee \in J^\vee := \hbar \, J$ (cf. the proof of Theorem 3.7), whence $\Delta(\mathrm{j}) = \mathrm{j} \otimes 1 + 1 \otimes \mathrm{j}$ for all $\mathrm{j} \in \mathfrak{h}$, so $F_\hbar[G]^\vee\Big|_{\hbar=0} = U(\mathfrak{h}) \equiv S(\mathfrak{h})$ as *Hopf* algebras too. Now, consider the linear map $\sigma : \mathfrak{g}^\times = J \big/ J^2 \longrightarrow \mathfrak{h} \; \big(\subset U(\mathfrak{h}) \big)$ given by $y_b \mapsto j_b^\vee$ ($b \in \mathcal{S}$). By construction this is clearly a vector space isomorphism, and also a *Lie algebra* isomorphism, since the Lie bracket is trivial on both sides (G has the trivial Poisson structure!). In addition, one has

$$\Big\langle u_1 \otimes u_2 \,,\, \delta_{\mathfrak{h}}\big(\sigma(y_b)\big) \Big\rangle = \Big\langle u_1 \otimes u_2 \,,\, \hbar^{-1} \big(\Delta - \Delta^{\mathrm{op}}\big)\big(\sigma(y_b)\big) \mod \hbar \Big\rangle =$$
$$= \Big\langle u_1 \otimes u_2 \,,\, \hbar^{-2} \big(\Delta - \Delta^{\mathrm{op}}\big)(j_b) \mod \hbar \Big\rangle = \Big\langle \big(u_1 \cdot u_2 - u_2 \cdot u_1\big) \,,\, \hbar^{-2} j_b \mod \hbar \Big\rangle =$$
$$= \Big\langle [u_1, u_2] \,,\, \hbar^{-2} j_b \mod \hbar \Big\rangle = \Big\langle u_1 \otimes u_2 \,,\, \hbar^{-2} \delta_{\mathfrak{g}^\times}(y_b) \mod \hbar \Big\rangle = \Big\langle u_1 \otimes u_2 \,,\, (\sigma \otimes \sigma)\big(\delta_{\mathfrak{g}^\times}(y_b)\big) \Big\rangle$$

for all $u_1, u_2 \in \mathfrak{g}$ (with $(u_1 \cdot u_2 - u_2 \cdot u_1) \in U(\mathfrak{g})$) and $b \in \mathcal{S}$, which is enough to prove that $\delta_{\mathfrak{h}} \circ \sigma = (\sigma \otimes \sigma) \circ \delta_{\mathfrak{g}^\times}$, i.e. σ is a Lie *bi*algebra morphism as well. Therefore the outcome is $F_\hbar[G]^\vee\Big|_{\hbar=0} = U(\mathfrak{g}^\times) \equiv S(\mathfrak{g}^\times)$ as *co-Poisson Hopf algebras*.

Another extreme case is when G is a *finite connected group scheme*: then, assuming for simplicity that $\Bbbk$ be perfect, we have $F[G] = \Bbbk[x_1, \dots, x_n] \Big/ \big(x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}} \big)$ for some $n, e_1, \dots, e_n \in \mathbb{N}$. The previous analysis, with minor changes, then shows that $F_\hbar[G]^\vee\Big|_{\hbar=0}$

is now a quotient of $U(\mathfrak{g}^\times) \equiv S(\mathfrak{g}^\times)$: namely, the x_i's take place of the j_b's, so the cosets of the $x_i^\vee$'s $(i=1,\dots,n)$ modulo $\hbar\, F_\hbar[G]^\vee$ generate $\mathfrak{g}^\times$, and we find $F_\hbar[G]^\vee\Big|_{\hbar=0} = U(\mathfrak{g}^\times)\Big/\big((x_1^\vee)^{p^{e_1}},\dots,(x_n^\vee)^{p^{e_n}}\big)$. Now, recall that for any Lie algebra $\mathfrak{h}$ we can consider $\mathfrak{h}^{[p]^\infty} := \Big\{ x^{[p]^n} := x^{p^n} \,\Big|\, x \in \mathfrak{h}\,, n \in \mathbb{N} \Big\}$, the *restricted Lie algebra generated by* $\mathfrak{h}$ inside $U(\mathfrak{h})$, with the p–operation given by $x^{[p]} := x^p$; then one always has $U(\mathfrak{h}) = \mathbf{u}\big(\mathfrak{h}^{[p]^\infty}\big)$. In the present case, the subset $\Big\{(x_1^\vee)^{p^{e_1}},\,\dots\,,(x_n^\vee)^{p^{e_n}}\Big\}$ generates a p–ideal $\mathcal{I}$ of $(\mathfrak{g}^\times)^{[p]^\infty}$, so $\mathfrak{g}^\times_{res} := \mathfrak{g}^{[p]^\infty}\Big/\mathcal{I}$ is a restricted Lie algebra too, with $\Big\{(x_1^\vee)^{p^{a_1}},\,\dots\,,(x_n^\vee)^{p^{a_n}} \,\Big|\, a_1 < e_1,\dots,a_n < e_n \Big\}$ as a $\Bbbk$–basis. Then the previous analysis proves $F_\hbar[G]^\vee\Big|_{\hbar=0} = \mathbf{u}\,(\mathfrak{g}^\times_{res}) \equiv S(\mathfrak{g}^\times)\Big/\Big(\Big\{(x_1^\vee)^{p^{e_1}},\,\dots\,,(x_n^\vee)^{p^{e_n}}\Big\}\Big)$ as co-Poisson Hopf algebras.

The general case is intermediate; we get it via the relation $F_\hbar[G]\Big/\hbar\, F_\hbar[G] = G_{\underline{J}}\big(F[G]\big)$.

Assume again for simplicity that $\Bbbk$ be perfect. Let $F[[G]]$ be the J–adic completion of $H = F[G]$. By standard results on algebraic groups (cf. [DG]) there is a subset $\{x_i\}_{i\in\mathcal{I}}$ of J such that $\big\{\,\overline{x}_i := x_i \mod J^2\,\big\}_{i\in\mathcal{I}}$ is a basis of $\mathfrak{g}^\times = J\big/J^2$ and $F[[G]] \cong \Bbbk\big[\big[\{x_i\}_{i\in\mathcal{I}}\big]\big]\Big/\Big(\Big\{x_i^{p^{n(x_i)}}\Big\}_{i\in\mathcal{I}_0}\Big)$ (the algebra of truncated formal power series), for some subset $\mathcal{I}_0 \subset \mathcal{I}$ and some $\big(n(x_i)\big)_{i\in\mathcal{I}_0} \in \mathbb{N}^{\mathcal{I}_0}$. Since $G_{\underline{J}}\big(F[G]\big) = G_{\underline{J}}\big(F[[G]]\big)$, we argue that $G_{\underline{J}}\big(F[G]\big) \cong \Bbbk\big[\{\overline{x}_i\}_{i\in\mathcal{I}}\big]\Big/\Big(\Big\{\overline{x}_i^{\,p^{n(x_i)}}\Big\}_{i\in\mathcal{I}_0}\Big)$; finally, since $\Bbbk\big[\{\overline{x}_i\}_{i\in\mathcal{I}}\big] \cong S(\mathfrak{g}^\times)$ we get $G_{\underline{J}}\big(F[G]\big) \cong S(\mathfrak{g}^\times)\Big/\Big(\Big\{\,\overline{x}^{\,p^{n(x)}}\Big\}_{x\in\mathcal{N}(F[G])}\Big)$ as $\Bbbk$–algebras, where $\mathcal{N}\big(F[G]\big)$ is the nilradical of $F[G]$ and $p^{n(x)}$ is the order of nilpotency of $x \in \mathcal{N}\big(F[G]\big)$.

Now, let $0 \neq \overline{\eta} \in J\big/J^2$, and let $\eta \in J$ be a lift of $\overline{\eta}$: then $\Delta(\eta) = \epsilon(\eta)\cdot 1\otimes 1 + \delta_1(\eta)\otimes 1 + 1\otimes\delta_1(\eta) + \delta_2(\eta) = \eta\otimes 1 + 1\otimes\eta + \delta_2(\eta)$, by the very definitions. But $\delta_2(\eta) \in J\otimes J$, hence $\overline{\delta_2(\eta)} = \overline{0} \in G_{\underline{J}}\big(F[G]\big)\otimes G_{\underline{J}}\big(F[G]\big)$: so $\Delta(\overline{\eta}\,) := \overline{\Delta(\eta)} = \overline{\eta}\otimes 1 + 1\otimes\overline{\eta}$. Therefore, all elements of $J\big/J^2 = \mathfrak{g}^\times$ are primitive: this implies that the previous isomorphism respects also the Hopf structure. As for the Lie cobracket of $G_{\underline{J}}\big(F[G]\big)$, by construction it is given by $\delta_{G_{\underline{J}}(F[G])}(\overline{x}\,) := \overline{\nabla(x)} = \overline{\Delta(x) - \Delta^{\mathrm{op}}(x)}$. Now, in the natural pairing between $F[G]$ and $U(\mathfrak{g})$, for all $x \in J$ we have $\big\langle \nabla(x), Y\otimes Z \big\rangle = \big\langle \Delta(x) - \Delta^{\mathrm{op}}(x), Y\otimes Z \big\rangle = \big\langle x, Y\,Z - Z\,Y \big\rangle = \big\langle x, [Y,Z] \big\rangle$, hence $\big\langle \delta_{G_{\underline{J}}(F[G])}(\overline{x}\,), Y\otimes Z \big\rangle = \big\langle x, [Y,Z] \big\rangle$ for all Y, $Z \in \mathfrak{g}$; similarly $\big\langle \delta_{\mathfrak{g}^\times}(\overline{x}\,), y\otimes z \big\rangle = \big\langle x, [Y,Z] \big\rangle$ for all Y, $Z \in \mathfrak{g}$. We then argue that $\delta_{G_{\underline{J}}(F[G])}(\overline{x}\,) = \delta_{\mathfrak{g}^\times}(\overline{x}\,)$ for all $x \in \mathfrak{g}^\times$, whence the two Lie cobrackets do correspond to one another in the isomorphism above. Since $F_\hbar[G]^\vee\Big|_{\hbar=0} = G_{\underline{J}}\big(F[G]\big)$, the outcome is that $F_\hbar[G]^\vee\Big|_{\hbar=0} \cong S(\mathfrak{g}^\times)\Big/\Big(\Big\{\,\overline{x}^{\,p^{n(x)}} \,\Big|\, x \in \mathcal{N}\big(F[G]\big)\Big\}\Big)$ as co-Poisson Hopf algebras.

Note also that the description of $F_\hbar[G]^\vee$ in the general case is exactly like the one we gave for the smooth case: one simply has to mod out the ideal generated by $\mathcal{N}\big(F[G]\big)^\vee := \hbar^{-1}\mathcal{N}\big(F[G]\big)$, i.e. (roughly) to set $(x_i^\vee)^{p^{n(x_i)}} = 0$ (with $x_i^\vee := \hbar^{-1}x_i$) for all $i \in \mathcal{I}$. By the way, to have this description we do not need $\Bbbk$ to be perfect. As for $F[G]^\vee := F[G] \big/ J^\infty$, it is known (cf. [Ab], Lemma 4.6.4) that $F[G]^\vee = F[G]$ whenever G is finite dimensional and there exists no $f \in F[G] \setminus \Bbbk$ which is separable algebraic over $\Bbbk$.

It is also interesting to consider $\big(F_\hbar[G]^\vee\big)'$. If $Char(\Bbbk) = 0$, then the proof of Proposition 4.3 does work, with no special simplifications, giving $\big(F_\hbar[G]^\vee\big)' = F_\hbar[G]$. If instead $Char(\Bbbk) = p > 0$, then the situation might change dramatically. Indeed, if the group G has height 1 — i.e., if $F[G] = \Bbbk\big[\{x_i\}_{i \in \mathcal{I}}\big]\Big/\big(\{x_i^p \,\big|\, i \in \mathcal{I}\}\big)$ as a $\Bbbk$–algebra — then the same analysis as in the characteristic zero case may be applied, with a few minor changes, whence one gets again $\big(F_\hbar[G]^\vee\big)' = F_\hbar[G]$. Otherwise, let $y \in J \setminus \{0\}$ be primitive and such that $y^p \neq 0$: for instance, this occurs for $F[G] = \Bbbk[x]$, i.e. $G \cong \mathbb{G}_a$, and $y = x$. Then y^p is primitive as well, hence $\delta_n(y^p) = 0$ for each $n > 1$. It follows that $0 \neq \hbar\,(y^\vee)^p \in \big(F_\hbar[G]^\vee\big)'$, whereas $\hbar\,(y^\vee)^p \notin F_\hbar[G]$, as follows from our previous description of $F_\hbar[G]^\vee$. Thus $\big(F_\hbar[G]^\vee\big)' \supsetneqq F_\hbar[G]^\vee$, a *counterexample* to Proposition 4.3.

What for $F_\hbar[G]'$ and $\widetilde{F[G]}$? Again, this depends on the group G under consideration. We give two simple examples, both "extreme", in a sense, and opposite to each other.

Let $G := \mathbb{G}_a = Spec\big(\Bbbk[x]\big)$, so $F[G] = F[\mathbb{G}_a] = \Bbbk[x]$ and $F_\hbar[\mathbb{G}_a] := R \otimes_\Bbbk \Bbbk[x] = R[x]$. Then since $\Delta(x) := x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$ we find $F_\hbar[\mathbb{G}_a]' = R[\hbar x]$ (like in §5.7 below: indeed, this is just a special instance, for $F[\mathbb{G}_a] = U(\mathfrak{g})$ where $\mathfrak{g}$ is the 1-dimensional Lie algebra). Moreover, iterating one gets easily $\big(F_\hbar[\mathbb{G}_a]'\big)' = R\big[\hbar^2 x\big]$, $\Big(\big(F_\hbar[\mathbb{G}_a]'\big)'\Big)' = R\big[\hbar^3 x\big]$, and in general $\bigg(\Big(\big(F_\hbar[\mathbb{G}_a\underbrace{]'\big)'\Big)'\cdots\bigg)'}_{n} = R\big[\hbar^n x\big] \cong R[x] = F_\hbar[\mathbb{G}_a]$ for all $n \in \mathbb{N}$.

Second, let $G := \mathbb{G}_m = Spec\Big(\Bbbk\big[z^{+1}, z^{-1}\big]\Big)$, that is $F[G] = F[\mathbb{G}_m] = \Bbbk\big[z^{+1}, z^{-1}\big]$ so that $F_\hbar[\mathbb{G}_m] := R \otimes_\Bbbk \Bbbk\big[z^{+1}, z^{-1}\big] = R\big[z^{+1}, z^{-1}\big]$. Then since $\Delta\big(z^{\pm 1}\big) := z^{\pm 1} \otimes z^{\pm 1}$ and $\epsilon\big(z^{\pm 1}\big) = 1$ we find $\Delta^n\big(z^{\pm 1}\big) = \big(z^{\pm 1}\big)^{\otimes n}$ and $\delta_n\big(z^{\pm 1}\big) = \big(z^{\pm 1} - 1\big)^{\otimes n}$ for all $n \in \mathbb{N}$. From that it follows easily $F_\hbar[\mathbb{G}_m]' = R \cdot 1$, the trivial possibility (see also §5.13 later on).

5.7 The enveloping algebra case. Let $\mathfrak{g}$ be any Lie algebra over the field $\Bbbk$, and $U(\mathfrak{g})$ its universal enveloping algebra with its standard Hopf structure. Assume $Char(\Bbbk) = 0$, and let $R = \Bbbk[\hbar]$ as in §5.1, and set $U_\hbar(\mathfrak{g}) := R \otimes_\Bbbk U(\mathfrak{g}) = \big(U(\mathfrak{g})\big)_\hbar$. Then $U_\hbar(\mathfrak{g})$ is trivially a QrUEA at $\hbar$, for $U_\hbar(\mathfrak{g}) \big/ \hbar\, U_\hbar(\mathfrak{g}) = U(\mathfrak{g})$, inducing on $\mathfrak{g}$ the trivial Lie cobracket. Therefore the dual Poisson group is nothing but $\mathfrak{g}^\star$ (the topological dual of $\mathfrak{g}$ w.r.t. the weak topology), an Abelian group w.r.t. addition, with $\mathfrak{g}$ as cotangent Lie bialgebra and function algebra $F[\mathfrak{g}^\star] = S(\mathfrak{g})$: the Hopf structure is the standard one, given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ (for all $x \in \mathfrak{g}$), and the Poisson structure is the one induced by $\{x, y\} := [x, y]$ for all x, $y \in \mathfrak{g}$. This is the so-called Kostant-Kirillov structure on $\mathfrak{g}^\star$.

Similarly, if $Char(\Bbbk) = p > 0$ and $\mathfrak{g}$ is any restricted Lie algebra over $\Bbbk$, let $\mathbf{u}(\mathfrak{g})$ be its restricted universal enveloping algebra, with its standard Hopf structure. Then if

$R = \Bbbk[\hbar]$ the Hopf R–algebra $U_\hbar(\mathfrak{g}) := R \otimes_\Bbbk \mathbf{u}(\mathfrak{g}) = \big(\mathbf{u}(\mathfrak{g})\big)_\hbar$ is a QrUEA at $\hbar$, because $\mathbf{u}_\hbar(\mathfrak{g}) \big/ \hbar\, \mathbf{u}_\hbar(\mathfrak{g}) = \mathbf{u}(\mathfrak{g})$, inducing on $\mathfrak{g}$ the trivial Lie cobracket: then the dual Poisson group is again $\mathfrak{g}^\star$, with cotangent Lie bialgebra $\mathfrak{g}$ and function algebra $F[\mathfrak{g}^\star] = S(\mathfrak{g})$ (the Poisson Hopf structure being as above). Recall also that $U(\mathfrak{g}) = \mathbf{u}\big(\mathfrak{g}^{[p]^\infty}\big)$ (cf. §5.6).

First we compute $\mathbf{u}_\hbar(\mathfrak{g})'$ (w.r.t. the prime $\hbar$) using (5.2), i.e. computing the filtration $\underline{D}$.

By the PBW theorem, once an ordered basis B of $\mathfrak{g}$ is fixed $\mathbf{u}(\mathfrak{g})$ admits as basis the set of ordered monomials in the elements of B whose degree (w.r.t. each element of B) is less than p; this yields a Hopf algebra filtration of $\mathbf{u}(\mathfrak{g})$ by the total degree, which we refer to as *the standard filtration*. Then from the very definitions a straightforward calculation shows that $\underline{D}$ coincides with the standard filtration. This together with (5.2) immediately implies $\mathbf{u}_\hbar(\mathfrak{g})' = \langle \tilde{\mathfrak{g}} \rangle = \langle \hbar\, \mathfrak{g} \rangle$: hereafter $\tilde{\mathfrak{g}} := \hbar\, \mathfrak{g}$, and similarly we set $\tilde{x} := \hbar\, x$ for all $x \in \mathfrak{g}$. Then the relations $x\, y - y\, x = [x,y]$ and $z^p = z^{[p]}$ in $\mathbf{u}(\mathfrak{g})$ yield $\tilde{x}\, \tilde{y} - \tilde{y}\, \tilde{x} = \hbar\, \widetilde{[x,y]} \equiv 0 \mod \hbar\, \mathbf{u}_\hbar(\mathfrak{g})'$ and $\tilde{z}^p = \hbar^{p-1} \widetilde{z^{[p]}} \equiv 0 \mod \hbar\, \mathbf{u}_\hbar(\mathfrak{g})'$; therefore from the presentation[4] $\mathbf{u}_\hbar(\mathfrak{g}) = T_R(\mathfrak{g}) \Big/ \big(\{\, x\, y - y\, x - [x,y]\,,\; z^p - z^{[p]} \bigm| x, y, z \in \mathfrak{g} \,\} \big)$ we get

$$\mathbf{u}_\hbar(\mathfrak{g})' = \langle \tilde{\mathfrak{g}} \rangle \xrightarrow{\ \hbar \to 0\ } \widetilde{\mathbf{u}(\mathfrak{g})} \;=\; T_\Bbbk(\tilde{\mathfrak{g}}) \bigg/ \bigg(\Big\{\, \tilde{x}\, \tilde{y} - \tilde{y}\, \tilde{x}\,,\; \tilde{z}^p \;\Big|\; \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{\mathfrak{g}} \,\Big\} \bigg) =$$

$$= T_\Bbbk(\mathfrak{g}) \Big/ \big(\{\, x\, y - y\, x\,,\; z^p \bigm| x, y, z \in \mathfrak{g} \,\} \big) = S_\Bbbk(\mathfrak{g}) \Big/ \big(\{\, z^p \bigm| z \in \mathfrak{g} \,\} \big) = F[\mathfrak{g}^\star] \Big/ \big(\{\, z^p \bigm| z \in \mathfrak{g} \,\} \big)$$

that is $\widetilde{\mathbf{u}(\mathfrak{g})} := G_{\underline{D}}\big(\mathbf{u}(\mathfrak{g})\big) = \mathbf{u}_\hbar(\mathfrak{g})' \big/ \hbar\, \mathbf{u}_\hbar(\mathfrak{g})' \cong F[\mathfrak{g}^\star] \Big/ \big(\{\, z^p \bigm| z \in \mathfrak{g} \,\} \big)$ *as Poisson Hopf algebras*. In particular, *this means that* $\widetilde{\mathbf{u}(\mathfrak{g})}$ *is the function algebra of, and* $\mathbf{u}_\hbar(\mathfrak{g})'$ *is a QFA (at* $\hbar$ *) for, a non-reduced algebraic Poisson group of dimension 0 and height 1, whose cotangent Lie bialgebra is* $\mathfrak{g}$*, hence which is dual to* $\mathfrak{g}$; thus, in a sense, part *(c)* of Theorem 2.2 is still valid in this (positive characteristic) case.

<u>*Remark:*</u> Note that this last result reminds the classical formulation of the analogue of Lie's Third Theorem in the context of group-schemes: *Given a restricted Lie algebra* $\mathfrak{g}$*, there exists a group-scheme G of dimension 0 and height 1 whose tangent Lie algebra is* $\mathfrak{g}$ (see e.g. [DG]). Here we have just given sort of a "dual Poisson-theoretic version" of this fact, in that our result sounds as follows: *Given a restricted Lie algebra* $\mathfrak{g}$*, there exists a Poisson group-scheme G of dimension 0 and height 1 whose cotangent Lie algebra is* $\mathfrak{g}$.

As a byproduct, since $U_\hbar(\mathfrak{g}) = \mathbf{u}_\hbar\big(\mathfrak{g}^{[p]^\infty}\big)$ we have also $U_\hbar(\mathfrak{g})' = \mathbf{u}_\hbar\big(\mathfrak{g}^{[p]^\infty}\big)'$, whence

$$U_\hbar(\mathfrak{g})' = \mathbf{u}_\hbar\big(\mathfrak{g}^{[p]^\infty}\big)' \xrightarrow{\ \hbar \to 0\ } S_\Bbbk\Big(\mathfrak{g}^{[p]^\infty}\Big) \bigg/ \Big(\{\, z^p \,\}_{z \in \mathfrak{g}^{[p]^\infty}} \Big) \;=\; F\Big[\big(\mathfrak{g}^{[p]^\infty}\big)^\star \Big] \bigg/ \Big(\{\, z^p \,\}_{z \in \mathfrak{g}^{[p]^\infty}} \Big).$$

Furthermore, $\mathbf{u}_\hbar(\mathfrak{g})' = \langle \tilde{\mathfrak{g}} \rangle$ implies that $I_{\mathbf{u}_\hbar(\mathfrak{g})'}$ is generated (as a two-sided ideal) by $\hbar R \cdot 1_{\mathbf{u}_\hbar(\mathfrak{g})} + R\, \tilde{\mathfrak{g}}$, hence $\hbar^{-1} I_{\mathbf{u}_\hbar(\mathfrak{g})'}$ is generated by $R \cdot 1 + R\, \mathfrak{g}$, thus $\big(\mathbf{u}_\hbar(\mathfrak{g})'\big)^\vee :=$

[4]Hereafter, $T_A(M)$, resp. $S_A(M)$, is the tensor, resp. symmetric algebra of an A–module M.

$\bigcup_{n \geq 0} \big(\hbar^{-1} I_{\mathbf{u}_\hbar(\mathfrak{g})'}\big)^n \; = \; \bigcup_{n \geq 0} \big(R \cdot 1 + R\,\mathfrak{g} \big)^n \; = \; \mathbf{u}_\hbar(\mathfrak{g})$; this means that also part *(b)* of Theorem 2.2 is still valid, though now $\mathit{Char}(\Bbbk) > 0$.

When $\mathit{Char}(\Bbbk) = 0$ and we look at $U(\mathfrak{g})$, the like argument applies: $\underline{D}$ coincides with the standard filtration of $U(\mathfrak{g})$ given by the total degree, via the PBW theorem. This and (5.2) immediately imply $U(\mathfrak{g})' = \langle \tilde{\mathfrak{g}} \rangle = \langle \hbar\,\mathfrak{g} \rangle$, so that from the presentation $U_\hbar(\mathfrak{g}) = T_R(\mathfrak{g}) \Big/ \big(\{\, x\,y - y\,x - [x,y] \,\}_{x,y,z \in \mathfrak{g}} \big)$ we get $U_\hbar(\mathfrak{g})' = T_R(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{ \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} - \hbar \cdot \widetilde{[x,y]} \Big\}_{\tilde{x},\tilde{y} \in \tilde{\mathfrak{g}}} \Big)$, whence we get at once

$$U_\hbar(\mathfrak{g})' = T_R(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{ \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} - \hbar \cdot \widetilde{[x,y]} \;\Big|\; \tilde{x},\tilde{y} \in \tilde{\mathfrak{g}} \Big\} \Big) \xrightarrow{\hbar \to 0} T_\Bbbk(\tilde{\mathfrak{g}}) \Big/ \big(\{\, \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} \mid \tilde{x},\tilde{y} \in \tilde{\mathfrak{g}} \,\} \big) =$$

$$= T_\Bbbk(\mathfrak{g}) \Big/ \big(\{\, x\,y - y\,x \mid x,y \in \mathfrak{g} \,\} \big) = S_\Bbbk(\mathfrak{g}) = F[\mathfrak{g}^\star]$$

i.e. $\widetilde{U(\mathfrak{g})} := G_{\underline{D}}\big(U(\mathfrak{g})\big) = U_\hbar(\mathfrak{g})' \big/ \hbar\, U_\hbar(\mathfrak{g})' \cong F[\mathfrak{g}^\star]$ *as Poisson Hopf algebras*, as predicted by Theorem 2.2*(c)*. Moreover, $U_\hbar(\mathfrak{g})' = \langle \tilde{\mathfrak{g}} \rangle = T(\tilde{\mathfrak{g}}) \Big/ \big(\{\, \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} = \hbar \cdot \widetilde{[x,y]} \mid \tilde{x},\tilde{y} \in \tilde{\mathfrak{g}} \,\} \big)$ implies that $I_{U_\hbar(\mathfrak{g})'}$ is generated by $\hbar\, R \cdot 1_{U_\hbar(\mathfrak{g})} + R\,\tilde{\mathfrak{g}}$: therefore $\hbar^{-1} I_{U_\hbar(\mathfrak{g})'}$ is generated by $R \cdot 1_{U_\hbar(\mathfrak{g})} + R\,\mathfrak{g}$, whence $\big(U_\hbar(\mathfrak{g})'\big)^\vee := \bigcup_{n \geq 0} \big(\hbar^{-1} I_{U_\hbar(\mathfrak{g})'}\big)^n = \bigcup_{n \geq 0} \big(R \cdot 1_{U_\hbar(\mathfrak{g})} + R\,\mathfrak{g} \big)^n = U_\hbar(\mathfrak{g})$, which agrees with Theorem 2.2*(b)*.

What for the functor $(\ )^\vee$? This heavily depends on the $\mathfrak{g}$ we start from!

First assume $\mathit{Char}(\Bbbk) = 0$. Let $\mathfrak{g}_{(1)} := \mathfrak{g}$, $\mathfrak{g}_{(k)} := \big[\mathfrak{g}, \mathfrak{g}_{(k-1)} \big]$ $(k \in \mathbb{N}_+)$, be the *lower central series* of $\mathfrak{g}$. Pick subsets B_1 , B_2 , ..., B_k , ... $(\subseteq \mathfrak{g})$ such that $B_k \mod \mathfrak{g}_{(k+1)}$ be a $\Bbbk$–basis of $\mathfrak{g}_{(k)} \big/ \mathfrak{g}_{(k+1)}$ (for all $k \in \mathbb{N}_+$), pick also a $\Bbbk$–basis B_∞ of $\mathfrak{g}_{(\infty)} := \bigcap_{k \in \mathbb{N}_+}$, and set $\partial(b) := k$ for any $b \in B_k$ and each $k \in \mathbb{N}_+ \cup \{\infty\}$. Then $B := \Big(\bigcup_{k \in \mathbb{N}_+} B_k \Big) \cup B_\infty$ is a $\Bbbk$–basis of $\mathfrak{g}$; we fix a total order on it. Applying the PBW theorem to this ordered basis of $\mathfrak{g}$ we get that J^n has basis the set of ordered monomials $\big\{\, b_1^{e_1} b_2^{e_2} \cdots b_s^{e_s} \;\big|\; s \in \mathbb{N}_+ \,, b_r \in B \,, \sum_{r=1}^s b_r\, \partial(b_r) \geq n \,\big\}$. Then one easily finds that $U_\hbar(\mathfrak{g})^\vee$ is generated by $\big\{\, \hbar^{-1} b \;\big|\; b \in B_1 \setminus B_2 \,\big\}$ (as a unital R–algebra) and it is the direct sum

$$U_\hbar(\mathfrak{g})^\vee = \bigg(\oplus_{\substack{s \in \mathbb{N}_+ \\ b_r \in B \setminus B_\infty}} R \big(\hbar^{-\partial(b_1)} b_1\big)^{e_1} \cdots \big(\hbar^{-\partial(b_s)} b_s\big)^{e_s} \bigg) \bigoplus \bigg(\oplus_{\substack{s \in \mathbb{N}_+,\, b_r \in B \\ \exists\, \bar{r} : b_{\bar{r}} \in B_\infty}} R\big[\hbar^{-1}\big]\, b_1^{e_1} \cdots b_s^{e_s} \bigg)$$

From this it follows at once that $U_\hbar(\mathfrak{g})^\vee \Big/ \hbar\, U_\hbar(\mathfrak{g})^\vee \cong U\big(\mathfrak{g}/\mathfrak{g}_{(\infty)}\big)$ via an isomorphism which maps $\hbar^{-\partial(b)} b \mod \hbar\, U_\hbar(\mathfrak{g})^\vee$ to $b \mod \mathfrak{g}_{(\infty)} \in \mathfrak{g}/\mathfrak{g}_{(\infty)} \subset U\big(\mathfrak{g}/\mathfrak{g}_{(\infty)}\big)$ for all $b \in B \setminus B_\infty$ and maps $\hbar^{-n} b \mod \hbar\, U_\hbar(\mathfrak{g})^\vee$ to 0 for all $b \in B \setminus B_\infty$ and all $n \in \mathbb{N}$.

Now assume $\mathit{Char}(\Bbbk) = p > 0$. Then in addition to the previous considerations one has to take into account the filtration of $\mathbf{u}(\mathfrak{g})$ induced by both the lower central series of $\mathfrak{g}$ *and* the p–filtration of $\mathfrak{g}$, that is $\mathfrak{g} \supseteq \mathfrak{g}^{[p]} \supseteq \mathfrak{g}^{[p]^2} \supseteq \cdots \supseteq \mathfrak{g}^{[p]^n} \supseteq \cdots$, where $\mathfrak{g}^{[p]^n}$ is the restricted Lie subalgebra generated by $\big\{\, x^{[p]^n} \;\big|\; x \in \mathfrak{g} \,\big\}$ and $x \mapsto x^{[p]}$ is the p–operation in $\mathfrak{g}$: these encode the J–filtration of $U(\mathfrak{g})$, hence of $H = \mathbf{u}_\hbar(\mathfrak{g})$, so permit to describe $H^\vee$.

In detail, for any restricted Lie algebra $\mathfrak{h}$, let $\mathfrak{h}_n := \Big\langle \bigcup_{(m\, p^k \geq n} \big(\mathfrak{h}_{(m)}\big)^{[p^k]} \Big\rangle$ for all $n \in \mathbb{N}_+$ (where $\langle X \rangle$ denotes the Lie subalgebra of $\mathfrak{h}$ generated by X) and $\mathfrak{h}_\infty := \bigcap_{n \in \mathbb{N}_+} \mathfrak{h}_n$:

we call $\big\{\mathfrak{h}_n\big\}_{n\in\mathbb{N}_+}$ *the p–lower central series of* $\mathfrak{h}$. It is a *strongly central series* of $\mathfrak{h}$, i.e. it is a central series (= decreasing filtration of ideals, each one centralizing the previous one) of $\mathfrak{h}$ such that $[\mathfrak{h}_m, \mathfrak{h}_n] \leq \mathfrak{h}_{m+n}$ for all m, n; in addition, it verifies $\mathfrak{h}_n^{[p]} \leq \mathfrak{h}_{n+1}$. When $\mathfrak{h}$ is Abelian $\big\{\mathfrak{h}_n\big\}_{n\in\mathbb{N}_+}$ coincides (after index rescaling) with the *p–power series* $\Big\{\mathfrak{h}^{[p^n]}\Big\}_{n\in\mathbb{N}}$.

Applying these tools to $\mathfrak{g} \subseteq \mathbf{u}(\mathfrak{g})$ the very definitions give $\mathfrak{g}_n \subseteq J^n$ (for all $n \in \mathbb{N}$) where $J := J_{\mathbf{u}(\mathfrak{g})}$: more precisely, if B is an ordered basis of $\mathfrak{g}$ then the (restricted) PBW theorem for $\mathbf{u}(\mathfrak{g})$ implies that $J^n \big/ J^{n+1}$ admits as $\Bbbk$–basis the set of ordered monomials of the form $x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_s}^{e_s}$ such that $\sum_{r=1}^{s} e_r \partial(x_{i_r}) = n$ where $\partial(x_{i_r}) \in \mathbb{N}$ is uniquely determined by the condition $x_{i_r} \in \mathfrak{g}_{\partial(x_{i_r})} \setminus \mathfrak{g}_{\partial(x_{i_r})+1}$ and each x_{i_k} is a fixed lift in $\mathfrak{g}$ of an element of a fixed ordered basis of $\mathfrak{g}_{\partial(x_{i_k})} \Big/ \mathfrak{g}_{\partial(x_{i_k})+1}$. This yields an explicit description of $\underline{J}$, hence of $\mathbf{u}(\mathfrak{g})^\vee$ and $\mathbf{u}_\hbar(\mathfrak{g})^\vee$, like before: in particular $\mathbf{u}_\hbar(\mathfrak{g})^\vee \Big/ \hbar\, \mathbf{u}_\hbar(\mathfrak{g})^\vee \cong \mathbf{u}\big(\mathfrak{g}/\mathfrak{g}_\infty\big)$.

Definition 5.8. *For any $\Bbbk$–coalgebra C, define $X \bigwedge Y := \Delta^{-1}\big(X \otimes C + C \otimes Y\big)$ for all subspaces X, Y of C. Set also $\bigwedge^1 X := X$ and $\bigwedge^{n+1} X := \big(\bigwedge^n X\big) \bigwedge X$ for all $n \in \mathbb{N}_+$, and also $\bigwedge^0 X := \Bbbk \cdot 1$ if C is a $\Bbbk$–bialgebra.*

Lemma 5.9. *Let H be a Hopf $\Bbbk$–algebra. Then $D_n = \bigwedge^{n+1}(\Bbbk \cdot 1)$ for all $n \in \mathbb{N}$.*

Proof. Definitions give $D_0 := Ker(\delta_1) = \Bbbk \cdot 1 = \bigwedge^1(\Bbbk \cdot 1)$. By coassociativity we have $D_n := Ker(\delta_{n+1}) = Ker\big((\delta_n \otimes \delta_1) \circ \delta_2\big) = Ker\big((\delta_n \otimes \delta_1) \circ \Delta\big) = \Delta^{-1}\big(Ker(\delta_n \otimes \delta_1)\big) = \Delta^{-1}\big(Ker(\delta_n) \otimes H + H \otimes Ker(\delta_1)\big) = \Delta^{-1}\big(D_{n-1} \otimes H + H \otimes D_0\big) = D_{n-1} \bigwedge D_0 = D_{n-1} \bigwedge (\Bbbk \cdot 1)$ for all $n \in \mathbb{N}_+$; so by induction $D_n = D_{n-1} \bigwedge (\Bbbk \cdot 1) = \big(\bigwedge^n(\Bbbk \cdot 1)\big) \bigwedge (\Bbbk \cdot 1) = \bigwedge^{n+1}(\Bbbk \cdot 1)$. $\square$

Definition 5.10.

(a) We call pre-restricted universal enveloping algebra *(in short, PrUEA) any $H \in \mathcal{HA}_\Bbbk$ which is down-filtered by $\underline{J}$ (that is, $\bigcap_{n\in\mathbb{N}} J^n = \{0\}$). We call $\mathcal{PrUEA}$ the full subcategory of $\mathcal{HA}_\Bbbk$ of all the PrUEAs.*

(b) We call pre-function algebra *(in short, PFA) any $H \in \mathcal{HA}_\Bbbk$ which is up-filtered by $\underline{D}$ (that is, $\bigcup_{n\in\mathbb{N}} D_n = H$). We call $\mathcal{PFA}$ the full subcategory of $\mathcal{HA}_\Bbbk$ of all the PFAs.*

The content of the notions of PrUEA and of PFA is revealed by parts *(a)* and *(b)* of next theorem, which collects the main results of this section.

Theorem 5.11. *("The Crystal Duality Principle")*

(a) The assignment $H \mapsto H^\vee := H \big/ J_H^\infty$, resp. $H \mapsto H' := \bigcup_{n\in\mathbb{N}} D_n$, defines a functor $(\ )^\vee \colon \mathcal{HA}_\Bbbk \longrightarrow \mathcal{HA}_\Bbbk$, resp. $(\ )' \colon \mathcal{HA}_\Bbbk \longrightarrow \mathcal{HA}_\Bbbk$, whose image is $\mathcal{PrUEA}$, resp. $\mathcal{PFA}$. More in general, the assignment $A \mapsto A^\vee := A \big/ J_A^\infty$, resp. $C \mapsto C' := \bigcup_{n\in\mathbb{N}} D_n(C)$, defines a surjective functor from augmented $\Bbbk$–algebras, resp. coaugmented $\Bbbk$–coalgebras, to augmented $\Bbbk$–algebras which are down-filtered by $\underline{J}$, resp. coaugmented $\Bbbk$–coalgebras which are up-filtered by $\underline{D}$; and similarly for $\Bbbk$–bialgebras.

(b) Let $H \in \mathcal{HA}_{\Bbbk}$ *. Then* $\widehat{H} := G_{\underline{J}}(H) \cong \mathcal{U}(\mathfrak{g})$ *, as graded co-Poisson Hopf algebras, for some restricted Lie bialgebra* $\mathfrak{g}$ *which is graded as a Lie algebra. In particular, if* $Char(\Bbbk) = 0$ *and* $\dim(H) \in \mathbb{N}$ *then* $\widehat{H} = \Bbbk \cdot 1$ *and* $\mathfrak{g} = \{0\}$ *.*

More in general, the same holds if $H = B$ *is a* $\Bbbk$*–bialgebra.*

(c) Let $H \in \mathcal{HA}_{\Bbbk}$ *. Then* $\widetilde{H} := G_{\underline{D}}(H) \cong F[G]$ *, as graded Poisson Hopf algebras, for some connected algebraic Poisson group G whose variety of closed points form a (pro)affine space. If* $Char(\Bbbk) = 0$ *then* $F[G] = \widetilde{H}$ *is a polynomial algebra, i.e.* $F[G] = \Bbbk\big[\{x_i\}_{i \in \mathcal{I}}\big]$ *(for some set* $\mathcal{I}$*); in particular, if* $\dim(H) \in \mathbb{N}$ *then* $\widetilde{H} = \Bbbk \cdot 1$ *and* $G = \{1\}$ *. If* $p := Char(\Bbbk) > 0$ *then G has dimension 0 and height 1, and if* $\Bbbk$ *is perfect then* $F[G] = \widetilde{H}$ *is a truncated polynomial algebra, i.e.* $F[G] = \Bbbk\big[\{x_i\}_{i \in \mathcal{I}}\big] \Big/ \big(\{x_i^p\}_{i \in \mathcal{I}}\big)$ *(for some set* $\mathcal{I}$*).*

More in general, the same holds if $H = B$ *is a* $\Bbbk$*–bialgebra.*

(d) For every $H \in \mathcal{HA}_{\Bbbk}$ *, there exist two 1-parameter families* $(H^\vee)_\hbar^\vee = \mathcal{R}_{\underline{J}}^\hbar(H^\vee)$ *and* $\big((H^\vee)_\hbar^\vee\big)'$ *in* $\mathcal{HA}_{\Bbbk}$ *giving deformations of* $H^\vee$ *with regular fibers*

$$\left.\begin{array}{ll} \textit{if } Char(\Bbbk) = 0\,, & U(\mathfrak{g}_-) \\ \textit{if } Char(\Bbbk) > 0\,, & \mathbf{u}(\mathfrak{g}_-) \end{array}\right\} = \widehat{H} \xleftarrow[(H^\vee)_\hbar^\vee]{\;0 \leftarrow \hbar \rightarrow 1\;} H^\vee \xleftarrow[((H^\vee)_\hbar^\vee)']{\;1 \leftarrow \hbar \rightarrow 0\;} \left\{\begin{array}{l} F[K_-] = F[G_-^\star] \\ F[K_-] \end{array}\right.$$

and two 1-parameter families $H_\hbar{}' = \mathcal{R}_{\underline{D}}^\hbar(H')$ *and* $(H_\hbar{}')^\vee$ *in* $\mathcal{HA}_{\Bbbk}$ *giving deformations of* H' *with regular fibers*

$$F[G_+] = \widetilde{H} \xleftarrow[H_\hbar{}']{\;0 \leftarrow \hbar \rightarrow 1\;} H' \xleftarrow[(H_\hbar{}')^\vee]{\;1 \leftarrow \hbar \rightarrow 0\;} \left\{\begin{array}{ll} U(\mathfrak{k}_+) = U(\mathfrak{g}_+^\times) & \textit{if } Char(\Bbbk) = 0 \\ \mathbf{u}(\mathfrak{k}_+) & \textit{if } Char(\Bbbk) > 0 \end{array}\right.$$

where G_+ *is like G in (c),* K_- *is a connected algebraic Poisson group,* $\mathfrak{g}_-$ *is like* $\mathfrak{g}$ *in (b),* $\mathfrak{k}_+$ *is a (restricted, if* $Char(\Bbbk) > 0$ *) Lie bialgebra,* $\mathfrak{g}_+^\times$ *is the cotangent Lie bialgebra to* G_+ *and* $G_-^\star$ *is a connected algebraic Poisson group whose cotangent Lie bialgebra is* $\mathfrak{g}_-$ *.*

(e) If $H = F[G]$ *is the function algebra of an algebraic Poisson group G, then* $\widehat{F[G]}$ *is a bi-Poisson Hopf algebra (see [KT], §1), namely*

$$\widehat{F[G]} \;\cong\; S(\mathfrak{g}^\times) \Big/ \Big(\Big\{ \overline{x}^{\,p^{n(x)}} \,\Big|\, x \in \mathcal{N}_{F[G]} \Big\} \Big) \;\cong\; U(\mathfrak{g}^\times) \Big/ \Big(\Big\{ \overline{x}^{\,p^{n(x)}} \,\Big|\, x \in \mathcal{N}_{F[G]} \Big\} \Big)$$

where $\mathcal{N}_{F[G]}$ *is the nilradical of* $F[G]$*,* $p^{n(x)}$ *is the order of nilpotency of* $x \in \mathcal{N}_{F[G]}$ *and the bi-Poisson Hopf structure of* $S(\mathfrak{g}^\times) \Big/ \Big(\Big\{ \overline{x}^{\,p^{n(x)}} \,\Big|\, x \in \mathcal{N}_{F[G]} \Big\} \Big)$ *is the quotient one from* $S(\mathfrak{g}^\times)$ *; in particular, if the group G is reduced then* $\widehat{F[G]} \cong S(\mathfrak{g}^\times) \cong U(\mathfrak{g}^\times)$ *.*

(f) If $Char(\Bbbk) = 0$ *and* $H = U(\mathfrak{g})$ *is the universal enveloping algebra of some Lie bialgebra* $\mathfrak{g}$*, then* $\widetilde{U(\mathfrak{g})}$ *is a bi-Poisson Hopf algebra, namely*

$$\widetilde{U(\mathfrak{g})} \;\cong\; S(\mathfrak{g}) \;=\; F[\mathfrak{g}^\star]$$

where the bi-Poisson Hopf structure on $S(\mathfrak{g})$ is the canonical one.

If $Char(\Bbbk) = p > 0$ and $H = \mathbf{u}(\mathfrak{g})$ is the restricted universal enveloping algebra of some restricted Lie bialgebra $\mathfrak{g}$, then $\widetilde{\mathbf{u}(\mathfrak{g})}$ is a bi-Poisson Hopf algebra, namely

$$\widetilde{\mathbf{u}(\mathfrak{g})} \;\cong\; S(\mathfrak{g}) \Big/ \big(\big\{ x^p \,\big|\, x \in \mathfrak{g} \big\} \big) \;=\; F[G^\star]$$

where the bi-Poisson Hopf structure on $S(\mathfrak{g}) \Big/ \big(\big\{ x^p \,\big|\, x \in \mathfrak{g} \big\} \big)$ is induced by the canonical one on $S(\mathfrak{g})$, and $G^\star$ is a connected algebraic Poisson group of dimension 0 and height 1 whose cotangent Lie bialgebra is $\mathfrak{g}$.

(g) Let $H, K \in \mathcal{HA}_\Bbbk$ and let $\pi \colon H \times K \longrightarrow \Bbbk$ be a Hopf pairing. Then π induce a filtered Hopf pairing $\pi_f : H^\vee \times K' \longrightarrow \Bbbk$, a graded Hopf pairing $\pi_{{}_G} : \widehat{H} \times \widetilde{K} \longrightarrow \Bbbk$, both perfect on the right, and Hopf pairings over $\Bbbk[\hbar]$ (notation of §5.1) $H_\hbar \times K_\hbar \longrightarrow \Bbbk[\hbar]$ and ${H_\hbar}^\vee \times {K_\hbar}' \longrightarrow \Bbbk[\hbar]$, the latter being perfect on the right. If in addition the pairing $\pi_f : H^\vee \times K' \longrightarrow \Bbbk$ is perfect, then $\pi_{{}_G}$ is perfect as well, and ${H_\hbar}^\vee$ and ${K_\hbar}'$ are dual to each other. The left-right symmetrical results hold too.

Proof. Parts *(a)* through *(c)* of the statement are proved by the analysis in §5.4, but for the naturality of $H \mapsto H^\vee$ and $H \mapsto H'$, which is however clear because, $\varphi\big(J_{{}_H}^\infty\big) \subseteq J_{{}_K}^\infty$ and $\varphi\big(D_n(H)\big) \subseteq D_n(K)$ for any morphism $\varphi \colon H \longrightarrow K$ within $\mathcal{HA}_\Bbbk$. In addition, for part *(b)* when $Char(\Bbbk) = 0$ and $\dim(H) \in \mathbb{N}$ we have to notice that $\widehat{H} = U(\mathfrak{g})$ is finite dimensional too, hence $\widehat{H} = U(\mathfrak{g}) = \Bbbk \cdot 1$ and $\mathfrak{g} = \{0\}$; similarly for *(c)* these assumptions imply that $\widetilde{H} = F[G]$ is finite dimensional too, so $\widetilde{H} = F[G] = \Bbbk \cdot 1$ and $G = \{1\}$, q.e.d. Finally, if $H = B$ is just a $\Bbbk$–bialgebra then both $\widehat{B} := G_{\underline{J}}(B)$ and $\widetilde{B} := G_{\underline{D}}(B)$ are *irreducible* graded $\Bbbk$–bialgebras: then by [Ab], Theorem 2.4.24, they are also graded Hopf algebras, whence we conclude as if B were a Hopf algebra.

Part *(d)* is proved by §5.5.

As for part *(e)*, it is almost entirely proved by the analysis in §5.6, noting also that in the case of $H = F[G]$ one has $S(\mathfrak{g}^\times) = U(\mathfrak{g}^\times)$ because $\mathfrak{g}^\times$ is Abelian. What is left to check is whatever refers to *bi-Poisson* structures. Indeed, the Lie bracket of $\mathfrak{g}^\times$ extends to a Poisson bracket which makes $S(\mathfrak{g}^\times)$ into a bi-Poisson Hopf algebra (see §5.1); then $\Big(\Big\{ \,\overline{x}^{\,p^{n(x)}} \Big\}_{x \in \mathcal{N}_{F[G]}} \Big)$ is a bi-Poisson Hopf ideal, thus $S(\mathfrak{g}^\times) \bigg/ \Big(\Big\{ \,\overline{x}^{\,p^{n(x)}} \Big\}_{x \in \mathcal{N}_{F[G]}} \Big)$ is a bi-Poisson Hopf algebra as well. But $\widehat{F[G]}$ also inherits a Poisson bracket from $F[G]$ which makes it into a bi-Poisson Hopf algebra too: it is then clear that the isomorphism $S(\mathfrak{g}^\times) \bigg/ \Big(\Big\{ \,\overline{x}^{\,p^{n(x)}} \Big\}_{x \in \mathcal{N}_{F[G]}} \Big) \cong \widehat{F[G]}$ is one of *bi-Poisson* Hopf algebras.

Similarly, part *(f)* is proved by the analysis in §5.7, noting also that both $\widetilde{U(\mathfrak{g})}$ and $S(\mathfrak{g}) = F[G^\star]$ are naturally bi-Poisson Hopf algebras, isomorphic to each other via the previously considered isomorphism. In addition, the same holds also for $\widetilde{\mathbf{u}(\mathfrak{g})}$ and $S(\mathfrak{g}) \Big/ \big(\big\{ x^p \,\big|\, x \in \mathfrak{g} \big\} \big) = F[G^\star]$, because $\big(\big\{ x^p \,\big|\, x \in \mathfrak{g} \big\} \big)$ is a bi-Poisson Hopf ideal of $S(\mathfrak{g})$.

Finally, we go for part *(g)*. Let $\pi \colon H \times K \longrightarrow \Bbbk$ be the Hopf pairing under study. Consider the filtrations $\underline{J} = \big\{ {J_{\scriptscriptstyle H}}^n \big\}_{n \in \mathbb{N}}$ and $\underline{D} = \big\{ D_n^{\scriptscriptstyle K} \big\}_{n \in \mathbb{N}}$. The key fact is that

$$D_n^{\scriptscriptstyle K} = \big({J_{\scriptscriptstyle H}}^{n+1} \big)^{\perp} \qquad \text{and} \qquad {J_{\scriptscriptstyle H}}^{n+1} \subseteq \big(D_n^{\scriptscriptstyle K} \big)^{\perp} \qquad\qquad \text{for all } \; n \in \mathbb{N} \, . \tag{5.6}$$

Indeed, if X is a subspace of a coalgebra C and C is in perfect "Hopf-like" pairing with an algebra A, one has $\bigwedge^n X = \big((X^\perp)^n \big)^\perp$ (cf. Definition 5.8) for all $n \in \mathbb{N}$, where the superscript $\perp$ means "orthogonal subspace" (either in A or in C) w.r.t. the pairing under exam (cf. [Ab] or [Mo]). Now, Lemma 5.9 gives $D_n^{\scriptscriptstyle K} = \bigwedge^{n+1} (\Bbbk \cdot 1_{\scriptscriptstyle K})$, thus $D_n^{\scriptscriptstyle K} = \bigwedge^{n+1} (\Bbbk \cdot 1_{\scriptscriptstyle K}) = \Big(\big((\Bbbk \cdot 1_{\scriptscriptstyle K})^\perp \big)^{n+1} \Big)^\perp = \big({J_{\scriptscriptstyle H}}^{n+1} \big)^\perp$ because $(\Bbbk \cdot 1_{\scriptscriptstyle K})^\perp = J_{\scriptscriptstyle H}$ (w.r.t. the pairing π above). Therefore $D_n^{\scriptscriptstyle K} = \big({J_{\scriptscriptstyle H}}^{n+1} \big)^\perp$, and this also implies ${J_{\scriptscriptstyle H}}^{n+1} \subseteq \big(D_n^{\scriptscriptstyle K} \big)^\perp$.

Now $K' := \bigcup_{n \in \mathbb{N}} D_n^{\scriptscriptstyle K} = \bigcup_{n \in \mathbb{N}} \big({J_{\scriptscriptstyle H}}^{n+1} \big)^\perp = \big(\bigcap_{n \in \mathbb{N}} {J_{\scriptscriptstyle H}}^{n+1} \big)^\perp = \big({J_{\scriptscriptstyle H}}^{\infty} \big)^\perp$. Thus π induces a Hopf pairing $\pi_f : H^\vee \times K' \longrightarrow \Bbbk$ as required, and by (5.6) this respects the filtrations on either side. Then by general theory π_f induces a graded Hopf pairing $\pi_{\scriptscriptstyle G}$ as required: in particular $\pi_{\scriptscriptstyle G}$ is well-defined because $D_n^{\scriptscriptstyle K} \subseteq \big({J_{\scriptscriptstyle H}}^{n+1} \big)^\perp$ and ${J_{\scriptscriptstyle H}}^{n+1} \subseteq \big(D_n^{\scriptscriptstyle K} \big)^\perp$ (for all $n \in \mathbb{N}_+$) by (5.6), and both π_f and $\pi_{\scriptscriptstyle G}$ are perfect on the right because all the inclusions $D_n^{\scriptscriptstyle K} \subseteq \big({J_{\scriptscriptstyle H}}^{n+1} \big)^\perp$ happen to be identities. Clearly by scalar extension π defines also a Hopf pairing $H_\hbar \times K_\hbar \longrightarrow \Bbbk[\hbar]$; then (5.6) and the description of ${H_\hbar}'$ and ${K_\hbar}^\vee$ in Lemma 5.2 directly imply that this yields another Hopf pairing ${H_\hbar}^\vee \times {K_\hbar}' \longrightarrow \Bbbk[\hbar]$ as claimed.

Finally when π_f is perfect it is easy to see that $\pi_{\scriptscriptstyle G}$ is perfect as well; note that this improves (5.6), for we have ${J_{\scriptscriptstyle H}}^{n+1} = \big(D_n^{\scriptscriptstyle K} \big)^\perp$ for all $n \in \mathbb{N}$. It is also clear that the pairing ${H_\hbar}^\vee \times {K_\hbar}' \longrightarrow \Bbbk[\hbar]$ is perfect as well, and that ${H_\hbar}^\vee$ and ${K_\hbar}'$ are dual to each other. $\square$

<u>*Remarks:*</u> *(a)* It is worth noticing that, though usually introduced in a different way, H' is an object which is pretty familiar to Hopf algebra theorists: indeed, it is the *connected component* of H (cf. [Ga5] for a proof); in particular, H is a PFA if and only if it is connected. Nevertheless, surprisingly enough the pretty remarkable property of its associated graded Hopf algebra $\widetilde{H} = G_{\underline{D}}(H)$ expressed by Theorem 5.11*(c)* seems to have been unknown so far (at least, to the author's knowledge)! Similarly, the "dual" construction of $H^\vee$ and the important property of its associated graded Hopf algebra $\widehat{H} = G_{\underline{J}}(H)$ stated in Theorem 5.11*(b)* seem to have escaped the specialists' attention.

(b) Part *(d)* of Theorem 5.11 is quite interesting for applications in physics. In fact, let H be a Hopf algebra which describes the symmetries of some physically meaningful system, but has no geometrical meaning (typically, when it is not commutative nor cocommutative, as it usually happens in quantum physics), and assume also $H' = H = H^\vee$. Then Theorem 5.11*(d)* yields a recipe to deform H to four different Hopf algebras bearing a geometrical meaning, which means having two Poisson groups and two Lie bialgebras attached to H, hence a rich "geometrical symmetry" (of Poisson type) underlying the physical system; if the ground field has characteristic zero (as usual) we simply have two pairs of mutually

dual Poisson groups together with their tangent Lie bialgebras. In §10 we'll give a nice application of this kind with the two pairs of groups strictly related, yet different.

5.12 The hyperalgebra case. Let G be an algebraic group, which for simplicity we assume to be finite-dimensional. By $Hyp\,(G)$ we mean the hyperalgebra associated to G, defined as $Hyp\,(G) := \big(F[G]^\bullet\big)_\epsilon = \big\{\, \phi \in F[G]^\circ \,\big|\, \phi(\mathfrak{m}_e{}^n) = 0 \,,\; \forall\; n \gg 0 \,\big\}$, that is the irreducible component of the *dual* Hopf algebra $F[G]^\circ$ containing $\epsilon = \epsilon_{{}_{F[G]}}$, which is a Hopf subalgebra of $F[G]^\circ$; in particular, $Hyp\,(G)$ is connected cocommutative. Recall that there's a natural Hopf algebra morphism $\Phi : U(\mathfrak{g}) \longrightarrow Hyp\,(G)$; if $Char\,(\Bbbk) = 0$ then Φ is an isomorphism, so $Hyp\,(G)$ identifies to $U(\mathfrak{g})$; if $Char\,(\Bbbk) > 0$ then Φ factors through $\mathbf{u}(\mathfrak{g})$ and the induced morphism $\overline{\Phi} : \mathbf{u}(\mathfrak{g}) \longrightarrow Hyp\,(G)$ is injective, so that $\mathbf{u}(\mathfrak{g})$ identifies with a Hopf subalgebra of $Hyp\,(G)$. Now we study $Hyp\,(G)'$, $Hyp\,(G)^\vee$, $\widetilde{Hyp\,(G)}$, $\widehat{Hyp\,(G)}$.

As $Hyp\,(G)$ is connected, letting $C_0 := Corad\big(Hyp\,(G)\big)$ be its coradical we have $Hyp\,(G) \,=\, \bigcup_{n \in \mathbb{N}} \bigwedge^{n+1} C_0 \,=\, \bigcup_{n \in \mathbb{N}} \bigwedge^{n+1} (\Bbbk \cdot 1) \,=\, \bigcup_{n \in \mathbb{N}} D_{n+1}\big(Hyp\,(G)\big) \,=:\, Hyp\,(G)'$. Now, Theorem 5.11*(c)* gives $\widetilde{Hyp\,(G)} := G_{\underline{D}}\big(Hyp\,(G)\big) = F[\varGamma\,]$ for some connected algebraic Poisson group $\varGamma$; Theorem 5.11*(e)* yields $\widehat{F[G]} \,\cong\, S(\mathfrak{g}^*) \Big/ \Big(\Big\{\, \overline{x}^{\,p^{n(x)}} \Big\}_{x \in \mathcal{N}_{F[G]}} \Big) \,=\, \mathbf{u}\bigg(P\bigg(S(\mathfrak{g}^*) \Big/ \Big(\Big\{\, \overline{x}^{\,p^{n(x)}} \Big\}_{x \in \mathcal{N}_{F[G]}} \Big) \bigg)\bigg) \,=\, \mathbf{u}\Big({(\mathfrak{g}^*)}^{p^\infty} \Big)$, with ${(\mathfrak{g}^*)}^{p^\infty} := Span\Big(\Big\{\, x^{p^n} \,\Big|\, x \in \mathfrak{g}^* \,, n \in \mathbb{N} \,\Big\} \Big) \subseteq \widehat{F[G]}$, and noting that $\mathfrak{g}^\times = \mathfrak{g}^*$. On the other hand, exactly like for $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$ respectively in case $Char\,(\Bbbk) = 0$ and $Char\,(\Bbbk) > 0$, the filtration $\underline{D}$ of $Hyp\,(G)$ is nothing but the natural filtration given by the order of differential operators: this implies immediately $Hyp\,(G)_\hbar{}' := \big(\Bbbk[\hbar\,] \otimes_\Bbbk Hyp\,(G)\big)' \,=\, \big\langle \big\{\, \hbar^n x^{(n)} \,\big|\, x \in \mathfrak{g} \,, n \in \mathbb{N} \,\big\} \big\rangle$, where hereafter notation like $x^{(n)}$ denotes the n–th divided power of $x \in \mathfrak{g}$ (recall that $Hyp\,(G)$ is generated as an algebra by all the $x^{(n)}$'s, some of which might be zero). It is then immediate to check that the graded Hopf pairing between $Hyp\,(G)_\hbar{}' \Big/ \hbar\, Hyp\,(G)_\hbar{}' = \widetilde{Hyp\,(G)} = F[\varGamma\,]$ and $\widehat{F[G]}$ given by Theorem 5.11*(g)* is perfect. From this we easily argue that the cotangent Lie bialgebra of $\varGamma$ is isomorphic to $\Big({(\mathfrak{g}^*)}^{p^\infty} \Big)^*$.

As for $Hyp\,(G)^\vee$ and $\widehat{Hyp\,(G)}$, the situation is much like for $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$, in that it strongly depends on the algebraic nature of G (cf. §5.7).

5.13 The CDP on group algebras and their duals. In this section, G is any abstract group. We divide the subsequent material in several subsections.

Group-related algebras. For any commutative unital ring $\mathbb{A}$, by $\mathbb{A}[G]$ we mean the group algebra of G over $\mathbb{A}$ and, when G is *finite*, we denote by $A_\mathbb{A}(G) := \mathbb{A}[G]^*$ (the linear dual of $\mathbb{A}[G]$) the function algebra of G over $\mathbb{A}$. Our purpose is to apply the Crystal Duality Principle to $\Bbbk[G]$ and $A_\Bbbk(G)$ with their standard Hopf algebra structure: hereafter $\Bbbk$ is a field and $R := \Bbbk[\hbar]$ as in §5.1, and we set $p := Char\,(\Bbbk)$.

Recall that $H := \mathbb{A}[G]$ admits G itself as a distinguished basis, with Hopf algebra structure given by $g \cdot_{H} \gamma := g \cdot_{G} \gamma$, $1_{H} := 1_{G}$, $\Delta(g) := g \otimes g$, $\epsilon(g) := 1$, $S(g) := g^{-1}$, for all $g, \gamma \in G$. Dually, $H := A_{\mathbb{A}}(G)$ has basis $\big\{ \varphi_g \,\big|\, g \in G \big\}$ dual to the basis G of $\mathbb{A}[G]$, with $\varphi_g(\gamma) := \delta_{g,\gamma}$ for all $g, \gamma \in G$; its Hopf algebra structure is given by $\varphi_g \cdot \varphi_\gamma := \delta_{g,\gamma} \varphi_g$, $1_{H} := \sum_{g \in G} \varphi_g$, $\Delta(\varphi_g) := \sum_{\gamma \cdot \ell = g} \varphi_\gamma \otimes \varphi_\ell$, $\epsilon(\varphi_g) := \delta_{g,1_G}$, $S(\varphi_g) := \varphi_{g^{-1}}$, for all $g, \gamma \in G$. In particular, $R[G] = R \otimes_{\Bbbk} \Bbbk[G]$ and $A_R[G] = R \otimes_{\Bbbk} A_{\Bbbk}[G]$. Our first result is

<u>*Theorem A*</u>*:* $\big(\Bbbk[G]\big)_\hbar{}' = R \cdot 1$, $\Bbbk[G]' = \Bbbk \cdot 1$ *and* $\widetilde{\Bbbk[G]} = \Bbbk \cdot 1 = F\big[\{*\}\big]$.

Proof. The claim follows easily from the formula $\delta_n(g) = (g-1)^{\otimes n}$, for $g \in G$, $n \in \mathbb{N}$. $\square$

<u>$R[G]^\vee$, $\Bbbk[G]^\vee$, $\widehat{\Bbbk[G]}$ *and the dimension subgroup problem*.</u> In contrast with the triviality result in Theorem A above, things are more interesting for $R[G]^\vee = \big(\Bbbk[G]\big)_\hbar^\vee$, $\Bbbk[G]^\vee$ and $\widehat{\Bbbk[G]}$. Note however that since $\Bbbk[G]$ is cocommutative the induced Poisson cobracket on $\widehat{\Bbbk[G]}$ is trivial, and the Lie cobracket of $\mathfrak{k}_G := P\Big(\widehat{\Bbbk[G]}\Big)$ is trivial as well.

Studying $\Bbbk[G]^\vee$ and $\widehat{\Bbbk[G]}$ amounts to study the filtration $\big\{ J^n \big\}_{n \in \mathbb{N}}$, with $J := Ker\,(\epsilon_{\Bbbk[G]})$, which is a classical topic. Indeed, for $n \in \mathbb{N}$ let $\mathcal{D}_n(G) := \big\{\, g \in G \,\big|\, (g-1) \in J^n \,\big\}$: this is a characteristic subgroup of G, called *the* n^{th} *dimension subgroup of* G. All these form a filtration inside G: characterizing it in terms of G is the *dimension subgroup problem*, which (for group algebras over fields) is completely solved (see [Pa], Ch. 11, §1, and [HB], and references therein); this also gives a description of $\big\{ J^n \big\}_{n \in \mathbb{N}_+}$. Thus we find ourselves within the domain of classical group theory: now we use the results which solve the dimension subgroup problem to argue a description of $\Bbbk[G]^\vee$, $\widehat{\Bbbk[G]}$ and $R[G]^\vee$, and later on we'll get from this a description of $\big(R[G]^\vee\big)'$ and its semiclassical limit too.

By construction, J has $\Bbbk$–basis $\big\{ \eta_g \,\big|\, g \in G \setminus \{1_G\} \big\}$, where $\eta_g := (g-1)$. Then $\Bbbk[G]^\vee$ is generated by $\big\{\, \eta_g \mod J^\infty \,\big|\, g \in G \setminus \{1_G\} \big\}$, and $\widehat{\Bbbk[G]}$ by $\big\{\, \overline{\eta_g} \,\big|\, g \in G \setminus \{1_G\} \big\}$: hereafter $\overline{x} := x \mod J^{n+1}$ for all $x \in J^n$, that is $\overline{x}$ is the element in $\widehat{\Bbbk[G]}$ which corresponds to $x \in \Bbbk[G]$. Moreover, $\overline{g} = \overline{1 + \eta_g} = \overline{1}$ for all $g \in G$; also, $\Delta\big(\overline{\eta_g}\big) = \overline{\eta_g} \otimes \overline{g} + 1 \otimes \overline{\eta_g} = \overline{\eta_g} \otimes 1 + 1 \otimes \overline{\eta_g}$: thus $\overline{\eta_g}$ is primitive, so $\big\{\, \overline{\eta_g} \,\big|\, g \in G \setminus \{1_G\} \big\}$ generates $\mathfrak{k}_G := P\Big(\widehat{\Bbbk[G]}\Big)$.

<u>*The Jennings-Hall theorem*.</u> The description of $\mathcal{D}_n(G)$ is given by the Jennings-Hall theorem, which we now recall. The construction involved strongly depends on whether $p := Char\,(\Bbbk)$ is zero or not, so we shall distinguish these two cases.

First assume $p = 0$. Let $G_{(1)} := G$, $G_{(k)} := (G, G_{(k-1)})$ $(k \in \mathbb{N}_+)$, form the *lower central series* of G; hereafter (X,Y) is the commutator subgroup of G generated by the set of commutators $\big\{ (x,y) := x\,y\,x^{-1}y^{-1} \,\big|\, x \in X, y \in Y \big\}$: this is a *strongly central series* in G, which means a central series $\big\{ G_k \big\}_{k \in \mathbb{N}_+}$ (= decreasing filtration of normal subgroups, each one centralizing the previous one) of G such that $(G_m, G_n) \leq G_{m+n}$ for all m, n. Then let $\sqrt{G_{(n)}} := \big\{ x \in G \,\big|\, \exists\, s \in \mathbb{N}_+ : x^s \in G_{(n)} \big\}$ for all $n \in \mathbb{N}_+$: these form a descending series of characteristic subgroups in G, such that each composition factor

$A^G_{(n)} := \sqrt{G_{(n)}} \Big/ \sqrt{G_{(n+1)}}$ is torsion-free Abelian. Therefore $\mathcal{L}_0(G) := \bigoplus_{n\in\mathbb{N}_+} A^G_{(n)}$ is a graded Lie ring, with Lie bracket $\big[\overline{g}, \overline{\ell}\,\big] := \overline{(g,\ell\,)}$ for all *homogeneous* $\overline{g}$, $\overline{\ell} \in \mathcal{L}_0(G)$, with obvious notation. It is easy to see that the map $\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_0(G) \longrightarrow \mathfrak{k}_G$, $\overline{g} \mapsto \overline{\eta_g}$, is an epimorphism *of graded Lie rings*: therefore *the Lie algebra* $\mathfrak{k}_G$ *is a quotient of* $\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_0(G)$; in fact, the above is an isomorphism, see below. We shall use notation $\partial(g) := n$ for all $g \in \sqrt{G_{(n)}} \setminus \sqrt{G_{(n+1)}}$.

For each $k \in \mathbb{N}_+$ pick in $A^G_{(k)}$ a subset $\overline{B}_k$ which is a $\mathbb{Q}$–basis of $\mathbb{Q} \otimes_{\mathbb{Z}} A^G_{(k)}$; for each $\overline{b} \in \overline{B}_k$, choose a fixed $b \in \sqrt{G_{(k)}}$ such that its coset in $A^G_{(k)}$ be $\overline{b}$, and denote by B_k the set of all such elements b. Let $B := \bigcup_{k\in\mathbb{N}_+} B_k$: we call such a set *t.f.l.c.s.-net* (= "torsion-free-lower-central-series-net") on G. Clearly $B_k = \Big(B \cap \sqrt{G_{(k)}}\,\Big) \setminus \Big(B \cap \sqrt{G_{(k+1)}}\,\Big)$ for all k. By an *ordered t.f.l.c.s.-net* is meant a t.f.l.c.s.-net B which is totally ordered in such a way that: *(i)* if $a \in B_m$, $b \in B_n$, $m < n$, then $a \preceq b$; *(ii)* for each k, every non-empty subset of B_k has a greatest element. An ordered t.f.l.c.s.-net always exists.

Now assume instead $p > 0$. The situation is similar, but we must also consider the p–power operation in the group G and in the restricted Lie algebra $\mathfrak{k}_G$. Starting from the lower central series $\big\{G_{(k)}\big\}_{k\in\mathbb{N}_+}$, define $G_{[n]} := \prod_{kp^\ell \geq n} \big(G_{(k)}\big)^{p^\ell}$ for all $n \in \mathbb{N}_+$ (hereafter, for any group $\varGamma$ we denote $\varGamma^{p^e}$ the subgroup generated by $\big\{\gamma^{p^e} \,\big|\, \gamma \in \varGamma\,\big\}$): this gives another strongly central series $\big\{G_{[n]}\big\}_{n\in\mathbb{N}_+}$ in G, with the additional property that $\big(G_{[n]}\big)^p \leq G_{[n+1]}$ for all n, called *the p–lower central series of G*. Then $\mathcal{L}_p(G) := \oplus_{n\in\mathbb{N}_+} G_{[n]} \big/ G_{[n+1]}$ is a graded restricted Lie algebra over $\mathbb{Z}_p := \mathbb{Z} \big/ p\,\mathbb{Z}$, with operations $\overline{g} + \overline{\ell} := \overline{g \cdot \ell}$, $\big[\overline{g}, \overline{\ell}\,\big] := \overline{(g,\ell\,)}$, $\overline{g}^{\,[p]} := \overline{g^p}$, for all $g, \ell \in G$ (cf. [HB], Ch. VIII, §9). Like before, we consider the map $\Bbbk \otimes_{\mathbb{Z}_p} \mathcal{L}_p(G) \longrightarrow \mathfrak{k}_G$, $\overline{g} \mapsto \overline{\eta_g}$, which now is an epimorphism *of graded restricted Lie $\mathbb{Z}_p$–algebras*, whose image spans $\mathfrak{k}_G$ over $\Bbbk$: therefore $\mathfrak{k}_G$ *is a quotient of* $\Bbbk \otimes_{\mathbb{Z}_p} \mathcal{L}_p(G)$; in fact, the above is an isomorphism, see below. Finally, we introduce also the notation $d(g) := n$ for all $g \in G_{[n]} \setminus G_{[n+1]}$.

For each $k \in \mathbb{N}_+$ choose a $\mathbb{Z}_p$–basis $\overline{B}_k$ of the $\mathbb{Z}_p$–vector space $G_{[k]} \big/ G_{[k+1]}$; for each $\overline{b} \in \overline{B}_k$, fix $b \in G_{[k]}$ such that $\overline{b} = b\, G_{[k+1]}$, and let B_k be the set of all such elements b. Let $B := \bigcup_{k\in\mathbb{N}_+} B_k$: such a set will be called a *p-l.c.s.-net* (= "p-lower-central-series-net"; the terminology in [HB] is "κ-net") on G. Of course $B_k = \big(B \cap G_{[k]}\big) \setminus \big(B \cap G_{[k+1]}\big)$ for all k. By an *ordered p-l.c.s.-net* we mean a p-l.c.s.-net B which is totally ordered in such a way that: *(i)* if $a \in B_m$, $b \in B_n$, $m < n$, then $a \preceq b$; *(ii)* for each k, every non-empty subset of B_k has a greatest element (like for $p = 0$). Again, p-l.c.s.-nets do exist.

We can now describe each $\mathcal{D}_n(G)$, hence also each graded summand $J^n \big/ J^{n+1}$ of $\widehat{\Bbbk[G]}$, in terms of the lower central series or the p–lower central series of G, more precisely in terms of a fixed ordered t.f.l.c.s.-net or p-l.c.s.-net. To unify notations, set $G_n := G_{(n)}$, $\theta(g) := \partial(g)$ if $p = 0$, and $G_n := G_{[n]}$, $\theta(g) := d(g)$ if $p > 0$, set $G_\infty := \bigcap_{n\in N_+} G_n$, let $B := \bigcup_{k\in\mathbb{N}_+} B_k$ be an ordered t.f.l.c.s.-net or p-l.c.s.-net according to whether $p = 0$ or $p > 0$, and set $\ell(0) := +\infty$ and $\ell(p) := p$ for $p > 0$. The key result we need is

Jennings-Hall theorem *(cf. [HB], [Pa] and references therein). Let $p := Char(\Bbbk)$.*

(a) For all $g \in G$, $\eta_g \in J^n \Longleftrightarrow g \in G_n$. Therefore $\mathcal{D}_n(G) = G_n$ for all $n \in \mathbb{N}_+$.

(b) For any $n \in \mathbb{N}_+$, the set of ordered monomials

$$\mathbb{B}_n := \Big\{ \overline{\eta_{b_1}}^{\,e_1} \cdots \overline{\eta_{b_r}}^{\,e_r} \;\Big|\; b_i \in B_{d_i}\,,\ e_i \in \mathbb{N}_+\,,\ e_i < \ell(p)\,,\ b_1 \precneqq \cdots \precneqq b_r\,,\ \textstyle\sum_{i=1}^{r} e_i\, d_i = n \Big\}$$

is a $\Bbbk$–basis of $J^n \big/ J^{n+1}$, and $\mathbb{B} := \{1\} \cup \bigcup_{n \in \mathbb{N}} \mathbb{B}_n$ is a $\Bbbk$–basis of $\widehat{\Bbbk[G]}$.

(c) $\big\{ \overline{\eta_b} \;\big|\; b \in B_n \big\}$ is a $\Bbbk$–basis of the n–th graded summand $\mathfrak{k}_G \cap \big(J^n \big/ J^{n+1}\big)$ of the graded restricted Lie algebra $\mathfrak{k}_G$, and $\big\{ \overline{\eta_b} \;\big|\; b \in B \big\}$ is a $\Bbbk$–basis of $\mathfrak{k}_G$.

(d) $\big\{ \overline{\eta_b} \;\big|\; b \in B_1 \big\}$ is a minimal set of generators of the (restricted) Lie algebra $\mathfrak{k}_G$.

(e) The map $\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_p(G) \longrightarrow \mathfrak{k}_G$, $\overline{g} \mapsto \overline{\eta_g}$, is an isomorphism of graded restricted Lie algebras. Therefore $\widehat{\Bbbk[G]} \cong \mathcal{U}\big(\Bbbk \otimes_{\mathbb{Z}} \mathcal{L}_p(G)\big)$ as Hopf algebras (notation of §1.1).

(f) $J^\infty = Span\big(\{ \eta_g \,|\, g \in G_\infty \}\big)$, whence $\Bbbk[G]^\vee \cong \bigoplus_{\overline{g} \in G/G_\infty} \Bbbk \cdot \overline{g} \cong \Bbbk\big[G/G_\infty\big]$. $\square$

Recall that $A\big[x, x^{-1}\big]$ (for any A) has A–basis $\big\{ (x-1)^n x^{-[n/2]} \,\big|\, n \in \mathbb{N} \big\}$, where $[q]$ is the integer part of $q \in \mathbb{Q}$. Then from Jennings-Hall theorem and (5.2) we argue

Proposition B. *Let $\chi_g := \hbar^{-\theta(g)} \eta_g$, for all $g \in \{G\} \setminus \{1\}$. Then*

$$R[G]^\vee = \Big(\bigoplus_{\substack{b_i \in B,\ 0 < e_i < \ell(p) \\ r \in \mathbb{N},\ b_1 \precneqq \cdots \precneqq b_r}} R \cdot \chi_{b_1}^{e_1}\, b_1^{-[e_1/2]} \cdots \chi_{b_r}^{e_r}\, b_r^{-[e_r/2]} \Big) \bigoplus R\big[\hbar^{-1}\big] \cdot J^\infty =$$

$$= \Big(\bigoplus_{\substack{b_i \in B,\ 0 < e_i < \ell(p) \\ r \in \mathbb{N},\ b_1 \precneqq \cdots \precneqq b_r}} R \cdot \chi_{b_1}^{e_1}\, b_1^{-[e_1/2]} \cdots \chi_{b_r}^{e_r}\, b_r^{-[e_r/2]} \Big) \bigoplus \Big(\textstyle\sum_{\gamma \in G_\infty} R\big[\hbar^{-1}\big] \cdot \eta_\gamma \Big);$$

If $J^\infty = J^n$ for some $n \in \mathbb{N}$ (iff $G_\infty = G_n$) we can drop the factors $b_1^{-[e_1/2]}, \dots, b_r^{-[e_r/2]}$. $\square$

Poisson groups from $\Bbbk[G]$. The previous discussion attached to the abstract group G the (maybe restricted) Lie algebra $\mathfrak{k}_G$ which, by Jennings-Hall theorem, is just the scalar extension of the Lie ring $\mathcal{L}_{Char(\Bbbk)}$ associated to G via the central series of the G_n's; in particular the functor $G \mapsto \mathfrak{k}_G$ is one considered since long in group theory.

Now, by Theorem 5.8*(d)* we know that $\big(R[G]^\vee\big)'$ is a QFA, with $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[\varGamma_G\big]$ for some connected Poisson group $\varGamma_G$. This defines a functor $G \mapsto \varGamma_G$ from abstract groups to connected Poisson groups, of dimension zero and height 1 if $p > 0$; in particular, this *$\varGamma_G$ is a new invariant for abstract groups.*

The description of $R[G]^\vee$ in Proposition B above leads us to an explicit description of $\big(R[G]^\vee\big)'$, hence of $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[\varGamma_G\big]$ and $\varGamma_G$ too. Indeed, direct inspection gives $\delta_n\big(\chi_g\big) = \hbar^{(n-1)\theta(g)} \chi_g^{\otimes n}$, so $\psi_g := \hbar\, \chi_g = \hbar^{1-\theta(g)} \eta_g \in \big(R[G]^\vee\big)' \setminus \hbar\, \big(R[G]^\vee\big)'$ for each $g \in G \setminus G_\infty$, whilst for $\gamma \in G_\infty$ we have $\eta_\gamma \in J^\infty$ which implies also $\eta_\gamma \in \big(R[G]^\vee\big)'$, and even $\eta_\gamma \in \bigcap_{n \in \mathbb{N}} \hbar^n \big(R[G]^\vee\big)'$. Therefore $\big(R[G]^\vee\big)'$ is generated by $\big\{ \psi_g \;\big|\; g \in G \setminus \{1\} \big\} \cup \big\{ \eta_\gamma \,\big|\, \gamma \in G_\infty \big\}$. Moreover, $g = 1 + \hbar^{\theta(g)-1} \psi_g \in \big(R[G]^\vee\big)'$ for every $g \in G \setminus G_\infty$, and $\gamma = 1 + (\gamma - 1) \in 1 + J^\infty \subseteq \big(R[G]^\vee\big)'$ for $\gamma \in G_\infty$. This and the previous analysis along with Proposition B prove next result, which in turn is the basis for Theorem D below.

<u>*Proposition C.*</u>

$$\big(R[G]^\vee\big)' \;=\; \Big(\bigoplus\nolimits_{\substack{b_i \in B,\; 0<e_i<\ell(p) \\ r\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_r}} R\cdot \psi_{b_1}^{\,e_1}\, b_1^{\,-[e_1/2]} \cdots \psi_{b_r}^{\,e_r}\, b_r^{\,-[e_r/2]}\Big) \bigoplus R\big[\hbar^{-1}\big]\cdot J^\infty \;=$$

$$=\; \Big(\bigoplus\nolimits_{\substack{b_i \in B,\; 0<e_i<\ell(p) \\ r\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_r}} R\cdot \psi_{b_1}^{\,e_1}\, b_1^{\,-[e_1/2]} \cdots \psi_{b_r}^{\,e_r}\, b_r^{\,-[e_r/2]}\Big) \bigoplus \Big(\sum\nolimits_{\gamma\in G_\infty} R\big[\hbar^{-1}\big]\cdot \eta_\gamma\Big)\;.$$

In particular, $\big(R[G]^\vee\big)' = R[G]$ *if and only if* $G_2 = \{1\} = G_\infty$ *. If in addition* $J^\infty = J^n$ *for some* $n\in\mathbb{N}$ *(iff* $G_\infty = G_n$*) then we can drop the factors* $b_1^{\,-[e_1/2]}, \dots, b_r^{\,-[e_r/2]}$ *.* □

<u>Theorem D</u>. *Let* $x_g := \psi_g \mod \hbar\,\big(R[G]^\vee\big)'$*,* $z_g := g \mod \hbar\,\big(R[G]^\vee\big)'$ *for all* $g \neq 1$*, and* $B_1 := \big\{\, b\in B \,\big|\, \theta(b) = 1 \big\}$*,* $B_> := \big\{\, b\in B \,\big|\, \theta(b) > 1 \big\}$*.*

(a) If $p = 0$*, then* $F\big[\varGamma_G\big] = \big(R[G]^\vee\big)'\Big|_{\hbar=0}$ *is a polynomial/Laurent polynomial algebra, namely* $F\big[\varGamma_G\big] = \Bbbk\Big[\big\{z_b^{\pm 1}\big\}_{b\in B_1} \cup \{x_b\}_{b\in B_>}\Big]$*, the* z_b*'s being group-like and the* x_b*'s being primitive. In particular* $\varGamma_G \cong \big(\mathbb{G}_m^{\times B_1}\big)\times\big(\mathbb{G}_a^{\times B_>}\big)$ *as algebraic groups, that is* $\varGamma_G$ *is a torus times a (pro)affine space.*

(b) If $p > 0$*, then* $F\big[\varGamma_G\big] = \big(R[G]^\vee\big)'\Big|_{\hbar=0}$ *is a truncated polynomial/Laurent polynomial algebra, namely* $F\big[\varGamma_G\big] = \Bbbk\Big[\big\{z_b^{\pm 1}\big\}_{b\in B_1} \cup \{x_b\}_{b\in B_>}\Big] \Big/ \Big(\big\{z_b^{\,p} - 1\big\}_{b\in B_1} \cup \big\{x_b^{\,p}\big\}_{b\in B_>}\Big)$*, the* z_b*'s being group-like and the* x_b*'s being primitive. In particular* $\varGamma_G \cong \big({\boldsymbol\mu_p}^{\times B_1}\big)\times\big({\boldsymbol\alpha_p}^{\times B_>}\big)$ *as algebraic groups of dimension zero and height 1.*

(c) The Poisson group $\varGamma_G$ *has cotangent Lie bialgebra* $\mathfrak{k}_G$*, that is* $\mathrm{coLie}\,(\varGamma_G) = \mathfrak{k}_G$*.*

Proof. (a) Definitions give $\partial(g\,\ell\,) \geq \partial(g) + \partial(\ell\,)$ for all $g, \ell \in G$, so that $[\psi_g, \psi_\ell] = \hbar^{1-\partial(g)-\partial(\ell)+\partial((g,\ell))}\, \psi_{(g,\ell)}\, g\,\ell \in \hbar\cdot\big(R[G]^\vee\big)'$, which proves (directly) that $\big(R[G]^\vee\big)'\Big|_{\hbar=0}$ is commutative! Moreover, the relation $1 = g^{-1}\,g = g^{-1}\big(1 + \hbar^{\partial(g)-1}\psi_g\big)$ (for any $g\in G$) yields $z_{g^{-1}} = z_g^{\,-1}$ iff $\partial(g) = 1$ and $z_{g^{-1}} = 1$ iff $\partial(g) > 1$. Noting also that $J^\infty \equiv 0 \mod \hbar\big(R[G]^\vee\big)'$ and $g = 1 + \hbar^{\partial(g)-1}\psi_g \equiv 1 \mod \hbar\big(R[G]^\vee\big)'$ for $g\in G\setminus G_\infty$, and also $\gamma = 1 + (\gamma - 1) \in 1 + J^\infty \equiv 1 \mod \hbar\big(R[G]^\vee\big)'$ for $\gamma\in G_\infty$, Proposition C gives

$$F\big[\varGamma_G\big] \;=\; \big(R[G]^\vee\big)'\Big|_{\hbar=0} = \Big(\bigoplus\nolimits_{\substack{b_i\in B_1,\; a_i\in\mathbb{Z} \\ s\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_s}} \Bbbk\cdot z_{b_1}^{a_1}\cdots z_{b_s}^{a_s}\Big) \bigotimes \Big(\bigoplus\nolimits_{\substack{b_i\in B_>,\; e_i\in\mathbb{N}_+ \\ r\in\mathbb{N},\; b_1 \precneqq \cdots \precneqq b_r}} \Bbbk\cdot x_{b_1}^{e_1}\cdots x_{b_r}^{e_r}\Big)$$

which means that $F\big[\varGamma_G\big]$ is a polynomial-Laurent polynomial algebra as claimed. Again definitions imply $\Delta(z_g) = z_g \otimes z_g$ for all $g\in G$ and $\Delta(x_g) = x_g \otimes 1 + 1 \otimes x_g$ iff $\partial(g) > 1$; thus the z_b's are group-like and the x_b's are primitive as claimed.

(b) The definition of d implies $d(g\,\ell\,) \geq d(g) + d(\ell\,)$ $(g, \ell\in G)$, whence we get $[\psi_g, \psi_\ell] = \hbar^{1-d(g)-d(\ell)+d((g,\ell))}\, \psi_{(g,\ell)}\, g\,\ell \in \hbar\cdot\big(R[G]^\vee\big)'$, proving that $\big(R[G]^\vee\big)'\Big|_{\hbar=0}$ is commutative. In addition $d(g^p) \geq p\, d(g)$, so $\psi_g^{\,p} = \hbar^{p\,(1-d(g))}\,\eta_g^{\,p} = \hbar^{p-1+d(g^p)-p\,d(g)}\,\psi_{g^p} \in \hbar\cdot\big(R[G]^\vee\big)'$,

whence $\big({\psi_g^{\,p}}\big|_{\hbar=0}\big)^p = 0$ inside $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[\varGamma_G\big]$, which proves that $\varGamma_G$ has dimension 0 and height 1. Finally $b^p = (1+\psi_b)^p = 1+{\psi_b}^p \equiv 1 \mod \hbar\,\big(R[G]^\vee\big)'$ for all $b \in B_1$, so $b^{-1} \equiv b^{p-1} \mod \hbar\,\big(R[G]^\vee\big)'$. Thus letting $x_g := \psi_g \mod \hbar\,\big(R[G]^\vee\big)'$ (for $g \neq 1$) we get

$$F\big[\varGamma_G\big] = \big(R[G]^\vee\big)'\Big|_{\hbar=0} = \Big(\bigoplus_{\substack{b_i \in B_1,\, 0<e_i<p \\ s \in \mathbb{N},\, b_1 \precneqq \cdots \precneqq b_s}} \Bbbk \cdot z_{b_1}^{e_1} \cdots z_{b_s}^{e_s}\Big) \bigotimes \Big(\bigoplus_{\substack{b_i \in B_>,\, 0<e_i<p \\ r \in \mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} \Bbbk \cdot x_{b_1}^{e_1} \cdots x_{b_r}^{e_r}\Big)$$

just like for *(a)* and also taking care that $z_b = x_b + 1$ and $z_b^{\,p} = 1$ for $b \in B_1$. Therefore $\big(R[G]^\vee\big)'\Big|_{\hbar=0}$ is a truncated polynomial/Laurent polynomial algebra as claimed. The properties of the x_b's and the z_b's w.r.t. the Hopf structure are then proved like for *(a)* again.

(c) The augmentation ideal $\mathfrak{m}_e$ of $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[\varGamma_G\big]$ is generated by $\{x_b\}_{b \in B}$; then $\hbar^{-1}\,[\psi_g, \psi_\ell] = \hbar^{\,\theta((g,\ell))-\theta(g)-\theta(\ell)}\,\psi_{(g,\ell)}\,\big(1 + \hbar^{\,\theta(g)-1}\psi_g\big)\,\big(1+\hbar^{\,\theta(\ell)-1}\psi_\ell\big)$ by the previous computation, whence at $\hbar = 0$ one has $\big\{x_g\,, x_\ell\big\} \equiv x_{(g,\ell)} \mod \mathfrak{m}_e^{\,2}$ if $\theta\big((g,\ell)\big) = \theta(g)+\theta(\ell)$, and $\big\{x_g\,, x_\ell\big\} \equiv 0 \mod \mathfrak{m}_e^{\,2}$ if $\theta\big((g,\ell)\big) > \theta(g)+\theta(\ell)$. This means that the cotangent Lie bialgebra $\mathfrak{m}_e \Big/ \mathfrak{m}_e^{\,2}$ of $\varGamma_G$ is isomorphic to $\mathfrak{k}_G$, as claimed. $\square$

Remarks: *(a)* Theorem D claims that the connected Poisson group $K_G^\star := \varGamma_G$ is *dual* to $\mathfrak{k}_G$ in the sense of §1.1. Since $R[G]^\vee\Big|_{\hbar=0} = \mathcal{U}(\mathfrak{k}_G)$ and $\big(R[G]^\vee\big)'\Big|_{\hbar=0} = F\big[K_G^\star\big]$, *this gives a close analogue, in positive characteristic, of the second half of Theorem 2.2(c).*

(b) Theorem D gives functorial recipes to attach to each abstract group G and each field $\Bbbk$ a connected Abelian algebraic Poisson group over $\Bbbk$, namely $G \mapsto \varGamma_G = K_G^\star$, explicitly described as algebraic group and such that $coLie\,(K_G^\star) = \mathfrak{k}_G$. *Every such $\varGamma_G$ (for given $\Bbbk$) is then an invariant of G, a new one to the author's knowledge. Indeed, it is perfectly equivalent to the well-known invariant* $\mathfrak{k}_G$ (over the same $\Bbbk$), because clearly $\varGamma_{G_1} \cong \varGamma_{G_2}$ implies $\mathfrak{k}_{G_1} \cong \mathfrak{k}_{G_2}$, whereas $\mathfrak{k}_{G_1} \cong \mathfrak{k}_{G_2}$ implies that $\varGamma_{G_1}$ and $\varGamma_{G_2}$ are isomorphic as algebraic groups — by Theorem D *(a–b)* — and bear isomorphic Poisson structures — by part *(c)* of Theorem D — whence $\varGamma_{G_1} \cong \varGamma_{G_2}$ as algebraic Poisson groups.

The case of $A_\Bbbk(G)$. Let's now dwell upon $H := A_\Bbbk(G)$, for a *finite* group G.

Let $\mathbb{A}$ be any commutative unital ring, and let $\Bbbk$, $R := \Bbbk[\hbar]$ be as before. By definition $A_{\mathbb{A}}(G) = \mathbb{A}[G]^*$, hence $\mathbb{A}[G] = A_{\mathbb{A}}(G)^*$, and we have a natural perfect Hopf pairing $A_{\mathbb{A}}(G) \times \mathbb{A}[G] \longrightarrow \mathbb{A}$. Our first result is one of triviality:

Theorem E. $A_R(G)^\vee = R \cdot 1 \oplus R\big[\hbar^{-1}\big]\,J = \big(A_R(G)^\vee\big)'$, $A_\Bbbk(G)^\vee = \Bbbk \cdot 1$, $\widehat{A_\Bbbk(G)} = A_R(G)^\vee\Big|_{\hbar=0} = \Bbbk \cdot 1 = \mathcal{U}(\mathbf{0})$ *and* $\big(A_R(G)^\vee\big)'\Big|_{\hbar=0} = \Bbbk \cdot 1 = F\big[\{*\}\big]$.

Proof. By construction $J := Ker\,(\epsilon_{A_\Bbbk(G)})$ has $\Bbbk$–basis $\big\{\varphi_g\big\}_{g \in G \setminus \{1_G\}} \cup \big\{\varphi_{1_G} - 1_{A_\Bbbk(G)}\big\}$, and since $\varphi_g = {\varphi_g}^2$ for all g and $(\varphi_{1_G} - 1)^2 = -(\varphi_{1_G} - 1)$ we have $J = J^\infty$, so $A_\Bbbk(G)^\vee = \Bbbk{\cdot}1$ and $\widehat{A_\Bbbk(G)} = \Bbbk{\cdot}1$. Similarly, $A_R(G)^\vee$ is generated by $\big\{\hbar^{-1}\varphi_g\big\}_{g \in G \setminus \{1_G\}} \cup \big\{\hbar^{-1}(\varphi_{1_G} -$

$1_{A_R(G)})\big\}$; moreover, $J = J^\infty$ implies $\hbar^{-n} J \subseteq A_R(G)^\vee$ for all n, whence $A_R(G)^\vee = R\,1 \oplus R[\hbar^{-1}]J$. Then $J_{A_R(G)^\vee} = R\big[\hbar^{-1}\big]J \subseteq \hbar\, A_R(G)^\vee$, which implies $\big(A_R(G)^\vee\big)' = A_R(G)^\vee$: in particular, $\big(A_R(G)^\vee\big)'\Big|_{\hbar=0} = A_R(G)^\vee\Big|_{\hbar=0} = \Bbbk \cdot 1$, as claimed. $\square$

Poisson groups from $A_\Bbbk(G)$. Now we look at $A_R(G)'$, $A_\Bbbk(G)'$ and $\widetilde{A_\Bbbk(G)}$. By construction $A_R(G)$ and $R[G]$ are in perfect Hopf pairing, and are free R–modules of *finite rank*. In this case, since Proposition 4.4 yields $A_R(G)' = \big(R[G]^\vee\big)^\bullet$ we have in fact $A_R(G)' = \big(R[G]^\vee\big)^\bullet = \big(R[G]^\vee\big)^*$: thus $A_R(G)'$ is the *dual* Hopf algebra to $R[G]^\vee$; then from Proposition B we can argue an explicit description of $A_R(G)'$, whence also of $\big(A_R(G)'\big)^\vee$. By Theorem 5.10*(g)* and its proof, namely that $A_\Bbbk(G)' = \big(J^\infty_{\Bbbk[G]}\big)^\perp$, there is a perfect filtered Hopf pairing $\Bbbk[G]^\vee \times A_\Bbbk(G)' \longrightarrow \Bbbk$ and a perfect graded Hopf pairing $\widehat{\Bbbk[G]} \times \widetilde{A_\Bbbk(G)} \longrightarrow \Bbbk$: thus $A_\Bbbk(G)' \cong \big(\Bbbk[G]^\vee\big)^*$ as filtered Hopf algebras and $\widetilde{A_\Bbbk(G)} \cong \big(\widehat{\Bbbk[G]}\big)^*$ as graded Hopf algebras. If $p = 0$ then $J = J^\infty$, as each $g \in G$ has finite order and $g^n = 1$ implies $g \in G_\infty$: then $\Bbbk[G]^\vee = \Bbbk \cdot 1 = \widehat{\Bbbk[G]}$, so $A_\Bbbk(G)' = \Bbbk \cdot 1 = \widetilde{A_\Bbbk(G)}$. If $p > 0$ instead, this analysis gives $\widetilde{A_\Bbbk(G)} = \big(\widehat{\Bbbk[G]}\big)^* = \big(\mathbf{u}(\mathfrak{k}_G)\big)^* = F[K_G]$, where K_G is a connected Poisson group of dimension 0, height 1 and tangent Lie bialgebra $\mathfrak{k}_G$. Thus

Theorem F.

(a) There is a second functorial recipe to attach to each finite abstract group a connected algebraic Poisson group of dimension zero and height 1 over any field $\Bbbk$ with $Char(\Bbbk) > 0$, namely $G \mapsto K_G := Spec\Big(\widetilde{A_\Bbbk(G)}\Big)$. This K_G is Poisson dual to $\varGamma_G$ of Theorem D in the sense of §1.1, in that $Lie(K_G) = \mathfrak{k}_G = coLie(\varGamma_G)$.

(b) If $p := Char(\Bbbk) > 0$, then $\big(A_R(G)'\big)^\vee\Big|_{\hbar=0} = \mathbf{u}\big(\mathfrak{k}_G^\times\big) = S\big(\mathfrak{k}_G^\times\big)\Big/\big(\big\{x^p \,\big|\, x \in \mathfrak{k}_G^\times\big\}\big)$.

Proof. Claim *(a)* is the outcome of the discussion above. Part *(b)* instead requires an explicit description of $\big(A_R(G)'\big)^\vee$. Since $A_R(G)' \cong \big(R[G]^\vee\big)^*$, from Proposition B we get $A_R(G)' = \Big(\bigoplus_{\substack{b_i \in B,\, 0<e_i<p \\ r \in \mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot \rho^{e_1,\dots,e_r}_{b_1,\dots,b_r}\Big)$ where each $\rho^{e_1,\dots,e_r}_{b_1,\dots,b_r}$ is defined by

$$\Big\langle\, \rho^{e_1,\dots,e_r}_{b_1,\dots,b_r} \,,\, \chi_{\beta_1}^{\varepsilon_1}\, \beta_1^{-[\varepsilon_1/2]} \cdots \chi_{\beta_s}^{\varepsilon_s}\, \beta_s^{-[\varepsilon_s/2]} \,\Big\rangle \;=\; \delta_{r,s} \prod\nolimits_{i=1}^{r} \delta_{b_i,\beta_i} \delta_{e_i,\varepsilon_i}$$

(for all $b_i, \beta_j \in B$ and $0 < e_i, \varepsilon_j < p$). Now, using notation of §1.3, $K_\infty \subseteq K'$ for any $K \in \mathcal{HA}$, whence $K' = \pi^{-1}\big(\overline{K}\,'\big)$ where $\pi \colon K \twoheadrightarrow K\big/K_\infty =: \overline{K}$ is the canonical projection. So let $K := R[G]^\vee$, $H := A_R(G)'$; Proposition B gives $K_\infty = R\big[\hbar^{-1}\big] \cdot J^\infty$ and provides at once a description of $\overline{K}$; from this and the previous description of H one sees also that in the present case K_∞ is exactly the right kernel of the natural pairing $H \times K \longrightarrow R$, which is perfect on the left, so that the induced pairing $H \times \overline{K} \longrightarrow R$ is perfect. By construction its specialization at $\hbar = 0$ is the natural pairing $F[K_G] \times \mathbf{u}(\mathfrak{k}_G) \longrightarrow \Bbbk$, which is perfect too. Then we can apply Proposition 4.4*(c)* (with $\overline{K}$ playing the rôle of K therein) which yields $\overline{K}\,' = \big(H^\vee\big)^\bullet = \Big(\big(A_R(G)'\big)^\vee\Big)^\bullet$. By construction, $\overline{K}\,' = \big(R[G]^\vee\big)'\Big/\big(R\big[\hbar^{-1}\big] \cdot J^\infty\big)$, and Proposition C describes the latter as $\overline{K}\,' = \Big(\bigoplus_{\substack{b_i \in B,\, 0 < e_i < p \\ r \in \mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot {\overline{\psi}_{b_1}}^{e_1} \cdots {\overline{\psi}_{b_r}}^{e_r} \Big)$, where $\overline{\psi}_{b_i} := \psi_{b_i} \mod R\big[\hbar^{-1}\big] \cdot J^\infty$ for all i; since $\overline{K}\,' = \Big(\big(A_R(G)' \big)^\vee \Big)^\bullet$ and $\psi_g = \hbar^{+1} \chi_g$, this analysis yields $\big(A_R(G)' \big)^\vee = \Big(\bigoplus_{\substack{b_i \in B,\, 0 < e_i < p \\ r \in \mathbb{N},\, b_1 \precneqq \cdots \precneqq b_r}} R \cdot \hbar^{-\sum_i e_i} \rho_{b_1,\dots,b_r}^{e_1,\dots,e_r} \Big) \cong \big(\overline{K}\,' \big)^*$, whence we get $\big(A_R(G)' \big)^\vee \Big|_{\hbar=0} \cong \big(\overline{K}\,' \big)^* \Big|_{\hbar=0} = \big(K' \big|_{\hbar=0} \big)^* = \Big(\big(R[G]^\vee \big)' \big|_{\hbar=0} \Big)^* \cong F\big[\varGamma_G\big]^* = \mathbf{u}\big(\mathfrak{k}_G^\times\big) = S\big(\mathfrak{k}_G^\times\big) \Big/ \big(\big\{ x^p \,\big|\, x \in \mathfrak{k}_G^\times \big\} \big)$ as claimed, the latter identity being trivial (as $\mathfrak{k}_G^\times$ is Abelian). $\square$

Remarks: (a) this K_G is another invariant for G, but again equivalent to $\mathfrak{k}_G$.

(b) Theorem F(b) is a positive characteristic analogue for $F_\hbar[G] = A_R(G)'$ of the first half of Theorem 2.2(c).

Examples:

(1) Finite Abelian p–groups. Let p be a prime number and $G := \mathbb{Z}_{p^{e_1}} \times \mathbb{Z}_{p^{e_2}} \times \cdots \times \mathbb{Z}_{p^{e_k}}$ $(k, e_1, \dots, e_k \in \mathbb{N})$, with $e_1 \geq e_2 \geq \cdots \geq e_k$. Let $\Bbbk$ be a field with $\mathit{Char}(\Bbbk) = p > 0$, and $R := \Bbbk[\hbar]$ as above, so that $\Bbbk[G]_\hbar = R[G]$.

First, $\mathfrak{k}_G$ is Abelian, because G is. Let g_i be a generator of $\mathbb{Z}_{p^{e_i}}$ (for all i), identified with its image in G. Since G is Abelian we have $G_{[n]} = G^{p^n}$ (for all n), and an ordered p-l.c.s.-net is $B := \bigcup_{r \in \mathbb{N}_+} B_r$ with $B_r := \Big\{ g_1^{p^r}, g_2^{p^r}, \dots, g_{j_r}^{p^r} \Big\}$ where j_r is uniquely defined by $e_{j_r} > r$, $e_{j_r+1} \leq r$. Then $\mathfrak{k}_G$ has $\Bbbk$–basis $\Big\{ \overline{\eta_{g_i^{p^{s_i}}}} \Big\}_{1 \leq i \leq k;\; 0 \leq s_i < e_i}$, and minimal set of generators (as a restricted Lie algebra) $\big\{ \overline{\eta_{g_1}}, \overline{\eta_{g_2}}, \dots, \overline{\eta_{g_k}} \big\}$, for the p–operation of $\mathfrak{k}_G$ is $\Big(\overline{\eta_{g_i^{p^s}}} \Big)^{[p]} = \overline{\eta_{g_i^{p^{s+1}}}}$, and the order of nilpotency of each $\overline{\eta_{g_i}}$ is exactly p^{e_i}, i.e. the order of g_i. In addition $J^\infty = \{0\}$ so $\Bbbk[G]^\vee = \Bbbk[G]$. The outcome is $\Bbbk[G]^\vee = \Bbbk[G]$ and

$$\widehat{\Bbbk[G]} \;=\; \mathbf{u}(\mathfrak{k}_G) \;=\; U(\mathfrak{k}_G) \Big/ \bigg(\Big\{ \big(\overline{\eta_{g_i^{p^s}}}\big)^p - \overline{\eta_{g_i^{p^{s+1}}}} \Big\}_{1 \leq i \leq k}^{0 \leq s < e_i} \bigcup \Big\{ \big(\overline{\eta_{g_i^{p^{e_i-1}}}}\big)^p \Big\}_{1 \leq i \leq k} \bigg)$$

whence $\widehat{\Bbbk[G]} \cong \Bbbk[x_1, \dots, x_k] \Big/ \Big(\Big\{ x_i^{p^{e_i}} \,\Big|\, 1 \leq i \leq k \Big\} \Big)$, via $\overline{\eta_{g_i^{p^s}}} \mapsto x_i^{p^s}$ (for all i, s).

As for $\Bbbk[G]_\hbar^\vee$, for all $r < e_i$ we have $d\big(g_i^{p^r}\big) = p^r$ and so $\chi_{g_i^{p^r}} = \hbar^{-p^r} \big(g_i^{p^r} - 1 \big)$ and $\psi_{g_i^{p^r}} = \hbar^{1-p^r} \big(g_i^{p^r} - 1 \big)$; since $G_{[\infty]} = \{1\}$ (or, equivalently, $J^\infty = \{0\}$) and everything is Abelian, from the general theory we conlude that both $\Bbbk[G]_\hbar^\vee$ and $\Big(\Bbbk[G]_\hbar^\vee \Big)'$ are truncated-polynomial algebras, in the $\chi_{g_i^{p^r}}$'s and in the $\psi_{g_i^{p^r}}$'s respectively, namely

$$\Bbbk[G]_\hbar^\vee \;=\; \Bbbk[\hbar]\Big[\big\{ \chi_{g_i^{p^s}} \big\}_{1 \leq i \leq k\,;\; 0 \leq s < e_i} \Big] \;\cong\; \Bbbk[\hbar]\big[y_1, \dots, y_k \big] \Big/ \Big(\Big\{ y_i^{p^{e_i}} \,\Big|\, 1 \leq i \leq k \Big\} \Big)$$

$$\Big(\Bbbk[G]_\hbar^\vee \Big)' \;=\; \Bbbk[\hbar]\Big[\big\{ \psi_{g_i^{p^s}} \big\}_{1 \leq i \leq k\,;\; 0 \leq s < e_i} \Big] \;\cong\; \Bbbk[\hbar]\Big[\big\{ z_{i,s} \big\}_{1 \leq i \leq k\,;\; 0 \leq s < e_i} \Big] \Big/ \Big(\Big\{ {z_{i,s}}^p \,\Big|\, 1 \leq i \leq k \Big\} \Big)$$

via the isomorphisms given by $\overline{\chi_{g_i^{p^s}}} \mapsto y_i^{p^s}$ and $\overline{\psi_{g_i^{p^s}}} \mapsto z_{i,s}$ (for all i, s). When $e_1 > 1$

this implies $\big(\Bbbk[G]_\hbar^\vee\big)' \underset{\neq}{\supseteq} \Bbbk[G]_\hbar$, that is a counterexample to Theorem 2.2*(b)*. Setting $\overline{\psi_{g_i^{p^s}}} := \psi_{g_i^{p^s}} \mod \hbar\, \big(\Bbbk[G]_\hbar^\vee\big)'$ (for all $1 \leq i \leq k$, $0 \leq s < e_i$) we have

$$F\big[\varGamma_G\big] \;=\; \big(\Bbbk[G]_\hbar^\vee\big)'\Big|_{\hbar=0} \;=\; \Bbbk\Big[\big\{\overline{\psi_{g_i^{p^s}}}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big] \;\cong\; \Bbbk\,\Big[\big\{\,w_{i,s}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big]\Big/\Big(\Big\{\,w_{i,s}^{\,p}\,\Big|\;1\leq i\leq k\,\Big\}\Big)$$

(via $\overline{\psi_{g_i^{p^s}}} \mapsto w_{i,s}$) as a $\Bbbk$–algebra. The Poisson bracket trivial, and the $w_{i,s}$'s are primitive for $s > 1$ and $\Delta(w_{i,1}) = w_{i,1} \otimes 1 + 1 \otimes w_{i,1} + w_{i,1} \otimes w_{i,1}$ for all $1 \leq i \leq k$. If instead $e_1 = \cdots = e_k = 1$, then $\big(\Bbbk[G]_\hbar^\vee\big)' = \Bbbk[G]_\hbar$. This is an analogue of Theorem 2.2*(b)*, although now $Char(\Bbbk) > 0$, in that in this case $\Bbbk[G]_\hbar$ is a QFA, with $\Bbbk[G]_\hbar\Big|_{\hbar=0} = \Bbbk[G] = F\big[\widehat{G}\,\big]$ where $\widehat{G}$ is the *group of characters* of G. But then $F\big[\widehat{G}\,\big] = \Bbbk[G] = \Bbbk[G]_\hbar\Big|_{\hbar=0} = \big(\Bbbk[G]_\hbar^\vee\big)'\Big|_{\hbar=0} = F\big[\varGamma_G\big]$ (by general analysis) which means that $\widehat{G}$ can be realized as a finite, connected, Poisson group-scheme of dimension 0 and height 1 dual to $\mathfrak{k}_G$, i.e. $\varGamma_G = K_G^\star$.

Finally, a direct easy calculation shows that — letting $\chi_g^* := \hbar^{\,d(g)}\,(\varphi_g - \varphi_1) \in A_\Bbbk(G)_\hbar'$ and $\psi_g^* := \hbar^{\,d(g)-1}\,(\varphi_g - \varphi_1) \in \big(A_\Bbbk(G)'\big)_\hbar^\vee$ (for all $g \in G \setminus \{1\}$) — we have also

$$A_\Bbbk(G)_\hbar' \;=\; \Bbbk[\hbar]\Big[\big\{\,\chi_{g_i^{p^s}}^*\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big] \;\cong\; \Bbbk[\hbar]\Big[\big\{Y_{i,j}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big]\Big/\Big(\big\{Y_{i,j}^{\;p}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big)$$

$$\big(A_\Bbbk(G)_\hbar'\big)^\vee \;=\; \Bbbk[\hbar]\Big[\big\{\,\psi_{g_i^{p^s}}^*\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big] \;\cong\; \Bbbk[\hbar]\Big[\big\{Z_{i,s}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big]\Big/\Big(\big\{Z_{i,s}^{\;p} - Z_{i,s}\big\}_{1\leq i\leq k}^{0\leq s<e_i}\Big)$$

via the isomorphisms given by $\chi_{g_i^{p^s}}^* \mapsto Y_{i,s}$ and $\psi_{g_i^{p^s}}^* \mapsto Z_{i,s}$, from which one also gets the analogous descriptions of $A_\Bbbk(G)_\hbar'\Big|_{\hbar=0} = \widetilde{A_\Bbbk(G)} = F[K_G]$ and of $\big(A_\Bbbk(G)_\hbar'\big)^\vee\Big|_{\hbar=0} = \mathbf{u}(\mathfrak{k}_G^\times)$.

(2) A non-Abelian p–group. Let p be a prime number, $\Bbbk$ be a field with $Char(\Bbbk) = p > 0$, and $R := \Bbbk[\hbar]$ as above, so that $\Bbbk[G]_\hbar = R[G]$.

Let $G := \mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$, that is the group with generators ν, τ and relations $\nu^p = 1$, $\tau^{p^2} = 1$, $\nu\,\tau\,\nu^{-1} = \tau^{1+p}$. In this case, $G_{[2]} = \cdots = G_{[p]} = \big\{1, \tau^p\big\}$, $G_{[p+1]} = \{1\}$, so we can take $B_1 = \{\nu\,,\tau\}$ and $B_p = \big\{\tau^p\big\}$ to form an ordered p-l.c.s.-net $B := B_1 \cup B_p$ w.r.t. the ordering $\nu \preceq \tau \preceq \tau^p$. Noting also that $J^\infty = \{0\}$ (for $G_{[\infty]} = \{1\}$), we have $\Bbbk[G]_\hbar^\vee \;=\; \bigoplus_{a,b,c=0}^{p-1} \Bbbk[\hbar] \cdot \chi_\nu^a\, \chi_\tau^b\, \chi_{\tau^p}^c \;=\; \bigoplus_{a,b,c=0}^{p-1} \Bbbk[\hbar]\, \hbar^{-a-b-c\,p} \cdot (\nu-1)^a\,(\tau-1)^b\,\big(\tau^p-1\big)^c$ as $\Bbbk[\hbar]$–modules, since $d(\nu) = 1 = d(\tau)$ and $d\big(\tau^p)\big) = p$, with $\Delta(\chi_g) = \chi_g \otimes 1 + 1 \otimes \chi_g + \hbar^{d(g)}\,\chi_g \otimes \chi_g$ for all $g \in B$. As a direct consequence we have also

$$\textstyle\bigoplus_{a,b,c=0}^{p-1} \Bbbk \cdot \overline{\chi_\nu}^{\;a}\,\overline{\chi_\tau}^{\;b}\,\overline{\chi_{\tau^p}}^{\;c} \;=\; \Bbbk[G]_\hbar^\vee\Big|_{\hbar=0} \;\cong\; \widehat{\Bbbk[G]} \;=\; \bigoplus_{a,b,c=0}^{p-1} \Bbbk \cdot \overline{\eta_\nu}^{\;a}\,\overline{\eta_\tau}^{\;b}\,\overline{\eta_{\tau^p}}^{\;c}\,.$$

The two relations $\nu^p = 1$ and $\tau^{p^2} = 1$ within G yield trivial relations inside $\Bbbk[G]$ and $\Bbbk[G]_\hbar$; instead, the relation $\nu\,\tau\,\nu^{-1} = \tau^{1+p}$ turns into $[\eta_\nu, \eta_\tau] = \eta_{\tau^p} \cdot \tau\,\nu$, which gives $[\chi_\nu, \chi_\tau] = \hbar^{p-2}\,\chi_{\tau^p} \cdot \tau\,\nu$ in $\Bbbk[G]_\hbar^\vee$. Therefore $\big[\,\overline{\chi_\nu}\,,\,\overline{\chi_\tau}\,\big] = \delta_{p,2}\,\overline{\chi_{\tau^p}}$. Since $\big[\,\overline{\chi_\tau}\,,\,\overline{\chi_{\tau^p}}\,\big] = 0 = \big[\,\overline{\chi_\nu}\,,\,\overline{\chi_{\tau^p}}\,\big]$ (because $\nu\,\tau^p\,\nu^{-1} = \big(\tau^{1+p}\big)^p = \tau^{p+p^2} = \tau^p$) and $\big\{\overline{\chi_\nu}\,,\,\overline{\chi_\tau}\,,\,\overline{\chi_{\tau^p}}\,\big\}$ is a $\Bbbk$–basis of $\mathfrak{k}_G = \mathcal{L}_p(G)$, we conclude that the latter has trivial or non-trivial Lie bracket according to whether $p \neq 2$ or $p = 2$. In addition, we have the relations $\chi_\nu^{\,p} = 0$, $\chi_{\tau^p}^{\,p} = 0$ and $\chi_\tau^{\,p} = \chi_{\tau^p}$: these give analogous relations in $\Bbbk[G]_\hbar^\vee\Big|_{\hbar=0}$, which read as formulas for the

p–operation of $\mathfrak{k}_G$, namely $\overline{\chi_\nu}^{\,[p]} = 0$, $\overline{\chi_{\tau^p}}^{\,[p]} = 0$, $\overline{\chi_\tau}^{\,[p]} = \chi_{\tau^p}$.

To sum up, we have a complete presentation for ${\Bbbk[G]_\hbar}^\vee$ by generators and relations, i.e.

$$ {\Bbbk[G]_\hbar}^\vee \;\cong\; \Bbbk[\hbar]\,\langle v_1, v_2, v_3\rangle \Big/ \!\!\left(\begin{matrix} v_1\,v_2 - v_2\,v_1 - \hbar^{p-2}\,v_3\,(1+\hbar\,v_\tau)\,(1+\hbar\,v_\nu) \\ v_1\,v_3 - v_3\,v_1\,, \quad v_1^{\,p}\,, \quad v_2^{\,p} - v_3\,, \quad v_3^{\,p}\,, \quad v_2\,v_3 - v_3\,v_2 \end{matrix} \right) $$

via $\chi_\nu \mapsto v_1$, $\chi_\tau \mapsto v_2$, $\chi_{\tau^p} \mapsto v_3$. Similarly (as a consequence) we have the presentation

$$ \widehat{\Bbbk[G]} \,=\, {\Bbbk[G]_\hbar}^\vee\Big|_{\hbar=0} \;\cong\; \Bbbk\,\langle y_1, y_2, y_3\rangle \Big/ \!\!\left(\begin{matrix} y_1\,y_2 - y_2\,y_1 - \delta_{p,2}\,y_3\,, \qquad y_2^{\,p} - y_3 \\ y_1\,y_3 - y_3\,y_1\,, \quad y_1^{\,p}\,, \quad y_3^{\,p}\,, \quad y_2\,y_3 - y_3\,y_2 \end{matrix} \right) $$

via $\overline{\chi_\nu} \mapsto y_1$, $\overline{\chi_\tau} \mapsto y_2$, $\overline{\chi_{\tau^p}} \mapsto y_3$, with p–operation as above and the y_i's being primitive

<u>Remark</u>: if $p \neq 2$ exactly the same result holds for $G = \mathbb{Z}_p \times \mathbb{Z}_{p^2}$, i.e. $\mathfrak{k}_{\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}} = \mathfrak{k}_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$: this shows that the restricted Lie bialgebra $\mathfrak{k}_G$ may be not enough to recover the group G.

As for $\big({\Bbbk[G]_\hbar}^\vee\big)'$, it is generated by $\psi_\nu = \nu - 1$, $\psi_\tau = \tau - 1$, $\psi_{\tau^p} = \hbar^{1-p}\big(\tau^p - 1\big)$, with relations $\psi_\nu^{\,p} = 0$, $\psi_\tau^{\,p} = \hbar^{p-1}\psi_{\tau^p}$, $\psi_{\tau^p}^{\,p} = 0$, $\psi_\nu\,\psi_\tau - \psi_\tau\,\psi_\nu = \hbar^{\,p-1}\psi_{\tau^p}\,(1+\psi_\tau)\,(1+\psi_\nu)$, $\psi_\tau\,\psi_{\tau^p} - \psi_{\tau^p}\,\psi_\tau = 0$, and $\psi_\nu\,\psi_{\tau^p} - \psi_{\tau^p}\,\psi_\nu = 0$. In particular $\big({\Bbbk[G]_\hbar}^\vee\big)' \supsetneqq \Bbbk[G]_\hbar$, and

$$ \big({\Bbbk[G]_\hbar}^\vee\big)' \;\cong\; \Bbbk[\hbar]\,\langle u_1, u_2, u_3\rangle \Big/ \!\!\left(\begin{matrix} u_1\,u_3 - u_3\,u_1\,, \quad u_2^{\,p} - \hbar^{p-1}u_3\,, \quad u_2\,u_3 - u_3\,u_2 \\ u_1^{\,p}\,, \quad u_1\,u_2 - u_2\,u_1 - \hbar^{\,p-1}u_3\,(1+u_2)\,(1+u_1)\,, \quad u_3^{\,p} \end{matrix} \right) $$

via $\psi_\nu \mapsto u_1$, $\psi_\tau \mapsto u_2$, $\psi_{\tau^p} \mapsto u_3$. Letting $z_1 := \psi_\nu\big|_{\hbar=0} + 1$, $z_2 := \psi_\tau\big|_{\hbar=0} + 1$ and $x_3 := \psi_{\tau^p}\big|_{\hbar=0}$ this gives $\big({\Bbbk[G]_\hbar}^\vee\big)'\Big|_{\hbar=0} = \Bbbk\big[z_1, z_2, x_3\big] \Big/ \big(z_1^{\,p} - 1, z_2^{\,p} - 1, x_3^{\,p}\big)$ as a $\Bbbk$–algebra, with the z_i's group-like, x_3 primitive (cf. Theorem D*(b)*), and Poisson bracket given by $\big\{z_1, z_2\big\} = \delta_{p,2}\,z_1\,z_2\,x_3$, $\big\{z_2, x_3\big\} = 0$ and $\big\{z_1, x_3\big\} = 0$. Thus $\big({\Bbbk[G]_\hbar}^\vee\big)'\Big|_{\hbar=0} = F[\varGamma_G]$ with $\varGamma_G \cong \boldsymbol{\mu}_p \times \boldsymbol{\mu}_p \times \boldsymbol{\alpha}_p$ as algebraic groups, with Poisson structure such that $\mathit{coLie}\,(\varGamma_G) \cong \mathfrak{k}_G$.

Since $G_\infty = \{1\}$ the general theory ensures that $A_\Bbbk(G)' = A_\Bbbk(G)$. We leave to the interested reader the task of computing the filtration $\underline{D}$ of $A_\Bbbk(G)$, and consequently describe $A_R(G)'$, $\big(A_R(G)'\big)^\vee$, $\widetilde{A_\Bbbk(G)}$ and the connected Poisson group $K_G := \mathit{Spec}\,\big(\widetilde{A_\Bbbk(G)}\big)$.

(3) An Abelian infinite group. Let $G = \mathbb{Z}^n$ (written multiplicatively, with generators $e_1, \dots, e_n$), then $\Bbbk[G] = \Bbbk[\mathbb{Z}^n] = \Bbbk\big[e_1^{\pm 1}, \dots, e_n^{\pm 1}\big]$ (the ring of Laurent polynomials). This is the function algebra of the algebraic group ${\mathbb{G}_m}^n$, i.e. the n–dimensional torus on $\Bbbk$ (which is exactly the character group of $\mathbb{Z}^n$), thus we get back to the function algebra case.

§ 6 First example: the Kostant-Kirillov structure

6.1 Classical and quantum setting. We study now another quantization of the Kostant-Kirillov structure. Let $\mathfrak{g}$ and $\mathfrak{g}^\star$ be as in §5.7, consider $\mathfrak{g}$ as a Lie bialgebra with trivial Lie cobracket and look at $\mathfrak{g}^\star$ as its dual Poisson group, hence its Poisson structure is exactly the Kostant-Kirillov one.

Take as ground ring $R := \Bbbk[\nu]$ (a PID): we shall consider the primes $\hbar = \nu$ and $\hbar = \nu - 1$, and we'll find quantum groups at either of them for both $\mathfrak{g}$ and $\mathfrak{g}^\star$.

To begin with, we assume $Char(\Bbbk) = 0$, and postpone to §6.4 the case $Char(\Bbbk) > 0$.

Let $\mathfrak{g}_\nu := \mathfrak{g}[\nu] = \Bbbk[\nu] \otimes_\Bbbk \mathfrak{g}$, endow it with the unique $\Bbbk[\nu]$–linear Lie bracket $[\ ,\ ]_\nu$ given by $[x,y]_\nu := \nu\,[x,y]$ for all $x, y \in \mathfrak{g}$, and define

$$H := U_{\Bbbk[\nu]}(\mathfrak{g}_\nu) = T_{\Bbbk[\nu]}(\mathfrak{g}_\nu) \Big/ \big(\{\, x \cdot y - y \cdot x - \nu\,[x,y] \;\big|\; x, y \in \mathfrak{g} \,\} \big),$$

the universal enveloping algebra of the Lie $\Bbbk[\nu]$–algebra $\mathfrak{g}_\nu$, endowed with its natural structure of Hopf algebra. Then H is a free $\Bbbk[\nu]$–algebra, so that $H \in \mathcal{HA}$ and $H_F := \Bbbk(\nu) \otimes_{\Bbbk[\nu]} H \in \mathcal{HA}_F$ (see §1.3); its specializations at $\nu = 1$ and at $\nu = 0$ are

$$H \Big/ (\nu-1)\,H \;=\; U(\mathfrak{g}) \qquad \text{as a } \textit{co-Poisson} \text{ Hopf algebra},$$
$$H \Big/ \nu\,H \;=\; S(\mathfrak{g}) \;=\; F[\mathfrak{g}^\star] \qquad \text{as a } \textit{Poisson} \text{ Hopf algebra};$$

in a more suggesting way, we can also express this with notation like $H \xrightarrow{\ \nu \to 1\ } U(\mathfrak{g})$, $H \xrightarrow{\ \nu \to 0\ } F[\mathfrak{g}^\star]$. Therefore, *$H$ is a QrUEA at $\hbar := (\nu-1)$ and a QFA at $\hbar := \nu$*; thus now we go and consider Drinfeld's functors for H at $(\nu-1)$ and at (ν).

6.2 Drinfeld's functors at (ν). Let $(\ )^{\vee_{(\nu)}} : \mathcal{HA} \longrightarrow \mathcal{HA}$ and $(\ )'^{(\nu)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ be the Drinfeld's functors at (ν) $\big(\in Spec\big(\Bbbk[\nu]\big) \big)$. By definitions $J := Ker\big(\epsilon\colon H \longrightarrow \Bbbk[\nu]\big)$ is nothing but the 2-sided ideal of $H := U(\mathfrak{g}_\nu)$ generated by $\mathfrak{g}_\nu$ itself; thus $H^{\vee_{(\nu)}}$, which by definition is the unital $\Bbbk[\nu]$–subalgebra of H_F generated by $J^{\vee_{(\nu)}} := \nu^{-1} J$, is just the unital $\Bbbk[\nu]$–subalgebra of H_F generated by ${\mathfrak{g}_\nu}^{\vee_{(\nu)}} := \nu^{-1}\,\mathfrak{g}_\nu$. Now consider the $\Bbbk[\nu]$–module isomorphism $(\ )^{\vee_{(\nu)}} : \mathfrak{g}_\nu \xrightarrow{\ \cong\ } {\mathfrak{g}_\nu}^{\vee_{(\nu)}} := \nu^{-1}\,\mathfrak{g}_\nu$ given by $z \mapsto z^\vee := \nu^{-1} z \in {\mathfrak{g}_\nu}^{\vee_{(\nu)}}$ for all $z \in \mathfrak{g}_\nu$; consider on $\mathfrak{g}_\nu := \Bbbk[\nu] \otimes_\Bbbk \mathfrak{g}$ the natural Lie algebra structure (with trivial Lie cobracket), given by scalar extension from $\mathfrak{g}$, and push it over ${\mathfrak{g}_\nu}^{\vee_{(\nu)}}$ via $(\ )^{\vee_{(\nu)}}$, so that ${\mathfrak{g}_\nu}^{\vee_{(\nu)}}$ is isomorphic to $\mathfrak{g}_\nu^{nat}$ (i.e. $\mathfrak{g}_\nu$ carrying the natural Lie bialgebra structure) as a Lie bialgebra. Consider $x^\vee, y^\vee \in {\mathfrak{g}_\nu}^{\vee_{(\nu)}}$ (with $x, y \in \mathfrak{g}_\nu$): then $H^{\vee_{(\nu)}} \ni \big(x^\vee y^\vee - y^\vee x^\vee \big) = \nu^{-2}\big(x\,y - y\,x \big) = \nu^{-2}\,[x,y]_\nu = \nu^{-2}\,\nu\,[x,y] = \nu^{-1}\,[x,y] = [x,y]^\vee =: \big[x^\vee, y^\vee \big] \in {\mathfrak{g}_\nu}^{\vee_{(\nu)}}$. Therefore we can conclude at once that $H^{\vee_{(\nu)}} = U\big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big) \cong U\big(\mathfrak{g}_\nu^{nat}\big)$.

As a first consequence, $\big(H^{\vee_{(\nu)}}\big)\Big|_{\nu=0} \cong U\big(\mathfrak{g}_\nu^{nat}\big) \Big/ \nu\,U\big(\mathfrak{g}_\nu^{nat}\big) = U\Big(\mathfrak{g}_\nu^{nat} / \nu\,\mathfrak{g}_\nu^{nat}\Big) = U(\mathfrak{g})$, that is $H^{\vee_{(\nu)}} \xrightarrow{\ \nu \to 0\ } U(\mathfrak{g})$, thus agreeing with the second half of Theorem 2.2*(c)*.

Second, look at $\big(H^{\vee_{(\nu)}}\big)'^{(\nu)}$. Since $H^{\vee_{(\nu)}} = U\big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)$, and $\delta_n(\eta) = 0$ for all $\eta \in U\big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)$ such that $\partial(\eta) < n$ (cf. the proof of Lemma 4.2*(d)*), it is easy to see that

$$\big(H^{\vee_{(\nu)}}\big)'^{(\nu)} = \big\langle \nu\,{\mathfrak{g}_\nu}^{\vee_{(\nu)}} \big\rangle = \big\langle \nu\,\nu^{-1} \mathfrak{g}_\nu \big\rangle = U(\mathfrak{g}_\nu) = H$$

(hereafter $\langle S \rangle$ is the subalgebra generated by S), so $\big(H^{\vee_{(\nu)}}\big)'^{(\nu)} = H$, which agrees with Theorem 2.2*(b)*. Finally, proceeding as in §5.7 we see that $H'^{(\nu)} = U(\nu\,\mathfrak{g}_\nu)$, whence $\big(H'^{(\nu)}\big)\Big|_{\nu=0} = \big(U(\nu\,\mathfrak{g}_\nu)\big)\Big|_{\nu=0} \cong S(\mathfrak{g}_{ab}) = F\big[\mathfrak{g}^\star_{\delta-ab}\big]$ where $\mathfrak{g}_{ab}$, resp. $\mathfrak{g}^\star_{\delta-ab}$, is simply $\mathfrak{g}$, resp. $\mathfrak{g}^\star$, endowed with the trivial Lie bracket, resp. cobracket, so that $\left(H'^{(\nu)}\right)\Big|_{\nu=0} \cong S(\mathfrak{g}_{ab}) = F\big[\mathfrak{g}^\star_{\delta-ab}\big]$ has trivial Poisson bracket. Similarly, we can iterate this procedure and find that all further images $\Big(\cdots\big((H)'^{(\nu)}\big)'^{(\nu)}\cdots\Big)'^{(\nu)}$ of the functor $(\ )'^{(\nu)}$ applied many times to H are all isomorphic, hence they all have the same specialization at (ν), namely

$$\bigg(\Big(\cdots\big((H)'^{(\nu)}\big)'^{(\nu)}\cdots\Big)'^{(\nu)}\bigg)\bigg|_{\nu=0} \cong\; S(\mathfrak{g}_{ab}) = F\big[\mathfrak{g}^\star_{\delta-ab}\big]\ .$$

6.3 Drinfeld's functors at $(\nu-1)$**.** Now consider $(\nu-1)\,\big(\in \mathit{Spec}\big(\Bbbk[\nu]\big)\,\big)$, and let $(\ )^{\vee(\nu-1)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ and $(\ )'^{(\nu-1)} : \mathcal{HA} \longrightarrow \mathcal{HA}$ be the corresponding Drinfeld's functors. Set ${\mathfrak{g}_\nu}'^{(\nu-1)} := (\nu-1)\,\mathfrak{g}_\nu$, let $\ : \mathfrak{g}_\nu \xrightarrow{\cong} {\mathfrak{g}_\nu}'^{(\nu-1)} := (\nu-1)\,\mathfrak{g}_\nu$ be the $\Bbbk[\nu]$–module isomorphism given by $z \mapsto z' := (\nu-1)\,z \in {\mathfrak{g}_\nu}'^{(\nu-1)}$ for all $z \in \mathfrak{g}_\nu$, and push over via it the Lie bialgebra structure of $\mathfrak{g}_\nu$ to an isomorphic Lie bialgebra structure on ${\mathfrak{g}_\nu}'^{(\nu-1)}$, whose Lie bracket will be denoted by $[\ ,\ ]_*$. Notice then that we have Lie bialgebra isomorphisms $\mathfrak{g} \cong \mathfrak{g}_\nu\big/(\nu-1)\,\mathfrak{g}_\nu \cong {\mathfrak{g}_\nu}'^{(\nu-1)}\big/(\nu-1)\,{\mathfrak{g}_\nu}'^{(\nu-1)}$.

Since $H := U(\mathfrak{g}_\nu)$ it is easy to see by direct computation that

$$H'^{(\nu-1)} = \big\langle(\nu-1)\,\mathfrak{g}_\nu\big\rangle = U\big({\mathfrak{g}_\nu}'^{(\nu-1)}\big)\,, \tag{6.1}$$

where ${\mathfrak{g}_\nu}'^{(\nu-1)}$ is considered as a Lie $\Bbbk[\nu]$–subalgebra of $\mathfrak{g}_\nu$. Now, if $x',\, y' \in {\mathfrak{g}_\nu}'^{(\nu-1)}$ (with $x,\, y \in \mathfrak{g}_\nu$), we have

$$x'\,y' - y'\,x' = (\nu-1)^2\big(x\,y - y\,x\big) = (\nu-1)^2\,[x,y]_\nu = (\nu-1)\,[x,y]_\nu{}' = (\nu-1)\,\big[x',y'\big]_*\,. \tag{6.2}$$

This and (6.1) show at once that $\left(H'^{(\nu-1)}\right)\Big|_{(\nu-1)=0} = S\Big({\mathfrak{g}_\nu}'^{(\nu-1)}\big/(\nu-1)\,{\mathfrak{g}_\nu}'^{(\nu-1)}\Big)$ as Hopf algebras, and also as *Poisson* algebras: indeed, the latter holds because the Poisson bracket $\{\ ,\ \}$ of $S\Big({\mathfrak{g}_\nu}'^{(\nu-1)}\big/(\nu-1)\,{\mathfrak{g}_\nu}'^{(\nu-1)}\Big)$ inherited from $H'^{(\nu-1)}$ (by specialization) is uniquely determined by its restriction to ${\mathfrak{g}_\nu}'^{(\nu-1)}\big/(\nu-1)\,{\mathfrak{g}_\nu}'^{(\nu-1)}$, and on the latter space we have $\{\ ,\ \} = [\ ,\ ]_* \mod (\nu-1)\,{\mathfrak{g}_\nu}'^{(\nu-1)}$ (by (6.2)). Finally, since ${\mathfrak{g}_\nu}'^{(\nu-1)}\big/(\nu-1)\,{\mathfrak{g}_\nu}'^{(\nu-1)} \cong \mathfrak{g}$ as Lie algebras we have $\left(H'^{(\nu-1)}\right)\Big|_{(\nu-1)=0} = S(\mathfrak{g}) = F[\mathfrak{g}^\star]$ as Poisson Hopf algebras, or, in short, $H'^{(\nu-1)} \xrightarrow{\ \nu\to 1\ } F[\mathfrak{g}^\star]$, as prescribed by the "first half" of Theorem 2.2*(c)*.

Second, look at $\big(H'^{(\nu-1)}\big)^{\vee(\nu-1)}$. Since $H'^{(\nu-1)} = U\big({\mathfrak{g}_\nu}'^{(\nu-1)}\big)$, we have that $J'^{(\nu-1)} := \mathit{Ker}\big(\epsilon\colon H'^{(\nu-1)} \longrightarrow \Bbbk[\nu]\big)$ is nothing but the 2-sided ideal of $H'^{(\nu-1)} = U\big({\mathfrak{g}_\nu}'^{(\nu-1)}\big)$ generated by ${\mathfrak{g}_\nu}'^{(\nu-1)}$; thus $\big(H'^{(\nu-1)}\big)^{\vee(\nu-1)}$, generated by $\big(J'^{(\nu-1)}\big)^{\vee(\nu-1)} := (\nu-1)^{-1} J'^{(\nu-1)}$ as a unital $\Bbbk[\nu]$–subalgebra of $\big(H'^{(\nu-1)}\big)_F = H_F$, is just the unital $\Bbbk[\nu]$–subalgebra of H_F generated by $(\nu-1)^{-1}{\mathfrak{g}_\nu}'^{(\nu-1)} = (\nu-1)^{-1}(\nu-1)\,\mathfrak{g}_\nu = \mathfrak{g}_\nu$, that is to say $\big(H'^{(\nu-1)}\big)^{\vee(\nu-1)} = U(\mathfrak{g}_\nu) = H$, according to Theorem 2.2*(b)*.

Finally, for $H^{\vee(\nu-1)}$ one has essentially the same feature as in §5.7, and the analysis therein can be applied again; the final result then will depend on the nature of $\mathfrak{g}$, in particular on its lower central series.

6.4 The case of positive characteristic. Let us consider now a field $\Bbbk$ such that $Char(\Bbbk) = p > 0$. Starting from $\mathfrak{g}$ and $R := \Bbbk[\nu]$ as in §6.1, define $\mathfrak{g}_\nu$ like therein, and consider $H := U_{\Bbbk[\nu]}(\mathfrak{g}_\nu) = U_R(\mathfrak{g}_\nu)$. Then we have again

$$H \Big/ (\nu - 1)\, H \;=\; U(\mathfrak{g}) \;=\; \mathbf{u}\Big(\mathfrak{g}^{[p]^\infty}\Big) \qquad \text{as a } \textit{co-Poisson} \text{ Hopf algebra}\,,$$
$$H \Big/ \nu\, H \;=\; S(\mathfrak{g}) \;=\; F[\mathfrak{g}^\star] \qquad \text{as a } \textit{Poisson} \text{ Hopf algebra}$$

so that H is a QrUEA at $\hbar := (\nu - 1)$ (for the restricted universal enveloping algebra $\mathbf{u}\Big(\mathfrak{g}^{[p]^\infty}\Big)$) and is a QFA at $\hbar := \nu$ (for the function algebra $F[\mathfrak{g}^\star]$); so now we study Drinfeld's functors for H at $(\nu - 1)$ and at (ν).

Exactly the same procedure as before shows again that $H^{\vee_{(\nu)}} = U\big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)$, from which it follows that $\big(H^{\vee_{(\nu)}}\big)\Big|_{\nu=0} \cong U(\mathfrak{g})$, i.e. in short $H^{\vee_{(\nu)}} \xrightarrow{\ \nu \rightarrow 0\ } U(\mathfrak{g})$, which is a result quite "parallel" to the second half of Theorem 2.2*(c)*.

Changes occur when looking at $\big(H^{\vee_{(\nu)}}\big)'_{(\nu)}$: since $H^{\vee_{(\nu)}} = U\big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big) = \mathbf{u}\Big(\big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)^{[p]^\infty}\Big)$ we have $\delta_n(\eta) = 0$ for all $\eta \in \mathbf{u}\Big(\big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)^{[p]^\infty}\Big)$ such that $\partial(\eta) < n$ w.r.t. the standard filtration of $\mathbf{u}\Big(\big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)^{[p]^\infty}\Big)$ (cf. the proof of Lemma 4.2*(d)*, which clearly adapts to the present situation): this implies

$$\big(H^{\vee_{(\nu)}}\big)'_{(\nu)} = \Big\langle \nu \cdot \big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)^{[p]^\infty} \Big\rangle \qquad \Big(\subset \mathbf{u}\Big(\nu \cdot \big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)^{[p]^\infty}\Big) \Big)$$

which is strictly *bigger* than H, because $\Big\langle \nu \cdot \big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)^{[p]^\infty} \Big\rangle = \Big\langle \sum_{n \geq 0} \nu \cdot \big({\mathfrak{g}_\nu}^{\vee_{(\nu)}}\big)^{[p]^n} \Big\rangle =$
$= \Big\langle \mathfrak{g}_\nu + \nu^{1-p} \left\{ x^p \,\middle|\, x \in \mathfrak{g}_\nu \right\} + \nu^{1-p^2} \Big\{ x^{p^2} \,\Big|\, x \in \mathfrak{g}_\nu \Big\} + \cdots \Big\rangle \supsetneqq U(\mathfrak{g}_\nu) = H$.

Finally, proceeding as above it is easy to see that $H'_{(\nu)} = \Big\langle \nu\, P\big(U(\mathfrak{g}_\nu)\big) \Big\rangle = \Big\langle \nu\, \mathfrak{g}^{[p]^\infty} \Big\rangle$ whence, letting $\tilde{\mathfrak{g}} := \nu\, \mathfrak{g}$ and $\tilde{x} := \nu\, x$ for all $x \in \mathfrak{g}$, we have

$$H'_{(\nu)} = T_R(\tilde{\mathfrak{g}}) \Big/ \bigg(\Big\{ \tilde{x}\, \tilde{y} - \tilde{y}\, \tilde{x} - \nu^2\, \widetilde{[x,y]}\,,\ \tilde{z}^p - \nu^{p-1} \widetilde{z^{[p]}} \;\Big|\; x, y, z \in \mathfrak{g} \Big\} \bigg)$$

so that $H'_{(\nu)} \xrightarrow{\ \nu \rightarrow 0\ } T_\Bbbk(\tilde{\mathfrak{g}}) \Big/ \bigg(\Big\{ \tilde{x}\, \tilde{y} - \tilde{y}\, \tilde{x}\,,\ \tilde{z}^p \;\Big|\; \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{\mathfrak{g}} \Big\} \bigg) = S_\Bbbk(\mathfrak{g}_{ab}) \Big/ \big(\left\{ z^p \,\middle|\, z \in \mathfrak{g} \right\} \big) =$
$= F[\mathfrak{g}^\star_{\delta-ab}] \Big/ \big(\left\{ z^p \,\middle|\, z \in \mathfrak{g} \right\} \big)$, that is $H'_{(\nu)}\Big|_{\nu=0} \cong F[\mathfrak{g}^\star_{\delta-\mathrm{ab}}] \Big/ \big(\left\{ z^p \,\middle|\, z \in \mathfrak{g} \right\} \big)$ *as Poisson Hopf algebras*, where $\mathfrak{g}_{ab}$ and $\mathfrak{g}^\star_{\delta-ab}$ are as above. Therefore $H'_{(\nu)}$ is a QFA (at $\hbar = \nu$) for a non-reduced algebraic Poisson group of height 1, whose cotangent Lie bialgebra is the vector space $\mathfrak{g}$ with trivial Lie bialgebra structure: this again yields somehow an analogue of part *(c)* of Theorem 2.2 for the present case. If we iterate, we find that all further images $\Big(\cdots \big((H)'_{(\nu)} \big)'_{(\nu)} \cdots \Big)'_{(\nu)}$ of the functor $(\ )'_{(\nu)}$ applied to H are all pairwise isomorphic, so

$$\Big(\cdots \big((H)'_{(\nu)} \big)'_{(\nu)} \cdots \Big)'_{(\nu)} \bigg|_{\nu=0} \;\cong\; S(\mathfrak{g}_{ab}) \Big/ \big(\left\{ z^p \,\middle|\, z \in \mathfrak{g} \right\} \big) = F\big[\mathfrak{g}^\star_{\delta-ab}\big] \Big/ \big(\left\{ z^p \,\middle|\, z \in \mathfrak{g} \right\} \big)\,.$$

Now for Drinfeld's functors at $(\nu-1)$. Up to minor changes, with the same procedure and notations as in §6.3 we get analogous results. First, an analogue of (6.1), namely $H'^{(\nu-1)} = \Big\langle (\nu-1) \cdot P\big(U(\mathfrak{g}_\nu)\big) \Big\rangle = \Big\langle (\nu-1)\,(\mathfrak{g}_\nu)^{[p]^\infty} \Big\rangle = \Big\langle \Big((\mathfrak{g}_\nu)^{[p]^\infty} \Big)'^{(\nu-1)} \Big\rangle$, holds and yields

$$H'^{(\nu-1)} = T_R\Big(\Big((\mathfrak{g}_\nu)^{[p]^\infty}\Big)'^{(\nu-1)}\Big) \Big/ \Big(\Big\{ x'\,y' - y'\,x' - (\nu-1)\,[x',y']_* \,,\, (x')^p - (\nu-1)^{p-1}\big(x^{[p]}\big)' \;\Big|\; x,y \in (\mathfrak{g}_\nu)^{[p]^\infty} \Big\} \Big)$$

and consequently $H'^{(\nu-1)}\Big|_{(\nu-1)=0} \cong S_{\Bbbk}(\mathfrak{g}) \Big/ \big(\{ x^p \mid x \in \mathfrak{g} \} \big) = F[\mathfrak{g}^\star] \Big/ \big(\{ x^p \mid x \in \mathfrak{g} \} \big)$ as Poisson Hopf algebras: in short, $H'^{(\nu-1)} \xrightarrow{\nu\to 1} F[\mathfrak{g}^\star] \Big/ \big(\{ x^p \mid x \in \mathfrak{g} \} \big)$.

Iterating, one finds again that all $\Big(\cdots \big((H)'^{(\nu)}\big)'^{(\nu-1)} \cdots \Big)'^{(\nu)}$ are pairwise isomorphic, so

$$\Big(\cdots \big((H)'^{(\nu-1)}\big)'^{((\nu-1)} \cdots \Big)'^{(\nu-1)}\Bigg|_{(\nu-1)=0} \cong S(\mathfrak{g}_{ab}) \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) = F\big[\mathfrak{g}^\star_{\delta-ab}\big] \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) .$$

Further on, one has $\big(H'^{(\nu-1)}\big)^{\vee(\nu-1)} = \Big\langle (\nu-1)\,(\mathfrak{g}_\nu)^{[p]^\infty} \Big\rangle^{\vee(\nu-1)} = \big\langle (\nu-1)^{-1} \cdot (\nu-1)\,\mathfrak{g}_\nu \big\rangle = \big\langle \mathfrak{g}_\nu \big\rangle = U_R(\mathfrak{g}_\nu) =: H$, which perfectly agrees with Theorem 2.2*(b)*.

Finally, as for $H^{\vee(\nu-1)}$ one has again the same feature as in §5.7: one has to apply the analysis therein, however, the p–filtration in this case is "harmless", since it is essentially encoded in the standard filtration of $U(\mathfrak{g})$. In any case the final result will depend on the properties of the lower central series of $\mathfrak{g}$.

Second, we assume in addition that $\mathfrak{g}$ be a *restricted* Lie algebra, and we consider $H := \mathbf{u}_{\Bbbk[\nu]}(\mathfrak{g}_\nu) = \mathbf{u}_R(\mathfrak{g}_\nu)$. Then we have

$$H \Big/ (\nu-1)\,H = \mathbf{u}(\mathfrak{g}) \qquad \text{as a } \textit{co-Poisson} \text{ Hopf algebra},$$

$$H \Big/ \nu\,H = S(\mathfrak{g}) \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) = F[\mathfrak{g}^\star] \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) \qquad \text{as a } \textit{Poisson} \text{ Hopf algebra}$$

which means that H is a QrUEA at $\hbar := (\nu-1)$ (for $\mathbf{u}(\mathfrak{g})$) and is a QFA at $\hbar := \nu$ (for $F[\mathfrak{g}^\star] \Big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big)$); we go and study Drinfeld's functors for H at $(\nu-1)$ and at (ν).

As for $H^{\vee(\nu)}$, it depends again on the p–operation of $\mathfrak{g}$, in short because the I–filtration of $\mathbf{u}_\nu(\mathfrak{g})$ depends on the p–filtration of $\mathfrak{g}$. In the previous case — i.e. when $\mathfrak{g} = \mathfrak{h}^{[p]^\infty}$ for some Lie algebra $\mathfrak{h}$ — the solution was a plain one, because the p–filtration of $\mathfrak{g}$ is "encoded" in the standard filtration of $U(\mathfrak{h})$; but the general case will be more complicated, and in consequence also the situation for $\big(H^{\vee(\nu)}\big)'^{(\nu)}$, since $H^{\vee(\nu)}$ will be different according to the nature of $\mathfrak{g}$. Instead, proceeding exactly like before one sees that $H'^{(\nu)} = \Big\langle \nu\, P\big(u(\mathfrak{g}_\nu)\big) \Big\rangle = \big\langle \nu\,\mathfrak{g} \big\rangle$, whence, letting $\tilde{\mathfrak{g}} := \nu\,\mathfrak{g}$ and $\tilde{x} := \nu\,x$ for all $x \in \mathfrak{g}$, we have

$$H'^{(\nu)} = T_{\Bbbk[\nu]}(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{ \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x} - \nu^2\,\widetilde{[x,y]} \,,\, \tilde{z}^p - \nu^{p-1}\,\widetilde{z^{[p]}} \;\Big|\; x,y,z \in \mathfrak{g} \Big\} \Big)$$

so that $H'^{(\nu)} \xrightarrow{\nu\to 0} T_{\Bbbk}(\tilde{\mathfrak{g}}) \Big/ \Big(\Big\{ \tilde{x}\,\tilde{y} - \tilde{y}\,\tilde{x}\,,\, \tilde{z}^p \,\Big|\, \tilde{x},\tilde{y},\tilde{z} \in \tilde{\mathfrak{g}} \Big\} \Big) = S_{\Bbbk}(\mathfrak{g}_{ab}) \big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) = $ $= F[\mathfrak{g}^\star_{\delta-ab}] \big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big)$, that is $H'^{(\nu)}\Big|_{\nu=0} \cong F[\mathfrak{g}^\star_{\delta-\mathrm{ab}}] \big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big)$ *as Poisson Hopf algebras* (using notation as before). Thus $H'^{(\nu)}$ is a QFA (at $\hbar = \nu$) for a non-reduced algebraic Poisson group of height 1, whose cotangent Lie bialgebra is $\mathfrak{g}$ with the trivial Lie bialgebra structure: so again we get an analogue of part of Theorem 2.2*(c)*. Moreover, iterating again one finds that all $\Big(\cdots \big((H)'^{(\nu)} \big)'^{(\nu-1)} \cdots \Big)'^{(\nu-1)}$ are pairwise isomorphic, so

$$\Big(\cdots \big((H)'^{(\nu-1)} \big)'^{((\nu-1)} \cdots \Big)'^{(\nu-1)} \bigg|_{(\nu-1)=0} \cong\; S(\mathfrak{g}_{ab}) \big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) = F\big[\mathfrak{g}^\star_{\delta-ab} \big] \big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) \, .$$

As for Drinfeld's functors at $(\nu-1)$, the situation is more similar to the previous case of $H = U_R(\mathfrak{g}_\nu)$. First $H'^{(\nu-1)} = \Big\langle (\nu-1) \cdot P\big(\mathbf{u}(\mathfrak{g}_\nu) \big) \Big\rangle = \big\langle (\nu-1)\, \mathfrak{g}_\nu \big\rangle =: \big\langle {\mathfrak{g}_\nu}'^{(\nu-1)} \big\rangle$, hence

$$H'^{(\nu-1)} = T_R\Big({\mathfrak{g}_\nu}'^{(\nu-1)} \Big) \Big/ \Big(\Big\{ x'\,y' - y'\,x' - (\nu-1)\,\big[x', y' \big]_* \,,\, (x')^p - (\nu-1)^{p-1} \big(x^{[p]} \big)' \;\Big|\; \Big| \; x, y \in \mathfrak{g}_\nu \Big\} \Big)$$

thus again $H'^{(\nu-1)}\Big|_{(\nu-1)=0} \cong S_{\Bbbk}(\mathfrak{g}) \big/ \big(\{ x^p \mid x \in \mathfrak{g} \} \big) = F[\mathfrak{g}^\star] \big/ \big(\{ x^p \mid x \in \mathfrak{g} \} \big)$ as Poisson Hopf algebras, that is $H'^{(\nu-1)} \xrightarrow{\nu\to 1} F[\mathfrak{g}^\star] \big/ \big(\{ x^p \mid x \in \mathfrak{g} \} \big)$. Iteration then shows that all $\Big(\cdots \big((H)'^{(\nu)} \big)'^{(\nu-1)} \cdots \Big)'^{(\nu)}$ are pairwise isomorphic, so that again

$$\Big(\cdots \big((H)'^{(\nu-1)} \big)'^{((\nu-1)} \cdots \Big)'^{(\nu-1)} \bigg|_{(\nu-1)=0} \cong\; S(\mathfrak{g}_{ab}) \big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) = F\big[\mathfrak{g}^\star_{\delta-ab} \big] \big/ \big(\{ z^p \mid z \in \mathfrak{g} \} \big) \, .$$

Further, we have $\big(H'^{(\nu-1)} \big)^{\vee(\nu-1)} = \big\langle (\nu-1)\, \mathfrak{g}_\nu \big\rangle^{\vee(\nu-1)} = \big\langle (\nu-1)^{-1} \cdot (\nu-1)\, \mathfrak{g}_\nu \big\rangle = \big\langle \mathfrak{g}_\nu \big\rangle = $ $= \mathbf{u}_R(\mathfrak{g}_\nu) =: H$, which agrees at all with Theorem 2.2*(b)*. Finally, $H^{\vee(\nu-1)}$ again has the same feature as in §5.7: in particular, in this case the final result will strongly depend *both* on the properties of the lower central series *and* of the p–filtration of $\mathfrak{g}$.

6.5 The hyperalgebra case. Let $\Bbbk$ be again a field with $Char(\Bbbk) = p > 0$. Like in §5.12, let G be an algebraic group (finite-dimensional, for simplicity), and let $Hyp(G) := \big(F[G]^\circ \big)_\epsilon = \big\{ \phi \in F[G]^\circ \,\big|\, \phi({\mathfrak{m}_e}^n) = 0 \,,\, \forall\, n \gg 0 \big\}$ be the hyperalgebra associated to G.

For each $\nu \in \Bbbk$, let $\mathfrak{g}_\nu := \big(\mathfrak{g}, [\ ,\ ]_\nu \big)$ be the Lie algebra given by $\mathfrak{g}$ endowed with the rescaled Lie bracket $[\ ,\ ]_\nu := \nu\,[\ ,\ ]_{\mathfrak{g}}$. By general theory, the algebraic group G is uniquely determined by a neighborhood of the identity together with the formal group law uniquely determined by $[\ ,\ ]_{\mathfrak{g}}$: similarly, a neighborhood of the identity of G together with $[\ ,\ ]_\nu$ uniquely determines a new connected algebraic group G_ν, whose hyperalgebra $Hyp(G_\nu)$

is an algebraic deformation of $Hyp\,(G)$; moreover, G_ν is birationally equivalent to G, and for $\nu \neq 0$ they are also isomorphic as algebraic groups, via an isomorphism induced by $\mathfrak{g} \cong \mathfrak{g}_\nu$, $x \mapsto \nu^{-1}x$ (however, this may not be the case when $\nu = 0$). Note that $Hyp\,(G_0)$ is clearly commutative, because G_0 is Abelian: indeed, we have

$$Hyp\,(G_0) \;=\; S_{\Bbbk}\Big(\mathfrak{g}^{(p)^\infty}\Big)\Big/\Big(\{\, x^p \,\}_{x \in \mathfrak{g}^{(p)^\infty}}\Big) \;=\; F\Big[\Big(\mathfrak{g}^{(p)^\infty}\Big)^{\star}\Big]\Big/\Big(\{\, y^p \,\}_{y \in \mathfrak{g}^{(p)^\infty}}\Big)$$

where $\mathfrak{g}^{(p)^\infty} := Span\Big(\Big\{\, x^{(p^n)} \,\Big|\, x \in \mathfrak{g}\,, n \in \mathbb{N} \Big\}\Big)$; here as usual $x^{(n)}$ denotes the n–th divided power of $x \in \mathfrak{g}$ (recall that $Hyp\,(G)$, hence also $Hyp\,(G_\nu)$, is generated as an algebra by all the $x^{(n)}$'s, some of which might be zero). So $Hyp\,(G_0) = F[\varGamma]$ where $\varGamma$ is a connected algebraic group of dimension zero and height 1: moreover, $\varGamma$ is a Poisson group, with cotangent Lie bialgebra $\mathfrak{g}^{(p)^\infty}$ and Poisson bracket induced by the Lie bracket of $\mathfrak{g}$.

Now think at ν as a parameter in $R := \Bbbk[\nu]$ (as in §6.1), and set $H := \Bbbk[\nu] \otimes_{\Bbbk} Hyp\,(G_\nu)$. Then we find a situation much similar to that of §6.1, which we shall shortly describe.

Namely, H is a free $\Bbbk[\nu]$–algebra, thus $H \in \mathcal{HA}$ and $H_F := \Bbbk(\nu) \otimes_{\Bbbk[\nu]} H \in \mathcal{HA}_F$ (see §1.3); its specialization at $\nu = 1$ is $H\big/(\nu-1)\,H \,=\, Hyp\,(G_1) \,=\, Hyp\,(G)$, and at $\nu = 0$ is $H\big/\nu\,H \,=\, Hyp\,(G_0) \,=\, F[\varGamma]$ (as a *Poisson* Hopf algebra), or $H \xrightarrow{\nu \to 1} Hyp\,(G)$ and $H \xrightarrow{\nu \to 0} F[\varGamma]$, i.e. H is a "quantum hyperalgebra" at $\hbar := (\nu - 1)$ and a QFA at $\hbar := \nu$. Now we study Drinfeld's functors for H at $\hbar = (\nu-1)$ and at $\hbar = \nu$.

First, a straightforward analysis like in §6.2 yields $H^{\vee_{(\nu)}} \cong \Bbbk[\nu] \otimes_{\Bbbk} Hyp\,(G)$ (induced by $\mathfrak{g} \cong \mathfrak{g}_\nu$, $x \mapsto \nu^{-1}x$) whence in particular $\big(H^{\vee_{(\nu)}}\big)\Big|_{\nu=0} \cong Hyp\,(G)$, that is $H^{\vee_{(\nu)}} \xrightarrow{\nu \to 0} Hyp\,(G)$. Second, one can also see (essentially, *mutatis mutandis*, like in §6.2) that $\big(H^{\vee_{(\nu)}}\big)'^{(\nu)} = H$, whence $\big(H^{\vee_{(\nu)}}\big)'^{(\nu)}\Big|_{\nu=0} = H\Big|_{\nu=0} = Hyp\,(G_0) \,=\, F[\varGamma]$.

At $\hbar = (\nu-1)$, we can see by direct computation that $H'^{(\nu-1)} = \Big\langle \big(\mathfrak{g}^{(p)^\infty}\big)'^{(\nu-1)} \Big\rangle$ where $\big(\mathfrak{g}^{(p)^\infty}\big)'^{(\nu-1)} := Span\Big(\Big\{\, (\nu-1)^{p^n} x^{(p^n)} \,\Big|\, x \in \mathfrak{g}\,, n \in \mathbb{N} \Big\}\Big)$. Indeed the structure of $H'^{(\nu-1)}$ depends only on the coproduct of H, in which ν plays no role; therefore we can do the same analysis as in the trivial deformation case (see §5.12): the filtration $\underline{D}$ of $Hyp\,(G_\nu)$ is just the natural filtration given by the order (of divided powers), and this yields the previous description of $H'^{(\nu-1)}$. When specializing at $\nu = 1$ we find

$$H'^{(\nu-1)}\Big/(\nu-1)\,H'^{(\nu-1)} \;\cong\; S_{\Bbbk}\Big(\mathfrak{g}^{(p)^\infty}\Big)\Big/\Big(\{\, x^p \,\}_{x \in \mathfrak{g}^{(p)^\infty}}\Big) \;=\; Hyp\,(G_0) \;=\; F[\varGamma]$$

as Poisson Hopf algebras: in a nutshell, $H'^{(\nu-1)}$ is a QFA, at $\hbar = \nu-1$, for the Poisson group $\varGamma$. Similarly $H'^{(\nu)} = \Big\langle \big(\mathfrak{g}^{(p)^\infty}\big)'^{(\nu)} \Big\rangle$ with $\big(\mathfrak{g}^{(p)^\infty}\big)'^{(\nu)} := Span\Big(\Big\{\, \nu^{p^n} x^{(p^n)} \,\Big|\, x \in \mathfrak{g}\,, n \in \mathbb{N} \Big\}\Big)$; thus on the upshot we have

$$H'^{(\nu)}\Big/\nu\,H'^{(\nu)} \;\cong\; S_{\Bbbk}\Big(\mathfrak{g}_{ab}^{(p)^\infty}\Big)\Big/\Big(\{\, x^p \,\}_{x \in \mathfrak{g}^{(p)^\infty}}\Big) \;=\; F\big[\varGamma_{ab}\big]$$

where $\mathfrak{g}_{ab}$ is simply $\mathfrak{g}$ with trivialized Lie bracket and Γ_{ab} is the *same algebraic group* as Γ but with *trivial* Poisson bracket: this comes essentially like in §6.2, roughly because $\big\{\overline{\nu\, x}\,,\overline{\nu\, y}\big\} := \big(\nu^{-1}[\nu\, x\,,\nu\, y\,]\big)\Big|_{\nu=0} = \big(\nu^{-1}\cdot\nu^3[x,y]_{\mathfrak{g}}\big)\Big|_{\nu=0} = \big(\nu\cdot\nu\,[x,y]_{\mathfrak{g}}\big)\Big|_{\nu=0} = 0 \ \ (x,y\in\mathfrak{g})$.

Finally, we have $\big({H'}_{(\nu-1)}\big)^{\vee_{(\nu-1)}} = \Big\langle\Big\{\,(\nu-1)^{p^n-1}\,x^{(p^n)}\,\Big|\,x\in\mathfrak{g}\,,\,n\in\mathbb{N}\Big\}\Big\rangle \subsetneqq H$ and $\big({H'}_{(\nu)}\big)^{\vee_{(\nu)}} = \Big\langle\Big\{\,\nu^{p^n-1}\,x^{(p^n)}\,\Big|\,x\in\mathfrak{g}\,,\,n\in\mathbb{N}\Big\}\Big\rangle \subsetneqq H$, by direct computation. For $H^{\vee_{(\nu-1)}}$ we have the same features as in §5.7: the analysis therein can be repeated, the upshot depending on the nature of G (or of $\mathfrak{g}$, essentially, in particular on its p–lower central series).

§ 7 Second example: quantum SL_2, SL_n, finite and affine Kac-Moody groups

7.1 The classical setting. Let $\Bbbk$ be any field of characteristic $p \geq 0$. Let $G := SL_2(\Bbbk) \equiv SL_2$; its tangent Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ is generated by f, h, e (the *Chevalley generators*) with relations $[h,e] = 2\,e$, $[h,f] = -2f$, $[e,f] = h$. The formulas $\delta(f) = h\otimes f - f\otimes h$, $\delta(h) = 0$, $\delta(e) = h\otimes e - e\otimes h$, define a Lie cobracket on $\mathfrak{g}$ which gives it a structure of Lie bialgebra, corresponding to a structure of Poisson group on G. These formulas give also a presentation of the co-Poisson Hopf algebra $U(\mathfrak{g})$ (with the standard Hopf structure). If $p > 0$, the p–operation in $\mathfrak{sl}_2$ is given by $e^{[p]} = 0$, $f^{[p]} = 0$, $h^{[p]} = h$.

On the other hand, $F[SL_2]$ is the unital associative commutative $\Bbbk$–algebra with generators a, b, c, d and the relation $ad - bc = 1$, and Poisson Hopf structure given by

$$\Delta(a) = a\otimes a + b\otimes c\,,\ \ \Delta(b) = a\otimes b + b\otimes d\,,\ \ \Delta(c) = c\otimes a + d\otimes c\,,\ \ \Delta(d) = c\otimes b + d\otimes d$$
$$\epsilon(a) = 1\,,\ \ \epsilon(b) = 0\,,\ \ \epsilon(c) = 0\,,\ \ \epsilon(d) = 1\,,\qquad S(a) = d\,,\ \ S(b) = -b\,,\ \ S(c) = -c\,,\ \ S(d) = a$$
$$\{a,b\} = b\,a\,,\quad \{a,c\} = c\,a\,,\quad \{b,c\} = 0\,,\quad \{d,b\} = -b\,d\,,\quad \{d,c\} = -c\,d\,,\quad \{a,d\} = 2\,b\,c\,.$$

The dual Lie bialgebra $\mathfrak{g}^* = {\mathfrak{sl}_2}^*$ is the Lie algebra with generators $\mathrm{f}, \mathrm{h}, \mathrm{e}$, and relations $[\mathrm{h},\mathrm{e}] = \mathrm{e}$, $[\mathrm{h},\mathrm{f}] = \mathrm{f}$, $[\mathrm{e},\mathrm{f}] = 0$, with Lie cobracket given by $\delta(\mathrm{f}) = 2(\mathrm{f}\otimes\mathrm{h} - \mathrm{h}\otimes\mathrm{f})$, $\delta(\mathrm{h}) = \mathrm{e}\otimes\mathrm{f} - \mathrm{f}\otimes\mathrm{e}$, $\delta(\mathrm{e}) = 2(\mathrm{h}\otimes\mathrm{e} - \mathrm{e}\otimes\mathrm{h})$ (we choose as generators $\mathrm{f} := f^*$, $\mathrm{h} := h^*$, $\mathrm{e} := e^*$, where $\big\{f^*,h^*,e^*\big\}$ is the basis of ${\mathfrak{sl}_2}^*$ which is the dual of the basis $\{f,h,e\}$ of $\mathfrak{sl}_2$). This again yields also a presentation of $U\big({\mathfrak{sl}_2}^*\big)$. If $p > 0$, the p–operation in ${\mathfrak{sl}_2}^*$ is given by $\mathrm{e}^{[p]} = 0$, $\mathrm{f}^{[p]} = 0$, $\mathrm{h}^{[p]} = \mathrm{h}$. The simply connected algebraic Poisson group whose tangent Lie bialgebra is ${\mathfrak{sl}_2}^*$ can be realized as the group of pairs of matrices (the left subscript s meaning "simply connected")

$${}_sSL_2^{\,*} = \left\{ \left(\begin{pmatrix} z^{-1} & 0 \\ y & z \end{pmatrix}, \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \right) \,\middle|\, x,y\in\Bbbk, z\in\Bbbk\setminus\{0\} \right\} \ \leq\ SL_2\times SL_2\ .$$

This group has centre $Z := \big\{(I,I),(-I,-I)\big\}$, so there is only one other (Poisson) group sharing the same Lie (bi)algebra, namely the quotient ${}_aSL_2^{\,*} := {}_sSL_2^{\,*}\Big/Z$ (the adjoint of ${}_sSL_2{}^*$, as the left subscript a means). Therefore $F\big[{}_sSL_2{}^*\big]$ is the unital associative commutative $\Bbbk$–algebra with generators x, $z^{\pm 1}$, y, with Poisson Hopf structure given by

$$\begin{gathered}\Delta(x) = x \otimes z^{-1} + z \otimes x\,, \qquad \Delta\big(z^{\pm 1}\big) = z^{\pm 1} \otimes z^{\pm 1}\,, \qquad \Delta(y) = y \otimes z^{-1} + z \otimes y \\ \epsilon(x) = 0\,, \quad \epsilon\big(z^{\pm 1}\big) = 1\,, \quad \epsilon(y) = 0\,, \qquad S(x) = -x\,, \quad S\big(z^{\pm 1}\big) = z^{\mp 1}\,, \quad S(y) = -y \\ \{x, y\} = \big(z^2 - z^{-2}\big)/2\,, \qquad \big\{z^{\pm 1}, x\big\} = \pm\, x\, z^{\pm 1}\,, \qquad \big\{z^{\pm 1}, y\big\} = \mp\, z^{\pm 1} y\end{gathered}$$

(N.B.: with respect to this presentation, we have $\mathrm{f} = \partial_y\big|_e$, $\mathrm{h} = z\,\partial_z\big|_e$, $\mathrm{e} = \partial_x\big|_e$, where e is the identity element of ${}_sSL_2{}^*$). Moreover, $F\big[{}_aSL_2{}^*\big]$ can be identified with the Poisson Hopf subalgebra of $F\big[{}_sSL_2{}^*\big]$ spanned by products of an even number of generators — i.e. monomials of even degree: this is generated, as a unital subalgebra, by xz, $z^{\pm 2}$, and $z^{-1}y$.

In general, we shall consider $\mathfrak{g} = \mathfrak{g}^\tau$ a semisimple Lie algebra, endowed with the Lie cobracket — depending on the parameter τ — given in [Ga1], §1.3; in the following we shall also retain from [*loc. cit.*] all the notation we need: in particular, we denote by Q, resp. P, the root lattice, resp. the weight lattice, of $\mathfrak{g}$, and by r the rank of $\mathfrak{g}$.

7.2 The[5] QrUEAs $U_q(\mathfrak{g})$. We turn now to quantum groups, starting with the $\mathfrak{sl}_2$ case. Let R be any domain, $\hbar \in R \setminus \{0\}$ an element such that $R/\hbar R = \Bbbk$; moreover, letting $q := \hbar + 1$ we assume that q be invertible in R, i.e. there exists $q^{-1} = (\hbar + 1)^{-1} \in R$. E.g., one can pick $R := \Bbbk\big[q, q^{-1}\big]$ for an indeterminate q and $\hbar := q - 1$, then $F(R) = \Bbbk(q)$.

Let $\mathbb{U}_q(\mathfrak{g}) = \mathbb{U}_q(\mathfrak{sl}_2)$ be the associative unital $F(R)$–algebra with (Chevalley-like) generators F, $K^{\pm 1}$, E, and relations

$$K K^{-1} = 1 = K^{-1} K\,,\; K^{\pm 1} F = q^{\mp 2} F K^{\pm 1}\,,\; K^{\pm 1} E = q^{\pm 2} E K^{\pm 1}\,,\; EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\,.$$

This is a Hopf algebra, with Hopf structure given by

$$\begin{gathered}\Delta(F) = F \otimes K^{-1} + 1 \otimes F\,, \qquad \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, \qquad \Delta(E) = E \otimes 1 + K \otimes E \\ \epsilon(F) = 0\,,\; \epsilon\big(K^{\pm 1}\big) = 1\,,\; \epsilon(E) = 0\,, \qquad S(F) = -FK,\; S\big(K^{\pm 1}\big) = K^{\mp 1},\; S(E) = -K^{-1} E\,.\end{gathered}$$

Then let $U_q(\mathfrak{g})$ be the R–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ generated by F, $H := \dfrac{K-1}{q-1}$, $\varGamma := \dfrac{K - K^{-1}}{q - q^{-1}}$, $K^{\pm 1}$, E. From the definition of $\mathbb{U}_q(\mathfrak{g})$ one gets a presentation of $U_q(\mathfrak{g})$ as the associative unital algebra with generators F, H, $\varGamma$, $K^{\pm 1}$, E and relations

$$\begin{gathered}K K^{-1} = 1 = K^{-1} K\,, \quad K^{\pm 1} H = H K^{\pm 1}\,, \quad K^{\pm 1} \varGamma = \varGamma K^{\pm 1}\,, \quad H \varGamma = \varGamma H \\ (q-1) H = K - 1\,, \quad \big(q - q^{-1}\big) \varGamma = K - K^{-1}\,, \quad H\big(1 + K^{-1}\big) = \big(1 + q^{-1}\big) \varGamma\,, \quad EF - FE = \varGamma \\ K^{\pm 1} F = q^{\mp 2} F K^{\pm 1}\,, \quad HF = q^{-2} FH - (q+1) F\,, \quad \varGamma F = q^{-2} F \varGamma - \big(q + q^{-1}\big) F \\ K^{\pm 1} E = q^{\pm 2} E K^{\pm 1}\,, \quad HE = q^{+2} EH + (q+1) E\,, \quad \varGamma E = q^{+2} E \varGamma + \big(q + q^{-1}\big) E\end{gathered}$$

[5]In §§7–9 we should use notation $U_{q-1}(\mathfrak{g})$ and $F_{q-1}[G]$, after Remark 1.5 (for $\hbar = q - 1$); instead, we write $U_q(\mathfrak{g})$ and $F_q[G]$ to be consistent with the standard notation in use for these quantum algebras.

and with a Hopf structure given by the same formulas as above for F, $K^{\pm 1}$, and E plus

$$\begin{array}{lll} \Delta(\mathit{\Gamma}) = \mathit{\Gamma} \otimes K + K^{-1} \otimes \mathit{\Gamma}\,, & \epsilon(\mathit{\Gamma}) = 0\,, & S(\mathit{\Gamma}) = -\mathit{\Gamma} \\ \Delta(H) = H \otimes 1 + K \otimes H\,, & \epsilon(H) = 0\,, & S(H) = -K^{-1} H\,. \end{array}$$

Note also that $K = 1+(q-1)H$ and $K^{-1} = K - \big(q-q^{-1}\big)\mathit{\Gamma} = 1+(q-1)H - \big(q-q^{-1}\big)\mathit{\Gamma}$, hence $U_q(\mathfrak{g})$ is generated even by F, H, $\mathit{\Gamma}$ and E alone. Further, notice that

$$\mathbb{U}_q(\mathfrak{g}) \;=\; \text{free } F(R)\text{–module over } \Big\{ F^a K^z E^d \,\Big|\, a, d \in \mathbb{N}, z \in \mathbb{Z} \Big\} \tag{7.1}$$

$$U_q(\mathfrak{g}) \;=\; R\text{–span of } \Big\{ F^a H^b \mathit{\Gamma}^c E^d \,\Big|\, a, b, c, d \in \mathbb{N} \Big\} \text{ inside } \mathbb{U}_q(\mathfrak{g}) \tag{7.2}$$

which implies that $F(R) \otimes_R U_q(\mathfrak{g}) = \mathbb{U}_q(\mathfrak{g})$. Moreover, definitions imply at once that $U_q(\mathfrak{g})$ is torsion-free, and also that it is a Hopf R–subalgebra of $\mathbb{U}_q(\mathfrak{g})$. Therefore $U_q(\mathfrak{g}) \in \mathcal{HA}$, and in fact $U_q(\mathfrak{g})$ is even a QrUEA, whose semiclassical limit is $U(\mathfrak{g}) = U(\mathfrak{sl}_2)$, with the generators F, $K^{\pm 1}$, H, $\mathit{\Gamma}$, E respectively mapping to f, 1, h, h, $e \in U(\mathfrak{sl}_2)$.

It is also possible to define a "simply connected" version of $\mathbb{U}_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$, obtained from the previous ones — referred to as the "adjoint (type) ones" — as follows. For $\mathbb{U}_q(\mathfrak{g})$, one simply adds a square root of $K^{\pm 1}$, call it $L^{\pm 1}$, as new generator; for $U_q(\mathfrak{g})$ one adds the new generators $L^{\pm 1}$ and also $D := \dfrac{L-1}{q-1}$. Then the same analysis as before shows that $U_q(\mathfrak{g})$ is another quantization (containing the "adjoint" one) of $U(\mathfrak{g})$.

In the general case of semisimple $\mathfrak{g}$, let $\mathbb{U}_q(\mathfrak{g})$ be the Lusztig-like quantum group — over R — associated to $\mathfrak{g} = \mathfrak{g}^\tau$ as in [Ga1], namely $\mathbb{U}_q(\mathfrak{g}) := U^M_{q,\varphi}(\mathfrak{g})$ with respect to the notation in [*loc. cit.*], where M is any intermediate lattice such that $Q \leq M \leq P$ (this is just a matter of choice, of the type mentioned in the statement of Theorem 2.2*(c)*): this is a Hopf algebra over $F(R)$, generated by elements F_i, M_i, E_i for $i = 1, \dots, r =: \mathit{rank}(\mathfrak{g})$. Then let $U_q(\mathfrak{g})$ be the unital R–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ generated by the elements F_i, $H_i := \dfrac{M_i - 1}{q-1}$, $\mathit{\Gamma}_i := \dfrac{K_i - K_i^{-1}}{q - q^{-1}}$, $M_i^{\pm 1}$, E_i, where the $K_i = M_{\alpha_i}$ are suitable product of M_j's, defined as in [Ga1], §2.2 (whence $K_i, K_i^{-1} \in U_q(\mathfrak{g})$). From [Ga1], §§2.5, 3.3, we have that $\mathbb{U}_q(\mathfrak{g})$ is the free $F(R)$–module with basis the set of monomials

$$\bigg\{ \prod_{\alpha \in \Phi^+} F_\alpha^{f_\alpha} \cdot \prod_{i=1}^n K_i^{z_i} \cdot \prod_{\alpha \in \Phi^+} E_\alpha^{e_\alpha} \,\bigg|\, f_\alpha, e_\alpha \in \mathbb{N}\,,\; z_i \in \mathbb{Z}\,,\;\; \forall\, \alpha \in \Phi^+,\, i = 1, \dots, n \bigg\}$$

while $U_q(\mathfrak{g})$ is the R–span inside $\mathbb{U}_q(\mathfrak{g})$ of the set of monomials

$$\bigg\{ \prod_{\alpha \in \Phi^+} F_\alpha^{f_\alpha} \cdot \prod_{i=1}^n H_i^{t_i} \cdot \prod_{j=1}^n \mathit{\Gamma}_j^{c_j} \cdot \prod_{\alpha \in \Phi^+} E_\alpha^{e_\alpha} \,\bigg|\, f_\alpha, t_i, c_j, e_\alpha \in \mathbb{N} \;\; \forall\, \alpha \in \Phi^+,\, i, j = 1, \dots, n \bigg\}$$

(hereafter, Φ^+ is the set of positive roots of $\mathfrak{g}$, each E_α, resp. F_α, is a root vector attached to $\alpha \in \Phi^+$, resp. to $-\alpha \in (-\Phi^+)$, and the products of factors indexed by Φ^+ are ordered with respect to a fixed convex order of Φ^+, see [Ga1]), whence (as for $n=2$) $U_q(\mathfrak{g})$ is a free R–module. In this case again $U_q(\mathfrak{g})$ is a QrUEA, with semiclassical limit $U(\mathfrak{g})$.

7.3 Computation of $U_q(\mathfrak{g})'$ and specialization $U_q(\mathfrak{g})' \xrightarrow{q \to 1} F[G^*]$. We begin with the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. From the definition of $U_q(\mathfrak{g}) = U_q(\mathfrak{sl}_2)$ we have ($\forall\, n \in \mathbb{N}$)

$$\delta_n(E) = (\mathrm{id}-\epsilon)^{\otimes n}\big(\Delta^n(E)\big) = (\mathrm{id}-\epsilon)^{\otimes n}\Big(\textstyle\sum_{s=1}^n K^{\otimes(s-1)} \otimes E \otimes 1^{\otimes(n-s)}\Big) =$$
$$= (\mathrm{id}-\epsilon)^{\otimes n}\big(K^{\otimes(n-1)} \otimes E\big) = (K-1)^{\otimes(n-1)} \otimes E = (q-1)^{n-1} \cdot H^{\otimes(n-1)} \otimes E$$

from which $\delta_n\big((q-1)E\big) \in (q-1)^n U_q(\mathfrak{g}) \setminus (q-1)^{n+1} U_q(\mathfrak{g})$, whence $(q-1)E \in U_q(\mathfrak{g})'$, whereas $E \notin U_q(\mathfrak{g})'$. Similarly, $(q-1)F \in U_q(\mathfrak{g})'$, whilst $F \notin U_q(\mathfrak{g})'$. As for generators H, $\varGamma$, $K^{\pm 1}$, we have $\Delta^n(H) = \sum_{s=1}^n K^{\otimes(s-1)} \otimes H \otimes 1^{\otimes(n-s)}$, $\Delta^n\big(K^{\pm 1}\big) = \big(K^{\pm 1}\big)^{\otimes n}$, $\Delta^n(\varGamma) = \sum_{s=1}^n K^{\otimes(s-1)} \otimes \varGamma \otimes \big(K^{-1}\big)^{\otimes(n-s)}$, hence for $\delta_n = (\mathrm{id}-\epsilon)^{\otimes n} \circ \Delta^n$ we have

$$\delta_n(H) = (q-1)^{n-1} \cdot H^{\otimes n}\,, \qquad \delta^n\big(K^{-1}\big) = (q-1)^n \cdot \big(-K^{-1}H\big)^{\otimes n}$$
$$\delta^n(K) = (q-1)^n \cdot H^{\otimes n}\,, \quad \delta^n(\varGamma) = (q-1)^{n-1} \cdot \sum_{s=1}^n (-1)^{n-s} H^{\otimes(s-1)} \otimes \varGamma \otimes \big(HK^{-1}\big)^{\otimes(n-s)}$$

for all $n \in \mathbb{N}$, so that $(q-1)H$, $(q-1)\varGamma$, $K^{\pm 1} \in U_q(\mathfrak{g})' \setminus (q-1)U_q(\mathfrak{g})'$. Therefore $U_q(\mathfrak{g})'$ contains the subalgebra U' generated by $(q-1)F$, K, K^{-1}, $(q-1)H$, $(q-1)\varGamma$, $(q-1)E$. On the other hand, using (7.2) a thorough — but straightforward — computation along the same lines as above shows that any element in $U_q(\mathfrak{g})'$ does necessarily lie in U' (details are left to the reader: everything follows from definitions and the formulas above for Δ^n). Thus $U_q(\mathfrak{g})'$ is nothing but the subalgebra of $U_q(\mathfrak{g})$ generated by $\dot{F} := (q-1)F$, K, K^{-1}, $\dot{H} := (q-1)H$, $\dot{\varGamma} := (q-1)\varGamma$, $\dot{E} := (q-1)E$; notice also that the generator $\dot{H}$ is unnecessary, for $\dot{H} = K-1$. As a consequence, $U_q(\mathfrak{g})'$ can be presented as the unital associative R–algebra with generators $\dot{F}$, $\dot{\varGamma}$, $K^{\pm 1}$, $\dot{E}$ and relations

$$KK^{-1} = 1 = K^{-1}K, \;\; K^{\pm 1}\dot{\varGamma} = \dot{\varGamma}K^{\pm 1}, \;\; \big(1+q^{-1}\big)\dot{\varGamma} = K - K^{-1}, \;\; \dot{E}\dot{F} - \dot{F}\dot{E} = (q-1)\dot{\varGamma}$$
$$K - K^{-1} = \big(1+q^{-1}\big)\dot{\varGamma}\,, \quad K^{\pm 1}\dot{F} = q^{\mp 2}\dot{F}K^{\pm 1}\,, \quad K^{\pm 1}\dot{E} = q^{\pm 2}\dot{E}K^{\pm 1}$$
$$\dot{\varGamma}\dot{F} = q^{-2}\dot{F}\dot{\varGamma} - (q-1)\big(q+q^{-1}\big)\dot{F}\,, \qquad \dot{\varGamma}\dot{E} = q^{+2}\dot{E}\dot{\varGamma} + (q-1)\big(q+q^{-1}\big)\dot{E}$$

with Hopf structure given by

$$\begin{array}{rlr}
\Delta\big(\dot{F}\big) = \dot{F} \otimes K^{-1} + 1 \otimes \dot{F}\,, & \epsilon\big(\dot{F}\big) = 0\,, & S\big(\dot{F}\big) = -\dot{F}K \\
\Delta\big(\dot{\varGamma}\big) = \dot{\varGamma} \otimes K + K^{-1} \otimes \dot{\varGamma}\,, & \epsilon\big(\dot{\varGamma}\big) = 0\,, & S\big(\dot{\varGamma}\big) = -\dot{\varGamma} \\
\Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, & \epsilon\big(K^{\pm 1}\big) = 1\,, & S\big(K^{\pm 1}\big) = K^{\mp 1} \\
\Delta\big(\dot{E}\big) = \dot{E} \otimes 1 + K \otimes \dot{E}\,, & \epsilon\big(\dot{E}\big) = 0\,, & S\big(\dot{E}\big) = -K^{-1}\dot{E}\,.
\end{array}$$

When $q \to 1$, an easy direct computation shows that this gives a presentation of the function algebra $F\big[{}_aSL_2^{\,*}\big]$, and the Poisson structure that $F\big[{}_aSL_2^{\,*}\big]$ inherits from this quantization process is exactly the one coming from the Poisson structure on ${}_aSL_2{}^*$: in fact, there is a Poisson Hopf algebra isomorphism

$$U_q(\mathfrak{g})' \Big/ (q-1)\, U_q(\mathfrak{g})' \xrightarrow{\;\cong\;} F\big[{}_aSL_2{}^*\big] \quad \Big(\subseteq F\big[{}_sSL_2{}^*\big] \Big)$$

given by: $\dot{E} \mod (q-1) \mapsto xz$, $K^{\pm 1} \mod (q-1) \mapsto z^{\pm 2}$, $\dot{H} \mod (q-1) \mapsto z^2 - 1$, $\dot{\varGamma} \mod (q-1) \mapsto \big(z^2 - z^{-2}\big)\big/2$, $\dot{F} \mod (q-1) \mapsto z^{-1} y$. In other words, $U_q(\mathfrak{g})'$ specializes to $F\big[{}_aSL_2{}^*\big]$ as a Poisson Hopf algebra. *Note that this was predicted by Theorem 2.2(c) when* $Char(\Bbbk) = 0$, *but our analysis now proved it also for* $Char(\Bbbk) > 0$.

Note that we got the *adjoint* Poisson group dual of $G = SL_2$, that is ${}_aSL_2{}^*$; a different choice of the initial QrUEA leads us to the *simply connected* one, i.e. ${}_sSL_2{}^*$. Indeed, if we start from the "simply connected" version of $U_q(\mathfrak{g})$ (see §7.2) the same analysis shows that $U_q(\mathfrak{g})'$ is like above but for containing also the new generators $L^{\pm 1}$, and similarly when specializing q at 1: thus we get the function algebra of a Poisson group which is a double covering of ${}_aSL_2{}^*$, namely ${}_sSL_2{}^*$. So changing the QrUEA quantizing $\mathfrak{g}$ we get two different QFAs, one for each of the two connected Poisson algebraic groups dual of SL_2, i.e. with tangent Lie bialgebra $\mathfrak{sl}_2{}^*$; this shows the dependence of the group $G^\star$ (here denoted G^* since $\mathfrak{g}^\times = \mathfrak{g}^*$) in Theorem 2.2*(c)* on the choice of the QrUEA (for a fixed $\mathfrak{g}$).

With a bit more careful study, exploiting the analysis in [Ga1], one can treat the general case too: we sketch briefly our arguments — restricting to the simply laced case, to simplify the exposition — leaving to the reader the (straightforward) task of filling in details.

So now let $\mathfrak{g} = \mathfrak{g}^\tau$ be a semisimple Lie algebra, as in §7.1, and let $U_q(\mathfrak{g})$ be the QrUEA introduced in §7.2: our aim again is to compute the QFA $U_q(\mathfrak{g})'$.

The same computations as for $\mathfrak{g} = \mathfrak{sl}(2)$ show that $\delta_n(H_i) = (q-1)^{n-1} \cdot H_i^{\otimes n}$ and $\delta^n(\varGamma_i) = (q-1)^{n-1} \cdot \sum_{s=1}^{n} (-1)^{n-s} H_i^{\otimes (s-1)} \otimes \varGamma_i \otimes \big(H_i K_i^{-1}\big)^{\otimes (n-s)}$, which gives

$$\dot{H}_i := (q-1) H_i \in U_q(\mathfrak{g})' \setminus (q-1)\, U_q(\mathfrak{g})' \quad \text{and} \quad \dot{\varGamma}_i := (q-1)\, \varGamma_i \in U_q(\mathfrak{g})' \setminus (q-1)\, U_q(\mathfrak{g})' .$$

As for root vectors, let $\dot{E}_\gamma := (q-1) E_\gamma$ and $\dot{F}_\gamma := (q-1) F_\gamma$ for all $\gamma \in \Phi^+$: using the same type of arguments as in [Ga1][6], §5.16, we can prove that $E_\alpha \notin U_q(\mathfrak{g})'$ but $\dot{E}_\alpha \in U_q(\mathfrak{g})' \setminus (q-1)\, U_q(\mathfrak{g})'$. In fact, let $\mathbb{U}_q(\mathfrak{b}_+)$ and $\mathbb{U}_q(\mathfrak{b}_-)$ be quantum Borel subalgebras, and $\mathfrak{U}_{\varphi,\geq}^M$, $\mathcal{U}_{\varphi,\geq}^M$, $\mathfrak{U}_{\varphi,\leq}^M$, $\mathcal{U}_{\varphi,\leq}^M$ their R–subalgebras defined in [Ga1], §2: then both $\mathbb{U}_q(\mathfrak{b}_+)$ and $\mathbb{U}_q(\mathfrak{b}_-)$ are Hopf subalgebras of $\mathbb{U}_q(\mathfrak{g})$; in addition, letting M' be the lattice between Q and P dual of M (in the sense of [Ga1], §1.1, there exists an $F(R)$–valued perfect Hopf pairing between $\mathbb{U}_q(\mathfrak{b}_\pm)$ and $\mathbb{U}_q(\mathfrak{b}_\mp)$ — one built up on M and the other on M' — such that $\mathfrak{U}_{\varphi,\geq}^M = \Big(\mathcal{U}_{\varphi,\leq}^{M'}\Big)^\bullet$, $\mathfrak{U}_{\varphi,\leq}^M = \Big(\mathcal{U}_{\varphi,\geq}^{M'}\Big)^\bullet$, $\mathcal{U}_{\varphi,\geq}^M = \Big(\mathfrak{U}_{\varphi,\leq}^{M'}\Big)^\bullet$, and $\mathcal{U}_{\varphi,\leq}^M = \Big(\mathfrak{U}_{\varphi,\geq}^{M'}\Big)^\bullet$.

[6] Note that in [Ga1] the assumption $Char(\Bbbk) = 0$ is made throughout: nevertheless, *this hypothesis is not necesary* for the analysis we are concerned with right now!

Now, $(q-q^{-1})E_\alpha \in \mathcal{U}^{\scriptscriptstyle M}_{\varphi,\geq} = \Big(\mathfrak{U}^{\scriptscriptstyle M'}_{\varphi,\leq}\Big)^{\bullet}$, hence — since $\mathfrak{U}^{\scriptscriptstyle M'}_{\varphi,\leq}$ is an algebra — we have $\Delta\Big(\big(q-q^{-1}\big)E_\alpha\Big) \in \Big(\mathfrak{U}^{\scriptscriptstyle M'}_{\varphi,\leq} \otimes \mathfrak{U}^{\scriptscriptstyle M'}_{\varphi,\leq}\Big)^{\bullet} = \Big(\mathfrak{U}^{\scriptscriptstyle M'}_{\varphi,\leq}\Big)^{\bullet} \otimes \Big(\mathfrak{U}^{\scriptscriptstyle M'}_{\varphi,\leq}\Big)^{\bullet} = \mathcal{U}^{\scriptscriptstyle M}_{\varphi,\geq} \otimes \mathcal{U}^{\scriptscriptstyle M}_{\varphi,\geq}$. Therefore, by definition of $\mathcal{U}^{\scriptscriptstyle M}_{\varphi,\geq}$ and by the PBW theorem for it and for $\mathfrak{U}^{\scriptscriptstyle M'}_{\varphi,\leq}$ (cf. [Ga1], §2.5) we have that $\Delta\Big(\big(q-q^{-1}\big)E_\alpha\Big)$ is an R–linear combination like $\Delta\Big(\big(q-q^{-1}\big)E_\alpha\Big) = \sum_r A_r^{(1)} \otimes A_r^{(2)}$ in which the $A_r^{(j)}$'s are monomials in the M_j's and in the $\overline{E}_\gamma$'s, where $\overline{E}_\gamma := \big(q-q^{-1}\big)E_\gamma$ for all $\gamma \in \Phi^+$: iterating, we find that $\Delta^\ell\Big(\big(q-q^{-1}\big)E_\alpha\Big)$ is an R–linear combination

$$\Delta^\ell\Big(\big(q-q^{-1}\big)E_\alpha\Big) \;=\; \textstyle\sum_r A_r^{(1)} \otimes A_r^{(2)} \otimes \cdots \otimes A_r^{(\ell)} \tag{7.3}$$

in which the $A_r^{(j)}$'s are again monomials in the M_j's and in the $\overline{E}_\gamma$'s. Now, we distinguish two cases: either $A_r^{(j)}$ does contain some $\overline{E}_\gamma \; \big(\in \big(q-q^{-1}\big)\, U_q(\mathfrak{g})\big)$, thus $\epsilon\Big(A_r^{(j)}\Big) = A_r^{(j)} \in (q-1)\, U_q(\mathfrak{g})$ whence $(\mathrm{id}-\epsilon)\Big(A_r^{(j)}\Big) = 0$; or $A_r^{(j)}$ does not contain any $\overline{E}_\gamma$ and is only a monomial in the M_t's, say $A_r^{(j)} = \prod_{t=1}^n M_t^{m_t}$: then $(\mathrm{id}-\epsilon)\Big(A_r^{(j)}\Big) = \prod_{t=1}^n M_t^{m_t} - 1 = \prod_{t=1}^n \big((q-1)\,H_t + 1\big)^{m_t} - 1 \in (q-1)\, U_q(\mathfrak{g})$. In addition, for some "Q–grading reasons" (as in [Ga1], §5.16), in each one of the summands in (7.3) the sum of all the γ's such that the (rescaled) root vectors $\overline{E}_\gamma$ occur in any of the factors $A_r^{(1)}$, $A_r^{(2)}$, …, $A_r^{(n)}$ must be equal to α: therefore, in each of these summands at least one factor $\overline{E}_\gamma$ does occur. The conclusion is that $\delta_\ell\big(\overline{E}_\alpha\big) \in \big(1+q^{-1}\big)(q-1)^\ell\, U_q(\mathfrak{g})^{\otimes \ell}$ (the factor $\big(1+q^{-1}\big)$ being there because at least one rescaled root vector $\overline{E}_\gamma$ occurs in each summand of $\delta_\ell\big(\overline{E}_\alpha\big)$, thus providing a coefficient $\big(q-q^{-1}\big)$ the term $\big(1+q^{-1}\big)$ is factored out of), whence $\delta_\ell\big(\dot{E}_\alpha\big) \in (q-1)^\ell\, U_q(\mathfrak{g})^{\otimes \ell}$. More precisely, we have also $\delta_\ell\big(\dot{E}_\alpha\big) \notin (q-1)^{\ell+1}\, U_q(\mathfrak{g})^{\otimes \ell}$, for we can easily check that $\Delta^\ell\big(\dot{E}_\alpha\big)$ is the sum of $M_\alpha \otimes M_\alpha \otimes \cdots \otimes M_\alpha \otimes \dot{E}_\alpha$ plus other summands which are R–linearly independent of this first term: but then $\delta_\ell\big(\dot{E}_\alpha\big)$ is the sum of $(q-1)^{\ell-1} H_\alpha \otimes H_\alpha \otimes \cdots \otimes H_\alpha \otimes \dot{E}_\alpha$ (where $H_\alpha := \frac{M_\alpha - 1}{q-1}$ is equal to an R–linear combination of products of M_j's and H_t's) plus other summands which are R–linearly independent of the first one, and since $H_\alpha \otimes H_\alpha \otimes \cdots \otimes H_\alpha \otimes \dot{E}_\alpha \notin (q-1)^2\, U_q(\mathfrak{g})^{\otimes \ell}$ we can conclude as claimed. Therefore $\delta_\ell\big(\dot{E}_\alpha\big) \in (q-1)^\ell\, U_q(\mathfrak{g})^{\otimes \ell} \setminus (q-1)^{\ell+1}\, U_q(\mathfrak{g})^{\otimes \ell}$, whence we get $\dot{E}_\alpha := (q-1)E_\alpha \in U_q(\mathfrak{g})' \setminus (q-1)\, U_q(\mathfrak{g})' \;\; \forall\, \alpha \in \Phi^+$. An entirely similar analysis yields also $\dot{F}_\alpha := (q-1)F_\alpha \in U_q(\mathfrak{g})' \setminus (q-1)\, U_q(\mathfrak{g})' \;\; \forall\, \alpha \in \Phi^+$.

Summing up, we have found that $U_q(\mathfrak{g})'$ contains for sure the subalgebra U' generated by $\dot{F}_\alpha$, $\dot{H}_i$, $\dot{\varGamma}_i$, $\dot{E}_\alpha$ for all $\alpha \in \Phi^+$ and all $i = 1, \dots, n$. On the other hand, using (7.2) a thorough — but straightforward — computation along the same lines as above shows that any element in $U_q(\mathfrak{g})'$ must lie in U' (details are left to the reader). Thus finally $U_q(\mathfrak{g})' = U'$, so we have a concrete description of $U_q(\mathfrak{g})'$.

Now compare $U' = U_q(\mathfrak{g})'$ with the algebra $\mathcal{U}^{\scriptscriptstyle M}_\varphi(\mathfrak{g})$ in [Ga1], §3.4 (for $\varphi = 0$), the latter being just the R–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ generated by the set $\big\{\, \overline{F}_\alpha, M_i, \overline{E}_\alpha \,\big|\, \alpha \in \Phi^+, i = 1, \dots, n \,\big\}$. First of all, by definition, we have $\mathcal{U}^{\scriptscriptstyle M}_\varphi(\mathfrak{g}) \subseteq U' = U_q(\mathfrak{g})'$; moreover,

$$\dot{F}_\alpha \equiv \frac{1}{2}\overline{F}_\alpha\,, \quad \dot{E}_\alpha \equiv \frac{1}{2}\overline{E}_\alpha\,, \quad \dot{\mathit{\Gamma}}_i \equiv \frac{1}{2}\big(K_i - K_i^{-1}\big) \mod (q-1)\,\mathcal{U}_\varphi^{\scriptscriptstyle M}(\mathfrak{g}) \qquad \forall\ \alpha,\, \forall\ i\,.$$

Then

$$\big(U_q(\mathfrak{g})'\big)_1 := U_q(\mathfrak{g})'\Big/(q-1)\,U_q(\mathfrak{g})' \,=\, \mathcal{U}_\varphi^{\scriptscriptstyle M}(\mathfrak{g})\Big/(q-1)\,\mathcal{U}_\varphi^{\scriptscriptstyle M}(\mathfrak{g}) \,\cong\, F\big[G_{\scriptscriptstyle M}^*\big]$$

where $G_{\scriptscriptstyle M}^*$ is the Poisson group dual of $G = G^\tau$ with centre $Z(G_{\scriptscriptstyle M}^*) \cong M\big/Q$ and fundamental group $\pi_1(G_{\scriptscriptstyle M}^*) \cong P\big/M$, and the isomorphism (of Poisson Hopf algebras) on the right is given by [Ga1], Theorem 7.4 (see also references therein for the original statement and proof of this result). In other words, $U_q(\mathfrak{g})'$ specializes to $F\big[G_{\scriptscriptstyle M}^*\big]$ *as a Poisson Hopf algebra*, as prescribed by Theorem 2.2. By the way, notice that in the present case the dependence of the dual group $G^\star = G_{\scriptscriptstyle M}^*$ on the choice of the initial QrUEA (for fixed $\mathfrak{g}$) — mentioned in the last part of the statement of Theorem 2.2*(c)* — is evident.

By the way, the previous discussion applies as well to the case of $\mathfrak{g}$ *an untwisted affine Kac-Moody algebra*: one just has to substitute any quotation from [Ga1] — referring to some result about *finite* Kac-Moody algebras — with a similar quotation from [Ga3] — referring to the corresponding analogous result about untwisted *affine* Kac-Moody algebras.

7.4 The identity $\big(U_q(\mathfrak{g})'\big)^\vee = U_q(\mathfrak{g})$. In the present section we check that part of Theorem 2.2*(b)* claiming that, when $p = 0$, one has $H \in \mathcal{QrUEA} \Longrightarrow \big(H'\big)^\vee = H$ for $H = U_q(\mathfrak{g})$ as above. In addition, our proof now will work for the case $p > 0$ as well. Of course, we start once again from $\mathfrak{g} = \mathfrak{sl}_2$.

Since $\epsilon\big(\dot{F}\big) = \epsilon\big(\dot{H}\big) = \epsilon\big(\dot{\mathit{\Gamma}}\big) = \epsilon\big(\dot{E}\big) = 0$, the ideal $J := Ker\big(\epsilon \colon U_q(\mathfrak{g})' \longrightarrow R\big)$ is generated by $\dot{F}$, $\dot{H}$, $\dot{\mathit{\Gamma}}$, and $\dot{E}$. This implies that J is the R–span of $\Big\{ \dot{F}^\varphi \dot{H}^\kappa \dot{\mathit{\Gamma}}^\gamma \dot{E}^\eta \,\Big|\, (\varphi,\kappa,\gamma,\eta) \in \mathbb{N}^4 \setminus \{(0,0,0,0)\} \Big\}$. Now $I := Ker\Big(U_q(\mathfrak{g})' \xrightarrow{\ \epsilon\ } R \xrightarrow{\ q\mapsto 1\ } \Bbbk\Big) = (q-1)\cdot U_q(\mathfrak{g})' + J$, therefore we get that $\big(U_q(\mathfrak{g})'\big)^\vee := \sum_{n\geq 0} \Big((q-1)^{-1} I\Big)^n$ is generated, as a unital R-subalgebra of $\mathbb{U}_q(\mathfrak{g})$, by the elements $(q-1)^{-1}\dot{F} = F$, $(q-1)^{-1}\dot{H} = H$, $(q-1)^{-1}\dot{\mathit{\Gamma}} = \mathit{\Gamma}$, $(q-1)^{-1}\dot{E} = E$, hence it coincides with $U_q(\mathfrak{g})$, q.e.d.

An entirely similar analysis works in the "adjoint" case as well; and also, *mutatis mutandis*, for the general semisimple or affine Kac-Moody case.

7.5 The quantum hyperalgebra $Hyp_q(\mathfrak{g})$. Let G be a semisimple (affine) algebraic group, with Lie algebra $\mathfrak{g}$, and let $\mathbb{U}_q(\mathfrak{g})$ be the quantum group considered in the previous sections. Lusztig introduced (cf. [Lu1-2]) a "quantum hyperalgebra", i.e. a Hopf subalgebra of $\mathbb{U}_q(\mathfrak{g})$ over $\mathbb{Z}\big[q,q^{-1}\big]$ whose specialization at $q = 1$ is exactly the Kostant's $\mathbb{Z}$–integer form $U_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$ from which one gets the hyperalgebra $Hyp\,(\mathfrak{g})$ over any field $\Bbbk$ of characteristic $p > 0$ by scalar extension, namely $Hyp\,(\mathfrak{g}) = \Bbbk \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g})$. In fact, to be precise one needs a suitable enlargement of the algebra given by Lusztig, which is given in [DL], §3.4, and denoted by $\varGamma(\mathfrak{g})$. Now we study Drinfeld's functors (at $\hbar = q-1$) on $\mathit{Hyp}_q(\mathfrak{g}) := R \otimes_{\mathbb{Z}[q,q^{-1}]} \varGamma(\mathfrak{g})$ (with R like in §7.2), taking as sample the case of $\mathfrak{g} = \mathfrak{sl}_2$.

Let $\mathfrak{g} = \mathfrak{sl}_2$. Let $\mathit{Hyp}_q^{\mathbb{Z}}(\mathfrak{g})$ be the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ (say the one of "adjoint type" defined like above *but over* $\mathbb{Z}\big[q,q^{-1}\big]$) generated the "quantum divided powers" $F^{(n)} := F^n \Big/ [n]_q!$, $\displaystyle \binom{K\,;\,c}{n} := \prod_{s=1}^{n} \frac{q^{c+1-s}K-1}{q^s-1}$, $E^{(n)} := E^n \Big/ [n]_q!$ (for all $n \in \mathbb{N}$, $c \in \mathbb{Z}$) and by K^{-1}, where $[n]_q! := \prod_{s=1}^{n} [s]_q$ and $[s]_q = \big(q^s - q^{-s}\big) \Big/ \big(q-q^{-1}\big)$ for all $n, s \in \mathbb{N}$. Then (cf. [DL]) this is a Hopf subalgebra of $\mathbb{U}_q(\mathfrak{g})$, and $\mathit{Hyp}_q^{\mathbb{Z}}(\mathfrak{g})\Big|_{q=1} \cong U_{\mathbb{Z}}(\mathfrak{g})$; therefore $\mathit{Hyp}_q(\mathfrak{g}) := R \otimes_{\mathbb{Z}[q,q^{-1}]} \mathit{Hyp}_q^{\mathbb{Z}}(\mathfrak{g})$ (for any R like in §7.2, with $\Bbbk := R \big/ \hbar R$ and $p := \mathit{Char}(\Bbbk)$) specializes at $q=1$ to the $\Bbbk$–hyperalgebra $\mathit{Hyp}(\mathfrak{g})$. Moreover, among all the $\binom{K;\,c}{n}$'s it is enough to take only those with $c=0$. *From now on we assume* $p>0$.

Using formulas for the iterated coproduct in [DL], Corollary 3.3 (which uses the opposite coproduct than ours, but this doesn't matter), and exploiting the PBW-like theorem for $\mathit{Hyp}_q(\mathfrak{g})$ (see [DL] again) we see by direct inspection that $\mathit{Hyp}_q(\mathfrak{g})'$ is the unital R–subalgebra of $\mathit{Hyp}_q(\mathfrak{g})$ generated by K^{-1} and the "rescaled quantum divided powers" $(q-1)^n F^{(n)}$, $(q-1)^n \binom{K;\,0}{n}$ and $(q-1)^n E^{(n)}$ for all $n \in \mathbb{N}$. Since $[n]_q!\Big|_{q=1} = n! = 0$ iff $p \,\Big|\, n$, we argue that $\mathit{Hyp}_q(\mathfrak{g})'\Big|_{q=1}$ is generated by the corresponding specializations of $(q-1)^{p^s} F^{(p^s)}$, $(q-1)^{p^s} \binom{K;\,0}{p^s}$ and $(q-1)^{p^s} E^{(p^s)}$ for all $s \in \mathbb{N}$: in particular this shows that the spectrum of $\mathit{Hyp}_q(\mathfrak{g})'\Big|_{q=1}$ has dimension 0 and height 1, and its cotangent Lie algebra $J \Big/ J^{\,2}$ — where J is the augmentation ideal of $\mathit{Hyp}_q(\mathfrak{g})'\Big|_{q=1}$ — has basis $\Big\{ (q-1)^{p^s} F^{(p^s)},\, (q-1)^{p^s} \binom{K;\,0}{p^s},\, (q-1)^{p^s} E^{(p^s)} \mod (q-1)\,\mathit{Hyp}_q(\mathfrak{g})' \mod J^{\,2} \,\Big|\, s \in \mathbb{N} \Big\}$. Furthermore, $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee$ is generated by $(q-1)^{p^s-1} F^{(p^s)}$, $(q-1)^{p^s-1} \binom{K;\,0}{p^s}$, K^{-1} and $(q-1)^{p^s-1} E^{(p^s)}$ for all $s \in \mathbb{N}$: in particular we have that $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee \subsetneqq \mathit{Hyp}_q(\mathfrak{g})$, and $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee\Big|_{q=1}$ is generated by the cosets modulo $(q-1)$ of the previous elements, which do form a basis of the restricted Lie bialgebra $\mathfrak{k}$ such that $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee\Big|_{q=1} = \mathbf{u}(\mathfrak{k})$.

We performed the previous study using the "adjoint" version of $U_q(\mathfrak{g})$ as starting point: instead, we can use as well its "simply connected" version, thus obtaining a "simply connected version of $\mathit{Hyp}_q(\mathfrak{g})$" which is defined exactly like before but for using $L^{\pm 1}$ instead of $K^{\pm 1}$ throughout; up to these changes, the analysis and its outcome will be exactly the same. Note that all quantum objects involved — namely, $\mathit{Hyp}_q(\mathfrak{g})$, $\mathit{Hyp}_q(\mathfrak{g})'$ and $\big(\mathit{Hyp}_q(\mathfrak{g})'\big)^\vee$ — will strictly contain the corresponding "adjoint" quantum objects; on the other hand, the semiclassical limit is the same in the case of $\mathit{Hyp}_q(\mathfrak{g})$ (giving $\mathit{Hyp}(\mathfrak{g})$, in both cases) and in the case of $\big(Hyp_q(\mathfrak{g})'\big)^\vee$ (giving $\mathbf{u}(\mathfrak{k})$, in both cases), whereas the semiclassical limit of $Hyp_q(\mathfrak{g})'$ in the "simply connected" case is a (countable) covering of that in the "adjoint" case.

The general case of semisimple or affine Kac-Moody $\mathfrak{g}$ can be dealt with similarly, with analogous outcome. Indeed, $Hyp_q^{\mathbb{Z}}(\mathfrak{g})$ is defined as the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbb{U}_q(\mathfrak{g})$ (defined like before *but over* $\mathbb{Z}\big[q,q^{-1}\big]$) generated by K_i^{-1} and the "quantum divided powers" (in the above sense) $F_i^{(n)}$, $\left({K_i\,;\,c \atop n}\right)$, $E_i^{(n)}$ for all $n \in \mathbb{N}$, $c \in \mathbb{Z}$ and $i = 1, \dots, \mathit{rank}(\mathfrak{g})$ (notation of §7.2, but now each divided power relative to i is built upon q_i, see [Ga1]). Then (cf. [DL]) this is a Hopf subalgebra of $\mathbb{U}_q(\mathfrak{g})$ with $Hyp_q^{\mathbb{Z}}(\mathfrak{g})\Big|_{q=1} \cong U_{\mathbb{Z}}(\mathfrak{g})$, so $Hyp_q(\mathfrak{g}) := R \otimes_{\mathbb{Z}[q,q^{-1}]} Hyp_q^{\mathbb{Z}}(\mathfrak{g})$ (for any R like before) specializes at $q=1$ to the $\Bbbk$–hyperalgebra $Hyp(\mathfrak{g})$; and among the $\left({K_i\,;\,c \atop n}\right)$'s it is enough to take those with $c = 0$.

Again a PBW-like theorem holds for $Hyp_q(\mathfrak{g})$ (see [DL]), where powers of root vectors are replaced by quantum divided powers like $F_\alpha^{(n)}$, $\left({K_i\,;\,c \atop n}\right) \cdot K_i^{-Ent(n/2)}$ and $E_\alpha^{(n)}$, for all positive roots α of $\mathfrak{g}$ (each divided power being relative to q_α, see [Ga1]) both in the finite and in the affine case. Using this and the same type of arguments as in §7.3 — i.e. the perfect graded Hopf pairing between quantum Borel subalgebras — we see by direct inspection that $Hyp_q(\mathfrak{g})'$ is the unital R–subalgebra of $Hyp_q(\mathfrak{g})$ generated by the K_i^{-1}'s and the "rescaled quantum divided powers" $(q_\alpha - 1)^n F_\alpha^{(n)}$, $(q_i - 1)^n \left({K_i\,;\,0 \atop n}\right)$ and $(q_\alpha - 1)^n E_\alpha^{(n)}$ for all $n \in \mathbb{N}$. Since $[n]_{q_\alpha}!\Big|_{q=1} = n! = 0$ iff $p\,\Big|\,n$, one argues like before that $Hyp_q(\mathfrak{g})'\Big|_{q=1}$ is generated by the corresponding specializations of $(q_\alpha - 1)^{p^s} F_\alpha^{(p^s)}$, $(q_i - 1)^{p^s} \left({K_i\,;\,0 \atop p^s}\right)$ and $(q_\alpha - 1)^{p^s} E_\alpha^{(p^s)}$ for all $s \in \mathbb{N}$ and all positive roots α: this shows that the spectrum of $Hyp_q(\mathfrak{g})'\Big|_{q=1}$ has (dimension 0 and) height 1, and its cotangent Lie algebra $J\Big/J^2$ (where J is the augmentation ideal of $Hyp_q(\mathfrak{g})'\Big|_{q=1}$) has basis $\Big\{ (q_\alpha - 1)^{p^s} F_\alpha^{(p^s)},\, (q_i - 1)^{p^s} \left({K_i\,;\,0 \atop p^s}\right),\, (q_\alpha - 1)^{p^s} E^{(p^s)} \mod (q-1) Hyp_q(\mathfrak{g})' \mod J^2 \,\Big|\, s \in \mathbb{N} \Big\}$. Moreover, $\big(Hyp_q(\mathfrak{g})'\big)^\vee$ is generated by $(q_\alpha - 1)^{p^s - 1} F_\alpha^{(p^s)}$, $(q_i - 1)^{p^s - 1} \left({K_i\,;\,0 \atop p^s}\right)$, K_i^{-1} and $(q_\alpha - 1)^{p^s - 1} E_\alpha^{(p^s)}$ for all s, i and α: in particular $\big(Hyp_q(\mathfrak{g})'\big)^\vee \subsetneqq Hyp_q(\mathfrak{g})$, and $\big(Hyp_q(\mathfrak{g})'\big)^\vee\Big|_{q=1}$ is generated by the cosets modulo $(q-1)$ of the previous elements, which in fact form a basis of the restricted Lie bialgebra $\mathfrak{k}$ such that $\big(Hyp_q(\mathfrak{g})'\big)^\vee\Big|_{q=1} = \mathbf{u}(\mathfrak{k})$.

7.6 The QFA $F_q[G]$. In this and the following sections we pass to look at Theorem 2.2 the other way round: namely, we start from QFAs and produce QrUEAs.

We begin with $G = SL_n$, with the standard Poisson structure, for which an especially explicit description of the QFA is available. Namely, let $F_q[SL_n]$ be the unital associative

R–algebra generated by $\{\, \rho_{ij} \mid i, j = 1, \dots, n \,\}$ with relations

$$\rho_{ij}\rho_{ik} = q\, \rho_{ik}\rho_{ij} \;, \qquad \rho_{ik}\rho_{hk} = q\, \rho_{hk}\rho_{ik} \qquad\qquad \forall\, j<k,\ i<h$$

$$\rho_{il}\rho_{jk} = \rho_{jk}\rho_{il} \;, \qquad \rho_{ik}\rho_{jl} - \rho_{jl}\rho_{ik} = \left(q - q^{-1}\right)\, \rho_{il}\rho_{jk} \qquad\qquad \forall\, i<j,\ k<l$$

$$det_q(\rho_{ij}) := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} \rho_{1,\sigma(1)}\rho_{2,\sigma(2)} \cdots \rho_{n,\sigma(n)} = 1 \,.$$

This is a Hopf algebra, with comultiplication, counit and antipode given by

$$\Delta(\rho_{ij}) = \sum_{k=1}^{n} \rho_{ik} \otimes \rho_{kj} \;, \quad \epsilon(\rho_{ij}) = \delta_{ij} \;, \quad S(\rho_{ij}) = (-q)^{i-j}\, det_q\Big((\rho_{hk})_{h \neq j}^{k \neq i} \Big)$$

for all $i, j = 1, \dots, n$. Let $\mathbb{F}_q[SL_n] := F(R) \otimes_R F_q[SL_n]$. The set of ordered monomials

$$M \;:=\; \left\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} \rho_{hk}^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \;\middle|\; N_{st} \in \mathbb{N} \;\; \forall\, s,t \;;\; \min\big\{ N_{1,1}, \dots, N_{n,n} \big\} = 0 \right\} \tag{7.4}$$

is an R–basis of $F_q[SL_n]$ and an $F(R)$–basis of $\mathbb{F}_q[SL_n]$ (cf. [Ga2], Theorem 7.4, and [Ga7], Theorem 2.1*(c)*). Moreover, $F_q[SL_n]$ is a QFA (at $\hbar = q-1$), with $F_q[SL_n] \xrightarrow{\;q \to 1\;} F[SL_n]$.

7.7 Computation of $F_q[G]^\vee$ and specialization $F_q[G]^\vee \xrightarrow{\;q \to 1\;} U(\mathfrak{g}^\times)$. In this section we compute $F_q[G]^\vee$ and its semiclassical limit (= specialization at $q = 1$). Note that

$$M' \;:=\; \left\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} (\rho_{hk} - 1)^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \;\middle|\; N_{st} \in \mathbb{N} \;\forall\, s,t \;;\; \min\big\{ N_{1,1}, \dots, N_{n,n} \big\} = 0 \right\}$$

is an R–basis of $F_q[SL_n]$ and an $F(R)$–basis of $\mathbb{F}_q[SL_n]$; then, from the definition of the counit, it follows that $M' \setminus \{1\}$ is an R–basis of $Ker\big(\epsilon : F_q[SL_n] \longrightarrow R\big)$. Now, by definition $I := Ker\Big(F_q[SL_n] \overset{\epsilon}{\twoheadrightarrow} R \overset{q \mapsto 1}{\twoheadrightarrow} \Bbbk \Big)$, whence $I = Ker(\epsilon) + (q-1) \cdot F_q[SL_n]$; therefore $\big(M' \setminus \{1\}\big) \cup \big\{ (q-1) \cdot 1 \big\}$ is an R–basis of I, hence $(q-1)^{-1} I$ has R–basis $(q-1)^{-1} \cdot \big(M' \setminus \{1\}\big) \cup \{1\}$. The outcome is that $F_q[SL_n]^\vee := \sum_{n \geq 0} \Big((q-1)^{-1} I \Big)^n$ is just the unital R–subalgebra of $\mathbb{F}_q[SL_n]$ generated by

$$\left\{ r_{ij} := \frac{\rho_{ij} - \delta_{ij}}{q-1} \;\middle|\; i, j = 1, \dots, n \right\}.$$

Then one can directly show that this is a Hopf algebra, and that $F_q[SL_n]^\vee \xrightarrow{\;q \to 1\;} U({\mathfrak{sl}_n}^*)$ as predicted by Theorem 2.2. Details can be found in [Ga2], §§ 2, 4, looking at the algebra $\widetilde{F}_q[SL_n]$ considered therein, up to the following changes. The algebra which is considered in [*loc. cit.*] has generators $\left(1 + q^{-1}\right)^{\delta_{ij}} \dfrac{\rho_{ij} - \delta_{ij}}{q - q^{-1}}$ ($i, j = 1, \dots, n$) instead of our r_{ij}'s (they coincide iff $i = j$) and also generators $\rho_{ii} = 1 + (q-1)\, r_{ii}$ ($i = 1, \dots, n$); then the presentation in §2.8 of [*loc. cit.*] must be changed accordingly; computing the specialization then goes exactly the same, and gives the same result — specialized generators are rescaled, though, compared with the standard ones given in [*loc. cit.*], §1.

We sketch the case of $n = 2$ (see also [FG]). Using notation $\mathrm{a} := \rho_{1,1}$, $\mathrm{b} := \rho_{1,2}$, $\mathrm{c} := \rho_{2,1}$, $\mathrm{d} := \rho_{2,2}$, we have the relations

$$\mathrm{a}\,\mathrm{b} = q\,\mathrm{b}\,\mathrm{a}\,, \qquad \mathrm{a}\,\mathrm{c} = q\,\mathrm{c}\,\mathrm{a}\,, \qquad \mathrm{b}\,\mathrm{d} = q\,\mathrm{d}\,\mathrm{b}\,, \qquad \mathrm{c}\,\mathrm{d} = q\,\mathrm{d}\,\mathrm{c}\,,$$
$$\mathrm{b}\,\mathrm{c} = \mathrm{c}\,\mathrm{b}\,, \qquad \mathrm{a}\,\mathrm{d} - \mathrm{d}\,\mathrm{a} = \big(q - q^{-1}\big)\mathrm{b}\,\mathrm{c}\,, \qquad \mathrm{a}\,\mathrm{d} - q\,\mathrm{b}\,\mathrm{c} = 1$$

holding in $F_q[SL_2]$ and in $\mathbb{F}_q[SL_2]$, with

$$\Delta(\mathrm{a}) = \mathrm{a} \otimes \mathrm{a} + \mathrm{b} \otimes \mathrm{c},\ \Delta(\mathrm{b}) = \mathrm{a} \otimes \mathrm{b} + \mathrm{b} \otimes \mathrm{d},\ \Delta(\mathrm{c}) = \mathrm{c} \otimes \mathrm{a} + \mathrm{d} \otimes \mathrm{c},\ \Delta(\mathrm{d}) = \mathrm{c} \otimes \mathrm{b} + \mathrm{d} \otimes \mathrm{d}$$
$$\epsilon(\mathrm{a}) = 1, \epsilon(\mathrm{b}) = 0, \epsilon(\mathrm{c}) = 0, \epsilon(\mathrm{d}) = 1, \quad S(\mathrm{a}) = \mathrm{d}, S(\mathrm{b}) = -q^{-1}\mathrm{b}, S(\mathrm{c}) = -q^{+1}\mathrm{c}, S(\mathrm{d}) = \mathrm{a}\,.$$

Then the elements $H_+ := r_{1,1} = \dfrac{\mathrm{a} - 1}{q - 1}$, $E := r_{1,2} = \dfrac{\mathrm{b}}{q - 1}$, $F := r_{2,1} = \dfrac{\mathrm{c}}{q - 1}$ and $H_- := r_{2,2} = \dfrac{\mathrm{d} - 1}{q - 1}$ generate $F_q[SL_2]^\vee$: these generators have relations

$$H_+E = q\,EH_+ + E\,, \quad H_+F = q\,FH_+ + F\,, \quad EH_- = q\,H_-E + E\,, \quad FH_- = q\,H_-F + F\,,$$
$$EF = FE\,, \quad H_+H_- - H_-H_+ = \big(q - q^{-1}\big)EF\,, \quad H_- + H_+ = (q-1)\big(q\,EF - H_+H_-\big)$$

and Hopf operations given by

$$\Delta(H_+) = H_+ \otimes 1 + 1 \otimes H_+ + (q-1)\big(H_+ \otimes H_+ + E \otimes F\big)\,, \quad \epsilon(H_+) = 0\,, \quad S(H_+) = H_-$$
$$\Delta(E) = E \otimes 1 + 1 \otimes E + (q-1)\big(H_+ \otimes E + E \otimes H_-\big)\,, \qquad \epsilon(E) = 0\,, \qquad S(E) = -q^{-1}E$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F + (q-1)\big(F \otimes H_+ + H_- \otimes F\big)\,, \qquad \epsilon(F) = 0\,, \qquad S(F) = -q^{+1}F$$
$$\Delta(H_-) = H_- \otimes 1 + 1 \otimes H_- + (q-1)\big(H_- \otimes H_- + F \otimes E\big)\,, \quad \epsilon(H_-) = 0\,, \quad S(H_-) = H_+$$

from which one easily checks that $F_q[SL_2]^\vee \xrightarrow{\ q \to 1\ } U(\mathfrak{sl}_2{}^*)$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$F_q[SL_2]^\vee \Big/ (q-1)\, F_q[SL_2]^\vee \xrightarrow{\ \cong\ } U(\mathfrak{sl}_2{}^*)$$

exists, given by: $H_\pm \mod (q-1) \mapsto \pm\mathrm{h}$, $E \mod (q-1) \mapsto \mathrm{e}$, $F \mod (q-1) \mapsto \mathrm{f}$; that is, $F_q[SL_2]^\vee$ specializes to $U(\mathfrak{sl}_2{}^*)$ *as a co-Poisson Hopf algebra*, q.e.d.

Finally, the general case of any semisimple group $G = G^\tau$, with the Poisson structure induced from the Lie bialgebra structure of $\mathfrak{g} = \mathfrak{g}^\tau$, can be treated in a different way. Following [Ga1], §§5–6, $\mathbb{F}_q[G]$ can be embedded into a (topological) Hopf algebra $\mathbb{U}_q(\mathfrak{g}^*) = \mathbb{U}_{q,\varphi}^{M}(\mathfrak{g}^*)$, so that the image of the integer form $F_q[G]$ lies into a suitable (topological) integer form $\mathcal{U}_{q,\varphi}^{M}(\mathfrak{g}^*)$ of $\mathbb{U}_q(\mathfrak{g}^*)$. Now, the analysis given in [*loc. cit.*], when carefully read, shows that $F_q[G]^\vee = \mathbb{F}_q[G] \cap \mathcal{U}_{q,\varphi}^{M}(\mathfrak{g}^*)^\vee$; moreover, the latter (intersection) algebra "almost" coincides — it is its closure in a suitable topology — with the integer form $\mathcal{F}_q[G]$ considered in [*loc. cit.*]: in particular, they have the same specialization at $q = 1$. Since in addition $\mathcal{F}_q[G]$ does specialize to $U(\mathfrak{g}^*)$, the same is true for $F_q[G]^\vee$, q.e.d.

The last point to stress is that, once more, the whole analysis above is valid for $p := \mathit{Char}(\Bbbk) \geq 0$, i.e. also for $p > 0$, which was not granted by Theorem 2.2.

7.8 The identity $\big(F_q[G]^\vee\big)' = F_q[G]$**.** In this section we verify the validity of that part of Theorem 2.2*(b)* claiming that $H \in \mathcal{QFA} \implies \big(H^\vee\big)' = H$ for $H = F_q[G]$ as above; moreover we show that this holds for $p > 0$ too. We begin with $G = SL_n$.

From $\Delta(\rho_{ij}) = \sum_{k=1}^n \rho_{i,k} \otimes \rho_{k,j}$, we get $\Delta^N(\rho_{ij}) = \sum_{k_1,\dots,k_{N-1}=1}^n \rho_{i,k_1} \otimes \rho_{k_1,k_2} \otimes \cdots \otimes \rho_{k_{N-1},j}$, by repeated iteration, whence a simple computation yields

$$\delta_N(r_{ij}) = \sum_{k_1,\dots,k_{N-1}=1}^n (q-1)^{-1} \cdot \big((q-1)\, r_{i,k_1} \otimes (q-1)\, r_{k_1,k_2} \otimes \cdots \otimes (q-1)\, r_{k_{N-1},j}\big) \qquad \forall\; i,j$$

so that

$$\delta_N\big((q-1)r_{ij}\big) \in (q-1)^N F_q[SL_n]^\vee \setminus (q-1)^{N+1} F_q[SL_n]^\vee \qquad \forall\; i,j\,. \tag{7.5}$$

Now, consider again the set $M' := \Big\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} (\rho_{hk}-1)^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \,\Big|\, N_{st} \in \mathbb{N}\; \forall\, s,t\,;$ $\min\big\{ N_{1,1}, \dots, N_{n,n} \big\} = 0 \Big\}$: since this is an R–basis of $F_q[SL_n]$, we have also that

$$M'' := \left\{ \prod_{i>j} r_{ij}^{N_{ij}} \prod_{h=k} r_{hk}^{N_{hk}} \prod_{l<m} r_{lm}^{N_{lm}} \,\middle|\, N_{st} \in \mathbb{N} \;\; \forall\, s,t\,;\; \min\big\{ N_{1,1}, \dots, N_{n,n} \big\} = 0 \right\}$$

is an R–basis of $F_q[SL_n]^\vee$. This and (7.5) above imply that $\big(F_q[SL_n]^\vee\big)'$ is the unital R–subalgebra of $\mathbb{F}_q[SL_n]$ generated by the set $\big\{ (q-1)r_{ij} \,\big|\, i,j = 1,\dots,n \big\}$; since $(q-1)\, r_{ij} = \rho_{ij} - \delta_{ij}$, the latter algebra does coincide with $F_q[SL_n]$, as expected.

For the general case of any semisimple group $G = G^\tau$, the result can be obtained again by looking at the immersions $\mathbb{F}_q[G] \subseteq \mathbb{U}_q(\mathfrak{g}^*)$ and $F_q[G] \subseteq \mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)$, and at the identity $F_q[G]^\vee = \mathbb{F}_q[G] \cap \mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)^\vee$ (cf. §7.6); if we try to compute $\Big(\mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)^\vee\Big)'$ (noting that $\big(\mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)\big)^\vee$ is a QrUEA), we have just to apply much the like methods as for $U_q(\mathfrak{g})'$, thus finding a similar result; then from this and the identity $F_q[G]^\vee = \mathbb{F}_q[G] \cap \mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)^\vee$ we eventually find $\Big(F_q[G]^\vee\Big)' = F_q[G]$, q.e.d.

We'd better point out once more that the previous analysis is valid for $p := \mathit{Char}(\Bbbk) \geq 0$, i.e. also for $p > 0$, so the outcome is stronger than what ensured by Theorem 2.2.

Remark: Formula (7.4) gives an explicit R–basis M of $F_q[SL_2]$. By direct computation one sees that $\delta_n(\mu) \in F_q[SL_2]^{\otimes n} \setminus (q-1)\, F_q[SL_2]^{\otimes n}$ for all $\mu \in M \setminus \{1\}$ and $n \in \mathbb{N}$, whence $F_q[SL_2]' = R{\cdot}1$, which implies $\big(F_q[SL_2]'\big)_F = F(R){\cdot}1 \subsetneqq \mathbb{F}_q[SL_2]$ (cf. the Remark after Corollary 4.6) and also $\big(F_q[SL_2]'\big)^\vee = R \cdot 1 \subsetneqq F_q[SL_2]$.

7.9 Drinfeld's functors and L–operators for $U_q(\mathfrak{g})$ when $\mathfrak{g}$ is classical. Let now $\Bbbk$ have characteristic zero, and let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over

$\Bbbk$ whose simple Lie subalgebra are all of classical type. It is known from [FRT2] that in this case $\mathbb{U}_q^P(\mathfrak{g})$ (where the subscript P means that we are taking a "simply-connected" quantum group) admits an alternative presentation, in which the generators are the so-called L–operators, denoted $l_{i,j}^{(\varepsilon)}$ with $\varepsilon = \pm 1$ and i, j ranging in a suitable set of indices (see [FRT2], §2). Now, if we consider instead the R–subalgebra H generated by the L–operators, we get at once from the very description of the relations between the $l_{i,j}^{(\varepsilon)}$'s given in [FRT2] that H is a *Hopf* R–subalgebra of $\mathbb{U}_q^P(\mathfrak{g})$, and more precisely it is a QFA for the connected simply-connected dual Poisson group $G^\star$.

When computing $H^\vee$, it is generated by the elements $(q-1)^{-1} l_{i,j}^{(\varepsilon)}$; even more, the elements $(q-1)^{-1} l_{i,i+1}^{(+)}$ and $(q-1)^{-1} l_{i+1,i}^{(-)}$ are enough to generate. Now, Theorem 12 in [FRT2] shows that these latter generators are simply multiples of the Chevalley generators of $U_q^P(\mathfrak{g})$ (in the sense of Jimbo, Drinfeld, etc.), by a coefficient $\pm q^s \big(1+q^{-1}\big)$, for some $s \in \mathbb{Z}$, times a "toral" generator: this proves directly that $H^\vee$ is a QrUEA associated to $\mathfrak{g}$, that is the dual Lie bialgebra of G^*, as prescribed by Theorem 2.2. Conversely, if we start from $U_q^P(\mathfrak{g})$, again Theorem 12 of [FRT2] shows that the $\big(q-q^{-1}\big)^{-1} l_{i,j}^{(\varepsilon)}$'s are quantum root vectors in $U_q^P(\mathfrak{g})$. Then when computing $U_q^P(\mathfrak{g})'$ we can shorten a lot the analysis in §5.3, because the explicit expression of the coproduct on the L–operators given in [FRT2] — roughly, Δ is given on them by a standard "matrix coproduct" — tells us directly that all the $\big(1+q^{-1}\big)^{-1} l_{i,j}^{(\varepsilon)}$'s do belong to $U_q^P(\mathfrak{g})'$, and again by a PBW argument we conclude that $U_q^P(\mathfrak{g})'$ is generated by these rescaled L–operators, i.e. the $\big(1+q^{-1}\big)^{-1} l_{i,j}^{(\varepsilon)}$.

Therefore, we can say in short that shifting from H to $H^\vee$ or from $U_q^P(\mathfrak{g})$ to $U_q^P(\mathfrak{g})'$ essentially amounts — up to rescaling by irrelevant factors (in that they do not vanish at $q=1$) — to switching from the presentation of $\mathbb{U}_q^P(\mathfrak{g})$ via L–operators (after [FRT2]) to the presentation of Serre-Chevalley type (after Drinfeld and Jimbo), and conversely. See also the analysis in [Ga7] for the cases $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathfrak{g} = \mathfrak{sl}_n$.

7.10 The cases $U_q(\mathfrak{gl}_n)$, $F_q[GL_n]$ and $F_q[M_n]$. In [Ga2], §5.2, a certain algebra $U_q(\mathfrak{gl}_n)$ is considered as a quantization of $\mathfrak{gl}_n$; due to their strict relationship, from the analysis we did for the case of $\mathfrak{sl}_n$ one can easily deduce a complete description of $U_q(\mathfrak{gl}_n)'$ and its specialization at $q=1$, and also verify that $\big(U_q(\mathfrak{gl}_n)'\big)^\vee = U_q(\mathfrak{gl}_n)$.

Similarly, we can consider the unital associative R–algebra $F_q[M_n]$ with generators ρ_{ij} (i, $j = 1, \dots, n$) and relations $\rho_{ij}\rho_{ik} = q\,\rho_{ik}\rho_{ij}$, $\rho_{ik}\rho_{hk} = q\,\rho_{hk}\rho_{ik}$ (for all $j<k$, $i<h$), $\rho_{il}\rho_{jk} = \rho_{jk}\rho_{il}$, $\rho_{ik}\rho_{jl} - \rho_{jl}\rho_{ik} = \big(q-q^{-1}\big)\,\rho_{il}\rho_{jk}$ (for all $i<j$, $k<l$) — i.e. like for SL_n, but for skipping the last relation. This is the celebrated standard quantization of $F[M_n]$, the function algebra of the variety M_n of $(n \times n)$–matrices over $\Bbbk$: it is a $\Bbbk$–bialgebra, whose structure is given by formulas $\Delta(\rho_{ij}) = \sum_{k=1}^n \rho_{ik} \otimes \rho_{kj}$, $\epsilon(\rho_{ij}) = \delta_{ij}$ (for all i, $j = 1, \dots, n$) again, but it is *not a Hopf algebra*. The quantum determinant $det_q(\rho_{ij}) := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} \rho_{1,\sigma(1)}\, \rho_{2,\sigma(2)} \cdots \rho_{n,\sigma(n)}$ is central in $F_q[M_n]$, so by standard theory we can extend $F_q[M_n]$ by adding a formal inverse to $det_q(\rho_{ij})$, thus

getting a larger algebra $F_q[GL_n] := F_q[M_n]\big[det_q(\rho_{ij})^{-1}\big]$: this is now a Hopf algebra, with antipode $S(\rho_{ij}) = (-q)^{i-j}\, det_q\Big((\rho_{hk})_{h\neq j}^{k\neq i}\Big)$ (for all $i, j = 1, \dots, n$), the well-known standard quantization of $F[GL_n]$, due to Manin (see [Ma]).

Applying Drinfeld's functor $(\ )^\vee$ w.r.t. $\hbar := (q-1)$ at $F_q[GL_n]$ we can repeat stepwise the analysis made for $F_q[SL_n]$: then we have that $F_q[GL_n]^\vee$ is generated by the r_{ij}'s and $(q-1)^{-1}\big(det_q(\rho_{ij})-1\big)$, the sole real difference being the lack of the relation $det_q(\rho_{ij}) = 1$, which implies one relation less among the r_{ij}'s inside $F_q[GL_n]^\vee$, hence also one relation less among their cosets modulo $(q-1)$. The outcome is pretty similar, in particular $F_q[GL_n]^\vee\Big|_{q=1} = U(\mathfrak{gl}_n^{\,*})$ (cf. [Ga2], §6.2). Even more, we can do the same with $F_q[M_n]$: things are even easier, because we have only the r_{ij}'s alone which generate $F_q[M_n]^\vee$, with no relation coming from the relation $det_q(\rho_{ij}) = 1$; nevertheless at $q = 1$ the relations among the cosets of the r_{ij}'s are exactly the same as in the case of $F_q[GL_n]^\vee\Big|_{q=1}$, whence we get $F_q[M_n]^\vee\Big|_{q=1} = U(\mathfrak{gl}_n^{\,*})$. *In particular, we get that $F_q[M_n]^\vee\Big|_{q=1}$ is a Hopf algebra, although both $F_q[M_n]$ and $F_q[M_n]^\vee$ are only bialgebras, not Hopf algebras: so this gives a non-trivial explicit example of what claimed in the first part of Theorem 3.7.*

Finally, an analysis of the relationship between Drinfeld functors and L–operators about $\mathbb{U}_q^P(\mathfrak{gl}_n)$ can be done again, exactly like in §7.9, leading to entirely similar results.

§ 8 Third example: quantum three-dimensional Euclidean group

8.1 The classical setting. Let $\Bbbk$ be any field of characteristic $p \geq 0$. Let $G := E_2(\Bbbk) \equiv E_2$, the three-dimensional Euclidean group; its tangent Lie algebra $\mathfrak{g} = \mathfrak{e}_2$ is generated by f, h, e with relations $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = 0$. The formulas $\delta(f) = h \otimes f - f \otimes h$, $\delta(h) = 0$, $\delta(e) = h \otimes e - e \otimes h$, make $\mathfrak{e}_2$ into a Lie bialgebra, hence E_2 into a Poisson group. These also give a presentation of the co-Poisson Hopf algebra $U(\mathfrak{e}_2)$ (with standard Hopf structure). If $p > 0$, we consider on $\mathfrak{e}_2$ the p–operation given by $e^{[p]} = 0$, $f^{[p]} = 0$, $h^{[p]} = h$.

On the other hand, the function algebra $F[E_2]$ is the unital associative commutative $\Bbbk$–algebra with generators b, $a^{\pm 1}$, c, with Poisson Hopf algebra structure given by

$$\Delta(b) = b \otimes a^{-1} + a \otimes b\,, \qquad \Delta\big(a^{\pm 1}\big) = a^{\pm 1} \otimes a^{\pm 1}\,, \qquad \Delta(c) = c \otimes a + a^{-1} \otimes c$$
$$\epsilon(b) = 0\,, \quad \epsilon\big(a^{\pm 1}\big) = 1\,, \quad \epsilon(c) = 0\,, \qquad S(b) = -b\,, \quad S\big(a^{\pm 1}\big) = a^{\mp 1}\,, \quad S(c) = -c$$
$$\big\{a^{\pm 1}, b\big\} = \pm a^{\pm 1} b\,, \qquad \big\{a^{\pm 1}, c\big\} = \pm a^{\pm 1} c\,, \qquad \{b, c\} = 0$$

We can realize E_2 as $E_2 = \big\{\, (b,a,c) \,\big|\, b, c \in k, a \in \Bbbk \setminus \{0\} \,\big\}$, with group operation

$$(b_1, a_1, c_1) \cdot (b_2, a_2, c_2) = \big(b_1 a_2^{-1} + a_1 b_2\,,\, a_1 a_2\,,\, c_1 a_2 + a_1^{-1} c_2\big)\,;$$

in particular the centre of E_2 is simply $Z := \big\{(0,1,0),(0,-1,0)\big\}$, so there is only one other connected Poisson group having $\mathfrak{e}_2$ as Lie bialgebra, namely the adjoint group ${}_aE_2 := E_2\Big/Z$ (the left subscript a stands for "adjoint"). Then $F[{}_aE_2]$ coincides with the Poisson Hopf subalgebra of $F[{}_aE_2]$ spanned by products of an even number of generators, i.e. monomials of even degree: as a unital subalgebra, this is generated by ba, $a^{\pm 2}$, and $a^{-1}c$.

The dual Lie bialgebra $\mathfrak{g}^* = {\mathfrak{e}_2}^*$ is the Lie algebra with generators f, h, e, and relations $[\mathrm{h},\mathrm{e}] = 2\mathrm{e}$, $[\mathrm{h},\mathrm{f}] = 2\mathrm{f}$, $[\mathrm{e},\mathrm{f}] = 0$, with Lie cobracket given by $\delta(\mathrm{f}) = \mathrm{f}\otimes\mathrm{h} - \mathrm{h}\otimes\mathrm{f}$, $\delta(\mathrm{h}) = 0$, $\delta(\mathrm{e}) = \mathrm{h}\otimes\mathrm{e} - \mathrm{e}\otimes\mathrm{h}$ (we choose as generators $\mathrm{f} := f^*$, $\mathrm{h} := 2h^*$, $\mathrm{e} := e^*$, where $\big\{f^*,h^*,e^*\big\}$ is the basis of ${\mathfrak{e}_2}^*$ which is the dual of the basis $\{f,h,e\}$ of $\mathfrak{e}_2$). If $p > 0$, the p–operation of ${\mathfrak{e}_2}^*$ is given by $\mathrm{e}^{[p]} = 0$, $\mathrm{f}^{[p]} = 0$, $\mathrm{h}^{[p]} = \mathrm{h}$. All this again gives a presentation of $U({\mathfrak{e}_2}^*)$ too. The simply connected algebraic Poisson group with tangent Lie bialgebra ${\mathfrak{e}_2}^*$ can be realized as the group of pairs of matrices

$$ {}_s{E_2}^* := \left\{ \left(\begin{pmatrix} z^{-1} & 0 \\ y & z \end{pmatrix}, \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \right) \,\middle|\, x,y \in k,\, z \in \Bbbk\setminus\{0\} \right\} ; $$

this group has centre $Z := \big\{(I,I),(-I,-I)\big\}$, so there is only one other (Poisson) group with Lie (bi)algebra ${\mathfrak{e}_2}^*$, namely the adjoint group ${}_a{E_2}^* := {}_s{E_2}^*\Big/Z$.

Therefore $F\big[{}_s{E_2}^*\big]$ is the unital associative commutative $\Bbbk$–algebra with generators x, $z^{\pm 1}$, y, with Poisson Hopf structure given by

$$ \begin{gathered} \Delta(x) = x\otimes z^{-1} + z\otimes x\,, \qquad \Delta\big(z^{\pm 1}\big) = z^{\pm 1}\otimes z^{\pm 1}\,, \qquad \Delta(y) = y\otimes z^{-1} + z\otimes y \\ \epsilon(x) = 0\,,\;\; \epsilon\big(z^{\pm 1}\big) = 1\,,\;\; \epsilon(y) = 0\,, \qquad S(x) = -x\,,\;\; S\big(z^{\pm 1}\big) = z^{\mp 1}\,,\;\; S(y) = -y \\ \{x,y\} = 0\,, \qquad \big\{z^{\pm 1},x\big\} = \pm z^{\pm 1}x\,, \qquad \big\{z^{\pm 1},y\big\} = \mp z^{\pm 1}y \end{gathered} $$

(N.B.: with respect to this presentation, we have $\mathrm{f} = \partial_y\big|_e$, $\mathrm{h} = z\,\partial_z\big|_e$, $\mathrm{e} = \partial_x\big|_e$, where e is the identity element of ${}_s{E_2}^*$). Moreover, $F\big[{}_a{E_2}^*\big]$ can be identified with the Poisson Hopf subalgebra of $F\big[{}_s{E_2}^*\big]$ spanned by products of an even number of generators, i.e. monomials of even degree: this is generated, as a unital subalgebra, by xz, $z^{\pm 2}$, and $z^{-1}y$.

8.2 The QrUEAs $U_q^s(\mathfrak{e}_2)$ and $U_q^a(\mathfrak{e}_2)$. We turn now to quantizations: the situation is much similar to the $\mathfrak{sl}_2$ case, so we follow the same pattern, but we stress a bit more the occurrence of different groups sharing the same tangent Lie bialgebra.

Let R be a domain and let $\hbar \in R\setminus\{0\}$ and $q := \hbar + 1 \in R$ be like in §7.2.

Let $\mathbb{U}_q(\mathfrak{g}) = \mathbb{U}_q^s(\mathfrak{e}_2)$ (where the superscript s stands for "simply connected") be the associative unital $F(R)$–algebra with generators F, $L^{\pm 1}$, E, and relations

$$ LL^{-1} = 1 = L^{-1}L\,, \quad L^{\pm 1}F = q^{\mp 1}FL^{\pm 1}\,, \quad L^{\pm 1}E = q^{\pm 1}EL^{\pm 1}\,, \quad EF = FE\,. $$

This is a Hopf algebra, with Hopf structure given by

$$ \begin{gathered} \Delta(F) = F\otimes L^{-2} + 1\otimes F\,, \qquad \Delta\big(L^{\pm 1}\big) = L^{\pm 1}\otimes L^{\pm 1}\,, \qquad \Delta(E) = E\otimes 1 + L^2\otimes E \\ \epsilon(F) = 0\,,\;\; \epsilon\big(L^{\pm 1}\big) = 1\,,\;\; \epsilon(E) = 0\,, \qquad S(F) = -FL^2,\;\; S\big(L^{\pm 1}\big) = L^{\mp 1},\;\; S(E) = -L^{-2}E\,. \end{gathered} $$

Then let $U_q^s(\mathfrak{e}_2)$ be the R–subalgebra of $\mathbb{U}_q^s(\mathfrak{e}_2)$ generated by F, $D_\pm := \dfrac{L^{\pm 1}-1}{q-1}$, E. From the definition of $\mathbb{U}_q^s(\mathfrak{e}_2)$ one gets a presentation of $U_q^s(\mathfrak{e}_2)$ as the associative unital algebra with generators F, $D_\pm$, E and relations

$$D_+E = qED_+ + E\,, \quad FD_+ = qD_+F + F\,, \quad ED_- = qD_-E + E\,, \quad D_-F = qFD_- + F$$
$$EF = FE\,, \qquad D_+D_- = D_-D_+\,, \qquad D_+ + D_- + (q-1)D_+D_- = 0$$

with a Hopf structure given by

$$\Delta(E) = E \otimes 1 + 1 \otimes E + 2(q-1)D_+ \otimes E + (q-1)^2 \cdot D_+^2 \otimes E$$
$$\Delta(D_\pm) = D_\pm \otimes 1 + 1 \otimes D_\pm + (q-1)\cdot D_\pm \otimes D_\pm$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F + 2(q-1)F \otimes D_- + (q-1)^2 \cdot F \otimes D_-^2$$
$$\begin{array}{ll} \epsilon(E) = 0\,, & S(E) = -E - 2(q-1)D_-E - (q-1)^2 D_-^2 E \\ \epsilon(D_\pm) = 0\,, & S(D_\pm) = D_\mp \\ \epsilon(F) = 0\,, & S(F) = -F - 2(q-1)FD_+ - (q-1)^2 FD_+^2\,. \end{array}$$

The "adjoint version" of $\mathbb{U}_q^s(\mathfrak{e}_2)$ is the unital subalgebra $\mathbb{U}_q^a(\mathfrak{e}_2)$ generated by F, $K^{\pm 1} := L^{\pm 2}$, E, which is clearly a Hopf subalgebra. It also has an R–integer form $U_q^a(\mathfrak{e}_2)$, the unital R–subalgebra generated by F, $H_\pm := \dfrac{K^{\pm 1}-1}{q-1}$, E: this has relations

$$EF = FE\,,\;\; H_+E = q^2EH_+ + (q+1)E\,,\;\; FH_+ = q^2H_+F + (q+1)F\,,\;\; H_+H_- = H_-H_+$$
$$EH_- = q^2H_-E + (q+1)E\,,\;\; H_-F = q^2FH_- + (q+1)F\,,\;\; H_+ + H_- + (q-1)H_+H_- = 0$$

and it is a Hopf subalgebra, with Hopf operations given by

$$\Delta(E) = E \otimes 1 + 1 \otimes E + (q-1)\cdot H_+ \otimes E\,, \qquad \epsilon(E) = 0\,, \qquad S(E) = -E - (q-1)H_-E$$
$$\Delta(H_\pm) = H_\pm \otimes 1 + 1 \otimes H_\pm + (q-1)\cdot H_\pm \otimes H_\pm\,, \qquad \epsilon(H_\pm) = 0\,, \qquad S(H_\pm) = H_\mp$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F + (q-1)\cdot F \otimes H_-\,, \qquad \epsilon(F) = 0\,, \qquad S(F) = -F - (q-1)FH_+\,.$$

It is easy to check that $U_q^s(\mathfrak{e}_2)$ is a QrUEA, whose semiclassical limit is $U(\mathfrak{e}_2)$: in fact, mapping the generators $F \mod (q-1)$, $D_\pm \mod (q-1)$, $E \mod (q-1)$ respectively to f, $\pm h/2$, $e \in U(\mathfrak{e}_2)$ gives an isomorphism $U_q^s(\mathfrak{e}_2)\big/(q-1)\,U_q^s(\mathfrak{e}_2) \xrightarrow{\cong} U(\mathfrak{e}_2)$ of co-Poisson Hopf algebras. Similarly, $U_q^a(\mathfrak{e}_2)$ is a QrUEA too, with semiclassical limit $U(\mathfrak{e}_2)$ again: here a co-Poisson Hopf algebra isomorphism $U_q^a(\mathfrak{e}_2)\big/(q-1)\,U_q^a(\mathfrak{e}_2) \cong U(\mathfrak{e}_2)$ is given mapping $F \mod (q-1)$, $H_\pm \mod (q-1)$, $E \mod (q-1)$ respectively to $f, \pm h, e \in U(\mathfrak{e}_2)$.

8.3 Computation of $U_q(\mathfrak{e}_2)'$ and specialization $U_q(\mathfrak{e}_2)' \xrightarrow{q\to 1} F\big[E_2^*\big]$. This section is devoted to compute $U_q^s(\mathfrak{e}_2)'$ and $U_q^a(\mathfrak{e}_2)'$, and their specialization at $q=1$: everything goes on as in §7.3, so we can be more sketchy. From definitions we have, for any $n \in \mathbb{N}$, $\Delta^n(E) = \sum_{s=1}^{n} K^{\otimes(s-1)} \otimes E \otimes 1^{\otimes(n-s)}$, so $\delta_n(E) = (K-1)^{\otimes(n-1)} \otimes E = (q-1)^{n-1} \cdot H_+^{\otimes(n-1)} \otimes E$, whence $\delta_n\big((q-1)E\big) \in (q-1)^n\, U_q^a(\mathfrak{e}_2) \setminus (q-1)^{n+1}\, U_q^a(\mathfrak{e}_2)$ thus $(q-1)E \in U_q^a(\mathfrak{e}_2)'$, whereas $E \notin U_q^a(\mathfrak{e}_2)'$. Similarly, we have $(q-1)F$, $(q-1)H_\pm \in U_q^a(\mathfrak{e}_2)' \setminus (q-1)\, U_q^a(\mathfrak{e}_2)'$. Therefore $U_q^a(\mathfrak{e}_2)$ contains the subalgebra U' generated by $\dot{F} := (q-1)F$, $\dot{H}_\pm := (q-1)H_\pm$, $\dot{E} := (q-1)E$. On the other hand, $U_q^a(\mathfrak{e}_2)'$ is clearly the R–span of the set $\Big\{ F^a H_+^b H_-^c E^d \,\Big|\, a,b,c,d \in \mathbb{N} \Big\}$: to be precise, the set

$$\Big\{ F^a H_+^b K^{-[b/2]} E^d \,\Big|\, a,b,d \in \mathbb{N} \Big\} = \Big\{ F^a H_+^b \big(1 + (q-1)H_-\big)^{[b/2]} E^d \,\Big|\, a,b,d \in \mathbb{N} \Big\}$$

is an R–basis of $U_q^a(\mathfrak{e}_2)'$; therefore, a straightforward computation shows that any element in $U_q^a(\mathfrak{e}_2)'$ does necessarily lie in U', thus $U_q^a(\mathfrak{e}_2)'$ coincides with U'. Moreover, since $\dot{H}_\pm = K^{\pm 1} - 1$, the unital algebra $U_q^a(\mathfrak{e}_2)'$ is generated by $\dot{F}$, $K^{\pm 1}$ and $\dot{E}$ as well.

The previous analysis — *mutatis mutandis* — ensures also that $U_q^s(\mathfrak{e}_2)'$ coincides with the unital R–subalgebra U'' of $\mathbb{U}_q^s(\mathfrak{e}_2)$ generated by $\dot{F} := (q-1)F$, $\dot{D}_\pm := (q-1)D_\pm$, $\dot{E} := (q-1)E$; in particular, $U_q^s(\mathfrak{e}_2)' \supset U_q^a(\mathfrak{e}_2)'$. Moreover, as $\dot{D}_\pm = L^{\pm 1} - 1$, the unital algebra $U_q^s(\mathfrak{e}_2)'$ is generated by $\dot{F}$, $L^{\pm 1}$ and $\dot{E}$ as well. Thus $U_q^s(\mathfrak{e}_2)'$ is the unital associative R–algebra with generators $\mathcal{F} := L\dot{F}$, $\mathcal{L}^{\pm 1} := L^{\pm 1}$, $\mathcal{E} := \dot{E}L^{-1}$ and relations

$$\mathcal{L}\mathcal{L}^{-1} = 1 = \mathcal{L}^{-1}\mathcal{L}\,, \qquad \mathcal{E}\mathcal{F} = \mathcal{F}\mathcal{E}\,, \qquad \mathcal{L}^{\pm 1}\mathcal{F} = q^{\mp 1}\mathcal{F}\mathcal{L}^{\pm 1}\,, \qquad \mathcal{L}^{\pm 1}\mathcal{E} = q^{\pm 1}\mathcal{E}\mathcal{L}^{\pm 1}$$

with Hopf structure given by

$$\begin{gathered} \Delta(\mathcal{F}) = \mathcal{F} \otimes \mathcal{L}^{-1} + \mathcal{L} \otimes \mathcal{F}\,, \qquad \Delta\big(\mathcal{L}^{\pm 1}\big) = \mathcal{L}^{\pm 1} \otimes \mathcal{L}^{\pm 1}\,, \qquad \Delta(\mathcal{E}) = \mathcal{E} \otimes \mathcal{L}^{-1} + \mathcal{L} \otimes \mathcal{E} \\ \epsilon(\mathcal{F}) = 0\,, \quad \epsilon\big(\mathcal{L}^{\pm 1}\big) = 1\,, \quad \epsilon(\mathcal{E}) = 0\,, \qquad S(\mathcal{F}) = -\mathcal{F}\,, \quad S\big(\mathcal{L}^{\pm 1}\big) = \mathcal{L}^{\mp 1}\,, \quad S(\mathcal{E}) = -\mathcal{E}\,. \end{gathered}$$

As $q \to 1$, this yields a presentation of the function algebra $F\big[{}_sE_2^{\,*}\big]$, and the Poisson bracket that $F\big[{}_sE_2^{\,*}\big]$ earns from this quantization process coincides with the one coming from the Poisson structure on ${}_sE_2^{\,*}$: namely, there is a Poisson Hopf algebra isomorphism

$$U_q^s(\mathfrak{e}_2)' \Big/ (q-1)\, U_q^s(\mathfrak{e}_2)' \xrightarrow{\;\cong\;} F\big[{}_sE_2^{\,*}\big]$$

given by $\mathcal{E} \mod (q-1) \mapsto x$, $\mathcal{L}^{\pm 1} \mod (q-1) \mapsto z^{\pm 1}$, $\mathcal{F} \mod (q-1) \mapsto y$. That is, $U_q^s(\mathfrak{e}_2)'$ specializes to $F\big[{}_sE_2^{\,*}\big]$ *as a Poisson Hopf algebra*, as predicted by Theorem 2.2.

In the "adjoint case", from the definition of U' and from $U_q^a(\mathfrak{e}_2)' = U'$ we find that $U_q^a(\mathfrak{e}_2)'$ is the unital associative R–algebra with generators $\dot{F}$, $K^{\pm 1}$, $\dot{E}$ and relations

$$K K^{-1} = 1 = K^{-1} K\,, \qquad \dot{E}\dot{F} = \dot{F}\dot{E}\,, \qquad K^{\pm 1}\dot{F} = q^{\mp 2}\dot{F}K^{\pm 1}\,, \qquad K^{\pm 1}\dot{E} = q^{\pm 2}\dot{E}K^{\pm 1}$$

with Hopf structure given by

$$\begin{gathered} \Delta\big(\dot{F}\big) = \dot{F} \otimes K^{-1} + 1 \otimes \dot{F}\,, \qquad \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, \qquad \Delta\big(\dot{E}\big) = \dot{E} \otimes 1 + K \otimes \dot{E} \\ \epsilon\big(\dot{F}\big) = 0\,,\ \epsilon\big(K^{\pm 1}\big) = 1\,,\ \epsilon\big(\dot{E}\big) = 0\,, \quad S\big(\dot{F}\big) = -\dot{F}K\,,\ S\big(K^{\pm 1}\big) = K^{\mp 1},\ S\big(\dot{E}\big) = -K^{-1}\dot{E}\,. \end{gathered}$$

The conclusion is that a Poisson Hopf algebra isomorphism

$$U_q^a(\mathfrak{e}_2)' \Big/ (q-1)\, U_q^a(\mathfrak{e}_2)' \xrightarrow{\ \cong\ } F\big[{}_aE_2{}^*\big] \quad \Big(\subset F\big[{}_sE_2{}^*\big] \Big)$$

exists, given by $\dot{E} \mod (q-1) \mapsto xz$, $K^{\pm 1} \mod (q-1) \mapsto z^{\pm 2}$, $\dot{F} \mod (q-1) \mapsto z^{-1}y$, i.e. $U_q^a(\mathfrak{e}_2)'$ specializes to $F\big[{}_aE_2{}^*\big]$ *as a Poisson Hopf algebra*, according to Theorem 2.2.

To finish with, note that *all this analysis (and its outcome) is entirely characteristic-free*.

8.4 The identity $\big(U_q(\mathfrak{e}_2)'\big)^\vee = U_q(\mathfrak{e}_2)$. The goal of this section is to check that part of Theorem 2.2*(b)* claiming that $H \in \mathcal{QrUEA} \Longrightarrow \big(H'\big)^\vee = H$ both for $H = U_q^s(\mathfrak{e}_2)$ and $H = U_q^a(\mathfrak{e}_2)$. *In addition, the proof below will work for $Char(\Bbbk) = 0$ and $Char(\Bbbk) > 0$ too, thus giving a stronger result than predicted by Theorem 2.2(b).*

First, $U_q^s(\mathfrak{e}_2)'$ is clearly a free R–module, with basis $\Big\{ \mathcal{F}^a \mathcal{L}^d \mathcal{E}^c \,\Big|\, a, c \in \mathbb{N}, d \in \mathbb{Z} \Big\}$, hence the set $\mathbb{B} := \Big\{ \mathcal{F}^a \big(\mathcal{L}^{\pm 1} - 1\big)^b \mathcal{E}^c \,\Big|\, a, b, c \in \mathbb{N} \Big\}$, is an R–basis as well. Second, as $\epsilon(\mathcal{F}) = \epsilon\big(\mathcal{L}^{\pm 1} - 1\big) = \epsilon(\mathcal{E}) = 0$, the ideal $J := Ker\Big(\epsilon : U_q^s(\mathfrak{e}_2)' \longrightarrow R\Big)$ is the span of $\mathbb{B} \setminus \{1\}$. Now $I := Ker\Big(U_q^s(\mathfrak{e}_2)' \xrightarrow{\ \epsilon\ } R \xrightarrow{\ q \mapsto 1\ } \Bbbk \Big) = J + (q-1) \cdot U_q^s(\mathfrak{e}_2)'$, therefore $\big(U_q^s(\mathfrak{e}_2)'\big)^\vee := \sum_{n \geq 0} \Big((q-1)^{-1} I \Big)^n$ is generated — as a unital R–subalgebra of $\mathbb{U}_q^s(\mathfrak{e}_2)$ — by $(q-1)^{-1}\mathcal{F} = LF$, $(q-1)^{-1}(\mathcal{L} - 1) = D_+$, $(q-1)^{-1}\big(\mathcal{L}^{-1} - 1\big) = D_-$, $(q-1)^{-1}\mathcal{E} = EL^{-1}$, hence by F, $D_\pm$, E, so it coincides with $U_q^s(\mathfrak{e}_2)$, q.e.d.

The situation is entirely similar for the adjoint case: one simply has to change $\mathcal{F}$, $\mathcal{L}^{\pm 1}$, $\mathcal{E}$ respectively with $\dot{F}$, $K^{\pm 1}$, $\dot{E}$, and $D_\pm$ with $H_\pm$, then everything goes through as above.

8.5 The quantum hyperalgebra $Hyp_q(\mathfrak{e}_2)$. Like for semisimple groups, we can define "quantum hyperalgebras" attached to $\mathfrak{e}_2$ mimicking what done in §7.5. Namely, we can first define a Hopf subalgebra of $\mathbb{U}_q^s(\mathfrak{e}_2)$ over $\mathbb{Z}\big[q, q^{-1}\big]$ whose specialization at $q = 1$ is exactly the Kostant-like $\mathbb{Z}$–integer form $U_\mathbb{Z}(\mathfrak{e}_2)$ of $U(\mathfrak{e}_2)$ (generated by divided powers, and giving the hyperalgebra $Hyp(\mathfrak{e}_2)$ over any field $\Bbbk$ by scalar extension, namely $Hyp(\mathfrak{e}_2) = \Bbbk \otimes_\mathbb{Z} U_\mathbb{Z}(\mathfrak{e}_2)$), and then take its scalar extension over R.

To be precise, let $Hyp_q^{s,\mathbb{Z}}(\mathfrak{e}_2)$ be the unital $\mathbb{Z}\big[q, q^{-1}\big]$–subalgebra of $\mathbb{U}_q^s(\mathfrak{e}_2)$ (defined like above *but over* $\mathbb{Z}\big[q, q^{-1}\big]$) generated by the "quantum divided powers" $F^{(n)} := F^n \Big/ [n]_q!$, $\left({L\,;\,c \atop n} \right) := \prod_{r=1}^{n} \dfrac{q^{c+1-r}L - 1}{q^r - 1}$, $E^{(n)} := E^n \Big/ [n]_q!$ (for all $n \in \mathbb{N}$ and $c \in \mathbb{Z}$, with notation of §7.5) and by L^{-1}. Comparing with the case of $\mathfrak{sl}_2$ one easily sees that this is a Hopf subalgebra of $\mathbb{U}_q^s(\mathfrak{e}_2)$, and $Hyp_q^{s,\mathbb{Z}}(\mathfrak{e}_2)\Big|_{q=1} \cong U_\mathbb{Z}(\mathfrak{e}_2)$; thus $Hyp_q^s(\mathfrak{e}_2) := R \otimes_{\mathbb{Z}[q,q^{-1}]} Hyp_q^{s,\mathbb{Z}}(\mathfrak{e}_2)$ (for any R like in §8.2, with $\Bbbk := R \big/ \hbar\, R$ and $p := Char(\Bbbk)$) specializes at $q = 1$ to the $\Bbbk$–hyperalgebra $Hyp(\mathfrak{e}_2)$. In addition, among all the $\left({L\,;\,c \atop n} \right)$'s it is enough to take only those with $c = 0$. *From now on we assume $p > 0$.*

Again a strict comparison with the $\mathfrak{sl}_2$ case — with some shortcuts, since the defining relations of $Hyp_q^{\,s}(\mathfrak{e}_2)$ are simpler! — shows us that $Hyp_q^{\,s}(\mathfrak{e}_2)'$ is the unital R–subalgebra of $Hyp_q^{\,s}(\mathfrak{e}_2)$ generated by L^{-1} and the "rescaled quantum divided powers" $(q-1)^n F^{(n)}$, $(q-1)^n \left(\begin{smallmatrix} L\,;\,0 \\ n \end{smallmatrix}\right)$ and $(q-1)^n E^{(n)}$ for all $n \in \mathbb{N}$. It follows that $Hyp_q^{\,s}(\mathfrak{e}_2)'\Big|_{q=1}$ is generated by the corresponding specializations of $(q-1)^{p^r} F^{(p^r)}$, $(q-1)^{p^r} \left(\begin{smallmatrix} L\,;\,0 \\ p^r \end{smallmatrix}\right)$ and $(q-1)^{p^r} E^{(p^r)}$ for all $r \in \mathbb{N}$: this proves that the spectrum of $Hyp_q^{\,s}(\mathfrak{e}_2)'\Big|_{q=1}$ has dimension 0 and height 1, and its cotangent Lie algebra has basis $\Big\{ (q-1)^{p^r} F^{(p^r)}, (q-1)^{p^r} \left(\begin{smallmatrix} L\,;\,0 \\ p^r \end{smallmatrix}\right),$ $(q-1)^{p^r} E^{(p^r)} \mod (q-1)\, Hyp_q^{\,s}(\mathfrak{g})' \mod J^2 \;\Big|\; r \in \mathbb{N} \Big\}$ (where J is the augmentation ideal of $Hyp_q^{\,s}(\mathfrak{e}_2)'\Big|_{q=1}$, so that $J \big/ J^2$ is the aforementioned cotangent Lie bialgebra). Moreover, $\big(Hyp_q^{\,s}(\mathfrak{e}_2)'\big)^\vee$ is generated by $(q-1)^{p^r-1} F^{(p^r)}$, $(q-1)^{p^r-1} \left(\begin{smallmatrix} L\,;\,0 \\ p^r \end{smallmatrix}\right)$, L^{-1} and $(q-1)^{p^r-1} E^{(p^r)}$ (for all $r \in \mathbb{N}$): in particular $\big(Hyp_q^{\,s}(\mathfrak{e}_2)'\big)^\vee \subsetneqq Hyp_q^{\,s}(\mathfrak{e}_2)$, and finally $\big(Hyp_q^{\,s}(\mathfrak{e}_2)'\big)^\vee\Big|_{q=1}$ is generated by the cosets modulo $(q-1)$ of the elements above, which in fact form a basis of the restricted Lie bialgebra $\mathfrak{k}$ such that $\big(Hyp_q^{\,s}(\mathfrak{e}_2)'\big)^\vee\Big|_{q=1} = \mathbf{u}(\mathfrak{k})$.

All this analysis was made starting from $\mathbb{U}_q^s(\mathfrak{e}_2)$, which gave "simply connected quantum objects". If we start instead from $\mathbb{U}_q^a(\mathfrak{e}_2)$, we get "adjoint quantum objects" following the same pattern but for replacing everywhere $L^{\pm 1}$ by $K^{\pm 1}$: apart from these changes, the analysis and its outcome will be exactly the same. Like for $\mathfrak{sl}_2$ (cf. §7.5), all the adjoint quantum objects — i.e. $Hyp_q^{\,a}(\mathfrak{e}_2)$, $Hyp_q^{\,a}(\mathfrak{e}_2)'$ and $\big(Hyp_q^{\,a}(\mathfrak{e}_2)'\big)^\vee$ — will be strictly contained in the corresponding simply connected quantum objects; nevertheless, the semiclassical limits will be the same in the case of $Hyp_q(\mathfrak{e}_2)$ (always yielding $Hyp(\mathfrak{e}_2)$) and in the case of $\big(Hyp_q(\mathfrak{e}_2)'\big)^\vee$ (giving $\mathbf{u}(\mathfrak{k})$, in both cases), while the semiclassical limit of $Hyp_q(\mathfrak{e}_2)'$ in the simply connected case will be a (countable) covering of that in the adjoint case.

8.6 The QFAs $F_q[E_2]$ and $F_q[{}_aE_2]$. In this and the following sections we look at Theorem 2.2 starting from QFAs, to get QrUEAs out of them.

We begin by introducing a QFA for the Euclidean groups E_2 and ${}_aE_2$. Let $F_q[E_2]$ be the unital associative R–algebra with generators $\mathrm{a}^{\pm 1}$, b, c and relations

$$\mathrm{a}\,\mathrm{b} = q\,\mathrm{b}\,\mathrm{a}\,, \qquad \mathrm{a}\,\mathrm{c} = q\,\mathrm{c}\,\mathrm{a}\,, \qquad \mathrm{b}\,\mathrm{c} = \mathrm{c}\,\mathrm{b}$$

endowed with the Hopf algebra structure given by

$$\Delta\big(\mathrm{a}^{\pm 1}\big) = \mathrm{a}^{\pm 1} \otimes \mathrm{a}^{\pm 1}\,, \qquad \Delta(\mathrm{b}) = \mathrm{b} \otimes \mathrm{a}^{-1} + \mathrm{a} \otimes \mathrm{b}\,, \qquad \Delta(\mathrm{c}) = \mathrm{c} \otimes \mathrm{a} + \mathrm{a}^{-1} \otimes \mathrm{c}$$
$$\epsilon\big(\mathrm{a}^{\pm 1}\big) = 1\,,\;\; \epsilon(\mathrm{b}) = 0\,,\;\; \epsilon(\mathrm{c}) = 0\,, \qquad S\big(\mathrm{a}^{\pm 1}\big) = \mathrm{a}^{\mp 1}\,,\;\; S(\mathrm{b}) = -q^{-1}\,\mathrm{b}\,,\;\; S(\mathrm{c}) = -q^{+1}\,\mathrm{c}\,.$$

Define $F_q[{}_aE_2]$ as the R–submodule of $F_q[E_2]$ spanned by the products of an even number of generators, i.e. monomials of even degree in $\mathrm{a}^{\pm 1}$, b, c: this is a unital subalgebra of $F_q[E_2]$, generated by $\beta := \mathrm{b}\,\mathrm{a}$, $\alpha^{\pm 1} := \mathrm{a}^{\pm 2}$, and $\gamma := \mathrm{a}^{-1}\mathrm{c}$. Let also $\mathbb{F}_q[E_2] := \big(F_q[E_2]\big)_F$ and $\mathbb{F}_q[{}_aE_2] := \big(F_q[{}_aE_2]\big)_F$, which have the same presentation than $F_q[E_2]$ and $F_q[{}_aE_2]$ but over $F(R)$. Essentially by definition, both $F_q[E_2]$ and $F_q[{}_aE_2]$ are QFAs (at $\hbar = q-1$), whose semiclassical limit is $F[E_2]$ and $F[{}_aE_2]$ respectively.

8.7 Computation of $F_q[E_2]^\vee$ and $F_q[{}_aE_2]^\vee$ and specializations $F_q[E_2]^\vee \xrightarrow{q\to 1} U(\mathfrak{g}^*)$ and $F_q[{}_aE_2]^\vee \xrightarrow{q\to 1} U(\mathfrak{g}^*)$. In this section we go and compute $F_q[G]^\vee$ and its semiclassical limit (i.e. its specialization at $q=1$) for both $G = E_2$ and $G = {}_aE_2$.

First, $F_q[E_2]$ is free over R, with basis $\Big\{ \mathrm{b}^b \mathrm{a}^a \mathrm{c}^c \,\Big|\, a \in \mathbb{Z}, b, c \in \mathbb{N} \Big\}$, so the set $\mathbb{B}_s := \Big\{ \mathrm{b}^b (\mathrm{a}^{\pm 1} - 1)^a \mathrm{c}^c \,\Big|\, a, b, c \in \mathbb{N} \Big\}$ is an R–basis as well. Second, since $\epsilon(\mathrm{b}) = \epsilon\big(\mathrm{a}^{\pm 1} - 1\big) = \epsilon(\mathrm{c}) = 0$, the ideal $J := Ker\Big(\epsilon \colon F_q[E_2] \longrightarrow R\Big)$ is the span of $\mathbb{B}_s \setminus \{1\}$. Now $I := Ker\Big(F_q[E_2] \xrightarrow{\ \epsilon\ } R \xrightarrow{q\mapsto 1} \Bbbk\Big) = J + (q-1)\cdot F_q[E_2]$, thus $F_q[E_2]^\vee := \sum_{n\geq 0} \Big((q-1)^{-1} I\Big)^n$ turns out to be the unital R–algebra (subalgebra of $\mathbb{F}_q[E_2]$) with generators $D_\pm := \dfrac{\mathrm{a}^{\pm 1} - 1}{q-1}$, $E := \dfrac{\mathrm{b}}{q-1}$, and $F := \dfrac{\mathrm{c}}{q-1}$ and relations

$$D_+E = qED_+ + E\,, \quad D_+F = qFD_+ + F\,, \quad ED_- = qD_-E + E\,, \quad FD_- = qD_-F + F$$
$$EF = FE\,, \qquad D_+D_- = D_-D_+\,, \qquad D_+ + D_- + (q-1)D_+D_- = 0$$

with a Hopf structure given by

$$\begin{aligned} \Delta(E) &= E \otimes 1 + 1 \otimes E + (q-1)\big(E \otimes D_- + D_+ \otimes E\big)\,, & \epsilon(E) &= 0\,, & S(E) &= -q^{-1}E \\ \Delta(D_\pm) &= D_\pm \otimes 1 + 1 \otimes D_\pm + (q-1)\cdot D_\pm \otimes D_\pm\,, & \epsilon(D_\pm) &= 0\,, & S(D_\pm) &= D_\mp \\ \Delta(F) &= F \otimes 1 + 1 \otimes F + (q-1)\big(F \otimes D_+ + D_- \otimes F\big)\,, & \epsilon(F) &= 0\,, & S(F) &= -q^{+1}F\,. \end{aligned}$$

This implies that $F_q[E_2]^\vee \xrightarrow{q\to 1} U(\mathfrak{e}_2{}^*)$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$F_q[E_2]^\vee \Big/ (q-1)\, F_q[E_2]^\vee \xrightarrow{\ \cong\ } U(\mathfrak{e}_2{}^*)$$

exists, given by $D_\pm \mod (q-1) \mapsto \pm\mathrm{h}/2$, $E \mod (q-1) \mapsto \mathrm{e}$, $F \mod (q-1) \mapsto \mathrm{f}$; thus $F_q[E_2]^\vee$ does specialize to $U(\mathfrak{e}_2{}^*)$ *as a co-Poisson Hopf algebra*, q.e.d.

Similarly, if we consider $F_q[{}_aE_2]$ the same analysis works again. In fact, $F_q[{}_aE_2]$ is free over R, with basis $\mathbb{B}_a := \Big\{ \beta^b (\alpha^{\pm 1} - 1)^a \gamma^c \,\Big|\, a, b, c \in \mathbb{N} \Big\}$; therefore, as above the ideal $J := Ker\Big(\epsilon \colon F_q[{}_aE_2] \to R\Big)$ is the span of $\mathbb{B}_a \setminus \{1\}$. Now, we have $I := Ker\Big(F_q[{}_aE_2] \xrightarrow{\ \epsilon\ } R \xrightarrow{q\mapsto 1} \Bbbk\Big) = J + (q-1)\cdot F_q[{}_aE_2]$, so $F_q[{}_aE_2]^\vee := \sum_{n\geq 0} \Big((q-1)^{-1} I\Big)^n$ is nothing but the unital R–algebra (subalgebra of $\mathbb{F}_q[{}_aE_2]$) with generators $H_\pm := \dfrac{\alpha^{\pm 1} - 1}{q-1}$, $E' := \dfrac{\beta}{q-1}$, and $F' := \dfrac{\gamma}{q-1}$ and relations

$$E'F' = q^{-2}F'E',\ H_+E' = q^2E'H_+ + (q+1)E',\ H_+F' = q^2F'H_+ + (q+1)F',\ H_+H_- = H_-H_+$$
$$E'H_- = q^2H_-E' + (q+1)E',\ F'H_- = q^2H_-F' + (q+1)F',\ H_+ + H_- + (q-1)H_+H_- = 0$$

with a Hopf structure given by

$$\Delta(E') = E' \otimes 1 + 1 \otimes E' + (q-1) \cdot H_+ \otimes E' \,, \qquad \epsilon(E') = 0 \,, \qquad S(E') = -E' - (q-1) H_- E'$$
$$\Delta(H_\pm) = H_\pm \otimes 1 + 1 \otimes H_\pm + (q-1) \cdot H_\pm \otimes H_\pm \,, \qquad \epsilon(H_\pm) = 0 \,, \qquad S(H_\pm) = H_\mp$$
$$\Delta(F') = F' \otimes 1 + 1 \otimes F' + (q-1) \cdot H_- \otimes F' \,, \qquad \epsilon(F') = 0 \,, \qquad S(F') = -F' - (q-1) H_+ F' \,.$$

This implies that $F_q[{}_aE_2]^\vee \xrightarrow{q \to 1} U(\mathfrak{e}_2{}^*)$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$F_q[{}_aE_2]^\vee \Big/ (q-1) \, F_q[{}_aE_2]^\vee \xrightarrow{\ \cong\ } U(\mathfrak{e}_2{}^*)$$

is given by $H_\pm \mod (q-1) \mapsto \pm \mathrm{h}$, $E' \mod (q-1) \mapsto \mathrm{e}$, $F' \mod (q-1) \mapsto \mathrm{f}$; so $F_q[{}_aE_2]^\vee$ too specializes to $U(\mathfrak{e}_2{}^*)$ *as a co-Poisson Hopf algebra*, as expected.

We finish noting that, once more, *this analysis (and its outcome) is characteristic-free.*

8.8 The identities $\big(F_q[E_2]^\vee\big)' = F_q[E_2]$ **and** $\big(F_q[{}_aE_2]^\vee\big)' = F_q[{}_aE_2]$**.** In this section we verify for the QFAs $H = F_q[E_2]$ and $H = F_q[{}_aE_2]$ the validity of the part of Theorem 2.2*(b)* claiming that $H \in \mathcal{QFA} \implies \big(H^\vee\big)' = H$. Once more, our arguments will prove this result for $Char(\Bbbk) \geq 0$, thus going beyond what forecasted by Theorem 2.2.

By induction we find formulas $\Delta^n(E) = \sum_{r+s+1=n} \mathrm{a}^{\otimes r} \otimes E \otimes \big(\mathrm{a}^{-1}\big)^{\otimes s}$, $\Delta^n(D_\pm) = \sum_{r+s+1=n} \big(\mathrm{a}^{\pm 1}\big)^{\otimes r} \otimes D_\pm \otimes 1^{\otimes s}$, and $\Delta^n(F) = \sum_{r+s+1=n} \big(\mathrm{a}^{-1}\big)^{\otimes r} \otimes E \otimes \mathrm{a}^{\otimes s}$: these imply

$$\delta_n(E) = \sum_{r+s+1=n} (\mathrm{a}-1)^{\otimes r} \otimes E \otimes \big(\mathrm{a}^{-1}-1\big)^{\otimes s} = (q-1)^{n-1} \sum_{r+s+1=n} D_+{}^{\otimes r} \otimes E \otimes D_-{}^{\otimes s}$$
$$\delta_n(D_\pm) = \big(\mathrm{a}^{\pm 1}-1\big)^{\otimes (n-1)} \otimes D_\pm = (q-1)^{n-1} D_\pm{}^{\otimes n}$$
$$\delta_n(F) = \sum_{r+s+1=n} \big(\mathrm{a}^{-1}-1\big)^{\otimes r} \otimes E \otimes (\mathrm{a}-1)^{\otimes s} = (q-1)^{n-1} \sum_{r+s+1=n} D_-{}^{\otimes r} \otimes E \otimes D_+{}^{\otimes s}$$

which gives $\dot{E} := (q-1)E$, $\dot{D}_\pm := (q-1)D_\pm$, $\dot{F} := (q-1)F \in \big(F_q[E_2]^\vee\big)' \setminus (q-1) \cdot \big(F_q[E_2]^\vee\big)'$. So $\big(F_q[E_2]^\vee\big)'$ contains the unital R–subalgebra A' generated (inside $\mathbb{F}_q[E_2]$) by $\dot{E}$, $\dot{D}_\pm$ and $\dot{F}$; but $\dot{E} = \mathrm{b}$, $\dot{D}_\pm = \mathrm{a}^{\pm 1} - 1$, and $\dot{F} = \mathrm{c}$, thus A' is just $F_q[E_2]$. Since $F_q[E_2]^\vee$ is the R–span of $\Big\{ E^e D_+^{d_+} D_-^{d_-} F^f \,\Big|\, e, d_+, d_-, f \in \mathbb{N} \Big\}$, one easily sees — using the previous formulas for Δ^n — that in fact $\big(F_q[E_2]^\vee\big)' = A' = F_q[E_2]$, q.e.d.

When dealing with the adjoint case, the previous arguments go through again: in fact, $\big(F_q[{}_aE_2]^\vee\big)'$ turns out to coincide with the unital R–subalgebra A'' generated (inside $\mathbb{F}_q[{}_aE_2]$) by $\dot{E}' := (q-1)E' = \beta$, $\dot{H}_\pm := (q-1)H_\pm = \alpha^{\pm 1} - 1$, and $\dot{F}' := (q-1)F' = \gamma$; but this is also generated by β, $\alpha^{\pm 1}$ and γ, thus it coincides with $F_q[{}_aE_2]$, q.e.d.

§ 9 Fourth example: quantum Heisenberg group

9.1 The classical setting. Let $\Bbbk$ be any field of characteristic $p \geq 0$. Let $G := H_n(\Bbbk) = H_n$, the $(2n+1)$–dimensional Heisenberg group; its tangent Lie algebra $\mathfrak{g} = \mathfrak{h}_n$ is generated by $\{ f_i, h, e_i \,|\, i = 1, \dots, n \}$ with relations $[e_i, f_j] = \delta_{ij} h$, $[e_i, e_j] = [f_i, f_j] = [h, e_i] = [h, f_j] = 0$ $(\forall\, i, j = 1, \dots n)$. The formulas $\delta(f_i) = h \otimes f_i - f_i \otimes h$, $\delta(h) = 0$, $\delta(e_i) = h \otimes e_i - e_i \otimes h$ $(\forall\, i = 1, \dots n)$ make $\mathfrak{h}_n$ into a Lie bialgebra, which yields H_n with a structure of Poisson group; these same formulas give also a presentation of the co-Poisson Hopf algebra $U(\mathfrak{h}_n)$ (with the standard Hopf structure). When $p > 0$ we consider on $\mathfrak{h}_n$ the p–operation uniquely defined by $e_i^{[p]} = 0$, $f_i^{[p]} = 0$, $h^{[p]} = h$ (for all $i = 1, \dots, n$), which makes it into a restricted Lie bialgebra. The group H_n is usually realized as the group of all square matrices $\big(a_{ij}\big)_{i,j=1,\dots,n+2;}$ such that $a_{ii} = 1 \;\forall\, i$ and $a_{ij} = 0 \;\forall\, i, j$ such that either $i > j$ or $1 \neq i < j$ or $i < j \neq n{+}2$; it can also be realized as $H_n = \Bbbk^n \times \Bbbk \times \Bbbk^n$ with group operation given by $\big(\underline{a}', c', \underline{b}'\big) \cdot \big(\underline{a}'', c'', \underline{b}''\big) = \big(\underline{a}' + \underline{a}'', c' + c'' + \underline{a}' * \underline{b}'', \underline{b}' + \underline{b}''\big)$, where we use vector notation $\underline{v} = (v_1, \dots, v_n) \in k^n$ and $\underline{a}' * \underline{b}'' := \sum_{i=1}^n a'_i b''_i$ is the standard scalar product in k^n; in particular the identity of H_n is $e = (\underline{0}, 0, \underline{0})$ and the inverse of a generic element is given by $\big(\underline{a}, c, \underline{b}\big)^{-1} = \big(-\underline{a}\,, -c + \underline{a} * \underline{b}\,, -\underline{b}\big)$. Therefore, the function algebra $F[H_n]$ is the unital associative commutative $\Bbbk$–algebra with generators $a_1, \dots, a_n, c, b_1, \dots, b_n$, and with Poisson Hopf algebra structure given by

$$\begin{gathered}
\Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i\,, \;\; \Delta(c) = c \otimes 1 + 1 \otimes c + \textstyle\sum_{\ell=1}^n a_\ell \otimes b_\ell\,, \;\; \Delta(b_i) = b_i \otimes 1 + 1 \otimes b_i \\
\epsilon(a_i) = 0\,, \;\; \epsilon(c) = 0\,, \;\; \epsilon(b_i) = 0\,, \qquad S(a_i) = -a_i\,, \;\; S(c) = -c + \textstyle\sum_{\ell=1}^n a_\ell b_\ell\,, \;\; S(b_i) = -b_i \\
\{a_i, a_j\} = 0\,, \qquad \{a_i, b_j\} = 0\,, \qquad \{b_i, b_j\} = 0\,, \qquad \{c\,, a_i\} = a_i\,, \qquad \{c\,, b_i\} = b_i
\end{gathered}$$

for all $i, j = 1, \dots, n$. (N.B.: with respect to this presentation, we have $f_i = \partial_{b_i}\big|_e$, $h = \partial_c\big|_e$, $e_i = \partial_{a_i}\big|_e$, where e is the identity element of H_n). The dual Lie bialgebra $\mathfrak{g}^* = {\mathfrak{h}_n}^*$ is the Lie algebra with generators f_i, h, e_i, and relations $[\mathrm{h}, \mathrm{e}_i] = \mathrm{e}_i$, $[\mathrm{h}, \mathrm{f}_i] = \mathrm{f}_i$, $[\mathrm{e}_i, \mathrm{e}_j] = [\mathrm{e}_i, \mathrm{f}_j] = [\mathrm{f}_i, \mathrm{f}_j] = 0$, with Lie cobracket given by $\delta(\mathrm{f}_i) = 0$, $\delta(\mathrm{h}) = \sum_{j=1}^n (\mathrm{e}_j \otimes \mathrm{f}_j - \mathrm{f}_j \otimes \mathrm{e}_j)$, $\delta(\mathrm{e}_i) = 0$ for all $i = 1, \dots, n$ (we take $\mathrm{f}_i := f_i^*$, $\mathrm{h} := h^*$, $\mathrm{e}_i := e_i^*$, where $\big\{ f_i^*, h^*, e_i^* \,|\, i = 1, \dots, n \big\}$ is the basis of ${\mathfrak{h}_n}^*$ which is the dual of the basis $\{ f_i, h, e_i \,|\, i = 1, \dots, n \}$ of $\mathfrak{h}_n$). This again gives a presentation of $U({\mathfrak{h}_n}^*)$ too. If $p > 0$ then ${\mathfrak{h}_n}^*$ is a restricted Lie bialgebra with respect to the p–operation given by $\mathrm{e}_i^{[p]} = 0$, $\mathrm{f}_i^{[p]} = 0$, $\mathrm{h}^{[p]} = \mathrm{h}$ (for all $i = 1, \dots, n$). The simply connected algebraic Poisson group with tangent Lie bialgebra ${\mathfrak{h}_n}^*$ can be realized (with $\Bbbk^\star := \Bbbk \setminus \{0\}$) as ${}_s{H_n}^* = \Bbbk^n \times \Bbbk^\star \times \Bbbk^n$, with group operation $\big(\underline{\dot{\alpha}}, \dot{\gamma}, \underline{\dot{\beta}}\big) \cdot \big(\underline{\check{\alpha}}, \check{\gamma}, \underline{\check{\beta}}\big) = \big(\check{\gamma}\underline{\dot{\alpha}} + \dot{\gamma}^{-1}\underline{\check{\alpha}}, \dot{\gamma}\check{\gamma}, \check{\gamma}\underline{\dot{\beta}} + \dot{\gamma}^{-1}\underline{\check{\beta}}\big)$; so the identity of ${}_s{H_n}^*$ is $e = (\underline{0}, 1, \underline{0})$ and the inverse is given by $\big(\underline{\alpha}, \gamma, \underline{\beta}\,\big)^{-1} = \big(-\underline{\alpha}, \gamma^{-1}, -\underline{\beta}\,\big)$. Its centre is $Z\big({}_s{H_n}^*\big) = \big\{(\underline{0}, 1, \underline{0}), (\underline{0}, -1, \underline{0})\big\} =: Z$, so there is only one other (Poisson) group with tangent Lie bialgebra ${\mathfrak{h}_n}^*$, that is the adjoint group ${}_a{H_n}^* := {}_s{H_n}^* \big/ Z$.

It is clear that $F\big[{}_s{H_n}^*\big]$ is the unital associative commutative $\Bbbk$–algebra with generators

$\alpha_1, \dots, \alpha_n, \gamma^{\pm 1}, \beta_1, \dots, \beta_n$, and with Poisson Hopf algebra structure given by

$$\begin{gathered}
\Delta(\alpha_i) = \alpha_i \otimes \gamma + \gamma^{-1} \otimes \alpha_i\,, \qquad \Delta\big(\gamma^{\pm 1}\big) = \gamma^{\pm 1} \otimes \gamma^{\pm 1}, \qquad \Delta(\beta_i) = \beta_i \otimes \gamma + \gamma^{-1} \otimes \beta_i \\
\epsilon(\alpha_i) = 0\,,\ \ \epsilon\big(\gamma^{\pm 1}\big) = 1\,,\ \ \epsilon(\beta_i) = 0\,, \qquad S(\alpha_i) = -\alpha_i\,,\ \ S\big(\gamma^{\pm 1}\big) = \gamma^{\mp 1},\ \ S(\beta_i) = -\beta_i \\
\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta_j\} = \{\beta_i, \beta_j\} = \{\alpha_i, \gamma\} = \{\beta_i, \gamma\} = 0\,, \qquad \{\alpha_i, \beta_j\} = \delta_{ij} \big(\gamma^2 - \gamma^{-2}\big)\big/2
\end{gathered}$$

for all $i, j = 1, \dots, n$ (N.B.: with respect to this presentation, we have $\mathrm{f}_i = \partial_{\beta_i}\big|_e$, $\mathrm{h} = \frac{1}{2}\, \gamma\, \partial_\gamma\big|_e$, $\mathrm{e}_i = \partial_{\alpha_i}\big|_e$, where e is the identity element of ${}_sH_n{}^*$), and $F\big[{}_aH_n{}^*\big]$ can be identified — as in the case of the Euclidean group — with the Poisson Hopf subalgebra of $F\big[{}_aH_n{}^*\big]$ which is spanned by products of an even number of generators: this is generated, as a unital subalgebra, by $\alpha_i \gamma$, $\gamma^{\pm 2}$, and $\gamma^{-1} \beta_i$ $\,(i = 1, \dots, n\,)$.

9.2 The QrUEAs $U_q^s(\mathfrak{h}_n)$ and $U_q^a(\mathfrak{h}_n)$. We switch now to quantizations. Once again, let R be a domain and let $\hbar \in R \setminus \{0\}$ and $q := 1 + \hbar \in R$ be like in §7.2.

Let $\mathbb{U}_q(\mathfrak{g}) = \mathbb{U}_q^s(\mathfrak{h}_n)$ be the unital associative $F(R)$–algebra with generators F_i, $L^{\pm 1}$, E_i $(i = 1, \dots, n\,)$ and relations

$$L L^{-1} = 1 = L^{-1} L\,, \quad L^{\pm 1} F = F L^{\pm 1}\,, \quad L^{\pm 1} E = E L^{\pm 1}\,, \quad E_i F_j - F_j E_i = \delta_{ij} \frac{L^2 - L^{-2}}{q - q^{-1}}$$

for all $i, j = 1, \dots, n$; we give it a structure of Hopf algebra, by setting $(\forall\, i, j = 1, \dots, n\,)$

$$\begin{gathered}
\Delta(E_i) = E_i \otimes 1 + L^2 \otimes E_i\,, \quad \Delta\big(L^{\pm 1}\big) = L^{\pm 1} \otimes L^{\pm 1}, \quad \Delta(F_i) = F_i \otimes L^{-2} + 1 \otimes F_i \\
\epsilon(E_i) = 0\,,\ \epsilon\big(L^{\pm 1}\big) = 1\,,\ \epsilon(F_i) = 0\,, \quad S(E_i) = -L^{-2} E_i,\ S\big(L^{\pm 1}\big) = L^{\mp 1},\ S(F_i) = -F_i L^2
\end{gathered}$$

Note that $\Big\{ \prod_{i=1}^n F_i^{a_i} \cdot L^z \cdot \prod_{i=1}^n E_i^{d_i} \,\Big|\, z \in \mathbb{Z},\ a_i, d_i \in \mathbb{N},\ \forall\, i \Big\}$ is an $F(R)$–basis of $\mathbb{U}_q^s(\mathfrak{h}_n)$.

Now, let $U_q^s(\mathfrak{h}_n)$ be the unital R–subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$ generated by $F_1, \dots, F_n$, $D := \dfrac{L - 1}{q - 1}$, $\varGamma := \dfrac{L - L^{-2}}{q - q^{-1}}$, $E_1, \dots, E_n$. Then $U_q^s(\mathfrak{h}_n)$ can be presented as the associative unital algebra with generators $F_1, \dots, F_n, L^{\pm 1}, D, \varGamma, E_1, \dots, E_n$ and relations

$$\begin{gathered}
D X = X D\,, \qquad L^{\pm 1} X = X L^{\pm 1}\,, \qquad \varGamma X = X \varGamma\,, \qquad E_i F_j - F_j E_i = \delta_{ij} \varGamma \\
L = 1 + (q - 1) D\,, \quad L^2 - L^{-2} = \big(q - q^{-1}\big) \varGamma\,, \quad D (L + 1) \big(1 + L^{-2}\big) = \big(1 + q^{-1}\big) \varGamma
\end{gathered}$$

for all $X \in \big\{ F_i, L^{\pm 1}, D, \varGamma, E_i \big\}_{i=1,\dots,n}$ and $i, j = 1, \dots, n$; furthermore, $U_q^s(\mathfrak{h}_n)$ is a Hopf subalgebra (over R), with

$$\begin{array}{lll}
\Delta(\varGamma) = \varGamma \otimes L^2 + L^{-2} \otimes \varGamma\,, & \epsilon(\varGamma) = 0\,, & S(\varGamma) = -\varGamma \\
\Delta(D) = D \otimes 1 + L \otimes D\,, & \epsilon(D) = 0\,, & S(D) = -L^{-1} D\,.
\end{array}$$

Moreover, from relations $L = 1 + (q - 1) D$ and $L^{-1} = L^3 - \big(q - q^{-1}\big) L \varGamma$ it follows that

$$U_q^s(\mathfrak{h}_n) \ = \ R\text{–span of } \left\{ \prod_{i=1}^n F_i^{a_i} \cdot D^b \varGamma^c \cdot \prod_{i=1}^n E_i^{d_i} \,\middle|\, a_i, b, c, d_i \in \mathbb{N},\ \forall\, i = 1, \dots, n \right\} \qquad (9.1)$$

The "adjoint version" of $\mathbb{U}_q^s(\mathfrak{h}_n)$ is the unital subalgebra $\mathbb{U}_q^a(\mathfrak{h}_n)$ generated by $F_i, K^{\pm 1} := L^{\pm 2}$, E_i $(i=1,\dots,n)$, which is clearly a Hopf subalgebra. It also has an R–integer form $U_q^a(\mathfrak{h}_n)$, namely the unital R–subalgebra generated by $F_1, \dots, F_n$, $K^{\pm 1}$, $H := \dfrac{K-1}{q-1}$, $\varGamma := \dfrac{K-K^{-1}}{q-q^{-1}}$, $E_1, \dots, E_n$: this has relations

$$\begin{gathered} HX = XH\,, \qquad K^{\pm 1}X = XK^{\pm 1}\,, \qquad \varGamma X = X\varGamma\,, \qquad E_iF_j - F_jE_i = \delta_{ij}\varGamma \\ K = 1 + (q-1)H\,, \qquad K - K^{-1} = \big(q-q^{-1}\big)\varGamma\,, \qquad H\big(1+K^{-1}\big) = \big(1+q^{-1}\big)\varGamma \end{gathered}$$

for all $X \in \big\{F_i, K^{\pm 1}, H, \varGamma, E_i\big\}_{i=1,\dots,n}$ and $i,j = 1,\dots,n$, and Hopf operations given by

$$\begin{array}{ccc} \Delta(E_i) = E_i \otimes 1 + K \otimes E_i\,, & \epsilon(E_i) = 0\,, & S(E_i) = -K^{-1}E_i \\ \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, & \epsilon\big(K^{\pm 1}\big) = 1\,, & S\big(K^{\pm 1}\big) = K^{\mp 1} \\ \Delta(H) = H \otimes 1 + K \otimes H\,, & \epsilon(H) = 0\,, & S(H) = -K^{-1}H \\ \Delta(\varGamma) = \varGamma \otimes K^{-1} + K \otimes \varGamma\,, & \epsilon(\varGamma) = 0\,, & S(\varGamma) = -\varGamma \\ \Delta(F_i) = F_i \otimes K^{-1} + 1 \otimes F_i\,, & \epsilon(F_i) = 0\,, & S(F_i) = -F_iK^{+1} \end{array}$$

for all $i = 1,\dots,n$. One can easily check that $U_q^s(\mathfrak{h}_n)$ is a QrUEA, with $U(\mathfrak{h}_n)$ as semiclassical limit: in fact, mapping the generators $F_i \mod (q-1)$, $L^{\pm 1} \mod (q-1)$, $D \mod (q-1)$, $\varGamma \mod (q-1)$, $E_i \mod (q-1)$ respectively to f_i, 1, $h/2$, h, $e_i \in U(\mathfrak{h}_n)$ yields a co-Poisson Hopf algebra isomorphism between $U_q^s(\mathfrak{h}_n)\Big/(q-1)\,U_q^s(\mathfrak{h}_n)$ and $U(\mathfrak{h}_n)$. Similarly, $U_q^a(\mathfrak{h}_n)$ is a QrUEA too, again with limit $U(\mathfrak{h}_n)$, for a co-Poisson Hopf algebra isomorphism between $U_q^a(\mathfrak{h}_n)\Big/(q-1)\,U_q^a(\mathfrak{h}_n)$ and $U(\mathfrak{h}_n)$ is given by mapping the generators $F_i \mod (q-1)$, $K^{\pm 1} \mod (q-1)$, $H \mod (q-1)$, $\varGamma \mod (q-1)$, $E_i \mod (q-1)$ respectively to f_i, 1, h, h, $e_i \in U(\mathfrak{h}_n)$.

9.3 Computation of $U_q(\mathfrak{h}_n)'$ and specialization $U_q(\mathfrak{h}_n)' \xrightarrow{\,q\to 1\,} F\big[{H_n}^*\big]$. Here we compute $U_q^s(\mathfrak{h}_n)'$ and $U_q^a(\mathfrak{h}_n)'$, and their semiclassical limits, along the pattern of §7.3.

Definitions give, for any $n \in \mathbb{N}$, $\Delta^n(E_i) = \sum_{s=1}^n \big(L^2\big)^{\otimes(s-1)} \otimes E_i \otimes 1^{\otimes(n-s)}$, hence $\delta_n(E_i) = (q-1)^{n-1} \cdot D^{\otimes(n-1)} \otimes E_i$ so $\delta_n\big((q-1)E\big) \in (q-1)^n\,U_q^s(\mathfrak{h}_n) \setminus (q-1)^{n+1}\,U_q^s(\mathfrak{h}_n)$ whence $\dot{E}_i := (q-1)\,E_i \in U_q^s(\mathfrak{h}_n)'$, whereas $E_i \notin U_q^s(\mathfrak{h}_n)'$; similarly, we have $\dot{F}_i := (q-1)\,F_i$, $L^{\pm 1}$, $\dot{D} := (q-1)\,D = L-1$, $\dot{\varGamma} := (q-1)\,\varGamma \in U_q^s(\mathfrak{h}_n)' \setminus (q-1)\,U_q^s(\mathfrak{h}_n)'$, for all $i = 1,\dots,n$. Therefore $U_q^s(\mathfrak{h}_n)'$ contains the subalgebra U' generated by $\dot{F}_i$, $L^{\pm 1}$, $\dot{D}$, $\dot{\varGamma}$, $\dot{E}_i$; we conclude that in fact $U_q^s(\mathfrak{h}_n)' = U'$: this is easily seen — like for SL_2 and for E_2 — using the formulas above along with (9.1). As a consequence, $U_q^s(\mathfrak{h}_n)'$ is the unital R–algebra with generators $\dot{F}_1, \dots, \dot{F}_n$, $L^{\pm 1}$, $\dot{D}$, $\dot{\varGamma}$, $\dot{E}_1, \dots, \dot{E}_n$ and relations

$$\begin{gathered} \dot{D}\dot{X} = \dot{X}\dot{D}\,, \qquad L^{\pm 1}\dot{X} = \dot{X}L^{\pm 1}\,, \qquad \dot{\varGamma}\dot{X} = \dot{X}\dot{\varGamma}\,, \qquad \dot{E}_i\dot{F}_j - \dot{F}_j\dot{E}_i = \delta_{ij}(q-1)\dot{\varGamma} \\ L = 1 + \dot{D}\,, \qquad L^2 - L^{-2} = \big(1+q^{-1}\big)\dot{\varGamma}\,, \qquad \dot{D}(L+1)\big(1+L^{-2}\big) = \big(1+q^{-1}\big)\dot{\varGamma} \end{gathered}$$

for all $\dot{X} \in \big\{\dot{F}_i, L^{\pm 1}, \dot{D}, \dot{\varGamma}, \dot{E}_i\big\}_{i=1,\dots,n}$ and $i, j = 1, \dots, n$, with Hopf structure given by

$$\begin{array}{cccc} \Delta\big(\dot{E}_i\big) = \dot{E}_i \otimes 1 + L^2 \otimes \dot{E}_i \, , & \epsilon\big(\dot{E}_i\big) = 0 \, , & S\big(\dot{E}_i\big) = -L^{-2} \dot{E}_i & \forall\, i = 1, \dots, n \\ \Delta\big(L^{\pm 1}\big) = L^{\pm 1} \otimes L^{\pm 1} \, , & \epsilon\big(L^{\pm 1}\big) = 1 \, , & S\big(L^{\pm 1}\big) = L^{\mp 1} & \\ \Delta\big(\dot{\varGamma}\big) = \dot{\varGamma} \otimes L^2 + L^{-2} \otimes \dot{\varGamma} \, , & \epsilon\big(\dot{\varGamma}\big) = 0 \, , & S\big(\dot{\varGamma}\big) = -\varGamma & \\ \Delta\big(\dot{D}\big) = \dot{D} \otimes 1 + L \otimes \dot{D} \, , & \epsilon\big(\dot{D}\big) = 0 \, , & S\big(\dot{D}\big) = -L^{-1} \dot{D} & \\ \Delta\big(\dot{F}_i\big) = \dot{F}_i \otimes L^{-2} + 1 \otimes \dot{F}_i \, , & \epsilon\big(F_i\big) = 0 \, , & S\big(\dot{F}_i\big) = -\dot{F}_i L^2 & \forall\, i = 1, \dots, n \, . \end{array}$$

A similar analysis shows that $U_q^a(\mathfrak{h}_n)'$ coincides with the unital R–subalgebra U'' of $U_q^a(\mathfrak{h}_n)$ generated by $\dot{F}_i$, $K^{\pm 1}$, $\dot{H} := (q-1)\,H$, $\dot{\varGamma}$, $\dot{E}_i$ $(i = 1, \dots, n)$; in particular, $U_q^a(\mathfrak{h}_n)' \subset U_q^s(\mathfrak{h}_n)'$. Therefore $U_q^a(\mathfrak{h}_n)'$ can be presented as the unital associative R–algebra with generators $\dot{F}_1, \dots, \dot{F}_n, \dot{H}, K^{\pm 1}, \dot{\varGamma}, \dot{E}_1, \dots, \dot{E}_n$ and relations

$$\dot{H}\dot{X} = \dot{X}\dot{H} \, , \qquad K^{\pm 1}\dot{X} = \dot{X}K^{\pm 1} \, , \qquad \dot{\varGamma}\dot{X} = \dot{X}\dot{\varGamma} \, , \qquad \dot{E}_i\dot{F}_j - \dot{F}_j\dot{E}_i = \delta_{ij}(q-1)\dot{\varGamma}$$
$$K = 1 + \dot{H} \, , \quad K - K^{-1} = \big(1 + q^{-1}\big)\dot{\varGamma} \, , \quad \dot{H}\big(1 + K^{-1}\big) = \big(1 + q^{-1}\big)\dot{\varGamma}$$

for all $\dot{X} \in \big\{\dot{F}_i, K^{\pm 1}, \dot{K}, \dot{\varGamma}, \dot{E}_i\big\}_{i=1,\dots,n}$ and $i, j = 1, \dots, n$, with Hopf structure given by

$$\begin{array}{cccc} \Delta\big(\dot{E}_i\big) = \dot{E}_i \otimes 1 + K \otimes \dot{E}_i \, , & \epsilon\big(\dot{E}_i\big) = 0 \, , & S\big(\dot{E}_i\big) = -K^{-1} \dot{E}_i & \forall\, i = 1, \dots, n \\ \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1} \, , & \epsilon\big(K^{\pm 1}\big) = 1 \, , & S\big(K^{\pm 1}\big) = K^{\mp 1} & \\ \Delta\big(\dot{\varGamma}\big) = \dot{\varGamma} \otimes K + K^{-1} \otimes \dot{\varGamma} \, , & \epsilon\big(\dot{\varGamma}\big) = 0 \, , & S\big(\dot{\varGamma}\big) = -\varGamma & \\ \Delta\big(\dot{H}\big) = \dot{H} \otimes 1 + K \otimes \dot{H} \, , & \epsilon\big(\dot{H}\big) = 0 \, , & S\big(\dot{H}\big) = -K^{-1} \dot{H} & \\ \Delta\big(\dot{F}_i\big) = \dot{F}_i \otimes K^{-1} + 1 \otimes \dot{F}_i \, , & \epsilon\big(F_i\big) = 0 \, , & S\big(\dot{F}_i\big) = -\dot{F}_i K & \forall\, i = 1, \dots, n \, . \end{array}$$

As $q \to 1$, the presentation above yields an isomorphism of Poisson Hopf algebras

$$U_q^s(\mathfrak{h}_n)' \Big/ (q-1)\, U_q^s(\mathfrak{h}_n)' \xrightarrow{\;\cong\;} F\big[{}_sH_n{}^*\big]$$

given by $\dot{E}_i \mod (q-1) \mapsto \alpha_i \gamma^{+1}$, $L^{\pm 1} \mod (q-1) \mapsto \gamma^{\pm 1}$, $\dot{D} \mod (q-1) \mapsto \gamma - 1$, $\dot{\varGamma} \mod (q-1) \mapsto \big(\gamma^2 - \gamma^{-2}\big)\Big/2$, $\dot{F}_i \mod (q-1) \mapsto \gamma^{-1}\beta_i$. In other words, the semiclassical limit of $U_q^s(\mathfrak{h}_n)'$ is $F\big[{}_sH_n{}^*\big]$, as predicted by Theorem 2.2*(c)* for $p = 0$. Similarly, when considering the "adjoint case", we find a Poisson Hopf algebra isomorphism

$$U_q^a(\mathfrak{h}_n)' \Big/ (q-1)\, U_q^a(\mathfrak{h}_n)' \xrightarrow{\;\cong\;} F\big[{}_aH_n{}^*\big] \quad \Big(\subset F\big[{}_sH_n{}^*\big] \Big)$$

given by $\dot{E}_i \mod (q-1) \mapsto \alpha_i \gamma^{+1}$, $K^{\pm 1} \mod (q-1) \mapsto \gamma^{\pm 2}$, $\dot{H} \mod (q-1) \mapsto \gamma^2 - 1$, $\dot{\varGamma} \mod (q-1) \mapsto \big(\gamma^2 - \gamma^{-2}\big)\Big/2$, $\dot{F}_i \mod (q-1) \mapsto \gamma^{-1}\beta_i$. That is to say, $U_q^a(\mathfrak{h}_n)'$ has semiclassical limit $F\big[{}_aH_n{}^*\big]$, as predicted by Theorem 2.2*(c)* for $p = 0$.

We stress the fact that *this analysis is characteristic-free*, so we get in fact that its outcome does hold for $p > 0$ as well, thus "improving" Theorem 2.2*(c)* (like in §§7–8).

9.4 The identity $\big(U_q(\mathfrak{h}_n)'\big)^\vee = U_q(\mathfrak{h}_n)$. In this section we verify the part of Theorem 2.2*(b)* claiming, for $p=0$, that $H \in \mathcal{Q}r\mathcal{UEA} \implies \big(H'\big)^\vee = H$, both for $H = U_q^s(\mathfrak{h}_n)$ and for $H = U_q^a(\mathfrak{h}_n)$. *In addition, the same arguments will prove such a result for $p>0$ too.*

To begin with, using (9.1) and the fact that $\dot{F}_i, \dot{D}, \dot{\varGamma}, \dot{E}_i \in \mathit{Ker}\Big(\epsilon \colon U_q^s(\mathfrak{h}_n)' \twoheadrightarrow R\Big)$ we get that $J := \mathit{Ker}(\epsilon)$ is the R–span of $\mathbb{M}\setminus\{1\}$, where $\mathbb{M}$ is the set in the right-hand-side of (9.1). Since $\big(U_q^s(\mathfrak{h}_n)'\big)^\vee := \sum_{n \geq 0} \Big((q-1)^{-1} I\Big)^n$ with $I := \mathit{Ker}\Big(U_q^s(\mathfrak{h}_n)' \overset{\epsilon}{\twoheadrightarrow} R \overset{q \mapsto 1}{\twoheadrightarrow} \Bbbk\Big) = J + (q-1) \cdot U_q^s(\mathfrak{h}_n)'$ we have that $\big(U_q^s(\mathfrak{h}_n)'\big)^\vee$ is generated — as a unital R–subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$ — by $(q-1)^{-1} \dot{F}_i = F_i$, $(q-1)^{-1} \dot{D} = D$, $(q-1)^{-1} \dot{\varGamma} = \varGamma$, $(q-1)^{-1} \dot{E}_i = E_i$ ($i = 1, \dots, n$), so it coincides with $U_q^s(\mathfrak{h}_n)$, q.e.d. In the adjoint case the procedure is similar: one changes $L^{\pm 1}$, resp. $\dot{D}$, with $K^{\pm 1}$, resp. $\dot{H}$, and everything works as before.

9.5 The quantum hyperalgebra $\mathit{Hyp}_q(\mathfrak{h}_n)$. Like in §§7.5 and 8.5, we can define "quantum hyperalgebras" associated to $\mathfrak{h}_n$. Namely, first we define a Hopf subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$ over $\mathbb{Z}\big[q,q^{-1}\big]$ whose specialization at $q=1$ is the natural Kostant-like $\mathbb{Z}$–integer form $U_{\mathbb{Z}}(\mathfrak{h}_n)$ of $U(\mathfrak{h}_n)$ (generated by divided powers, and giving the hyperalgebra $\mathit{Hyp}\,(\mathfrak{h}_n)$ over any field $\Bbbk$ by scalar extension), and then take its scalar extension over R.

To be precise, let $\mathit{Hyp}_q^{s,\mathbb{Z}}(\mathfrak{h}_n)$ be the unital $\mathbb{Z}\big[q,q^{-1}\big]$–subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$ (defined like above *but over* $\mathbb{Z}\big[q,q^{-1}\big]$) generated by the "quantum divided powers" $F_i^{(m)} := F_i^m \Big/ [m]_q!$, $\left(\begin{matrix} L\,;\,c \\ m \end{matrix}\right) := \prod_{r=1}^n \dfrac{q^{c+1-r} L - 1}{q^r - 1}$, $E_i^{(m)} := E_i^m \Big/ [m]_q!$ (for all $m \in \mathbb{N}$, $c \in \mathbb{Z}$ and $i = 1, \dots, n$, with notation of §7.5) and by L^{-1}. Comparing with the case of $\mathfrak{sl}_2$ — noting that for each i the quadruple (F_i, L, L^{-1}, E_i) generates a copy of $\mathbb{U}_q^s(\mathfrak{sl}_2)$ — we see at once that this is a Hopf subalgebra of $\mathbb{U}_q^s(\mathfrak{h}_n)$, and $\mathit{Hyp}_q^{s,\mathbb{Z}}(\mathfrak{h}_n)\Big|_{q=1} \cong U_{\mathbb{Z}}(\mathfrak{h}_n)$; thus $\mathit{Hyp}_q^s(\mathfrak{h}_n) := R \otimes_{\mathbb{Z}[q,q^{-1}]} \mathit{Hyp}_q^{s,\mathbb{Z}}(\mathfrak{h}_n)$ (for any R like in §8.2, with $\Bbbk := R \big/ \hbar\, R$ and $p := \mathit{Char}(\Bbbk)$) specializes at $q=1$ to the $\Bbbk$–hyperalgebra $\mathit{Hyp}\,(\mathfrak{h}_n)$. Moreover, among all the $\left(\begin{smallmatrix} L\,;\,c \\ n \end{smallmatrix}\right)$'s it is enough to take only those with $c=0$. *From now on we assume $p>0$.*

Pushing forward the close comparison with the case of $\mathfrak{sl}_2$ we also see that $\mathit{Hyp}_q^s(\mathfrak{h}_n)'$ is the unital R–subalgebra of $\mathit{Hyp}_q^s(\mathfrak{h}_n)$ generated by L^{-1} and the "rescaled quantum divided powers" $(q-1)^m F_i^{(m)}$, $(q-1)^m \left(\begin{smallmatrix} L\,;\,0 \\ m \end{smallmatrix}\right)$ and $(q-1)^m E_i^{(m)}$, for all $m \in \mathbb{N}$ and $i = 1, \dots, n$. It follows that $\mathit{Hyp}_q^s(\mathfrak{h}_n)'\Big|_{q=1}$ is generated by the specializations at $q=1$ of $(q-1)^{p^r} F_i^{(p^r)}$, $(q-1)^{p^r} \left(\begin{smallmatrix} L\,;\,0 \\ p^r \end{smallmatrix}\right)$ and $(q-1)^{p^r} E_i^{(p^r)}$, for all $r \in \mathbb{N}$, $i = 1, \dots, n$: this proves directly that the spectrum of $\mathit{Hyp}_q^s(\mathfrak{h}_n)'\Big|_{q=1}$ has dimension 0 and height 1, and its cotangent Lie algebra has basis $\Big\{ (q-1)^{p^r} F_i^{(p^r)}, (q-1)^{p^r} \left(\begin{smallmatrix} L\,;\,0 \\ p^r \end{smallmatrix}\right), (q-1)^{p^r} E_i^{(p^r)} \mod (q-1)\, \mathit{Hyp}_q^s(\mathfrak{g})' \mod J^2 \;\Big|\; r \in \mathbb{N},\, i = 1, \dots, n \Big\}$ (with J being the augmentation ideal of $Hyp_q^{\,s}(\mathfrak{h}_n)'\Big|_{q=1}$, so that $J\Big/J^{\,2}$ is the aforementioned cotangent Lie bialgebra). Finally, $\big(Hyp_q^{\,s}(\mathfrak{h}_n)'\big)^\vee$ is generated by $(q-1)^{p^r-1}F_i^{(p^r)}$, $(q-1)^{p^r-1}\left(\begin{smallmatrix} L\,;\,0 \\ p^r \end{smallmatrix}\right)$, L^{-1} and $(q-1)^{p^r-1}E_i^{(p^r)}$ (for $r \in \mathbb{N}$, $i=1,\dots,n$): in particular $\big(Hyp_q^{\,s}(\mathfrak{h}_n)'\big)^\vee \subsetneqq Hyp_q^{\,s}(\mathfrak{h}_n)$, and $\big(Hyp_q^{\,s}(\mathfrak{h}_n)'\big)^\vee\Big|_{q=1}$ is generated by the cosets modulo $(q-1)$ of the elements above, which form indeed a basis of the restricted Lie bialgebra $\mathfrak{k}$ such that $\big(Hyp_q^{\,s}(\mathfrak{h}_n)'\big)^\vee\Big|_{q=1} = \mathbf{u}(\mathfrak{k})$.

The previous analysis stems from $\mathbb{U}_q^s(\mathfrak{h}_n)$, and so gives "simply connected quantum objects". Instead we can start from $\mathbb{U}_q^a(\mathfrak{h}_n)$, thus getting "adjoint quantum objects", moving along the same pattern but for replacing $L^{\pm 1}$ by $K^{\pm 1}$ throughout: apart from this, the analysis and its outcome are exactly the same. Like for $\mathfrak{sl}_2$ (cf. §7.5), all the adjoint quantum objects — i.e. $Hyp_q^{\,a}(\mathfrak{h}_n)$, $Hyp_q^{\,a}(\mathfrak{h}_n)'$ and $\big(Hyp_q^{\,a}(\mathfrak{h}_n)'\big)^\vee$ — will be strictly contained in the corresponding simply connected quantum objects; however, the semiclassical limits will be the same in the case of $Hyp_q(\mathfrak{g})$ (giving $Hyp(\mathfrak{h}_n)$, in both cases) and in the case of $\big(Hyp_q(\mathfrak{g})'\big)^\vee$ (always yielding $\mathbf{u}(\mathfrak{k})$), whereas the semiclassical limit of $Hyp_q(\mathfrak{g})'$ in the simply connected case will be a (countable) covering of the limit in the adjoint case.

9.6 The QFA $F_q[H_n]$. Now we look at Theorem 2.2 the other way round, i.e. from QFAs to QrUEAs. We begin by introducing a QFA for the Heisenberg group.

Let $F_q[H_n]$ be the unital associative R–algebra with generators $\mathrm{a}_1, \dots, \mathrm{a}_n, \mathrm{c}, \mathrm{b}_1, \dots, \mathrm{b}_n$, and relations (for all $i,j=1,\dots,n$)

$$\mathrm{a}_i\mathrm{a}_j = \mathrm{a}_j\mathrm{a}_i\,,\ \ \mathrm{a}_i\mathrm{b}_j = \mathrm{b}_j\mathrm{a}_i\,,\ \ \mathrm{b}_i\mathrm{b}_j = \mathrm{b}_j\mathrm{b}_i\,,\ \ \mathrm{c}\,\mathrm{a}_i = \mathrm{a}_i\,\mathrm{c} + (q-1)\,\mathrm{a}_i\,,\ \ \mathrm{c}\,\mathrm{b}_j = \mathrm{b}_j\,\mathrm{c} + (q-1)\,\mathrm{b}_j$$

with a Hopf algebra structure given by (for all $i,j=1,\dots,n$)

$$\Delta(\mathrm{a}_i) = \mathrm{a}_i \otimes 1 + 1 \otimes \mathrm{a}_i\,, \quad \Delta(\mathrm{c}) = \mathrm{c} \otimes 1 + 1 \otimes \mathrm{c} + \sum_{j=1}^n \mathrm{a}_\ell \otimes \mathrm{b}_\ell\,, \quad \Delta(\mathrm{b}_i) = \mathrm{b}_i \otimes 1 + 1 \otimes \mathrm{b}_i$$

$$\epsilon(\mathrm{a}_i) = 0\,,\ \ \epsilon(\mathrm{c}) = 0\,,\ \ \epsilon(\mathrm{b}_i) = 0\,, \quad S(\mathrm{a}_i) = -\mathrm{a}_i\,,\ \ S(\mathrm{c}) = -\mathrm{c} + \sum_{j=1}^n \mathrm{a}_\ell\mathrm{b}_\ell\,,\ \ S(\mathrm{b}_i) = -\mathrm{b}_i$$

and let also $\mathbb{F}_q[H_n]$ be the $F(R)$–algebra obtained from $F_q[H_n]$ by scalar extension. Then $\mathbb{B} := \Big\{ \prod_{i=1}^n \mathrm{a}_i^{a_i} \cdot \mathrm{c}^c \cdot \prod_{j=1}^n \mathrm{b}_j^{b_j} \,\Big|\, a_i, c, b_j \in \mathbb{N}\;\forall\, i,j \Big\}$ is an R–basis of $F_q[H_n]$, hence an $F(R)$–basis of $\mathbb{F}_q[H_n]$. Moreover $F_q[H_n]$ is a QFA (at $\hbar = q-1$) with semiclassical limit $F[H_n]$.

9.7 Computation of $F_q[H_n]^\vee$ and specialization $F_q[H_n]^\vee \xrightarrow{\;q \to 1\;} U({\mathfrak{h}_n}^*)$. This section is devoted to compute $F_q[H_n]^\vee$ and its semiclassical limit (at $q=1$).

Definitions imply that $\mathbb{B} \setminus \{1\}$ is an R–basis of $J := Ker\Big(\epsilon \colon F_q[H_n] \twoheadrightarrow R\Big)$, so $\big(\mathbb{B} \setminus \{1\}\big) \cup \big\{(q-1)\cdot 1\big\}$ is an R–basis of $I := Ker\Big(F_q[H_n] \xrightarrow{\;\epsilon\;}\!\!\!\!\rightarrow R \xrightarrow{\;q \mapsto 1\;}\!\!\!\!\rightarrow \Bbbk\Big)$, for $I = J + (q-1)\cdot F_q[H_n]$. Therefore $F_q[H_n]^\vee := \sum_{n \geq 0} \Big((q-1)^{-1} I\Big)^n$ is nothing but the unital R–algebra (subalgebra of $\mathbb{F}_q[H_n]$) with generators $E_i := \dfrac{\mathrm{a}_i}{q-1}$, $H := \dfrac{\mathrm{c}}{q-1}$, and $F_i := \dfrac{\mathrm{b}_i}{q-1}$ ($i = 1, \dots, n$) and relations (for all $i, j = 1, \dots, n$)

$$E_i E_j = E_j E_i \,, \quad E_i F_j = F_j E_i \,, \quad F_i F_j = F_j F_i \,, \quad H E_i = E_i H + E_i \,, \quad H F_j = F_j H + F_j$$

with Hopf algebra structure given by (for all $i, j = 1, \dots, n$)

$$\Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i \,, \;\; \Delta(H) = H \otimes 1 + 1 \otimes H + (q-1) \sum_{j=1}^n E_j \otimes F_j \,, \;\; \Delta(F_i) = F_i \otimes 1 + 1 \otimes F_i$$

$$\epsilon(E_i) = \epsilon(H) = \epsilon(F_i) = 0 \,, \quad S(E_i) = -E_i \,, \quad S(H) = -H + (q-1) \sum_{j=1}^n E_j F_j \,, \quad S(F_i) = -F_i \,.$$

At $q = 1$ this implies that $F_q[H_n]^\vee \xrightarrow{\; q \to 1 \;} U(\mathfrak{h}_n{}^*)$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$F_q[H_n]^\vee \Big/ (q-1) \, F_q[H_n]^\vee \xrightarrow{\;\cong\;} U(\mathfrak{h}_n{}^*)$$

exists, given by $E_i \mod (q-1) \mapsto \pm \mathrm{e}_i$, $H \mod (q-1) \mapsto \mathrm{h}$, $F_i \mod (q-1) \mapsto \mathrm{f}_i$, for all $i, j = 1, \dots, n$; so $F_q[H_n]^\vee$ specializes to $U(\mathfrak{h}_n{}^*)$ *as a co-Poisson Hopf algebra*, q.e.d.

9.8 The identity $\big(F_q[H_n]^\vee\big)' = F_q[H_n]$. Finally, we check the validity of the part of Theorem 2.2*(b)* claiming, when $p = 0$, that $H \in \mathcal{QFA} \implies \big(H^\vee\big)' = H$ for the QFA $H = F_q[H_n]$. Once more the proof works for all $p \geq 0$, so we do improve Theorem 2.2*(b)*.

First of all, from definitions induction gives, for all $m \in \mathbb{N}$,

$$\Delta^m(E_i) = \sum_{r+s=m-1} 1^{\otimes r} \otimes E_i \otimes 1^{\otimes s} \,, \qquad \Delta^m(F_i) = \sum_{r+s=m-1} 1^{\otimes r} \otimes F_i \otimes 1^{\otimes s} \qquad \forall\, i = 1, \dots, n$$

$$\Delta^m(H) = \sum_{r+s=m-1} 1^{\otimes r} \otimes H \otimes 1^{\otimes s} + \sum_{i=1}^m \sum_{\substack{j,k=1 \\ j<k}}^m 1^{\otimes (j-1)} \otimes E_i \otimes 1^{\otimes (k-j-1)} \otimes F_i \otimes 1^{\otimes (m-k)}$$

so that $\delta_m(E_i) = \delta_\ell(H) = \delta_m(F_i) = 0$ for all $m > 1$, $\ell > 2$ and $i = 1, \dots, n$; moreover, for $\dot{E}_i := (q-1) E_i = \mathrm{a}_i$, $\dot{H} := (q-1) H = \mathrm{c}$, $\dot{F}_i := (q-1) F_i = \mathrm{b}_i$ $(i = 1, \dots, n)$ one has

$$\delta_1\big(\dot{E}_i\big) = (q-1) E_i \,, \; \delta_1\big(\dot{H}\big) = (q-1) H, \; \delta_1\big(\dot{F}_i\big) = (q-1) F_i \in (q-1) \, F_q[H_n]^\vee \setminus (q-1)^2 F_q[H_n]^\vee$$
$$\delta_2\big(\dot{H}\big) \; = \; (q-1)^2 \textstyle\sum_{i=1}^n E_i \otimes F_i \in (q-1)^2 \big(F_q[H_n]^\vee\big)^{\otimes 2} \setminus (q-1)^3 \big(F_q[H_n]^\vee\big)^{\otimes 2} \,.$$

The outcome is that $\dot{E}_i = \mathrm{a}_i, \dot{H} = \mathrm{c}, \dot{F}_i = \mathrm{b}_i \in \big(F_q[H_n]^\vee\big)'$, so the latter algebra contains the one generated by these elements, that is $F_q[H_n]$. Even more, $F_q[H_n]^\vee$ is clearly the R–span of the set $\mathbb{B}^\vee := \Big\{ \prod_{i=1}^n E_i^{a_i} \cdot H^c \cdot \prod_{j=1}^n F_j^{b_j} \,\Big|\, a_i, c, b_j \in \mathbb{N} \; \forall\, i, j \Big\}$, so from this and the previous formulas for Δ^n one gets that $\big(F_q[H_n]^\vee\big)' = F_q[H_n]$, q.e.d.

§ 10 Fifth example: non-commutative Hopf algebra of formal diffeomorphisms

10.1 The goal: from "quantum symmetries" to "classical (geometrical) symmetries". The purpose of this section is to give a highly significant example of how the global quantum duality principle — more precisely, the crystal duality principle of §5 — may be applied. We consider a concrete sample, taken from the theory of non-commutative renormalization of quantum electro-dynamics (=QED) performed by Brouder and Frabetti in [BF2]. This is just one of several possible examples of the same type: indeed, several cases of Hopf algebras built out of combinatorial data have been introduced in last years both in (co)homological theories (see for instance [LR] and [Fo1–3], and references therein) and in renormalization studies (starting with [CK1]). In most cases these Hopf algebras are neither commutative nor cocommutative, and our discussion apply almost *verbatim* to them, giving analogous results. So the present analysis of the "toy model" Hopf algebra of [BF2], can be taken more in general as a pattern for all those cases. See also [Ga6].

Note that the Hopf algebras under study are usually thought of as "generalized symmetries" (or "quantum symmetries", in physicists' terminology); well, the crystal duality principle tells us how to get out of them — via 1-parameter deformations! — "classical geometric symmetries", i.e., Poisson groups and Lie bialgebras; in other words, in a sense this method yields the classical geometrical counterparts of a quantum symmetry object.

10.2 The classical data. Let $\Bbbk$ be a fixed field of characteristic zero.

Consider the set $\mathcal{G}^{\text{dif}} := \big\{\, x + \sum_{n\geq 1} a_n\, x^{n+1} \,\big|\, a_n \in \Bbbk \;\forall\; n \in \mathbb{N}_+ \,\big\}$ of all formal series starting with x: endowed with the composition product, this is a group, which can be seen as the group of all "formal diffeomorphisms" $f : \Bbbk \longrightarrow \Bbbk$ such that $f(0) = 0$ and $f'(0) = 1$ (i.e. tangent to the identity), also known as the *Nottingham group* (see, e.g., [Ca] and references therein). In fact, $\mathcal{G}^{\text{dif}}$ is an infinite dimensional (pro)affine algebraic group, whose function algebra $F\big[\mathcal{G}^{\text{dif}}\big]$ is generated by the coordinate functions a_n $(n \in \mathbb{N}_+)$. Giving to each a_n the weight[7] $\partial(a_n) := n$, we have that $F\big[\mathcal{G}^{\text{dif}}\big]$ is an $\mathbb{N}$–*graded* Hopf algebra, with polynomial structure $F\big[\mathcal{G}^{\text{dif}}\big] = \Bbbk[a_1, a_2, \dots, a_n, \dots]$ and Hopf algebra structure given by

$$\Delta(a_n) = a_n \otimes 1 + 1 \otimes a_n + \sum\nolimits_{m=1}^{n-1} a_m \otimes Q^m_{n-m}(a_*)\,, \qquad \epsilon(a_n) = 0$$

$$S(a_n) = -a_n - \sum\nolimits_{m=1}^{n-1} a_m\, S\big(Q^m_{n-m}(a_*)\big) = -a_n - \sum\nolimits_{m=1}^{n-1} S(a_m)\, Q^m_{n-m}(a_*)$$

where $Q^\ell_t(a_*) := \sum_{k=1}^{t} \binom{\ell+1}{k} P^{(k)}_t(a_*)$ and $P^{(k)}_t(a_*) := \sum_{\substack{j_1,\dots,j_k>0 \\ j_1+\cdots+j_k=t}} a_{j_1} \cdots a_{j_k}$ (the symmetric monic polynomial of weight m and degree k in the indeterminates a_j's) for all m, k, $\ell \in \mathbb{N}_+$, and the formula for $S(a_n)$ gives the antipode by recursion. From now on, to simplify notation we shall use notation $\mathcal{G} := \mathcal{G}^{\text{dif}}$ and $\mathcal{G}_\infty := \mathcal{G} = \mathcal{G}^{\text{dif}}$. Note also that

[7] We say *weight* instead of *degree* because we save the latter term for the degree of polynomials.

the tangent Lie algebra of $\mathcal{G}^{\mathrm{dif}}$ is just the Lie subalgebra $W_1^{\geq 1} = Span\big(\{\, d_n \,|\, n \in \mathbb{N}_+ \,\}\big)$ of the one-sided Witt algebra $W_1 := Der\big(\Bbbk[t]\big) = Span\big(\big\{\, d_n := t^{n+1} \frac{d}{dt} \,\big|\, n \in \mathbb{N} \cup \{-1\} \,\big\}\big)$.

In addition, for all $\nu \in \mathbb{N}_+$ the subset $\mathcal{G}^\nu := \big\{\, f \in \mathcal{G} \,\big|\, a_n(f) = 0, \ \forall\ n \leq \nu \,\big\}$ is a normal subgroup of $\mathcal{G}$; the corresponding quotient group $\mathcal{G}_\nu := \mathcal{G}\big/\mathcal{G}^\nu$ is unipotent, with dimension ν and function algebra $F\big[\mathcal{G}_\nu\big]$ (isomorphic to) the Hopf subalgebra of $F\big[\mathcal{G}\big]$ generated by $a_1, \dots, a_\nu$. In fact, the $\mathcal{G}^\nu$'s form exactly the lower central series of $\mathcal{G}$ (cf. [Je2]). Moreover, $\mathcal{G}$ is (isomorphic to) the inverse (or projective) limit of these quotient groups $\mathcal{G}_\nu$ ($\nu \in \mathbb{N}_+$), hence $\mathcal{G}$ is pro-unipotent; conversely, $F[\mathcal{G}]$ is the direct (or inductive) limit of the direct system of its graded Hopf subalgebras $F[\mathcal{G}_\nu]$ ($\nu \in \mathbb{N}_+$). Finally, the set $\mathcal{G}^{\mathrm{odd}} := \big\{\, f \in \mathcal{G}^{\mathrm{dif}} \,\big|\, a_{2n+1}(f) = 0 \ \forall\ n \in \mathbb{N}_+ \,\big\}$ is another normal subgroup of $\mathcal{G}^{\mathrm{dif}}$ (the group of *odd* formal diffeomorphisms[8] after [CK3]), whose function algebra $F\big[\mathcal{G}^{\mathrm{odd}}\big]$ is (isomorphic to) the quotient Hopf algebra $F\big[\mathcal{G}^{\mathrm{dif}}\big]\Big/\Big(\big\{a_{2n-1}\big\}_{n\in\mathbb{N}_+}\Big)$. The latter has the following description: denoting again the cosets of the a_{2n}'s with the like symbol, we have $F\big[\mathcal{G}^{\mathrm{odd}}\big] = \Bbbk[a_2, a_4, \dots, a_{2n}, \dots]$ with Hopf algebra structure

$$\Delta(a_{2n}) = a_{2n} \otimes 1 + 1 \otimes a_{2n} + \sum\nolimits_{m=1}^{n-1} a_{2m} \otimes \bar{Q}^m_{n-m}(a_{2*}) \,, \qquad \epsilon(a_{2n}) = 0$$
$$S(a_{2n}) = -a_{2n} - \sum\nolimits_{m=1}^{n-1} a_{2m}\, S\big(\bar{Q}^m_{n-m}(a_*)\big) = -a_{2n} - \sum\nolimits_{m=1}^{n-1} S(a_{2m})\, \bar{Q}^m_{n-m}(a_{2*})$$

where $\bar{Q}^\ell_t(a_{2*}) := \sum_{k=1}^t \binom{2\ell+1}{k} \bar{P}^{(k)}_t(a_{2*})$ and $\bar{P}^{(k)}_t(a_{2*}) := \sum_{\substack{j_1,\dots,j_k>0 \\ j_1+\cdots+j_k=t}} a_{2j_1} \cdots a_{2j_k}$ for all $m, k, \ell \in \mathbb{N}_+$. For each $\nu \in \mathbb{N}_+$ we can consider also the normal subgroup $\mathcal{G}^\nu \cap \mathcal{G}^{\mathrm{odd}}$ and the corresponding quotient $\mathcal{G}^{\mathrm{odd}}_\nu := \mathcal{G}^{\mathrm{odd}}\big/\big(\mathcal{G}^\nu \cap \mathcal{G}^{\mathrm{odd}}\big)$: then $F\big[\mathcal{G}^{\mathrm{odd}}_\nu\big]$ is (isomorphic to) the quotient Hopf algebra $F\big[\mathcal{G}^{\mathrm{odd}}\big]\Big/\Big(\big\{a_{2n-1}\big\}_{(2n-1)\in\mathbb{N}_\nu}\Big)$, in particular it is the Hopf subalgebra of $F\big[\mathcal{G}^{\mathrm{odd}}\big]$ generated by $a_2, \dots, a_{2\,[\nu/2]}$. All the $F\big[\mathcal{G}^{\mathrm{odd}}_\nu\big]$'s are graded Hopf (sub)algebras forming a direct system with direct limit $F\big[\mathcal{G}^{\mathrm{odd}}\big]$; conversely, the $\mathcal{G}^{\mathrm{odd}}_\nu$'s form an inverse system with inverse limit $\mathcal{G}^{\mathrm{odd}}$. In the sequel we write $\mathcal{G}^+ := \mathcal{G}^{\mathrm{odd}}$ and $\mathcal{G}^+_\nu := \mathcal{G}^{\mathrm{odd}}_\nu$.

For each $\nu \in \mathbb{N}_+$, set $\mathbb{N}_\nu := \{1, \dots, \nu\}$; set also $\mathbb{N}_\infty := \mathbb{N}_+$. For each $\nu \in \mathbb{N}_+ \cup \{\infty\}$, let $\mathcal{L}_\nu = \mathcal{L}(\mathbb{N}_\nu)$ be the free Lie algebra over $\Bbbk$ generated by $\{x_n\}_{n\in\mathbb{N}_\nu}$ and let $U_\nu = U(\mathcal{L}_\nu)$ be its universal enveloping algebra; let also $V_\nu = V(\mathbb{N}_\nu)$ be the $\Bbbk$–vector space with basis $\{x_n\}_{n\in\mathbb{N}_\nu}$, and let $T_\nu = T(V_\nu)$ be its associated tensor algebra. Then there are canonical identifications $U(\mathcal{L}_\nu) = T(V_\nu) = \Bbbk\big\langle \{\, x_n \,|\, n \in \mathbb{N}_\nu \,\} \big\rangle$, the latter being the unital $\Bbbk$–algebra of non-commutative polynomials in the set of indeterminates $\{x_n\}_{n\in\mathbb{N}_\nu}$, and $\mathcal{L}_\nu$ is just the Lie subalgebra of $U_\nu = T_\nu$ generated by $\{x_n\}_{n\in\mathbb{N}_\nu}$. Moreover, $\mathcal{L}_\nu$ has a basis B_ν made of Lie monomials in the x_n's ($n \in \mathbb{N}_\nu$), like $[x_{n_1}, x_{n_2}]$, $[[x_{n_1}, x_{n_2}], x_{n_3}]$, $[[[x_{n_1}, x_{n_2}], x_{n_3}], x_{n_4}]$, etc.: details can be found e.g. in [Re], Ch. 4–5. In the sequel I shall use these identifications with no further mention. We consider on $U(\mathcal{L}_\nu)$ the standard Hopf algebra structure given by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$, $S(x) = -x$ for all $x \in \mathcal{L}_\nu$, which is also

[8] The fixed-point set of the group homomorphism $\Phi : \mathcal{G} \to \mathcal{G}$, $f \mapsto \Phi(f)$ $x \mapsto$ $\Phi(f)$ $(x) := -f(-x)$

determined by the same formulas for $x \in \{x_n\}_{n \in \mathbb{N}_\nu}$ alone. By construction $\nu \leq \mu$ implies $\mathcal{L}_\nu \subseteq \mathcal{L}_\mu$, whence the $\mathcal{L}_\nu$'s form a direct system (of Lie algebras) whose direct limit is exactly $\mathcal{L}_\infty$; similarly, $U(\mathcal{L}_\infty)$ is the direct limit of all the $U(\mathcal{L}_\nu)$'s. Finally, with $\mathbb{B}_\nu$ we shall mean the obvious PBW-like basis of $U(\mathcal{L}_\nu)$ w.r.t. some fixed total order $\preceq$ of B_ν, namely $\mathbb{B}_\nu := \big\{ x_{\underline{b}} \,\big|\, \underline{b} = b_1 \cdots b_k \,;\, b_1, \dots, b_k \in B_\nu \,;\, b_1 \preceq \cdots \preceq b_k \big\}$.

The same construction applies to define the corresponding "odd" objects, based on $\{x_n\}_{n \in \mathbb{N}_\nu^+}$, with $\mathbb{N}_\nu^+ := \mathbb{N}_\nu \cap 2\,\mathbb{N}$, instead of $\{x_n\}_{n \in \mathbb{N}_\nu}$ (for each $\nu \in \mathbb{N} \cup \{\infty\}$). Thus we have $\mathcal{L}_\nu^+ = \mathcal{L}(\mathbb{N}_\nu^+)$, $U_\nu^+ = U(\mathcal{L}_\nu^+)$, $V_\nu^+ = V(\mathbb{N}_\nu^+)$, $T_\nu^+ = T(V_\nu^+)$, with the obvious canonical identifications $U(\mathcal{L}_\nu^+) = T(V_\nu^+) = \Bbbk\big\langle \{ x_n \,|\, n \in \mathbb{N}_\nu^+ \} \big\rangle$; moreover, $\mathcal{L}_\nu^+$ has a basis B_ν^+ made of Lie monomials in the x_n's ($n \in \mathbb{N}_\nu^+$), etc. The $\mathcal{L}_\nu^+$'s form a direct system whose direct limit is $\mathcal{L}_\infty^+$, and $U(\mathcal{L}_\infty^+)$ is the direct limit of all the $U(\mathcal{L}_\nu^+)$'s.

Warning: in the sequel, we shall often deal with subsets $\{\mathbf{y}_b\}_{b \in B_\nu}$ (of some algebra) in bijection with B_ν, the fixed basis of $\mathcal{L}_\nu$. Then we shall write things like $\mathbf{y}_\lambda$ with $\lambda \in \mathcal{L}_\nu$: this means we extend the bijection $\{\mathbf{y}_b\}_{b \in B_\nu} \cong B_\nu$ to $\mathit{Span}\big(\{\mathbf{y}_b\}_{b \in B_\nu}\big) \cong \mathcal{L}_\nu$ by linearity, so that $\mathbf{y}_\lambda \cong \sum_{b \in B_\nu} c_b\, b$ iff $\lambda = \sum_{b \in B_\nu} c_b\, b$ ($c_b \in \Bbbk$). The same kind of convention will be applied with B_ν^+ instead of B_ν and $\mathcal{L}_\nu^+$ instead of $\mathcal{L}_\nu$.

10.3 The noncommutative Hopf algebra of formal diffeomorphisms. For all $\nu \in \mathbb{N}_+ \cup \{\infty\}$, let $\mathcal{H}_\nu$ be the Hopf $\Bbbk$–algebra given as follows: as a $\Bbbk$–algebra it is simply $\mathcal{H}_\nu := \Bbbk\big\langle \{ \mathbf{a}_n \,|\, n \in \mathbb{N}_\nu \} \big\rangle$ (the $\Bbbk$–algebra of non-commutative polynomials in the set of indeterminates $\{\mathbf{a}_n\}_{n \in \mathbb{N}_\nu}$), and its Hopf algebra structure is given by (for all $n \in \mathbb{N}_\nu$)

$$\begin{gathered} \Delta(\mathbf{a}_n) = \mathbf{a}_n \otimes 1 + 1 \otimes \mathbf{a}_n + \sum\nolimits_{m=1}^{n-1} \mathbf{a}_m \otimes Q_{n-m}^m(\mathbf{a}_*)\,, \qquad \epsilon(\mathbf{a}_n) = 0 \\ S(\mathbf{a}_n) = -\mathbf{a}_n - \sum\nolimits_{m=1}^{n-1} \mathbf{a}_m\, S\big(Q_{n-m}^m(\mathbf{a}_*)\big) = -\mathbf{a}_n - \sum\nolimits_{m=1}^{n-1} S(\mathbf{a}_m)\, Q_{n-m}^m(\mathbf{a}_*) \end{gathered} \tag{10.1}$$

(notation like in §10.2) where the latter formula yields the antipode by recursion. Moreover, $\mathcal{H}_\nu$ is in fact an $\mathbb{N}$–*graded Hopf algebra*, once generators have been given degree — in the sequel called *weight* — by the rule $\partial(\mathbf{a}_n) := n$ (for all $n \in \mathbb{N}_\nu$). By construction the various $\mathcal{H}_\nu$'s (for all $\nu \in \mathbb{N}_+$) form a direct system, whose direct limit is $\mathcal{H}_\infty$: the latter was originally introduced[9] in [BF2], §5.1 (with $\Bbbk = \mathbb{C}$), under the name $\mathcal{H}^{\text{dif}}$.

Similarly, for all $\nu \in \mathbb{N}_+ \cup \{\infty\}$ we set $\mathcal{K}_\nu := \Bbbk\big\langle \{ \mathbf{a}_n \,|\, n \in \mathbb{N}_\nu^+ \} \big\rangle$ (where $\mathbb{N}_\nu^+ := \mathbb{N}_\nu \cap (2\,\mathbb{N})$): this bears a Hopf algebra structure given by (for all $2\,n \in \mathbb{N}_\nu^+$)

$$\begin{gathered} \Delta(\mathbf{a}_{2n}) = \mathbf{a}_{2n} \otimes 1 + 1 \otimes \mathbf{a}_{2n} + \sum\nolimits_{m=1}^{n-1} \mathbf{a}_{2m} \otimes \bar{Q}_{n-m}^m(\mathbf{a}_{2*})\,, \qquad \epsilon(\mathbf{a}_{2n}) = 0 \\ S(\mathbf{a}_{2n}) = -\mathbf{a}_{2n} - \sum\nolimits_{m=1}^{n-1} \mathbf{a}_{2m}\, S\big(\bar{Q}_{n-m}^m(\mathbf{a}_{2*})\big) = -\mathbf{a}_{2n} - \sum\nolimits_{m=1}^{n-1} S(\mathbf{a}_{2m})\, Q_{n-m}^m(\mathbf{a}_{2*}) \end{gathered}$$

(notation of §10.2). Indeed, this is an $\mathbb{N}$–graded Hopf algebra where generators have degree — called *weight* — given by $\partial(\mathbf{a}_n) := n$ (for all $n \in \mathbb{N}_\nu^+$). All the $\mathcal{K}_\nu$'s form a direct

[9] However, the formulas in [BF2] give the *opposite* coproduct, hence change the antipode accordingly; we made the present choice to make these formulas "fit well" with those for $F\,\mathcal{G}^{\text{dif}}$ (see below).

system with direct limit $\mathcal{K}_\infty$. Finally, for each $\nu \in \mathbb{N}_\nu^+$ there is a graded Hopf algebra epimorphism $\mathcal{H}_\nu \relbar\joinrel\twoheadrightarrow \mathcal{K}_\nu$ given by $\mathbf{a}_{2n} \mapsto \mathbf{a}_{2n}$, $\mathbf{a}_{2m+1} \mapsto 0$ for all $2n, 2m+1 \in \mathbb{N}_\nu$.

Definitions and §10.2 imply that

$$\big(\mathcal{H}_\nu\big)_{ab} := \mathcal{H}_\nu \Big/ \big(\big[\mathcal{H}_\nu, \mathcal{H}_\nu\big]\big) \;\cong\; F\big[\mathcal{G}_\nu\big]\,, \qquad \text{via} \qquad \mathbf{a}_n \mapsto a_n \quad \forall\, n \in \mathbb{N}_\nu$$

as $\mathbb{N}$–graded Hopf algebras: in other words, the abelianization of $\mathcal{H}_\nu$ is nothing but $F\big[\mathcal{G}_\nu\big]$. Thus in a sense one can think at $\mathcal{H}_\nu$ as a *non-commutative version* (indeed, the "coarsest" one) of $F\big[\mathcal{G}_\nu\big]$, hence as a "quantization" of $\mathcal{G}_\nu$ itself: however, this is *not* a quantization in the sense we mean in this paper, for $F\big[\mathcal{G}_\nu\big]$ is attained through abelianization, not through specialization (of some deformation parameter). Similarly we have also

$$\big(\mathcal{K}_\nu\big)_{ab} := \mathcal{K}_\nu \Big/ \big(\big[\mathcal{K}_\nu, \mathcal{K}_\nu\big]\big) \;\cong\; F\big[\mathcal{G}_\nu^+\big]\,, \qquad \text{via} \qquad \mathbf{a}_{2n} \mapsto a_{2n} \quad \forall\, 2n \in \mathbb{N}_\nu^+$$

as $\mathbb{N}$–graded Hopf algebras: in other words, the abelianization of $\mathcal{K}_\nu$ is just $F\big[\mathcal{G}_\nu^+\big]$.

Note that $\big(\mathcal{H}_\nu\big)^\vee = \mathcal{H}_\nu = \big(\mathcal{H}_\nu\big)'$ (notation of §5.1) because $\mathcal{H}_\nu$ is graded and connected: therefore applying the crystal duality principle to $\mathcal{H}_\nu$ we'll end up with (5.5), which means we can deform $\mathcal{H}_\nu$ in four different ways to Hopf algebras bearing some (Poisson-type) geometrical content; and similarly for $\mathcal{K}_\nu$. In particular we'll describe the Poisson groups G_+ and $G_-^\star$, and their cotangent Lie bialgebras $\mathfrak{g}_+^\times$ and $\mathfrak{g}_-$, attached to $\mathcal{H}_\nu$ and to $\mathcal{K}_\nu$ in this way. We perform the analysis explicitly for $\mathcal{H}_\nu$; the case $\mathcal{K}_\nu$ is the like, and we leave to the reader the easy task to fill in details.

We follow the recipe in §§5.1–4. Let's drop the subscript ν (which stands fixed) and write $\mathcal{H} := \mathcal{H}_\nu$. Let $R := \Bbbk[\hbar]$, and set $\mathcal{H}_\hbar := \mathcal{H}[\hbar] \equiv \Bbbk[\hbar] \otimes_\Bbbk \mathcal{H}$: this is a Hopf algebra over $\Bbbk[\hbar]$, namely $\mathcal{H}_\hbar = \Bbbk[\hbar]\big\langle \{\, \mathbf{a}_n \,|\, n \in \mathbb{N}_\nu \,\} \big\rangle$ with Hopf structure given by (10.1) again. More precisely, we have $\mathcal{H}[\hbar] \in \mathcal{HA}$ w.r.t. the ground ring $R := \Bbbk[\hbar]$ (a PID). Then $F(R) = \Bbbk(\hbar)$, and $\big(\mathcal{H}_\hbar\big)_F := \Bbbk(\hbar) \otimes_{\Bbbk[\hbar]} \mathcal{H}_\hbar = \Bbbk(\hbar) \otimes_\Bbbk \mathcal{H} = \mathcal{H}(\hbar) = \Bbbk(\hbar)\big\langle \{\, \mathbf{a}_n \,|\, n \in \mathbb{N}_\nu \,\} \big\rangle$.

10.4 Drinfeld's algebra $\mathcal{H}_\hbar^{\,\vee} := \big(\mathcal{H}[\hbar]\big)^\vee$. By the method in §5 leading to the Crystal Duality Principle, we can apply Drinfeld's functors at the prime $\hbar \in \Bbbk[\hbar]$ to $\mathcal{H}_\hbar := \mathcal{H}[\hbar]$. We begin with $\mathcal{H}_\hbar^{\,\vee} := \sum_{n \geq 0} \hbar^{-n} J^n \;\big(\subseteq \big(\mathcal{H}_\hbar\big)_F = \mathcal{H}(\hbar)\big)$, where $J := \mathit{Ker}\big(\epsilon_{\mathcal{H}_\hbar} : \mathcal{H}_\hbar \longrightarrow \Bbbk[\hbar]\big)$. We'll describe $\mathcal{H}_\hbar^{\,\vee}$ explicitly, thus checking that it is really a QrUEA, as predicted by Theorem 2.2*(a)*; then we'll look at its specialization at $\hbar = 1$, and finally we'll study $\big(\mathcal{H}_\hbar^{\,\vee}\big)'$ and its specializations at $\hbar = 0$ and $\hbar = 1$. The outcome will be an explicit description of the diagram of deformations (5.3) for $H = \mathcal{H}$ $(= \mathcal{H}_\nu)$.

For all $n \in \mathbb{N}_\nu$, set $\mathbf{x}_n := \hbar^{-1} \mathbf{a}_n$. Then clearly $\mathcal{H}_\hbar^{\,\vee}$ is the $\Bbbk[\hbar]$–subalgebra of $\mathcal{H}(\hbar)$ generated by the set $\{\mathbf{x}_n\}_{n \in \mathbb{N}_\nu}$, and thus $\mathcal{H}_\hbar^{\,\vee} = \Bbbk[\hbar]\big\langle \{\, \mathbf{x}_n \,|\, n \in \mathbb{N}_\nu \,\} \big\rangle$. Moreover,

$$\Delta(\mathbf{x}_n) = \mathbf{x}_n \otimes 1 + 1 \otimes \mathbf{x}_n + \sum\nolimits_{m=1}^{n-1} \sum\nolimits_{k=1}^{m} \hbar^k \binom{n-m+1}{k} \mathbf{x}_{n-m} \otimes P_m^{(k)}(\mathbf{x}_*) \,, \quad \epsilon(\mathbf{x}_n) = 0$$
$$S(\mathbf{x}_n) = -\mathbf{x}_n - \sum\nolimits_{m=1}^{n-1} \sum\nolimits_{k=1}^{m} \hbar^k \binom{n-m+1}{k} \mathbf{x}_{n-m} \, S\big(P_m^{(k)}(\mathbf{x}_*)\big) = \tag{10.2}$$
$$= -\mathbf{x}_n - \sum\nolimits_{m=1}^{n-1} \sum\nolimits_{k=1}^{m} \hbar^k \binom{n-m+1}{k} S(\mathbf{x}_{n-m}) \, P_m^{(k)}(\mathbf{x}_*)$$

for all $n \in \mathbb{N}_\nu$, due to (10.1); from this one sees by hands that the following holds:

Proposition 10.5. *Formulas (10.2) make* ${\mathcal{H}_\hbar}^\vee = \Bbbk[\hbar]\big\langle \{ \mathbf{x}_n, \,|\, n \in \mathbb{N}_\nu \} \big\rangle$ *into a graded Hopf* $\Bbbk[\hbar]$*–algebra, embedded into* $\mathcal{H}(\hbar) := \Bbbk(\hbar) \otimes_{\Bbbk} \mathcal{H}$ *as a graded Hopf subalgebra. Moreover,* ${\mathcal{H}_\hbar}^\vee$ *is a deformation of* $\mathcal{H}$*, for its specialization at* $\hbar = 1$ *is isomorphic to* $\mathcal{H}$*, i.e.*

$${\mathcal{H}_\hbar}^\vee\Big|_{\hbar=1} := {\mathcal{H}_\hbar}^\vee \Big/ (\hbar-1)\, {\mathcal{H}_\hbar}^\vee \cong \mathcal{H} \quad \textit{via} \quad \mathbf{x}_n \mod (\hbar-1)\, {\mathcal{H}_\hbar}^\vee \mapsto \mathbf{a}_n \quad (\forall\ n \in \mathbb{N}_\nu)$$

as graded Hopf algebras over $\Bbbk$. $\square$

<u>Remark</u>*:* The previous result shows that $\mathcal{H}_\hbar$ *is a deformation of* $\mathcal{H}$*, which is "recovered" as specialization limit (of* $\mathcal{H}_\hbar$*) at* $\hbar = 1$. The next result instead shows that $\mathcal{H}_\hbar$ *is also a deformation of* $U(\mathcal{L}_\nu)$*, which is "recovered" as specialization limit at* $\hbar = 0$. Altogether, this gives the left-hand-side of (5.3) for $H = \mathcal{H} := \mathcal{H}_\nu$, with $\mathfrak{g}_- = \mathcal{L}_\nu$.

Theorem 10.6. ${\mathcal{H}_\hbar}^\vee$ *is a QrUEA at* $\hbar = 0$*. Namely, the specialization limit of* ${\mathcal{H}_\hbar}^\vee$ *at* $\hbar = 0$ *is* ${\mathcal{H}_\hbar}^\vee\Big|_{\hbar=0} := {\mathcal{H}_\hbar}^\vee \Big/ \hbar\, {\mathcal{H}_\hbar}^\vee \cong U(\mathcal{L}_\nu)$ *via* $\mathbf{x}_n \mod \hbar\, {\mathcal{H}_\hbar}^\vee \mapsto x_n$ *for all* $n \in \mathbb{N}_\nu$*, thus inducing on* $U(\mathcal{L}_\nu)$ *the structure of co-Poisson Hopf algebra uniquely given by the Lie bialgebra structure on* $\mathcal{L}_\nu$ *given by* $\delta(x_n) = \sum_{\ell=1}^{n-1} (\ell+1)\, x_\ell \wedge x_{n-\ell}$ *(for all* $n \in \mathbb{N}_\nu$*)*[10]*. In particular in the diagram (5.3) for* $H = \mathcal{H}\ (= \mathcal{H}_\nu)$ *we have* $\mathfrak{g}_- = \mathcal{L}_\nu$*.*

Finally, the grading d *given by* $d(x_n) := 1 \ (n \in \mathbb{N}_+)$ *makes* ${\mathcal{H}_\hbar}^\vee\Big|_{\hbar=0} \cong U(\mathcal{L}_\nu)$ *into a graded co-Poisson Hopf algebra; similarly, the grading* ∂ *given by* $\partial(x_n) := n \ (n \in \mathbb{N}_+)$ *makes* ${\mathcal{H}_\hbar}^\vee\Big|_{\hbar=0} \cong U(\mathcal{L}_\nu)$ *into a graded Hopf algebra and* $\mathcal{L}_\nu$ *into a graded Lie bialgebra.*

Proof. First observe that since ${\mathcal{H}_\hbar}^\vee = \Bbbk[\hbar]\big\langle \{ \mathbf{x}_n \,|\, n \in \mathbb{N}_\nu \} \big\rangle$ and $U(\mathcal{L}_\nu) = T(V_\nu) = \Bbbk\big\langle \{ x_n \,|\, n \in \mathbb{N}_\nu \} \big\rangle$ mapping $\mathbf{x}_n \mod \hbar\, {\mathcal{H}_\hbar}^\vee \mapsto x_n$ $(\forall\ n \in \mathbb{N}_\nu)$ does really define an isomorphism *of algebras* $\Phi \colon {\mathcal{H}_\hbar}^\vee \Big/ \hbar\, {\mathcal{H}_\hbar}^\vee \cong U(\mathcal{L}_\nu)$. Second, formulas (10.2) give

$$\Delta(\mathbf{x}_n) \equiv \mathbf{x}_n \otimes 1 + 1 \otimes \mathbf{x}_n \mod \hbar \left({\mathcal{H}_\hbar}^\vee \otimes {\mathcal{H}_\hbar}^\vee \right)$$
$$\epsilon(\mathbf{x}_n) \equiv 0 \mod \hbar\, \Bbbk[\hbar] \,, \qquad S(\mathbf{x}_n) \equiv -\mathbf{x}_n \mod \hbar\, {\mathcal{H}_\hbar}^\vee$$

[10]Hereafter, I use notation $a \wedge b := a \otimes b - b \otimes a$.

for all $n \in \mathbb{N}_\nu$; comparing with the standard Hopf structure of $U(\mathcal{L}_\nu)$ this shows that Φ is in fact an isomorphism *of Hopf algebras* too. Finally, as ${\mathcal{H}_\hbar}^\vee\Big|_{\hbar=0}$ is cocommutative, a Poisson co-bracket is defined on it by the standard recipe in Remark 1.5: applying it yields

$$\delta(x_n) \;:=\; \big(\hbar^{-1}\big(\Delta(\mathbf{x}_n)-\Delta^{\mathrm{op}}(\mathbf{x}_n)\big)\big) \mod \hbar\Big({\mathcal{H}_\hbar}^\vee \otimes {\mathcal{H}_\hbar}^\vee\Big) \;=\; \\ =\; \textstyle\sum_{m=1}^{n-1} \binom{n-m+1}{1} x_{n-m} \wedge P_m^{(1)}(x_*) \;=\; \sum_{\ell=1}^{n-1} (\ell+1)\, x_\ell \wedge x_{n-\ell} \qquad \forall\; n \in \mathbb{N}_\nu\,. \;\square$$

10.7 Drinfeld's algebra $\big({\mathcal{H}_\hbar}^\vee\big)'$. I look now at the other Drinfeld's functor (at $\hbar$), and consider $\big({\mathcal{H}_\hbar}^\vee\big)' := \Big\{\eta \in {\mathcal{H}_\hbar}^\vee \;\Big|\; \delta_n(\eta) \in \hbar^n \big({\mathcal{H}_\hbar}^\vee\big)^{\otimes n}\;\forall\, n \in \mathbb{N}\Big\}$ $\big(\subseteq {\mathcal{H}_\hbar}^\vee\big)$. Theorem 2.2 tells us that $\big({\mathcal{H}_\hbar}^\vee\big)'$ *is a Hopf* $\Bbbk[\hbar]$*–subalgebra of* ${\mathcal{H}_\hbar}^\vee$*, and the specialization of* $\big({\mathcal{H}_\hbar}^\vee\big)'$ *at* $\hbar=0$*, that is* $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0} := \big({\mathcal{H}_\hbar}^\vee\big)'\Big/\hbar\big({\mathcal{H}_\hbar}^\vee\big)'$*, is the function algebra of a connected algebraic Poisson group* $G_{\mathcal{L}_\nu}^\star$ *dual to* $G_{\mathcal{L}_\nu}$, the latter being the connected simply-connected Poisson algebraic group with tangent Lie bialgebra $\mathcal{L}_\nu$. In other words, $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0}$ must be isomorphic (as a Poisson Hopf algebra) to $F\big[G_{\mathcal{L}_\nu}^\star\big]$, where $G_{\mathcal{L}_\nu}^\star$ is connected and has cotangent Lie bialgebra $\mathit{Lie}\big(G_{\mathcal{L}_\nu}^\star\big) = \mathcal{L}_\nu$. Therefore we must prove that $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0}$ is a commutative Hopf $\Bbbk$–algebra, it has no non-trivial idempotents, and $\Big(\textit{co-Lie}\big(G_{\mathcal{L}_\nu}^\star\big) :=\Big)$ $J_0\big/J_0^{\,2} \cong \mathcal{L}_\nu$ as Lie bialgebras, where $J_0 := \mathit{Ker}\Big(\epsilon\colon \big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0} \!\longrightarrow \Bbbk\Big)$. We prove all this directly, via explicit description of $\big({\mathcal{H}_\hbar}^\vee\big)'$ and its specialization at $\hbar=0$.

Step I: A direct check shows that $\tilde{\mathbf{x}}_n := \hbar\,\mathbf{x}_n = \mathbf{a}_n \in \big({\mathcal{H}_\hbar}^\vee\big)'$*, for all* $n \in \mathbb{N}_\nu$. Indeed, we have of course $\delta_0(\tilde{\mathbf{x}}_n) = \epsilon(\tilde{\mathbf{x}}_n) \in \hbar^0\,{\mathcal{H}_\hbar}^\vee$ and $\delta_1(\tilde{\mathbf{x}}_n) = \tilde{\mathbf{x}}_n - \epsilon(\tilde{\mathbf{x}}_n) \in \hbar^1\,{\mathcal{H}_\hbar}^\vee$. Moreover, $\delta_2(\tilde{\mathbf{x}}_n) = \sum_{m=1}^{n-1} \tilde{\mathbf{x}}_{n-m} \otimes Q_m^{n-m}(\tilde{\mathbf{x}}_*) = \sum_{m=1}^{n-1}\sum_{k=1}^{m} \hbar^{k+1}\binom{n-m+1}{k}\mathbf{x}_{n-m} \otimes P_m^{(k)}(\mathbf{x}_*) \in \hbar^2\Big({\mathcal{H}_\hbar}^\vee \otimes {\mathcal{H}_\hbar}^\vee\Big)$. Since in general $\delta_\ell = \big(\delta_{\ell-1} \otimes \mathrm{id}\big)\circ\delta_2$ for all $\ell \in \mathbb{N}_+$, we have

$$\delta_\ell(\tilde{\mathbf{x}}_n) = \big(\delta_{\ell-1}\otimes\mathrm{id}\big)\big(\delta_2(\tilde{\mathbf{x}}_n)\big) \;=\; \sum_{m=1}^{n-1}\sum_{k=1}^{m} \hbar^k \binom{n-m+1}{k}\, \delta_{\ell-1}(\mathbf{x}_{n-m}) \otimes P_m^{(k)}(\mathbf{x}_*)$$

whence induction gives $\delta_\ell(\tilde{\mathbf{x}}_n) \in \hbar^\ell\,\big({\mathcal{H}_\hbar}^\vee\big)^{\otimes\ell}$ for all $\ell \in \mathbb{N}$, thus $\tilde{\mathbf{x}}_n \in \big({\mathcal{H}_\hbar}^\vee\big)'$, q.e.d.

Step II: By Theorem 2.2*(a)* we have that $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0}$ is commutative: this means $[a,b] \equiv 0 \mod \hbar\big({\mathcal{H}_\hbar}^\vee\big)'$, that is $[a,b] \in \hbar\big({\mathcal{H}_\hbar}^\vee\big)'$ hence also $\hbar^{-1}[a,b] \in \big({\mathcal{H}_\hbar}^\vee\big)'$, for all a, $b \in \big({\mathcal{H}_\hbar}^\vee\big)'$. *In particular,* we get $\widetilde{[\mathbf{x}_n,\mathbf{x}_m]} := \hbar\,[\mathbf{x}_n,\mathbf{x}_m] = \hbar^{-1}[\tilde{\mathbf{x}}_n,\tilde{\mathbf{x}}_m] \in \big({\mathcal{H}_\hbar}^\vee\big)'$ for all n, $m \in \mathbb{N}_\nu$, whence iterating (and recalling $\mathcal{L}_\nu$ is generated by the $\mathbf{x}_n$'s) *we get* $\tilde{\mathbf{x}} := \hbar\,\mathbf{x} \in \big({\mathcal{H}_\hbar}^\vee\big)'$ *for every* $\mathbf{x} \in \mathcal{L}_\nu$. Hereafter we identify $\mathcal{L}_\nu$ with its image via the embedding $\mathcal{L}_\nu \hookrightarrow U(\mathcal{L}_\nu) = \Bbbk\big\langle\{x_n\}_{n\in\mathbb{N}_\nu}\big\rangle \hookrightarrow \Bbbk[\hbar]\big\langle\{\mathbf{x}_n\}_{n\in\mathbb{N}_\nu}\big\rangle = {\mathcal{H}_\hbar}^\vee$ given by $x_n \mapsto \mathbf{x}_n$ $(n \in \mathbb{N}_\nu)$.

Step III: The previous step showed that, if we embed $\mathcal{L}_\nu \hookrightarrow U(\mathcal{L}_\nu) \hookrightarrow {\mathcal{H}_\hbar}^\vee$ via $x_n \mapsto \mathbf{x}_n$ (for all $n \in \mathbb{N}_\nu$) we find $\widetilde{\mathcal{L}_\nu} := \hbar\,\mathcal{L}_\nu \subseteq \big({\mathcal{H}_\hbar}^\vee\big)'$. *Let* $\big\langle\widetilde{\mathcal{L}_\nu}\big\rangle$ *be the* $\Bbbk[\hbar]$*–subalgebra of* $\big({\mathcal{H}_\hbar}^\vee\big)'$ *generated by* $\widetilde{\mathcal{L}_\nu}$*: then* $\big\langle\widetilde{\mathcal{L}_\nu}\big\rangle \subseteq \big({\mathcal{H}_\hbar}^\vee\big)'$, because $\big({\mathcal{H}_\hbar}^\vee\big)'$ is a subalgebra. In particular, if $\mathbf{b}_b \in {\mathcal{H}_\hbar}^\vee$ is the image of any $b \in B_\nu$ (cf. §10.2) we have $\widetilde{\mathbf{b}}_b := \hbar\,\mathbf{b}_b \in \big({\mathcal{H}_\hbar}^\vee\big)'$.

Step IV: Conversely to Step III, we have $\big\langle \widetilde{\mathcal{L}_\nu} \big\rangle \supseteq \big({\mathcal{H}_\hbar}^\vee\big)'$. In fact, let $\eta \in \big({\mathcal{H}_\hbar}^\vee\big)'$; then there are unique $d \in \mathbb{N}$, $\eta_+ \in {\mathcal{H}_\hbar}^\vee \setminus \hbar\,{\mathcal{H}_\hbar}^\vee$ such that $\eta = \hbar^d \eta_+$; set also $\bar{y} := y \mod \hbar\, {H_\hbar}^\vee \in {H_\hbar}^\vee \big/ \hbar\, {H_\hbar}^\vee$ for all $y \in {H_\hbar}^\vee$. As ${\mathcal{H}_\hbar}^\vee = \Bbbk[\hbar]\big\langle \{\, \mathbf{x}_n \,|\, n \in \mathbb{N}_\nu \,\}\big\rangle$ there is a unique $\hbar$–adic expansion of η_+, namely $\eta_+ = \eta_0 + \hbar\,\eta_1 + \cdots + \hbar^s\,\eta_s = \sum_{k=0}^s \hbar^k\,\eta_k$ with all $\eta_k \in \Bbbk\big\langle \{\, \mathbf{x}_n \,|\, n \in \mathbb{N}_\nu \,\}\big\rangle$ and $\eta_0 \neq 0$. Then $\bar{\eta}_+ = \bar{\eta}_0 := \eta_0 \mod \hbar\, {\mathcal{H}_\hbar}^\vee$; thus Lemma 4.2*(d)* gives $\partial(\bar{\eta}_0) \leq d$, where now $\partial(\bar{\eta}_0)$ denotes the degree of $\bar{\eta}_0$ for the standard filtration of $U(\mathcal{L}_\nu)$. By the PBW theorem, $\partial(\bar{\eta}_0)$ is also the degree of $\bar{\eta}_0$ as a polynomial in the $\bar{\mathbf{x}}_b$'s, hence also of η_0 as a polynomial in the $\mathbf{x}_b$'s ($b \in B_\nu$): then $\hbar^d\,\eta_0 \in \big\langle \widetilde{\mathcal{L}_\nu} \big\rangle \subseteq \big({\mathcal{H}_\hbar}^\vee\big)'$ (using *Step III*), hence we find

$$\eta_{(1)} := \hbar^{d+1}\left(\eta_1 + \hbar\,\eta_2 + \cdots + \hbar^{s-1}\,\eta_s\right) = \eta - \hbar^d\,\eta_0 \in \big({\mathcal{H}_\hbar}^\vee\big)'.$$

Thus we can apply our argument again, with $\eta_{(1)}$ instead of η. Iterating we find $\partial(\bar{\eta}_k) \leq d + k$, whence $\hbar^{d+k}\,\eta_k \in \big\langle \widetilde{\mathcal{L}_\nu} \big\rangle \ \Big(\subseteq \big({\mathcal{H}_\hbar}^\vee\big)'\Big)$ for all k, thus $\eta = \sum_{k=0}^s \hbar^{d+k}\,\eta_k \in \big\langle \widetilde{\mathcal{L}_\nu} \big\rangle$, q.e.d.

An entirely similar analysis clearly works with $\mathcal{K}_\hbar$ taking the role of $\mathcal{H}_\hbar$, with similar results (*mutatis mutandis*). On the upshot, we get the following description:

Theorem 10.8. *(a) With notation of Step II in §10.7 (and* $[a,c] := a\,c - c\,a$*), we have*

$$\big({\mathcal{H}_\hbar}^\vee\big)' = \Big\langle \widetilde{\mathcal{L}_\nu} \Big\rangle = \Bbbk[\hbar]\Big\langle \big\{\, \widetilde{\mathbf{b}}_b \,\big\}_{b \in B_\nu} \Big\rangle \bigg/ \!\!\!\bigg(\Big\{ \Big[\widetilde{\mathbf{b}}_{b_1}, \widetilde{\mathbf{b}}_{b_2}\Big] - \hbar\,\widetilde{\big[\mathbf{b}_{b_1}, \mathbf{b}_{b_2}\big]} \,\Big|\, \forall\, b_1, b_2 \in B_\nu \,\Big\} \bigg).$$

(b) $\big({\mathcal{H}_\hbar}^\vee\big)'$ *is a graded Hopf* $\Bbbk[\hbar]$*–subalgebra of* ${\mathcal{H}_\hbar}^\vee$*, and* $\mathcal{H}$ *is naturally embedded into* $\big({\mathcal{H}_\hbar}^\vee\big)'$ *as a graded Hopf subalgebra via* $\mathcal{H} \lhook\joinrel\longrightarrow \big({\mathcal{H}_\hbar}^\vee\big)'$, $\mathbf{a}_n \mapsto \tilde{\mathbf{x}}_n$ *(for all* $n \in \mathbb{N}_\nu$*).*

(c) $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0} := \big({\mathcal{H}_\hbar}^\vee\big)' \big/ \hbar\,\big({\mathcal{H}_\hbar}^\vee\big)' = F\big[G_{\mathcal{L}_\nu}^\star\big]$*, where* $G_{\mathcal{L}_\nu}^\star$ *is an infinite dimensional connected Poisson algebraic group with cotangent Lie bialgebra isomorphic to* $\mathcal{L}_\nu$ *(with the graded Lie bialgebra structure of Theorem 10.6). Indeed,* $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0}$ *is the free Poisson (commutative) algebra over* $\mathbb{N}_\nu$*, generated by all the* $\tilde{\mathbf{x}}_n\big|_{\hbar=0}$ *(*$n \in \mathbb{N}_\nu$*) with Hopf structure given by (10.1) with* $\tilde{\mathbf{x}}_*$ *instead of* $\mathbf{a}_*$*. Thus* $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0}$ *is the polynomial algebra* $\Bbbk\big[\{\,\beta_b\,\}_{b \in B_\nu}\big]$ *generated by a set of indeterminates* $\{\,\beta_b\,\}_{b \in B_\nu}$ *in bijection with the basis* B_ν *of* $\mathcal{L}_\nu$*, so* $G_{\mathcal{L}_\nu}^\star \cong \mathbb{A}_\Bbbk^{B_\nu}$ *(a (pro)affine* $\Bbbk$*–space) as algebraic varieties. Finally,* $F\big[G_{\mathcal{L}_\nu}^\star\big] = \big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0} \cong \Bbbk\big[\{\,\beta_b\,\}_{b \in B_\nu}\big]$ *bears the natural* algebra *grading* d *of polynomial algebras and the* Hopf *algebra grading inherited from* $\big({\mathcal{H}_\hbar}^\vee\big)'$*, respectively given by* $d\big(\widetilde{\mathbf{b}}_b\big) = 1$ *and* $\partial\big(\widetilde{\mathbf{b}}_b\big) = \sum_{i=1}^k n_i$ *for all* $b = [[\cdots[[x_{n_1}, x_{n_2}], x_{n_3}], \cdots], x_{n_k}] \in B_\nu$*.*

(d) $F\big[\mathcal{G}_\nu\big]$ *is naturally embedded into* $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0} = F\big[G_{\mathcal{L}_\nu}^\star\big]$ *as a graded Hopf subalgebra via* $\mu\colon F\big[\mathcal{G}_\nu\big] \lhook\joinrel\longrightarrow \big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0} = F\big[G_{\mathcal{L}_\nu}^\star\big]$, $a_n \mapsto \Big(\tilde{\mathbf{x}}_n \mod \hbar\,\big({\mathcal{H}_\hbar}^\vee\big)'\Big)$ *(for all* $n \in \mathbb{N}_\nu$*); moreover,* $F\big[\mathcal{G}_\nu\big]$ *freely generates* $F\big[G_{\mathcal{L}_\nu}^\star\big]$ *as a* Poisson *algebra. Thus there is an algebraic group epimorphism* $\mu_*\colon G_{\mathcal{L}_\nu}^\star \relbar\joinrel\twoheadrightarrow \mathcal{G}_\nu$*, that is* $G_{\mathcal{L}_\nu}^\star$ *is an extension of* $\mathcal{G}_\nu$*.*

(e) Mapping $\Big(\tilde{\mathbf{x}}_n \mod \hbar\,\big({\mathcal{H}_\hbar}^\vee\big)'\Big) \mapsto a_n$ *(for all* $n \in \mathbb{N}_\nu$*) gives a well-defined graded*

Hopf algebra epimorphism $\pi : F\big[G_{\mathcal{L}_\nu}^\star\big] \relbar\joinrel\twoheadrightarrow F\big[\mathcal{G}_\nu\big]$ *. Thus there is an algebraic group monomorphism* $\pi_* : \mathcal{G}_\nu \lhook\joinrel\relbar\joinrel\rightarrow G_{\mathcal{L}_\nu}^\star$ *, that is* $\mathcal{G}_\nu$ *is an algebraic subgroup of* $G_{\mathcal{L}_\nu}^\star$ *.*

(f) The map μ *is a section of* π*, hence* π_* *is a section of* μ_* *. Thus* $G_{\mathcal{L}_\nu}^\star$ *is a semidirect product of algebraic groups, namely* $G_{\mathcal{L}_\nu}^\star = \mathcal{G}_\nu \ltimes \mathcal{N}_\nu$ *where* $\mathcal{N}_\nu := \mathrm{Ker}\,(\mu_*) \trianglelefteq G_{\mathcal{L}_\nu}^\star$ *.*

(g) The analogues of statements (a)–(f) hold with $\mathcal{K}$ *instead of* $\mathcal{H}$ *, with* X^+ *instead of* X *for all* $X = \mathcal{L}_\nu, B_\nu, \mathbb{N}_\nu, \mu, \pi, \mathcal{N}_\nu$ *, and with* $G_{\mathcal{L}_\nu^+}^\star$ *instead of* $G_{\mathcal{L}_\nu}^\star$*.*

Proof. *(a)* This part follows directly from *Step III* and *Step IV* in §10.7.

(b) To show that $\big({\mathcal{H}_\hbar}^\vee\big)'$ is a graded Hopf subalgebra we use its presentation in *(a)*. But first observe that by construction $\mathbf{a}_n = \tilde{\mathbf{x}}_n$ (for all $n \in \mathbb{N}_\nu$), so $\mathcal{H}$ embeds into $\big({\mathcal{H}_\hbar}^\vee\big)'$ via an embedding which is compatible with the Hopf operations: then this will be a Hopf algebra monomorphism, *up to proving that* $\big({\mathcal{H}_\hbar}^\vee\big)'$ *is a Hopf subalgebra* (of ${\mathcal{H}_\hbar}^\vee$).

Now, $\epsilon_{{\mathcal{H}_\hbar}^\vee}$ obviously restricts to give a counit for $\big({\mathcal{H}_\hbar}^\vee\big)'$. Second, we show that $\Delta\Big(\big({\mathcal{H}_\hbar}^\vee\big)'\Big) \subseteq \big({\mathcal{H}_\hbar}^\vee\big)' \otimes \big({\mathcal{H}_\hbar}^\vee\big)'$, so Δ restricts to a coproduct for $\big({\mathcal{H}_\hbar}^\vee\big)'$. Indeed, each $b \in B_\nu$ is a Lie monomial, say $b = [[[\dots[x_{n_1}, x_{n_2}], x_{n_3}], \dots], x_{n_k}]$ for some k, n_1, ..., $n_k \in \mathbb{N}_\nu$, where k is its Lie degree: by induction on k we'll prove $\Delta\big(\widetilde{\mathbf{b}}_b\big) \in \big({\mathcal{H}_\hbar}^\vee\big)' \otimes \big({\mathcal{H}_\hbar}^\vee\big)'$ (with $\widetilde{\mathbf{b}}_b := \hbar\, \mathbf{b}_b = \hbar\, [[[\dots[\mathbf{x}_{n_1}, \mathbf{x}_{n_2}], \mathbf{x}_{n_3}], \dots], \mathbf{x}_{n_k}]$).

If $k = 1$ then $b = x_n$ for some $n \in \mathbb{N}_\nu$. Then $\widetilde{\mathbf{b}}_b = \hbar\, \mathbf{x}_n = \mathbf{a}_n$ and

$$\Delta\Big(\widetilde{\mathbf{b}}_b\Big) = \Delta(\mathbf{a}_n) = \mathbf{a}_n \otimes 1 + 1 \otimes \mathbf{a}_n + \sum_{m=1}^{n-1} \mathbf{a}_{n-m} \otimes Q_m^{n-m}(\mathbf{a}_*) \in \mathcal{H}^{\mathrm{dif}} \otimes \mathcal{H}^{\mathrm{dif}} \subseteq \big({\mathcal{H}_\hbar}^\vee\big)' \otimes \big({\mathcal{H}_\hbar}^\vee\big)' .$$

If $k > 1$ then $b = [b^-, x_n]$ for some $n \in \mathbb{N}_\nu$ and some $b^- \in B_\nu$ expressed by a Lie monomial of degree $k-1$. Then $\widetilde{\mathbf{b}}_b = \hbar\,[\mathbf{b}^-, \mathbf{x}_n] = \Big[\widetilde{\mathbf{b}}^-, \mathbf{x}_n\Big]$ and

$$\begin{aligned}
\Delta\Big(\widetilde{\mathbf{b}}_b\Big) \; &= \; \Delta\Big(\Big[\widetilde{\mathbf{b}}^-, \mathbf{x}_n\Big]\Big) \; = \; \Big[\Delta\Big(\widetilde{\mathbf{b}}^-\Big), \Delta(\mathbf{x}_n)\Big] \; = \; \hbar^{-1} \Big[\Delta\Big(\widetilde{\mathbf{b}}^-\Big), \Delta(\mathbf{a}_n)\Big] \; = \\
&= \; \hbar^{-1} \left[\sum\nolimits_{(\widetilde{\mathbf{b}}^-)} \widetilde{\mathbf{b}}^-_{(1)} \otimes \widetilde{\mathbf{b}}^-_{(2)} \, , \; \mathbf{a}_n \otimes 1 + 1 \otimes \mathbf{a}_n + \sum\nolimits_{m=1}^{n-1} \mathbf{a}_{n-m} \otimes Q_m^{n-m}(\mathbf{a}_*) \right] \; = \\
= \; & \sum\nolimits_{(\widetilde{\mathbf{b}}^-)} \hbar^{-1} \Big[\widetilde{\mathbf{b}}^-_{(1)} \, , \mathbf{a}_n \Big] \otimes \widetilde{\mathbf{b}}^-_{(2)} \; + \sum\nolimits_{(\widetilde{\mathbf{b}}^-)} \widetilde{\mathbf{b}}^-_{(1)} \otimes \hbar^{-1} \Big[\widetilde{\mathbf{b}}^-_{(2)} \, , \mathbf{a}_n \Big] \; + \\
+ \; & \sum_{(\widetilde{\mathbf{b}}^-)} \sum_{m=1}^{n-1} \bigg(\hbar^{-1} \Big[\widetilde{\mathbf{b}}^-_{(1)} \, , \mathbf{a}_{n-m} \Big] \otimes \widetilde{\mathbf{b}}^-_{(2)} \, Q_m^{n-m}(\mathbf{a}_*) \; + \; \widetilde{\mathbf{b}}^-_{(1)} \, \mathbf{a}_{n-m} \otimes \hbar^{-1} \Big[\widetilde{\mathbf{b}}^-_{(2)} \, , \, Q_m^{n-m}(\mathbf{a}_*) \Big] \bigg)
\end{aligned}$$

where we used the standard Σ–notation for $\Delta\Big(\widetilde{\mathbf{b}}^-\Big) = \sum_{(\widetilde{\mathbf{b}}^-)} \widetilde{\mathbf{b}}^-_{(1)} \otimes \widetilde{\mathbf{b}}^-_{(2)}$. By inductive hypothesis we have $\widetilde{\mathbf{b}}^-_{(1)}$, $\widetilde{\mathbf{b}}^-_{(2)} \in \big({\mathcal{H}_\hbar}^\vee\big)'$; then since also $\mathbf{a}_\ell \in \big({\mathcal{H}_\hbar}^\vee\big)'$ for all ℓ and since $\big({\mathcal{H}_\hbar}^\vee\big)'$ is commutative modulo $\hbar$ we have

$$\hbar^{-1} \Big[\widetilde{\mathbf{b}}^-_{(1)} \, , \mathbf{a}_n \Big] \; , \; \hbar^{-1} \Big[\widetilde{\mathbf{b}}^-_{(2)} \, , \mathbf{a}_n \Big] \; , \; \hbar^{-1} \Big[\widetilde{\mathbf{b}}^-_{(1)} \, , \mathbf{a}_{n-m} \Big] \; , \; \hbar^{-1} \Big[\widetilde{\mathbf{b}}^-_{(2)} \, , \, Q_m^{n-m}(\mathbf{a}_*) \Big] \; \in \; \big({\mathcal{H}_\hbar}^\vee\big)'$$

for all n and $(n-m)$ above: so the previous formula gives $\Delta\big(\widetilde{\mathbf{b}}_b\big) \in \big({\mathcal{H}_\hbar}^\vee\big)' \otimes \big({\mathcal{H}_\hbar}^\vee\big)'$, q.e.d.

Finally, for the antipode we proceed as above. Let $b \in B_\nu$ be the Lie monomial $b = [[[\dots[x_{n_1}, x_{n_2}], x_{n_3}], \dots], x_{n_k}]$, so $\widetilde{\mathbf{b}}_b = \hbar\, \mathbf{b}_b = \hbar\, [[[\dots[\mathbf{x}_{n_1}, \mathbf{x}_{n_2}], \mathbf{x}_{n_3}], \dots], \mathbf{x}_{n_k}]$. We prove that $S\big(\widetilde{\mathbf{b}}_b\big) \in \big({\mathcal{H}_\hbar}^\vee\big)'$ by induction on the degree k.

If $k=1$ then $b = x_n$ for some n, so $\widetilde{\mathbf{b}}_b = \hbar\, \mathbf{x}_n = \mathbf{a}_n$ and

$$S\Big(\widetilde{\mathbf{b}}_b\Big) = S(\mathbf{a}_n) = -\mathbf{a}_n - \sum\nolimits_{m=1}^{n-1} \mathbf{a}_{n-m}\, S\big(Q_m^{n-m}(\mathbf{a}_*)\big) \ \in\ \mathcal{H}^{\mathrm{dif}} \subseteq \big({\mathcal{H}_\hbar}^\vee\big)' , \quad \text{q.e.d.}$$

If $k>1$ then $b = [b^-, x_n]$ for some $n \in \mathbb{N}_\nu$ and some $b^- \in B_\nu$ which is a Lie monomial of degree $k-1$. Then $\widetilde{\mathbf{b}}_b = \hbar\, [\mathbf{b}^-, \mathbf{x}_n] = \Big[\widetilde{\mathbf{b}}^-, \mathbf{x}_n\Big] = \hbar^{-1} \Big[\widetilde{\mathbf{b}}^-, \mathbf{a}_n\Big]$ and so

$$S\Big(\widetilde{\mathbf{b}}_b\Big) \;=\; S\Big(\Big[\widetilde{\mathbf{b}}^-, \mathbf{x}_n\Big]\Big) \;=\; \hbar^{-1} \Big[S(\mathbf{a}_n), S\big(\widetilde{\mathbf{b}}^-\big)\Big] \;\in\; \hbar^{-1} \Big[\big({\mathcal{H}_\hbar}^\vee\big)', \big({\mathcal{H}_\hbar}^\vee\big)'\Big] \;\subseteq\; \big({\mathcal{H}_\hbar}^\vee\big)'$$

using the fact $S(\mathbf{a}_n) = S\big(\widetilde{\mathbf{x}}_n\big) = S\big(\widetilde{\mathbf{b}}_{x_n}\big) \in \big({\mathcal{H}_\hbar}^\vee\big)'$ (by the case $k=1$) along with the inductive assumption $S\big(\widetilde{\mathbf{b}}^-\big) \in \big({\mathcal{H}_\hbar}^\vee\big)'$ and the commutativity of $\big({\mathcal{H}_\hbar}^\vee\big)'$ modulo $\hbar$.

(c) As a consequence of *(a)*, $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0}$ is a *polynomial* $\Bbbk$–algebra, namely

$$\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0} \;=\; \Bbbk\Big[\big\{\, \beta_b \,\big\}_{b\in B}\Big] \qquad \text{with} \qquad \beta_b := \widetilde{\mathbf{b}}_b \mod \hbar\, \big({\mathcal{H}_\hbar}^\vee\big)' \quad \text{for all } \ b \in B_\nu \,.$$

So $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0}$ is the algebra of regular functions $F[\varGamma]$ of some (affine) algebraic variety $\varGamma$; as $\big({\mathcal{H}_\hbar}^\vee\big)'$ is a Hopf algebra the same is true for $\big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0} = F[\varGamma]$, so $\varGamma$ is an (affine) algebraic group; and since $F[\varGamma] = \big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0}$ is a specialization limit of $\big({\mathcal{H}_\hbar}^\vee\big)'$, it is endowed with a Poisson structure too, hence $\varGamma$ is a *Poisson* (affine) algebraic group.

We compute the cotangent Lie bialgebra of $\varGamma$. First, $\mathfrak{m}_e := \mathit{Ker}\,\big(\epsilon_{F[\varGamma]}\big) = \Big(\big\{\, \beta_b \,\big\}_{b\in B_\nu}\Big)$ (the ideal generated by the β_b's) by construction, so $\mathfrak{m}_e^2 = \Big(\big\{\, \beta_{b_1} \beta_{b_2} \,\big\}_{b_1, b_2 \in B_\nu}\Big)$. Therefore the cotangent Lie bialgebra $Q\big(F[\varGamma]\big) := \mathfrak{m}_e \Big/ \mathfrak{m}_e^2$ as a $\Bbbk$–vector space has basis $\big\{\, \overline{\beta}_b \,\big\}_{b\in B_\nu}$ where $\overline{\beta}_b := \beta_b \mod \mathfrak{m}_e^2$ for all $b \in B_\nu$. For its Lie bracket we have (cf. Remark 1.5)

$$\begin{aligned}
\big[\, \overline{\beta}_{b_1}, \overline{\beta}_{b_2} \big] \;&:=\; \big\{\beta_{b_1}, \beta_{b_2}\big\} \mod \mathfrak{m}_e^2 \;=\; \Big(\hbar^{-1} \big[\, \widetilde{\mathbf{b}}_{b_1}, \widetilde{\mathbf{b}}_{b_2} \big] \mod \hbar\, \big({\mathcal{H}_\hbar}^\vee\big)'\Big) \mod \mathfrak{m}_e^2 \;= \\
&= \Big(\hbar^{-1} \hbar^2 \,\big[\, \mathbf{b}_{b_1}, \mathbf{b}_{b_2} \big] \mod \hbar\, \big({\mathcal{H}_\hbar}^\vee\big)'\Big) \mod \mathfrak{m}_e^2 \;=\; \Big(\hbar\, \mathbf{b}_{[b_1, b_2]} \mod \hbar\, \big({\mathcal{H}_\hbar}^\vee\big)'\Big) \mod \mathfrak{m}_e^2 \;= \\
&= \Big(\widetilde{\mathbf{b}}_{[b_1, b_2]} \mod \hbar\, \big({\mathcal{H}_\hbar}^\vee\big)'\Big) \mod \mathfrak{m}_e^2 \;=\; \beta_{[b_1, b_2]} \mod \mathfrak{m}_e^2 \;=\; \overline{\beta}_{[b_1, b_2]} \,,
\end{aligned}$$

thus the $\Bbbk$–linear map $\Psi : \mathcal{L}_\nu \longrightarrow \mathfrak{m}_e \big/ \mathfrak{m}_e^2$ defined by $b \mapsto \overline{\beta}_b$ for all $b \in B_\nu$ is a *Lie algebra isomorphism*. As for the Lie cobracket, using the general identity $\delta = \Delta - \Delta^{\mathrm{op}}$

mod $\left(\mathfrak{m}_e^2 \otimes F[\Gamma] + F[\Gamma] \otimes \mathfrak{m}_e^2\right)$ (written mod $\widehat{\mathfrak{m}_e^2}$ for short) we get, for all $n \in \mathbb{N}_\nu$,

$$\delta\big(\overline{\beta}_{x_n}\big) = \big(\Delta - \Delta^{\mathrm{op}}\big)\big(\beta_{x_n}\big) \bmod \widehat{\mathfrak{m}_e^2} = \Big(\big(\Delta - \Delta^{\mathrm{op}}\big)(\tilde{\mathbf{x}}_n) \bmod \hbar\Big(\big({\mathcal{H}_\hbar}^\vee\big)' \otimes \big({\mathcal{H}_\hbar}^\vee\big)'\Big)\Big) \bmod \widehat{\mathfrak{m}_e^2} =$$

$$= \left(\left(\mathbf{a}_n \wedge 1 \,+\, 1 \wedge \mathbf{a}_n \,+\, \sum\nolimits_{m=1}^{n-1} \mathbf{a}_{n-m} \wedge Q_m^{n-m}(\mathbf{a}_*)\right) \bmod \hbar\left({\mathcal{H}_\hbar}' \otimes {\mathcal{H}_\hbar}'\right)\right) \bmod \widehat{\mathfrak{m}_e^2} =$$

$$= \left(\beta_{x_n} \wedge 1 \,+\, 1 \wedge \beta_{x_n} \,+\, \sum\nolimits_{m=1}^{n-1} \beta_{x_{n-m}} \wedge Q_m^{n-m}(\beta_{x_*})\right) \bmod \widehat{\mathfrak{m}_e^2} =$$

$$= \left(\sum\nolimits_{m=1}^{n-1} \beta_{x_{n-m}} \wedge Q_m^{n-m}(\beta_{x_*})\right) \bmod \widehat{\mathfrak{m}_e^2} =$$

$$= \left(\sum\nolimits_{m=1}^{n-1} \sum\nolimits_{k=1}^{m} \binom{n-m+1}{k} \beta_{x_{n-m}} \wedge P_m^{(k)}(\beta_{x_*})\right) \bmod \widehat{\mathfrak{m}_e^2} =$$

$$= \left(\sum\nolimits_{m=1}^{n-1} \binom{n-m+1}{1} \beta_{x_{n-m}} \wedge P_m^{(1)}(\beta_{x_*})\right) \bmod \widehat{\mathfrak{m}_e^2} =$$

$$= \sum\nolimits_{m=1}^{n-1} \binom{n-m+1}{1} \overline{\beta}_{x_{n-m}} \wedge \overline{\beta}_{x_m} = \sum\nolimits_{\ell=1}^{n-1} (\ell+1)\, \overline{\beta}_{x_\ell} \wedge \overline{\beta}_{x_{n-\ell}}$$

because — among other things — one has $P_m^{(k)}(\beta_{x_*}) \in \mathfrak{m}_e^2$ for all $k > 1$: therefore

$$\delta\big(\overline{\beta}_{x_n}\big) = \sum\nolimits_{\ell=1}^{n-1} (\ell+1)\, \overline{\beta}_{x_\ell} \wedge \overline{\beta}_{x_{n-\ell}} \qquad \forall\ n \in \mathbb{N}_\nu\,. \tag{10.3}$$

Since $\mathcal{L}_\nu$ is generated (as a Lie algebra) by the x_n's, the last formula shows that the map $\Psi : \mathcal{L}_\nu \longrightarrow \mathfrak{m}_e \big/ \mathfrak{m}_e^2$ given above is also an isomorphism *of Lie bialgebras*, q.e.d.

Finally, the statements about gradings of $\left({\mathcal{H}_\hbar}^\vee\right)'\Big|_{\hbar=0}$ should be trivially clear.

(d) The part about Hopf algebras is a direct consequence of *(a)* and *(b)*, noting that the $\tilde{\mathbf{x}}_n$'s commute modulo $\hbar\left({\mathcal{H}_\hbar}^\vee\right)'$, since $\left({\mathcal{H}_\hbar}^\vee\right)'\Big|_{\hbar=0}$ is commutative. Then, taking spectra (i.e. sets of characters of each Hopf algebras) we get (functorially) an algebraic group morphism $\mu_* : G_{\mathcal{L}_\nu}^\star \longrightarrow \mathcal{G}_\nu$, which in fact happens to be *onto* because, due to the special polynomial form of these algebras, each character of $F\big[\mathcal{G}_\nu\big]$ does extend to a character of $F\big[G_{\mathcal{L}_\nu}^\star\big]$, hence the former does arise from restriction of the latter.

(e) Due to the explicit description of $F\big[G_{\mathcal{L}_\nu}^\star\big]$ coming from *(a)* and *(b)*, mapping $\Big(\tilde{\mathbf{x}}_n \bmod \hbar\left({\mathcal{H}_\hbar}^\vee\right)'\Big) \mapsto a_n$ (for all $n \in \mathbb{N}_\nu$) clearly yields a well-defined Hopf algebra epimorphism $\pi : F\big[G_{\mathcal{L}_\nu}^\star\big] \twoheadrightarrow F\big[\mathcal{G}_\nu\big]$ (w.r.t. the trivial Poisson bracket on the right-hand-side) is again a routine matter. Then taking spectra gives a monomorphism $\pi_* : \mathcal{G}_\nu \hookrightarrow G_{\mathcal{L}_\nu}^\star$ of algebraic groups as required.

(f) The map μ is a section of π by construction. Then clearly π_* is a section of μ_*, which implies $G_{\mathcal{L}_\nu}^\star = \mathcal{G}_\nu \ltimes \mathcal{N}_\nu$ (with $\mathcal{N}_\nu := \mathit{Ker}(\mu_*) \trianglelefteq G_{\mathcal{L}_\nu}^\star$) by general theory.

(g) This ought to be clear from the whole discussion, for all arguments apply again — *mutatis mutandis* — when starting with $\mathcal{K}$ instead of $\mathcal{H}$; details are left to the reader. $\square$

Remark: Roughly speaking, we can say that the extension $F\big[\mathcal{G}_\nu\big] \lhook\joinrel\longrightarrow F\big[G_{\mathcal{L}_\nu}^\star\big]$ is performed simply by adding to $F\big[\mathcal{G}_\nu\big]$ a *free* Poisson structure, which happens to be compatible with the Hopf structure. Then the Poisson bracket starting from the "elementary" coordinates a_n (for $n \in \mathbb{N}_\nu$) freely generates *new* coordinates $\{a_{n_1}, a_{n_2}\}$, $\big\{\{a_{n_1}, a_{n_2}\}, a_{n_3}\big\}$, etc., thus enlarging $F\big[\mathcal{G}_\nu\big]$ and generating $F\big[G_{\mathcal{L}_\nu}^\star\big]$. At the group level, this means that $\mathcal{G}_\nu$ *freely Poisson-generates* the Poisson group $G_{\mathcal{L}_\nu}^\star$: technically speaking, new 1-parameter subgroups, which are build up in a *"Poisson-free"* manner from those attached to the a_n's, are freely "pasted" to $\mathcal{G}_\nu$, thus expanding it and so building up $G_{\mathcal{L}_\nu}^\star$. Then the algebraic group epimorphism $G_{\mathcal{L}_\nu}^\star \overset{\mu_*}{\relbar\joinrel\twoheadrightarrow} \mathcal{G}_\nu$ is just a "forgetful map": it kills the new 1-parameter subgroups and is injective (hence an isomorphism) on the subgroup generated by the old ones. On the other hand, definitions imply that $F\big[G_{\mathcal{L}_\nu}^\star\big]\Big/\Big(\big\{F\big[G_{\mathcal{L}_\nu}^\star\big], F\big[G_{\mathcal{L}_\nu}^\star\big]\big\}\Big) \cong F\big[\mathcal{G}_\nu\big]$, and with this identification the map $F\big[G_{\mathcal{L}_\nu}^\star\big] \overset{\pi}{\relbar\joinrel\twoheadrightarrow} F\big[\mathcal{G}_\nu\big]$ is just the canonical map, which "mods out" all Poisson brakets $\{f_1, f_2\}$, for $f_1, f_2 \in F\big[G_{\mathcal{L}_\nu}^\star\big]$.

10.9 Specialization limits. So far, we have already pointed out (by Proposition 10.5, Theorem 10.6, Theorem 10.8*(c)*) the following specialization limits of ${\mathcal{H}_\hbar}^\vee$ and $\big({\mathcal{H}_\hbar}^\vee\big)'$:

$$ {\mathcal{H}_\hbar}^\vee \xrightarrow{\ \hbar\to 1\ } \mathcal{H}\ , \qquad {\mathcal{H}_\hbar}^\vee \xrightarrow{\ \hbar\to 0\ } U(\mathcal{L}_\nu)\ , \qquad \big({\mathcal{H}_\hbar}^\vee\big)' \xrightarrow{\ \hbar\to 0\ } F\big[G_{\mathcal{L}_\nu}^\star\big] $$

as graded Hopf $\Bbbk$–algebras, with some (co-)Poisson structures in the last two cases. As for the specialization limit of $\big({\mathcal{H}_\hbar}^\vee\big)'$ at $\hbar = 1$, Theorem 10.8 implies that it is $\mathcal{H}$. Indeed, by Theorem 10.8*(b)* $\mathcal{H}$ embeds into $\big({\mathcal{H}_\hbar}^\vee\big)'$ via $\mathbf{a}_n \mapsto \tilde{\mathbf{x}}_n$ (for all $n \in \mathbb{N}_\nu$): then

$$ [\mathbf{a}_n, \mathbf{a}_m] \,=\, \big[\tilde{\mathbf{x}}_n, \tilde{\mathbf{x}}_m\big] \,=\, \hbar\,\widetilde{[\mathbf{x}_n, \mathbf{x}_m]} \,\equiv\, \widetilde{[\mathbf{x}_n, \mathbf{x}_m]} \mod (\hbar-1)\,\big({\mathcal{H}_\hbar}^\vee\big)' \qquad (\forall\, n, m \in \mathbb{N}_\nu) $$

whence, due to the presentation of $\big({\mathcal{H}_\hbar}^\vee\big)'$ by generators and relations in Theorem 10.8*(a)*,

$$ \big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=1} \,:=\, \big({\mathcal{H}_\hbar}^\vee\big)'\Big/(\hbar-1)\,\big({\mathcal{H}_\hbar}^\vee\big)' \,=\, \Bbbk\big\langle \overline{\tilde{\mathbf{x}}}_1, \overline{\tilde{\mathbf{x}}}_2, \dots, \overline{\tilde{\mathbf{x}}}_n, \dots \big\rangle \,=\, \Bbbk\big\langle \overline{\mathbf{a}}_1, \overline{\mathbf{a}}_2, \dots, \overline{\mathbf{a}}_n, \dots \big\rangle $$

(where $\overline{\mathbf{c}} := \mathbf{c} \mod (\hbar-1)\,\big({\mathcal{H}_\hbar}^\vee\big)'$) as $\Bbbk$–algebras, and the Hopf structure is exactly the one of $\mathcal{H}$ because it is given by the like formulas on generators. In a nutshell, we have

$$ \big({\mathcal{H}_\hbar}^\vee\big)' \xrightarrow{\ \hbar\to 1\ } \mathcal{H} $$

as Hopf $\Bbbk$–algebras. Therefore we got the bottom part of the diagram of deformations (5.5), corresponding to (5.3), for $H = \mathcal{H}$ $(:= \mathcal{H}_\nu)$: it is

$$ U(\mathcal{L}_\nu) = {\mathcal{H}_\hbar}^\vee\Big|_{\hbar=0} \xleftarrow[{\mathcal{H}_\hbar}^\vee]{\ 0\leftarrow\hbar\to 1\ }\!\!\!\!\rightarrow {\mathcal{H}_\hbar}^\vee\Big|_{\hbar=1} = \mathcal{H} = \big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=1} \xleftarrow[({\mathcal{H}_\hbar}^\vee)']{\ 1\leftarrow\hbar\to 0\ }\!\!\!\!\rightarrow \big({\mathcal{H}_\hbar}^\vee\big)'\Big|_{\hbar=0} = F\big[G_{\mathcal{L}_\nu}^\star\big] $$

or simply $U(\mathcal{L}_\nu) \xleftrightarrow[{\mathcal{H}_\hbar}^\vee]{\ 0\leftarrow\hbar\to 1\ } \mathcal{H} \xleftrightarrow[({\mathcal{H}_\hbar}^\vee)']{\ 1\leftarrow\hbar\to 0\ } F\big[G_{\mathcal{L}_\nu}^\star\big]$. Therefore $\mathcal{H}$ *is intermediate* *between the (Poisson-type) "geometrical symmetries" $U(\mathcal{L}_\nu)$ and $F\big[G_{\mathcal{L}_\nu}^\star\big]$*, hence the geometrical meaning of the latters should shed some light on it; in turn, the physical meaning of $\mathcal{H}$ should have some reflect on the physical meaning of both $U(\mathcal{L}_\nu)$ and $F\big[G_{\mathcal{L}_\nu}^\star\big]$.

10.10 Drinfeld's algebra $\mathcal{H}_\hbar{}' := \big(\mathcal{H}[\hbar]\big)'$. From now on we shall deal with Drinfeld's functors in the opposite order: first $(\ )'$, and then $(\ )^\vee$. Like in §2.1, define $\mathcal{H}_\hbar{}' := \big\{ \eta \in \mathcal{H}_\hbar \,\big|\, \delta_n(\eta) \in \hbar^n {\mathcal{H}_\hbar}^{\otimes n} \ \ \forall\, n \in \mathbb{N} \big\} \ \ \big(\subseteq \mathcal{H}_\hbar \big)$. We shall describe $\mathcal{H}_\hbar{}'$ explicitly, thus checking that it is really a QFA, as predicted by Theorem 2.2*(a)*; then we'll look at its specialization at $\hbar = 0$ and at $\hbar = 1$, and finally we'll study $\big(\mathcal{H}_\hbar{}'\big)^\vee$ and its specializations at $\hbar = 0$ and $\hbar = 1$. The outcome will be an explicit description of (5.4) for $H = \mathcal{H}$ $(= \mathcal{H}_\nu$, with $\nu \in \mathbb{N} \cup \{\infty\}$ fixed as before).

Let $\underline{D} := \underline{D}(\mathcal{H}) = \big\{D_n\big\}_{n \in \mathbb{N}} = \big\{ \mathit{Ker}\big(\delta_n : \mathcal{H} \longrightarrow (\mathcal{H}^{\mathrm{dif}})^{\otimes n}\big) \big\}_{n \in \mathbb{N}_\nu}$ be the Hopf algebra filtration of $\mathcal{H}$ as considered in §5.1. Then by Lemma 5.2, we have

$$\mathcal{H}_\hbar{}' \; = \; \mathcal{R}_{\underline{D}}^\hbar(\mathcal{H}) \; := \; \Bbbk[\hbar] \cdot D_0 + \hbar\, \Bbbk[\hbar] \cdot D_1 + \cdots + \hbar^n\, \Bbbk[\hbar] \cdot D_n + \cdots$$

so we only need to compute the filtration $\underline{D}$. The idea is to describe it in combinatorial terms, based on the non-commutative polynomial nature of $\mathcal{H}$.

As before, we proceed in steps.

10.11 Gradings and filtrations: Let ∂_- be the unique Lie algebra grading of $\mathcal{L}_\nu$ given by $\partial_-(\boldsymbol{\alpha}_n) := n - 1 + \delta_{n,1}$ (for all $n \in \mathbb{N}_\nu$). Let also d be the standard Lie algebra grading associated with the central lower series of $\mathcal{L}_\nu$: in down-to-earth terms, d is defined by $d\big([\cdots[[x_{s_1}, x_{s_2}], \dots x_{s_k}]\big) = k - 1$ on any Lie monomial of $\mathcal{L}_\nu$. Since both ∂_- and d are Lie algebra gradings, their difference $(\partial_- - d)$ is a Lie algebra grading too. Let $\big\{F_n\big\}_{n \in \mathbb{N}}$ be the Lie algebra filtration associated with the grading $(\partial_- - d)$; then the down-shifted filtration $\underline{T} := \big\{\, T_n := F_{n-1} \,\big\}_{n \in \mathbb{N}}$ is again a Lie algebra filtration of $\mathcal{L}_\nu$. There is a unique algebra filtration on $U(\mathcal{L}_\nu)$ extending $\underline{T}$, which we denote by $\underline{\varTheta} = \big\{\varTheta_n\big\}_{n \in \mathbb{N}}$; as a matter of notation, we set also $\varTheta_{-1} := \{0\}$. Finally, for each $y \in U(\mathcal{L}_\nu) \setminus \{0\}$ there is a unique $\tau(y) \in \mathbb{N}$ with $y \in \varTheta_{\tau(y)} \setminus \varTheta_{\tau(y)-1}$; in particular, we have $\tau(b) = \partial_-(b) - d(b)$, $\tau(b\, b') = \tau(b) + \tau(b')$ and $\tau\big([b, b']\big) = \tau(b) + \tau(b') - 1$ for all $b, b' \in B_\nu$.

We can explicitly describe $\underline{\varTheta}$. Indeed, let us fix any total order $\preceq$ on the basis B_ν of §10.2: then $\mathcal{X} := \Big\{ \underline{b} := b_1 \cdots b_k \,\Big|\, k \in \mathbb{N},\ b_1, \dots, b_k \in B_\nu,\ b_1 \preceq \cdots \preceq b_k \Big\}$ is a $\Bbbk$–basis of $U(\mathcal{L}_\nu)$, by the PBW theorem. It follows that $\varTheta$ induces a set-theoretic filtration $\underline{\mathcal{X}} = \big\{\mathcal{X}_n\big\}_{n \in \mathbb{N}}$ of $\mathcal{X}$ with $\mathcal{X}_n := \mathcal{X} \cap \varTheta_n = \Big\{ \underline{b} := b_1 \cdots b_k \,\Big|\, k \in \mathbb{N},\ b_1, \dots, b_k \in B_\nu,\ b_1 \preceq \cdots \preceq b_k,\ \tau(\underline{b}) = \tau(b_1) + \cdots + \tau(b_k) \leq n \Big\}$, and also that $\varTheta_n = \mathit{Span}\big(\mathcal{X}_n\big)$ for all $n \in \mathbb{N}$.

Let us define $\boldsymbol{\alpha}_1 := \mathbf{a}_1$ and $\boldsymbol{\alpha}_n := \mathbf{a}_n - {\mathbf{a}_1}^n$ for all $n \in \mathbb{N}_\nu \setminus \{1\}$. This "change of variables" — which switch from the $\mathbf{a}_n$'s to their "differentials", in a sense — will be the key to achieve a complete description of the filtration $\underline{D}$; in turn, this will pass through a close comparison among $\mathcal{H}$ and $U(\mathcal{L}_\nu)$.

By definition $\mathcal{H} = \mathcal{H}_\nu$ is the free associative algebra over $\{\mathbf{a}_n\}_{n \in \mathbb{N}_\nu}$, hence — by definition of the $\boldsymbol{\alpha}$'s — also over $\{\boldsymbol{\alpha}_n\}_{n \in \mathbb{N}_\nu}$; so we have an algebra isomorphism $\Phi : \mathcal{H} \xhookrightarrow{\cong} U(\mathcal{L}_\nu)$ given by $\boldsymbol{\alpha}_n \mapsto x_n$ ($\forall\ n \in \mathbb{N}_\nu$). Via Φ we pull back all data and results about gradings, filtrations, PBW bases and so on mentioned above for $U(\mathcal{L}_\nu)$; in particular we set $\boldsymbol{\alpha}_{\underline{b}} := \Phi(x_{\underline{b}}) = \boldsymbol{\alpha}_{b_1} \cdots \boldsymbol{\alpha}_{b_k}$ (for all $b_1, \dots, b_k \in B_\nu$), $\mathcal{A}_n := \Phi(\mathcal{X}_n)$ (for all $n \in \mathbb{N}$) and $\mathcal{A} := \Phi(\mathcal{X}) = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$. For gradings on $\mathcal{H}$ we stick to the like notation, i.e. ∂_-, d and τ, and similarly for the filtration $\underline{\Theta}$.

Finally, for all $a \in \mathcal{H} \setminus \{0\}$ we set also $\kappa(a) := k$ iff $a \in D_k \setminus D_{k-1}$ (with $D_{-1} := \{0\}$).

Our goal is to prove an identity of filtrations, namely $\underline{D} = \underline{\Theta}$, or equivalently $\kappa = \tau$. In fact, this would give to the Hopf filtration $\underline{D}$, which is defined intrinsically in Hopf algebraic terms, an explicit *combinatorial* description, namely the one of Θ explained above.

Lemma 10.12. *For all $\ell, t \in \mathbb{N}$, $t \geq 1$, we have (notation of §10.11)*

$$Z_t^\ell(\boldsymbol{\alpha}_*) := \Big(Q_t^\ell(\mathbf{a}_*) - \tbinom{\ell+t}{t} \mathbf{a}_1{}^t \Big) \in \Theta_{t-1} \quad \textit{and} \quad Q_t^\ell(\mathbf{a}_*) \in \Theta_t \setminus \Theta_{t-1} \,.$$

Proof. When $t = 1$ definitions give $Q_1^\ell(\mathbf{a}_*) = (\ell+1)\, \mathbf{a}_1 \in \Theta_1$ and so $Z_1^\ell(\boldsymbol{\alpha}_*) = (\ell+1)\, \mathbf{a}_1 - \binom{\ell+1}{1} \mathbf{a}_1 = 0 \in \Theta_0$, for all $\ell \in \mathbb{N}$. Similarly, when $\ell = 0$ we have $Q_t^0(\mathbf{a}_*) = \mathbf{a}_t \in \Theta_t$ and so $Z_t^0(\boldsymbol{\alpha}_*) = \mathbf{a}_t - \binom{1}{1} \mathbf{a}_1{}^t = \boldsymbol{\alpha}_t \in \Theta_{t-1}$ (by definition), for all $t \in \mathbb{N}_+$.

When $\ell > 0$ and $t > 1$, we can prove the claim using two independent methods.

<u>First method</u>: The very definitions imply that the following recurrence formula holds:

$$Q_t^\ell(\mathbf{a}_*) \;=\; Q_t^{\ell-1}(\mathbf{a}_*) \;+\; \sum\nolimits_{s=1}^{t-1} Q_{t-s}^{\ell-1}(\mathbf{a}_*)\, \mathbf{a}_s + \mathbf{a}_t \qquad \forall \quad \ell \geq 1 \,,\; t \geq 2 \,.$$

From this formula we argue

$$\begin{aligned}
Z_t^\ell(\boldsymbol{\alpha}_*) \;:=\; & Q_t^\ell(\mathbf{a}_*) \;-\; \tbinom{\ell+t}{t} \mathbf{a}_1{}^t \;=\; Q_t^{\ell-1}(\mathbf{a}_*) \;+\; \textstyle\sum_{s=1}^{t-1} Q_{t-s}^{\ell-1}(\mathbf{a}_*)\, \mathbf{a}_s \;+\; \mathbf{a}_t \;-\; \tbinom{\ell+t}{t} \mathbf{a}_1{}^t \;= \\
=\; & Z_t^{\ell-1}(\mathbf{a}_*) \;+\; \tbinom{\ell-1+t}{t} \mathbf{a}_1{}^t \;+\; \textstyle\sum_{s=1}^{t-1} \Big(Z_{t-s}^{\ell-1}(\mathbf{a}_*) \;+\; \tbinom{\ell-1+t-s}{t-s} \mathbf{a}_1{}^{t-s} \Big)\, \mathbf{a}_s + \mathbf{a}_t - \tbinom{\ell+t}{t} \mathbf{a}_1{}^t \;= \\
=\; & Z_t^{\ell-1}(\mathbf{a}_*) \;+\; \tbinom{\ell-1+t}{t} \boldsymbol{\alpha}_1{}^t \;+\; \textstyle\sum_{s=1}^{t-1} Z_{t-s}^{\ell-1}(\mathbf{a}_*) \big(\boldsymbol{\alpha}_s + \boldsymbol{\alpha}_1{}^s \big) \;+ \\
& \qquad + \textstyle\sum_{s=1}^{t-1} \tbinom{\ell-1+t-s}{t-s} \boldsymbol{\alpha}_1{}^{t-s} \big(\boldsymbol{\alpha}_s + \boldsymbol{\alpha}_1{}^s \big) \;+\; \big(\boldsymbol{\alpha}_t + \boldsymbol{\alpha}_1{}^t \big) \;-\; \tbinom{\ell+t}{t} \boldsymbol{\alpha}_1{}^t \;= \\
=\; & Z_t^{\ell-1}(\mathbf{a}_*) \;+\; \textstyle\sum_{s=1}^{t-1} Z_{t-s}^{\ell-1}(\mathbf{a}_*) \big(\boldsymbol{\alpha}_s + \boldsymbol{\alpha}_1{}^s \big) \;+\; \textstyle\sum_{s=1}^{t-1} \tbinom{\ell-1+t-s}{t-s} \boldsymbol{\alpha}_1{}^{t-s}\, \boldsymbol{\alpha}_s \;+\; \boldsymbol{\alpha}_t \;+ \\
& \qquad + \textstyle\sum_{s=1}^{t-1} \tbinom{\ell-1+t-s}{t-s} \boldsymbol{\alpha}_1{}^{t-s}\, \boldsymbol{\alpha}_1{}^s \;+\; \boldsymbol{\alpha}_1{}^t \;+\; \tbinom{\ell-1+t}{t} \boldsymbol{\alpha}_1{}^t \;-\; \tbinom{\ell+t}{t} \boldsymbol{\alpha}_1{}^t \;= \\
=\; & Z_t^{\ell-1}(\mathbf{a}_*) \;+\; \textstyle\sum_{s=1}^{t-1} Z_{t-s}^{\ell-1}(\mathbf{a}_*) \big(\boldsymbol{\alpha}_s + \boldsymbol{\alpha}_1{}^s \big) \;+\; \textstyle\sum_{s=1}^{t-1} \tbinom{\ell-1+t-s}{t-s} \boldsymbol{\alpha}_1{}^{t-s}\, \boldsymbol{\alpha}_s \;+\; \boldsymbol{\alpha}_t \;+ \\
& \qquad + \Bigg(\textstyle\sum_{r=0}^{t} \tbinom{\ell-1+r}{\ell-1} \;-\; \tbinom{\ell+t}{t} \Bigg) \boldsymbol{\alpha}_1{}^t \;= \\
=\; & Z_t^{\ell-1}(\mathbf{a}_*) \;+\; \textstyle\sum_{s=1}^{t-1} Z_{t-s}^{\ell-1}(\mathbf{a}_*) \big(\boldsymbol{\alpha}_s + \boldsymbol{\alpha}_1{}^s \big) \;+\; \textstyle\sum_{s=1}^{t-1} \tbinom{\ell-1+t-s}{t-s} \boldsymbol{\alpha}_1{}^{t-s}\, \boldsymbol{\alpha}_s \;+\; \boldsymbol{\alpha}_t
\end{aligned}$$

because of the classical identity $\binom{\ell+t}{\ell} = \sum_{r=0}^{t} \binom{\ell-1+r}{\ell-1}$. Then induction upon ℓ and the very definitions allow to conclude that all summands in the final sum belong to Θ_{t-1}, hence $Z_t^\ell(\boldsymbol{\alpha}_*) \in \Theta_{t-1}$ as well. Finally, this implies $Q_t^\ell(\mathbf{a}_*) \;=\; Z_t^\ell(\boldsymbol{\alpha}_*) \;+\; \binom{\ell+t}{t} \boldsymbol{\alpha}_1{}^t \in \Theta_t \setminus \Theta_{t-1}$.

Second method: $Q_t^\ell(\mathbf{a}_*) := \sum_{s=1}^t \binom{\ell+1}{s} P_t^{(s)}(\mathbf{a}_*) = \sum_{s=1}^t \binom{\ell+1}{s} \sum_{\substack{j_1,\dots,j_s>0 \\ j_1+\cdots+j_s=t}} \mathbf{a}_{j_1}\cdots\mathbf{a}_{j_s}$, by definition; then expanding the $\mathbf{a}_j$'s as $\mathbf{a}_1 = \boldsymbol{\alpha}_1$ and $\mathbf{a}_j = \boldsymbol{\alpha}_j + {\boldsymbol{\alpha}_1}^j$ (for $j>1$) we find that $Q_t^\ell(\mathbf{a}_*) = Q_t^\ell\big(\boldsymbol{\alpha}_* + {\boldsymbol{\alpha}_1}^*\big)$ is a linear combination of monomials $\boldsymbol{\alpha}_{(j_1)}\cdots\boldsymbol{\alpha}_{(j_s)}$ with $j_1,\dots,j_s>0$, $j_1+\cdots+j_s=t$, $\boldsymbol{\alpha}_{(j_r)} \in \big\{\boldsymbol{\alpha}_{j_r}, {\boldsymbol{\alpha}_1}^{j_r}\big\}$ for all r. Let Q_- be the linear combination of those monomials such that $\big(\boldsymbol{\alpha}_{(j_1)},\boldsymbol{\alpha}_{(j_2)},\dots,\boldsymbol{\alpha}_{(j_s)}\big) \neq \big({\boldsymbol{\alpha}_1}^{j_1},{\boldsymbol{\alpha}_1}^{j_2},\dots,{\boldsymbol{\alpha}_1}^{j_s}\big)$; for the remaining monomials we have $\boldsymbol{\alpha}_{j_1}\cdot\boldsymbol{\alpha}_{j_2}\cdots\boldsymbol{\alpha}_{j_s} = {\boldsymbol{\alpha}_1}^{j_1+\cdots+j_s} = {\boldsymbol{\alpha}_1}^t$, hence their linear combination giving $Q_+ := Q_t^\ell(\mathbf{a}_*) - Q_-$ is a multiple of ${\boldsymbol{\alpha}_1}^t$, say $Q_+ = N\,{\boldsymbol{\alpha}_1}^t$.

Now we compute this coefficient N. First, by construction N is nothing but $N = Q_t^\ell(1_*) = Q_t^\ell(1,1,\dots,1,\dots)$ where the latter means the (positive integer) value of the polynomial Q_t^ℓ when all its indeterminates are set equal to 1. Thus we compute $Q_t^\ell(1_*)$.

Recall that the Q_t^ℓ's enter in the definition of the coproduct of $F\big[\mathcal{G}^{\mathrm{dif}}\big]$: the latter is dual to the (composition) product of series in $\mathcal{G}^{\mathrm{dif}}$, thus if $\{a_n\}_{n\in\mathbb{N}_+}$ and $\{b_n\}_{n\in\mathbb{N}_+}$ are two countable sets of commutative indeterminates then

$$\Big(x + \textstyle\sum_{n=1}^{+\infty} a_n\,x^{n+1}\Big) \circ \Big(x + \sum_{m=1}^{+\infty} b_m\,x^{m+1}\Big) :=$$
$$:= \left(\left(x + \sum_{m=1}^{+\infty} b_m\,x^{m+1}\right) + \sum_{n=1}^{+\infty} a_n \left(x + \sum_{m=1}^{+\infty} b_m\,x^{m+1}\right)^{n+1}\right) = x + \sum_{k=0}^{+\infty} c_k\,x^{k+1}$$

with $c_k = Q_k^0(b_*) + \sum_{r=1}^k a_r\cdot Q_{k-r}^r(b_*)$ (cf. §10.2). Specializing $a_\ell = 1$ and $a_r = 0$ for all $r\neq\ell$ we get $c_{t+\ell} = Q_{t+\ell}^0(b_*) + Q_t^\ell(b_*) = b_{t+\ell} + Q_t^\ell(b_*)$. In particular setting $b_* = 1_*$ we have that $1 + Q_t^\ell(1_*)$ is the coefficient $c_{\ell+t}$ of $x^{\ell+t+1}$ in the series

$$\big(x + x^{\ell+1}\big) \circ \Big(x + \textstyle\sum_{m=1}^{+\infty} x^{m+1}\Big) =$$
$$= \big(x + x^{\ell+1}\big) \circ \big(x\cdot(1-x)^{-1}\big) = x\cdot(1-x)^{-1} + \big(x\cdot(1-x)^{-1}\big)^{\ell+1} =$$
$$= \textstyle\sum_{m=0}^{+\infty} x^{m+1} + x^{\ell+1}\Big(\sum_{m=0}^{+\infty} x^m\Big)^{\ell+1} = \sum_{m=0}^{+\infty} x^{m+1} + x^{\ell+1}\sum_{n=0}^{+\infty}\binom{\ell+n}{\ell} x^n =$$
$$= \textstyle\sum_{s=0}^{\ell-1} x^{s+1} + \sum_{s=\ell}^{+\infty}\Big(1+\binom{s}{\ell}\Big)\,x^{s+1} \quad ;$$

therefore $1 + Q_t^\ell(1_*) = c_{\ell+t} = 1 + \binom{\ell+t}{\ell}$, whence $Q_t^\ell(1_*) = \binom{\ell+t}{\ell}$. As an alternative approach, one can prove that $Q_t^\ell(1_*) = \binom{\ell+t}{\ell}$ by induction using the recurrence formula $Q_t^\ell(\mathbf{x}_*) = Q_t^{\ell-1}(\mathbf{x}_*) + \sum_{s=1}^{t-1} Q_{t-s}^{\ell-1}(\mathbf{x}_*)\,\mathbf{x}_s + \mathbf{x}_t$ and the identity $\binom{\ell+t}{\ell} = \sum_{s=0}^t \binom{\ell+t-1}{\ell-1}$.

The outcome is $N = Q_t^\ell(1_*) = \binom{\ell+t}{\ell}$ (for all t,ℓ), thus $Q_t^\ell(\mathbf{a}_*) - \binom{\ell+t}{\ell}\mathbf{a}_t = Q_- + Q_+ - \binom{\ell+t}{\ell}\mathbf{a}_t = Q_- + N\,\mathbf{a}_t - \binom{\ell+t}{\ell}\mathbf{a}_t = Q_-$. Now, by definition $\tau(\boldsymbol{\alpha}_{j_r}) = j_r - 1$ and $\tau\big({\boldsymbol{\alpha}_1}^{j_r}\big) = j_r$. Therefore if $\boldsymbol{\alpha}_{(j_r)} \in \big\{\boldsymbol{\alpha}_{j_r}, {\boldsymbol{\alpha}_1}^{j_r}\big\}$ (for all $r = 1,\dots,s$) and $(\boldsymbol{\alpha}_{(j_1)},\boldsymbol{\alpha}_{(j_2)},\dots,\boldsymbol{\alpha}_{(j_s)}) \neq \big({\boldsymbol{\alpha}_1}^{j_1},{\boldsymbol{\alpha}_1}^{j_2},\dots,{\boldsymbol{\alpha}_1}^{j_s}\big)$, then $\tau\big(\boldsymbol{\alpha}_{(j_1)}\cdots\boldsymbol{\alpha}_{(j_s)}\big) \leq j_1 + \cdots + j_s - 1 = t-1$. Then by construction $\tau(Q_-) \leq t-1$, whence, since $Z_t^\ell(\boldsymbol{\alpha}_*) := Q_t^\ell(\mathbf{a}_*) - \binom{\ell+t}{\ell}\mathbf{a}_t = Q_-$, we get also $\tau\big(Z_t^\ell(\boldsymbol{\alpha}_*)\big) \leq t-1$, i.e. $Z_t^\ell(\boldsymbol{\alpha}_*) \in \varTheta_{t-1}$, so $Q_t^\ell(\mathbf{a}_*) = Z_t^\ell(\boldsymbol{\alpha}_*) + \binom{\ell+t}{t}{\boldsymbol{\alpha}_1}^t \in \varTheta_t \setminus \varTheta_{t-1}$. $\square$

Proposition 10.13. *$\underline{\Theta}$ is a Hopf algebra filtration of $\mathcal{H}$.*

Proof. By construction (cf. §10.11) $\underline{\Theta}$ is an algebra filtration; so to check it is *Hopf* too we are left only to show that $(\star)\ \Delta(\Theta_n) \subseteq \sum_{r+s=n} \Theta_r \otimes \Theta_s$ (for all $n \in \mathbb{N}$), for then $S(\Theta_n) \subseteq \Theta_n$ (for all n) will follow from that by recurrence (and Hopf algebra axioms).

By definition $\Theta_0 = \Bbbk \cdot 1_{\mathcal{H}}$; then $\Delta(1_{\mathcal{H}}) = 1_{\mathcal{H}} \otimes 1_{\mathcal{H}}$ proves $(\star)$ for $n=0$. For $n=1$ definitions tell that Θ_1 is nothing but the direct sum of Θ_0 with the (free) Lie (sub)algebra (of $\mathcal{H}$) generated by $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$. Since $\Delta(\boldsymbol{\alpha}_1) = \boldsymbol{\alpha}_1 \otimes 1 + 1 \otimes \boldsymbol{\alpha}_1$ and $\Delta(\boldsymbol{\alpha}_2) = \boldsymbol{\alpha}_2 \otimes 1 + 1 \otimes \boldsymbol{\alpha}_2$ (directly from definitions) and since

$$\Delta\big([x,y]\big) \;=\; \big[\Delta(x),\Delta(y)\big] \;=\; \textstyle\sum_{(x),(y)} \big([x_{(1)},y_{(1)}] \otimes x_{(2)}y_{(2)} + x_{(1)}y_{(1)} \otimes [x_{(2)},y_{(2)}]\big)$$

(for all $x, y \in \mathcal{H}$) we argue that $(\star)$ holds for $n=1$ too.

Further on, for every $n > 1$ we have (setting $Q_0^n(\mathbf{a}_*) = 1 = \mathbf{a}_0$ for short)

$$\begin{aligned}\Delta(\boldsymbol{\alpha}_n) \;=\; \Delta(\mathbf{a}_n) - \Delta\big({\mathbf{a}_1}^n\big) \;=\; \textstyle\sum_{k=0}^n \mathbf{a}_k \otimes Q_{n-k}^k(\mathbf{a}_*) - \sum_{k=0}^n \binom{n}{k} {\mathbf{a}_1}^k \otimes {\mathbf{a}_1}^{n-k} \;=& \\ =\; \textstyle\sum_{k=2}^n \boldsymbol{\alpha}_k \otimes Q_{n-k}^k(\mathbf{a}_*) \;+\; \sum_{k=0}^{n-1} {\boldsymbol{\alpha}_1}^k \otimes Z_{n-k}^k(\boldsymbol{\alpha}_*)&\end{aligned}$$

hence $\Delta(\boldsymbol{\alpha}_n) \in \sum_{r+s=n-1} \Theta_r \otimes \Theta_s$ due to Lemma 10.12 (and to $\boldsymbol{\alpha}_m \in \Theta_{m-1}$ for $m > 1$).

Finally, as $\Delta\big([x,y]\big) = \big[\Delta(x),\Delta(y)\big] = \sum_{(x),(y)} \big([x_{(1)},y_{(1)}] \otimes x_{(2)}y_{(2)} + x_{(1)}y_{(1)} \otimes [x_{(2)},y_{(2)}]\big)$ and similarly $\Delta(x\,y) = \Delta(x)\Delta(y) = \sum_{(x),(y)} x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}$ (for $x, y \in \mathcal{H}$), we have that Δ does not increase $(\partial_- - d)$: as Θ is exactly the (algebra) filtration induced by $(\partial_- - d)$, it is a Hopf algebra filtration as well. $\square$

Lemma 10.14. *(notation of §10.11)*

(a) $\kappa(a) \leq \partial(a)$ *for every* $a \in \mathcal{H} \setminus \{0\}$ *which is* $\partial(a)$*–homogeneous.*

(b) $\kappa(a\,a') \leq \kappa(a) + \kappa(a')$ *and* $\kappa\big([a,a']\big) < \kappa(a) + \kappa(a')$ *for all* $a, a' \in \mathcal{H} \setminus \{0\}$.

(c) $\kappa(\boldsymbol{\alpha}_n) = \partial_-(\boldsymbol{\alpha}_n) = \tau(\boldsymbol{\alpha}_n)$ *for all* $n \in \mathbb{N}_\nu$.

(d) $\kappa\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]\big) = \partial_-(\boldsymbol{\alpha}_r) + \partial_-(\boldsymbol{\alpha}_s) - 1 = \tau\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]\big)$ *for all* $r,\ s \in \mathbb{N}_\nu$ *with* $r \neq s$.

(e) $\kappa(\boldsymbol{\alpha}_b) = \partial_-(\boldsymbol{\alpha}_b) - d(\boldsymbol{\alpha}_b) + 1 = \tau(\boldsymbol{\alpha}_b)$ *for every* $b \in B_\nu$.

(f) $\kappa(\boldsymbol{\alpha}_{b_1}\boldsymbol{\alpha}_{b_2}\cdots\boldsymbol{\alpha}_{b_\ell}) = \tau(\boldsymbol{\alpha}_{b_1}\boldsymbol{\alpha}_{b_2}\cdots\boldsymbol{\alpha}_{b_\ell})$ *for all* $b_1, b_2, \dots, b_\ell \in B_\nu$.

(g) $\kappa\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big) = \kappa(\boldsymbol{\alpha}_{b_1}) + \kappa(\boldsymbol{\alpha}_{b_2}) - 1 = \tau\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big)$, *for all* $b_1, b_2 \in B_\nu$.

Proof. *(a)* Let $a \in \mathcal{H} \setminus \{0\}$ be $\partial(a)$–homogeneous. Since $\mathcal{H}$ is graded, we have $\partial\big(\delta_\ell(a)\big) = \partial(a)$ for all ℓ; moreover, $\delta_\ell(a) \in J^{\otimes \ell}$ (with $J := \mathit{Ker}(\epsilon_{\mathcal{H}})$) by definition, and $\partial(y) > 0$ for each ∂–homogeneous $y \in J \setminus \{0\}$. Then $\delta_\ell(a) = 0$ for all $\ell > \partial(a)$, whence the claim.

(b) This is just a reformulation of Lemma 3.4*(c)*.

(c) By part *(a)* we have $\kappa(\mathbf{a}_n) \leq \partial(\mathbf{a}_n) = n$. Moreover, by definition $\delta_2(\mathbf{a}_n) = \sum_{k=1}^{n-1} \mathbf{a}_k \otimes Q_{n-k}^k(\mathbf{a}_*)$, thus $\delta_n(\mathbf{a}_n) = (\delta_{n-1} \otimes \delta_1)\big(\delta_2(\mathbf{a}_n)\big) = \sum_{k=1}^{n-1} \delta_{n-1}(\mathbf{a}_k) \otimes \delta_1\big(Q_{n-k}^k(\mathbf{a}_*)\big)$ by coassociativity. Since $\delta_\ell(\mathbf{a}_m) = 0$ for $\ell > m$, $Q_1^{n-1}(\mathbf{a}_*) = n\,\mathbf{a}_1$ and $\delta_1(\mathbf{a}_1) = \mathbf{a}_1$, we

have $\delta_n(\mathbf{a}_n) = \delta_{n-1}(\mathbf{a}_{n-1}) \otimes (n\,\mathbf{a}_1)$, thus by induction $\delta_n(\mathbf{a}_n) = n!\,{\mathbf{a}_1}^{\otimes n}$ $(\neq 0)$, whence $\kappa(\mathbf{a}_n) = n$. But also $\delta_n({\mathbf{a}_1}^n) = n!\,{\mathbf{a}_1}^{\otimes n}$. Thus $\delta_n(\boldsymbol{\alpha}_n) = \delta_n(\mathbf{a}_n) - \delta_n({\mathbf{a}_1}^n) = 0$ for $n > 1$.

Clearly $\kappa(\boldsymbol{\alpha}_1) = 1$. For the general case, for all $\ell \geq 2$ we have

$$\delta_{\ell-1}(\mathbf{a}_\ell) \;=\; (\delta_{\ell-2} \otimes \delta_1)\big(\delta_2(\mathbf{a}_\ell)\big) \;=\; \sum\nolimits_{k=1}^{\ell-1} \delta_{\ell-2}(\mathbf{a}_k) \otimes \delta_1\big(Q^k_{\ell-1-k}(\mathbf{a}_*)\big)$$

which, thanks to the previous analysis, gives

$$\begin{aligned} \delta_{\ell-1}(\mathbf{a}_\ell) \;&=\; \delta_{\ell-2}(\mathbf{a}_{\ell-2}) \otimes \left((\ell-1)\,\mathbf{a}_2 + \binom{\ell-1}{2} {\mathbf{a}_1}^2 \right) \,+\, \delta_{\ell-2}(\mathbf{a}_{\ell-1}) \otimes \ell\,\mathbf{a}_1 \;= \\ &=\; (\ell-1)! \cdot {\mathbf{a}_1}^{\otimes(\ell-2)} \otimes \left(\mathbf{a}_2 + \frac{\ell-1}{2} \cdot {\mathbf{a}_1}^2 \right) \,+\, \ell \cdot \delta_{\ell-2}(\mathbf{a}_{\ell-1}) \otimes \mathbf{a}_1 \,. \end{aligned}$$

Iterating we get, for all $\ell \geq 2$ (with $\binom{-1}{2} := 0$, and changing indices)

$$\delta_{\ell-1}(\mathbf{a}_\ell) \;=\; \sum\nolimits_{m=1}^{\ell-1} \frac{\ell\,!}{m+1} \cdot {\mathbf{a}_1}^{\otimes(m-1)} \otimes \left(\mathbf{a}_2 + \frac{m-1}{2} \cdot {\mathbf{a}_1}^2 \right) \otimes {\mathbf{a}_1}^{\otimes(\ell-1-m)} \,.$$

On the other hand, we have also

$$\delta_{\ell-1}\big({\mathbf{a}_1}^\ell\big) \;=\; \sum\nolimits_{m=1}^{\ell-1} \frac{\ell\,!}{2} \cdot {\mathbf{a}_1}^{\otimes(m-1)} \otimes {\mathbf{a}_1}^2 \otimes {\mathbf{a}_1}^{\otimes(\ell-1-m)} \,.$$

Therefore, for $\delta_{n-1}(\boldsymbol{\alpha}_n) = \delta_{n-1}(\mathbf{a}_n) - \delta_{n-1}({\mathbf{a}_1}^n)$ (for all $n \in \mathbb{N}_\nu$, $n \geq 2$) the outcome is

$$\begin{aligned} \delta_{n-1}(\boldsymbol{\alpha}_n) \;&=\; \sum\nolimits_{m=1}^{n-1} \frac{n!}{m+1} \cdot {\mathbf{a}_1}^{\otimes(m-1)} \otimes \big(\mathbf{a}_2 - {\mathbf{a}_1}^2\big) \otimes {\mathbf{a}_1}^{\otimes(n-1-m)} \;= \\ &=\; \sum\nolimits_{m=1}^{n-1} \frac{n!}{m+1} \cdot {\boldsymbol{\alpha}_1}^{\otimes(m-1)} \otimes \boldsymbol{\alpha}_2 \otimes {\boldsymbol{\alpha}_1}^{\otimes(n-1-m)} \quad ; \end{aligned} \tag{10.5}$$

in particular $\delta_{n-1}(\boldsymbol{\alpha}_n) \neq 0$, whence $\boldsymbol{\alpha}_n \notin D_{n-2}$ and so $\kappa(\boldsymbol{\alpha}_n) = n-1$, q.e.d.

(d) Let $r \neq 1 \neq s$. From *(b)–(c)* we get $\kappa\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]\big) < \kappa(\boldsymbol{\alpha}_r) + \kappa(\boldsymbol{\alpha}_s) = r+s-2$. In addition, we prove now that $\delta_{r+s-3}\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]\big) \neq 0$, which yields *(d)*. Lemma 3.4*(b)* gives

$$\delta_{r+s-3}\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]\big) \;=\; \sum_{\substack{\Lambda \cup Y = \{1,\dots,r+s-3\} \\ \Lambda \cap Y \neq \emptyset}} \big[\delta_\Lambda(\boldsymbol{\alpha}_r), \delta_Y(\boldsymbol{\alpha}_s)\big] \;=\; \sum_{\substack{\Lambda \cup Y = \{1,\dots,r+s-3\} \\ \Lambda \cap Y \neq \emptyset,\, |\Lambda| = r-1,\, |Y| = s-1}} \big[j_\Lambda\big(\delta_{r-1}(\boldsymbol{\alpha}_r)\big), j_Y\big(\delta_{s-1}(\boldsymbol{\alpha}_s)\big)\big] \,.$$

Using (10.5) in the form $\delta_{\ell-1}(\mathbf{a}_\ell) = \sum_{m=1}^{\ell-1} \frac{\ell\,!}{2} \cdot \boldsymbol{\alpha}_2 \otimes {\boldsymbol{\alpha}_1}^{\otimes(\ell-2)} + \boldsymbol{\alpha}_1 \otimes \eta_\ell$ (for some $\eta_\ell \in \mathcal{H}$), and counting how many Λ's and Y's exist with $1 \in \Lambda$ and $\{1,2\} \subseteq Y$, and — conversely — how many of them exist with $\{1,2\} \subseteq \Lambda$ and $1 \in Y$, we argue

$$\delta_{r+s-3}\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]\big) \;=\; c_{r,s} \cdot [\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes \boldsymbol{\alpha}_2 \otimes {\boldsymbol{\alpha}_1}^{\otimes(r+s-5)} + \boldsymbol{\alpha}_1 \otimes \varphi_1 + \boldsymbol{\alpha}_2 \otimes \varphi_2 + [\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes \boldsymbol{\alpha}_1 \otimes \psi$$

for some $\varphi_1, \varphi_2 \in \mathcal{H}^{\otimes(r+s-4)}$, $\psi \in \mathcal{H}^{\otimes(r+s-5)}$, and with

$$c_{r,s} = \frac{r!}{2} \cdot \frac{s!}{3} \cdot \binom{r+s-5}{r-2} - \frac{s!}{2} \cdot \frac{r!}{3} \cdot \binom{s+r-5}{s-2} = \frac{2}{3}\binom{r}{2}\binom{s}{2}(s-r)(r+s-5)! \neq 0 .$$

In particular $\delta_{r+s-3}\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]\big) = c_{r,s} \cdot [\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes \boldsymbol{\alpha}_2 \otimes {\boldsymbol{\alpha}_1}^{\otimes(r+s-5)} + \textit{l.i.t.}$, where "*l.i.t.*" stands for some further *terms* which are *linearly independent* of $[\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes \boldsymbol{\alpha}_2 \otimes {\boldsymbol{\alpha}_1}^{\otimes(r+s-5)}$ and $c_{r,s} \neq 0$. Then $\delta_{r+s-3}\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_s]\big) \neq 0$, q.e.d.

Finally, if $r > 1 = s$ (and similarly if $r = 1 < s$) things are simpler. Indeed, again *(b)* and *(c)* together give $\kappa\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_1]\big) < \kappa(\boldsymbol{\alpha}_r) + \kappa(\boldsymbol{\alpha}_1) = (r-1) + 1 = r$, and we prove that $\delta_{r-1}\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_1]\big) \neq 0$. Like before, Lemma 3.4*(b)* gives (since $\delta_1(\boldsymbol{\alpha}_1) = \boldsymbol{\alpha}_1$)

$$\delta_{r-1}\big([\boldsymbol{\alpha}_r, \boldsymbol{\alpha}_1]\big) = \sum_{\substack{\Lambda \cup Y = \{1,2,\dots,r-1\} \\ \Lambda \cap Y \neq \emptyset,\, |\Lambda| = r-1,\, |Y| = 1}} \big[\delta_\Lambda(\boldsymbol{\alpha}_r), \delta_Y(\boldsymbol{\alpha}_1)\big] = \sum_{k=1}^{r-1} \Big[\delta_{r-1}(\boldsymbol{\alpha}_r), 1^{\otimes(k-1)} \otimes \boldsymbol{\alpha}_1 \otimes 1^{\otimes(r-1-k)}\Big] =$$

$$= \sum_{m=1}^{r-1} \frac{r!}{m+1} \cdot {\boldsymbol{\alpha}_1}^{\otimes(m-1)} \otimes [\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1] \otimes {\boldsymbol{\alpha}_1}^{\otimes(n-1-m)} \neq 0 , \quad \text{q.e.d.}$$

(e) We perform induction upon $d(b)$: the cases $d(b) = 0$ and $d(b) = 1$ are dealt with in parts *(c)* and *(d)* of the claim, thus we can assume $d(b) \geq 2$, so that $b = \big[b', x_\ell\big]$ for some $\ell \in \mathbb{N}_\nu$ and some other $b' \in B_\nu$ with $d(b') = d(b) - 1$; then $\tau(\boldsymbol{\alpha}_b) = \tau\big([\boldsymbol{\alpha}_{b'}, \boldsymbol{\alpha}_\ell]\big) = \tau(\boldsymbol{\alpha}_{b'}) + \tau(\boldsymbol{\alpha}_\ell) - 1$, directly from definitions. Moreover $\tau(\boldsymbol{\alpha}_\ell) = \kappa(\boldsymbol{\alpha}_\ell)$ by part *(c)*, and $\tau(\boldsymbol{\alpha}_{b'}) = \kappa(\boldsymbol{\alpha}_{b'})$ by inductive assumption.

From *(b)* we have $\kappa(\boldsymbol{\alpha}_b) = \kappa\big([\boldsymbol{\alpha}_{b'}, \boldsymbol{\alpha}_\ell]\big) \leq \kappa(\boldsymbol{\alpha}_{b'}) + \kappa(\boldsymbol{\alpha}_\ell) - 1 = \tau(\boldsymbol{\alpha}_{b'}) + \tau(\boldsymbol{\alpha}_\ell) - 1 = \tau(\boldsymbol{\alpha}_b)$, i. e. $\kappa(\boldsymbol{\alpha}_b) \leq \tau(\boldsymbol{\alpha}_b)$; we must prove the converse, for which it is enough to show

$$\delta_{\tau(\boldsymbol{\alpha}_b)}(\boldsymbol{\alpha}_b) = c_b \cdot [\cdots [\underbrace{[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2], \boldsymbol{\alpha}_2], \dots, \boldsymbol{\alpha}_2}_{d(b)+1}] \otimes \boldsymbol{\alpha}_2 \otimes {\boldsymbol{\alpha}_1}^{\otimes(\tau(\boldsymbol{\alpha}_b)-2)} + \textit{l.i.t.} \tag{10.6}$$

for some $c_b \in \Bbbk \setminus \{0\}$, where "*l.i.t.*" means the same as before.

Since $\tau(\boldsymbol{\alpha}_b) = \tau\big([\boldsymbol{\alpha}_{b'}, \boldsymbol{\alpha}_\ell]\big) = \tau(\boldsymbol{\alpha}_{b'}) + \ell - 2$, computation via Lemma 3.4*(b)* gives

$$\delta_{\tau(\boldsymbol{\alpha}_b)}(\boldsymbol{\alpha}_b) = \delta_{\tau(\boldsymbol{\alpha}_b)}\big([\boldsymbol{\alpha}_{b'}, \boldsymbol{\alpha}_\ell]\big) = \sum_{\substack{\Lambda \cup Y = \{1,\dots,\tau(\boldsymbol{\alpha}_b)\} \\ \Lambda \cap Y \neq \emptyset}} \big[\delta_\Lambda(\boldsymbol{\alpha}_{b'}), \delta_Y(\boldsymbol{\alpha}_\ell)\big] =$$

$$= \sum_{\substack{\Lambda \cup Y = \{1,\dots,\tau(\boldsymbol{\alpha}_b)\},\ \Lambda \cap Y \neq \emptyset \\ |\Lambda| = \tau(\boldsymbol{\alpha}_{b'}),\ |Y| = \ell - 1}} \Big[j_\Lambda\big(\delta_{\tau(\boldsymbol{\alpha}_{b'})}(\boldsymbol{\alpha}_{b'})\big), j_Y\big(\delta_{\ell-1}(\boldsymbol{\alpha}_\ell)\big)\Big] =$$

$$= \sum_{\substack{\Lambda \cup Y = \{1,\dots,\tau(\boldsymbol{\alpha}_b)\},\ \Lambda \cap Y \neq \emptyset \\ |\Lambda| = \tau(\boldsymbol{\alpha}_{b'}),\ \ |Y| = \ell - 1}} \Big[j_\Lambda\big(c_{b'} [\cdots \underbrace{[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2], \dots, \boldsymbol{\alpha}_2}_{d(b')+1}] \otimes \boldsymbol{\alpha}_2 \otimes {\boldsymbol{\alpha}_1}^{\otimes(\tau(\boldsymbol{\alpha}_{b'})-2)}\big),\, j_Y\big(\tfrac{\ell\,!}{2}\, \boldsymbol{\alpha}_2 \otimes {\boldsymbol{\alpha}_1}^{\otimes(\ell-2)}\big)\Big] + \textit{l.i.t.} =$$

$$= c_{b'} \cdot \frac{\ell\,!}{2} \cdot \binom{\tau(\boldsymbol{\alpha}_b) - 2}{\ell - 2} \cdot [\,[\cdots [\underbrace{[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2], \boldsymbol{\alpha}_2], \dots, \boldsymbol{\alpha}_2], \boldsymbol{\alpha}_2}_{d(b')+1+1 = d(b)+1}\,] \otimes \boldsymbol{\alpha}_2 \otimes {\boldsymbol{\alpha}_1}^{\otimes(\tau(\boldsymbol{\alpha}_b)-2)} + \textit{l.i.t.}$$

(using induction about $\boldsymbol{\alpha}_{b'}$); this proves (10.6) with $c_b = c_{b'} \cdot \frac{\ell\,!}{2} \cdot \binom{\tau(\boldsymbol{\alpha}_b)-2}{\ell-2} \neq 0$.

Thus (10.6) holds, yielding $\delta_{\tau(\boldsymbol{\alpha}_b)}(\boldsymbol{\alpha}_b) \neq 0$, hence $\kappa(\boldsymbol{\alpha}_b) \geq \tau(\boldsymbol{\alpha}_b)$, q.e.d.

(f) The case $\ell = 1$ is proved by part *(e)*, so we can assume $\ell > 1$. By part *(b)* and the case $\ell = 1$ we have $\kappa\left(\boldsymbol{\alpha}_{b_1}\boldsymbol{\alpha}_{b_2}\cdots\boldsymbol{\alpha}_{b_\ell}\right) \leq \sum_{i=1}^{\ell} \kappa\left(\boldsymbol{\alpha}_{b_i}\right) = \sum_{i=1}^{\ell} \tau(\boldsymbol{\alpha}_{b_i}) = \tau(\boldsymbol{\alpha}_{b_1}\boldsymbol{\alpha}_{b_2}\cdots\boldsymbol{\alpha}_{b_\ell})$; so we must only prove the converse inequality. We begin with $\ell = 2$ and $d(b_1) = d(b_2) = 0$, so $\boldsymbol{\alpha}_{b_1} = \boldsymbol{\alpha}_r$, $\boldsymbol{\alpha}_{b_2} = \boldsymbol{\alpha}_s$, for some $r, s \in \mathbb{N}_\nu$.

If $r = s = 1$ then $\kappa(\boldsymbol{\alpha}_r) = \kappa(\boldsymbol{\alpha}_s) = \kappa(\boldsymbol{\alpha}_1) = 1$, by part *(c)*. Then

$$\delta_2(\boldsymbol{\alpha}_1\,\boldsymbol{\alpha}_1) \;=\; \delta_2(\mathbf{a}_1\,\mathbf{a}_1) \;=\; (\mathrm{id}-\epsilon)^{\otimes 2}\Delta\big({\mathbf{a}_1}^2\big) \;=\; 2\cdot\mathbf{a}_1\otimes\mathbf{a}_1 \;=\; 2\cdot\boldsymbol{\alpha}_1\otimes\boldsymbol{\alpha}_1 \;\neq\; 0$$

so that $\kappa(\boldsymbol{\alpha}_1\,\boldsymbol{\alpha}_1) \geq 2 = \kappa(\boldsymbol{\alpha}_1) + \kappa(\boldsymbol{\alpha}_1)$, hence $\kappa(\boldsymbol{\alpha}_1\,\boldsymbol{\alpha}_1) = \kappa(\boldsymbol{\alpha}_1) + \kappa(\boldsymbol{\alpha}_1)$, q.e.d.

If $r > 1 = s$ (and similarly if $r = 1 < s$) then $\kappa(\boldsymbol{\alpha}_r) = r-1$, $\kappa(\boldsymbol{\alpha}_s) = \kappa(\boldsymbol{\alpha}_1) = 1$, by part *(c)*. Then Lemma 3.4*(b)* gives

$$\begin{aligned}
\delta_r(\boldsymbol{\alpha}_r\,\boldsymbol{\alpha}_1) \;&=\; \sum_{\substack{\Lambda\cup Y=\{1,\dots,r\}\\ |\Lambda|=r-1\,,\ |Y|=1}} \delta_\Lambda(\boldsymbol{\alpha}_r)\,\delta_Y(\boldsymbol{\alpha}_1) \;= \\
&= \sum_{m=1}^{r}\sum_{k<m} \frac{r!}{m+1}\cdot\Big({\boldsymbol{\alpha}_1}^{\otimes(k-1)}\otimes 1\otimes{\boldsymbol{\alpha}_1}^{\otimes(m-1-k)}\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(r-1-m)}\Big)\times \\
&\qquad\qquad \times\Big(1^{\otimes(k-1)}\otimes\boldsymbol{\alpha}_1\otimes 1^{\otimes(r-k)}\Big) \;+ \\
&+ \sum_{m=1}^{r}\sum_{k>m} \frac{r!}{m+1}\cdot\Big({\boldsymbol{\alpha}_1}^{\otimes(m-1)}\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(k-1-m)}\otimes 1\otimes{\boldsymbol{\alpha}_1}^{\otimes(r-1-k)}\Big)\times \\
&\qquad\qquad \times\Big(1^{\otimes(k-1)}\otimes\boldsymbol{\alpha}_1\otimes 1^{\otimes(r-k)}\Big) \;= \\
&= \sum\nolimits_{m=1}^{r} \frac{r!}{m+1}\cdot{\boldsymbol{\alpha}_1}^{\otimes(m-1)}\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(r-1-m)} \;\neq\; 0
\end{aligned}$$

so that $\kappa(\boldsymbol{\alpha}_r\,\boldsymbol{\alpha}_1) \geq r = \kappa(\boldsymbol{\alpha}_r) + \kappa(\boldsymbol{\alpha}_1)$ and so $\kappa(\boldsymbol{\alpha}_r\,\boldsymbol{\alpha}_1) = \kappa(\boldsymbol{\alpha}_r) + \kappa(\boldsymbol{\alpha}_1)$, q.e.d.

Finally let $r, s > 1$ (and $r \neq s$). Then $\kappa(\boldsymbol{\alpha}_r) = r-1$, $\kappa(\boldsymbol{\alpha}_s) = s-1$, by part *(c)*; then Lemma 3.4*(b)* gives

$$\delta_{r+s-2}\big(\boldsymbol{\alpha}_r\,\boldsymbol{\alpha}_s\big) \;=\; \sum_{\substack{\Lambda\cup Y=\{1,\dots,r+s-2\}\\ |\Lambda|=r-1\,,\ |Y|=s-1}} \delta_\Lambda(\boldsymbol{\alpha}_r)\cdot\delta_Y(\boldsymbol{\alpha}_s) \;=\; \sum_{\substack{\Lambda\cup Y=\{1,\dots,r+s-2\}\\ |\Lambda|=r-1\,,\ |Y|=s-1}} j_\Lambda\big(\delta_{r-1}(\boldsymbol{\alpha}_r)\big)\cdot j_Y\big(\delta_{s-1}(\boldsymbol{\alpha}_s)\big)\,.$$

Using (10.5) in the form $\delta_{t-1}(\mathbf{a}_t) = \sum_{m=1}^{t-1} \frac{t\,!}{2}\cdot\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(t-2)} + \boldsymbol{\alpha}_1\otimes\eta_t$ (for some $\eta_t \in \mathcal{H}$ and $t \in \{r,s\}$) and counting how many Λ's and Y's exist with $1 \in \Lambda$ and $2 \in Y$ and viceversa — actually, it is a matter of counting $(r-2, s-2)$-shuffles — we argue

$$\delta_{r+s-2}\big(\boldsymbol{\alpha}_r\,\boldsymbol{\alpha}_s\big) \;=\; e_{r,s}\cdot\boldsymbol{\alpha}_2\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(r+s-4)} \;+\; \boldsymbol{\alpha}_1\otimes\varphi$$

for some $\varphi \in \mathcal{H}^{\otimes(r+s-3)}$ and with

$$e_{r,s} \;=\; \frac{r!}{2}\cdot\frac{s!}{2}\cdot\left(\binom{r+s-4}{r-2} + \binom{s+r-4}{s-2}\right) \;=\; \frac{r!\,s!}{2}\cdot\binom{r+s-4}{r-2} \;\neq\; 0\,.$$

In particular $\delta_{r+s-2}\big(\boldsymbol{\alpha}_r\,\boldsymbol{\alpha}_s\big) \,=\, e_{r,s}\cdot \boldsymbol{\alpha}_2\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(r+s-4)} \,+\, \textit{l.i.t.}$, where "*l.i.t.*" stands again for some further *terms* which are *linearly independent* of $\boldsymbol{\alpha}_2\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(r+s-4)}$ and $e_{r,s}\neq 0$. Then $\delta_{r+s-2}\big(\boldsymbol{\alpha}_r\,\boldsymbol{\alpha}_s\big)\neq 0$, so $\kappa(\boldsymbol{\alpha}_r\,\boldsymbol{\alpha}_1)\geq r+s-2=\kappa(\boldsymbol{\alpha}_r)+\kappa(\boldsymbol{\alpha}_1)$, q.e.d.

Now let again $\ell=2$ but $d(b_1),d(b_2)>0$. Set $\kappa_i:=\kappa(\boldsymbol{\alpha}_{b_i})$ for $i=1,2$. Applying (10.6) to $b=b_1$ and $b=b_2$ (and reminding $\tau\equiv\kappa$) gives

$$\delta_{\kappa_1+\kappa_2}(\boldsymbol{\alpha}_{b_1}\,\boldsymbol{\alpha}_{b_2}) \;=\; \sum_{\Lambda\cup Y=\{1,\dots,\kappa_1+\kappa_2\}} \delta_\Lambda(\boldsymbol{\alpha}_{b_1})\,\delta_Y(\boldsymbol{\alpha}_{b_2}) \;=\; \sum_{\substack{\Lambda\cup Y=\{1,\dots,\kappa_1+\kappa_2\}\\ |\Lambda|=\kappa_1,\,|Y|=\kappa_2}} j_\Lambda\big(\delta_{\kappa_1}(\boldsymbol{\alpha}_{b_1})\big)\,j_Y\big(\delta_{\kappa_2}(\boldsymbol{\alpha}_{b_2})\big) \;=$$

$$= \sum_{\substack{\Lambda\cup Y=\{1,\dots,\kappa_1+\kappa_2\}\\ |\Lambda|=\kappa_1,\,|Y|=\kappa_2}} j_\Lambda\Big(c_{b_1}\cdot[\cdots[\underbrace{[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2],\boldsymbol{\alpha}_2],\dots,\boldsymbol{\alpha}_2}_{d(b_1)+1}]\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(\kappa_1-2)} \,+\, \textit{l.i.t.}\Big)\times$$

$$\times\, j_Y\Big(c_{b_2}\cdot[\cdots[\underbrace{[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2],\boldsymbol{\alpha}_2],\dots,\boldsymbol{\alpha}_2}_{d(b_2)+1}]\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(\kappa_2-2)} \,+\, \textit{l.i.t.}\Big) \;=$$

$$=\; c_{b_1}\,c_{b_2}\cdot 2\binom{\kappa_1+\kappa_2-4}{\kappa_1-2}\times$$

$$\times\,[\cdots[\underbrace{[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2],\boldsymbol{\alpha}_2],\dots,\boldsymbol{\alpha}_2}_{d(b_1)+1}]\otimes[\cdots[\underbrace{[\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1],\boldsymbol{\alpha}_2],\dots,\boldsymbol{\alpha}_2}_{d(b_2)+1}]\otimes\boldsymbol{\alpha}_2\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(\kappa_1+\kappa_2-4)} + \textit{l.i.t.}$$

which proves the claim for $\ell=2$. In addition, we can take this last result as the basis of induction (on ℓ) to prove the following: for all $\underline{b}:=(b_1,\dots,b_\ell)\in {B_\nu}^\ell$, one has

$$\delta_{|\underline{\kappa}|}\bigg(\prod_{i=1}^{\ell}\boldsymbol{\alpha}_{b_i}\bigg) = c_{\underline{b}}\,\bigg(\bigotimes_{i=1}^{\ell}[\cdots[\underbrace{[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2],\boldsymbol{\alpha}_2],\dots,\boldsymbol{\alpha}_2}_{d(b_i)+1}]\bigg)\otimes{\boldsymbol{\alpha}_2}^{\otimes\ell}\otimes{\boldsymbol{\alpha}_1}^{\otimes(|\underline{\kappa}|-2\,\ell)} + \textit{l.i.t.} \qquad (10.7)$$

for some $c_{\underline{b}}\in\Bbbk\setminus\{0\}$, with $|\underline{\kappa}|:=\sum_{i=1}^{\ell}\kappa_i$ and $\kappa_i:=\kappa(\boldsymbol{\alpha}_{b_i})$ ($i=1,\dots,\ell$). The induction step, from ℓ to $(\ell+1)$, amounts to compute (with $\kappa_{\ell+1}:=\kappa(\boldsymbol{\alpha}_{b_{\ell+1}})$)

$$\delta_{|\underline{\kappa}|+\kappa_{\ell+1}}(\boldsymbol{\alpha}_{b_1}\cdots\boldsymbol{\alpha}_{b_\ell}\cdot\boldsymbol{\alpha}_{b_{\ell+1}}) \;=\; \sum_{\Lambda\cup Y=\{1,\dots,|\underline{\kappa}|+\kappa_{\ell+1}\}} \delta_\Lambda(\boldsymbol{\alpha}_{b_1}\cdots\boldsymbol{\alpha}_{b_\ell})\,\delta_Y(\boldsymbol{\alpha}_{b_{\ell+1}}) \;=$$

$$=\; \sum_{\substack{\Lambda\cup Y=\{1,\dots,|\underline{\kappa}|+\kappa_{\ell+1}\}\\ |\Lambda|=|\underline{\kappa}|,\,|Y|=\kappa_{\ell+1}}} j_\Lambda\big(\delta_{|\underline{\kappa}|}(\boldsymbol{\alpha}_{b_1}\cdots\boldsymbol{\alpha}_{b_\ell})\big)\cdot j_Y\big(\delta_{\kappa_{\ell+1}}(\boldsymbol{\alpha}_{b_{\ell+1}})\big) \;=$$

$$= \sum_{\substack{\Lambda\cup Y=\{1,\dots,|\underline{\kappa}|+\kappa_{\ell+1}\}\\ |\Lambda|=|\underline{\kappa}|,\,|Y|=\kappa_{\ell+1}}} j_\Lambda\Bigg(c_{\underline{b}}\cdot\bigg(\bigotimes_{i=1}^{\ell}[\cdots[\underbrace{[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2],\boldsymbol{\alpha}_2],\dots,\boldsymbol{\alpha}_2}_{d(b_i)+1}]\bigg)\otimes{\boldsymbol{\alpha}_2}^{\otimes\ell}\otimes{\boldsymbol{\alpha}_1}^{\otimes(|\underline{\kappa}|-2\,\ell)} + \textit{l.i.t.}\Bigg)\times$$

$$\times\, j_Y\Big(c_{b_{\ell+1}}\cdot[\cdots[\underbrace{[\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2],\boldsymbol{\alpha}_2],\dots,\boldsymbol{\alpha}_2}_{d(b_{\ell+1})+1}]\otimes\boldsymbol{\alpha}_2\otimes{\boldsymbol{\alpha}_1}^{\otimes(\kappa_{\ell+1}-2)} \,+\, \textit{l.i.t.}\Big) \;=$$

$$= \; c_{\underline{b}}\, c_{b_{\ell+1}} \cdot (\ell+1) \binom{|\underline{\kappa}| + \kappa_{\ell+1} - 2\,(\ell+1)}{|\underline{\kappa}| - 2\,\ell} \cdot \Bigg(\bigotimes_{i=1}^{\ell} [\cdots[\underbrace{[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2], \boldsymbol{\alpha}_2], \ldots, \boldsymbol{\alpha}_2}_{d(b_i)+1}] \Bigg) \otimes$$

$$\otimes\, [\cdots[\underbrace{[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2], \boldsymbol{\alpha}_2], \ldots, \boldsymbol{\alpha}_2}_{d(b_{\ell+1})+1}] \otimes \boldsymbol{\alpha}_2{}^{\otimes(\ell+1)} \otimes \boldsymbol{\alpha}_1{}^{\otimes(|\underline{\kappa}|+\kappa_{\ell+1}-2\,(\ell+1))} \; + \; \textit{l.i.t.}$$

which proves (10.7) for $(\underline{b}\,, b_{\ell+1})$ with $c_{(\underline{b}\,, b_{\ell+1})} = c_{\underline{b}}\, c_{b_{\ell+1}} \cdot (\ell+1) \binom{|\underline{\kappa}|+\kappa_{\ell+1}-2\,(\ell+1)}{|\underline{\kappa}|-2\,\ell} \neq 0$. Finally, (10.7) yields $\delta_{|\underline{\kappa}|}(\boldsymbol{\alpha}_{b_1} \cdots \boldsymbol{\alpha}_{b_\ell}) \neq 0$, so $\kappa(\boldsymbol{\alpha}_{b_1} \cdots \boldsymbol{\alpha}_{b_\ell}) \geq \kappa(\boldsymbol{\alpha}_{b_1}) + \cdots + \kappa(\boldsymbol{\alpha}_{b_\ell})$, q.e.d.

(g) Part *(d)* proves the claim for $d(b_1) = d(b_2) = 0$, that is $b_1, b_2 \in \{x_n\}_{n \in \mathbb{N}}$. Moreover, when $b_2 = x_n \in \{x_m\}_{m \in \mathbb{N}_\nu}$ we can replicate the proof of part *(d)* to show that $\kappa\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big) = \kappa\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_n]\big) = \partial_-\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_n]\big) - d\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_n]\big)$: but the latter is exactly $\tau\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big)$, q.e.d. Everything is similar if $b_1 = x_n \in \{x_m\}_{m \in \mathbb{N}_\nu}$.

Now let $b_1, b_2 \in B_\nu \setminus \{x_n\}_{n \in \mathbb{N}_\nu}$. Then *(b)* gives $\kappa\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big) \leq \kappa(\boldsymbol{\alpha}_{b_1}) + \kappa(\boldsymbol{\alpha}_{b_2}) - 1 = \tau\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big)$. Applying (10.6) to $b = b_1$ and $b = b_2$ we get, for $\kappa_i := \kappa(\boldsymbol{\alpha}_{b_i})$ $(i = 1, 2)$

$$\delta_{\kappa_1+\kappa_2-1}\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big) \;=\; \sum_{\substack{\Lambda \cup Y = \{1,\ldots,\kappa_1+\kappa_2-1\} \\ \Lambda \cap Y \neq \emptyset}} \big[\delta_\Lambda(\boldsymbol{\alpha}_{b_1}), \delta_Y(\boldsymbol{\alpha}_{b_2})\big] \;=$$

$$= \sum_{\substack{\Lambda \cup Y = \{1,\ldots,\kappa_1+\kappa_2-1\} \\ |\Lambda| = \kappa_1,\, |Y| = \kappa_2}} \Big[j_\Lambda\big(\delta_{\kappa_1}(\boldsymbol{\alpha}_{b_1})\big), j_Y\big(\delta_{\kappa_2}(\boldsymbol{\alpha}_{b_2})\big)\Big] \;=$$

$$= \sum_{\substack{\Lambda \cup Y = \{1,\ldots,\kappa_1+\kappa_2\} \\ |\Lambda| = \kappa_1,\, |Y| = \kappa_2}} \Big[j_\Lambda\Big(c_{b_1} \cdot [\cdots[\underbrace{[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2], \boldsymbol{\alpha}_2], \ldots, \boldsymbol{\alpha}_2}_{d(b_1)+1}] \otimes \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1{}^{\otimes(\kappa_1-2)} \; + \; \textit{l.i.t.} \Big) \times$$

$$\times\; j_Y\Big(c_{b_2} \cdot [\cdots[\underbrace{[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2], \boldsymbol{\alpha}_2], \ldots, \boldsymbol{\alpha}_2}_{d(b_2)+1}] \otimes \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1{}^{\otimes(\kappa_2-2)} \; + \; \textit{l.i.t.} \Big)\Big] \;=$$

$$= \; c_{b_1}\, c_{b_2} \cdot 2 \binom{\kappa_1 + \kappa_2 - 4}{\kappa_1 - 2} \times$$

$$\times \Big[[\cdots[\underbrace{[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2], \boldsymbol{\alpha}_2], \ldots, \boldsymbol{\alpha}_2}_{d(b_1)+1}], [\cdots[\underbrace{[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2], \boldsymbol{\alpha}_2], \ldots, \boldsymbol{\alpha}_2}_{d(b_2)+1}]\Big] \otimes \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_1{}^{\otimes(\kappa_1+\kappa_2-4)} + \textit{l.i.t.}$$

(note that $d(b_i) \geq 1$ because $b_i \notin \big\{ x_n \,\big|\, n \in \mathbb{N}_\nu \big\}$ for $i = 1, 2$). In particular this means $\delta_{\kappa_1+\kappa_2-1}\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big) \neq 0$, thus $\kappa\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big) \geq \kappa(\boldsymbol{\alpha}_{b_1}) + \kappa(\boldsymbol{\alpha}_{b_2}) - 1 = \tau\big([\boldsymbol{\alpha}_{b_1}, \boldsymbol{\alpha}_{b_2}]\big)$. $\square$

Lemma 10.15. *Let V be a $\Bbbk$–vector space, and $\psi \in Hom_{\Bbbk}\big(V, V \wedge V\big)$. Let $\mathcal{L}(V)$ be the free Lie algebra over V, and $\psi_{d\mathcal{L}} \in Hom_{\Bbbk}\big(\mathcal{L}(V), \mathcal{L}(V) \wedge \mathcal{L}(V)\big)$ the unique extension of ψ from V to $\mathcal{L}(V)$ by derivations, i.e. such that $\psi_{d\mathcal{L}}\big|_V = \psi$ and $\psi_{d\mathcal{L}}\big([x,y]\big) = \big[x \otimes 1 + 1 \otimes x, \psi_{d\mathcal{L}}(y)\big] + \big[\psi_{d\mathcal{L}}(x),\, y \otimes 1 + 1 \otimes y\,\big] = x.\psi_{d\mathcal{L}}(y) - y.\psi_{d\mathcal{L}}(x)$ in the $\mathcal{L}(V)$–module $\mathcal{L}(V) \wedge \mathcal{L}(V)$, $\forall\, x, y \in \mathcal{L}(V)$. Let $K := Ker(\psi)$: then $Ker\big(\psi_{d\mathcal{L}}\big) = \mathcal{L}(K)$, the free Lie algebra over K.*

Proof. For each $z \in \mathcal{L}(V)$ set $z^\otimes := z \otimes 1 + 1 \otimes z$. Let I be a complement of K inside V, so that $V = K \oplus I$, and $\psi\big|_I$ is injective while $\psi\big|_K = 0_K$. Let B_K and B_I be bases of K and I respectively; then there is a basis of $\mathcal{L}(V)$ made of Lie monomials of the form $x_{\underline{i}} := \big[\big[\cdots\big[\cdots\big[\big[x_{i_1}, x_{i_2}\big], x_{i_3}\big]\ldots, x_{i_s}\big]\ldots, x_{i_{k-1}}\big], x_{i_k}\big]$ (with $\underline{i} = (i_1, i_2, i_3, \ldots, i_s, \ldots, i_{k-1}, i_k)$) for some $x_{i_r} \in B_K \cup B_I$: for these Lie monomials definitions yield $\psi_{d\mathcal{L}}(x_{\underline{i}}) = \sum_{x_{i_s} \in B_I} \big[\big[\cdots\big[\cdots\big[\big[x_{i_1}^\otimes, x_{i_2}^\otimes\big], x_{i_3}^\otimes\big]\ldots, \psi(x_{i_s})\big]\ldots, x_{i_{k-1}}^\otimes\big], x_{i_k}^\otimes\big]$. In addition, since $\psi\big|_I$ is injective the set $\big\{\, z^\otimes \,\big\}_{z \in B_I \cup B_K} \cup \big\{\psi(b)\,\big\}_{b \in B_I}$ is linearly independent. Then the set of all Lie monomials $y_{\underline{i}} := \big[\big[\cdots\big[\cdots\big[\big[y_{i_1}^\bullet, y_{i_2}^\bullet\big], y_{i_3}^\bullet\big]\ldots, y_{i_s}^\bullet\big]\ldots, y_{i_{k-1}}^\bullet\big], y_{i_k}^\bullet\big]$ with the same $\underline{i}$'s which give the basis of $\mathcal{L}(V)$ and with $y_j^\bullet \in \big\{x_j^\otimes, \psi(x_j)\big\}$ is again a linearly independent set inside $\mathcal{L}(V) \wedge \mathcal{L}(V)$. Therefore, for a general $x = \sum_{\underline{i}} c_{\underline{i}}\, x_{\underline{i}} \in \mathcal{L}(V)$ we have $\psi_{d\mathcal{L}}(x) = \sum_{\underline{i}} c_{\underline{i}} \sum_{x_{i_s} \in B_I} \big[\big[\cdots\big[\cdots\big[\big[x_{i_1}^\otimes, x_{i_2}^\otimes\big], x_{i_3}^\otimes\big]\ldots, \psi(x_{i_s})\big]\ldots, x_{i_{k-1}}^\otimes\big], x_{i_k}^\otimes\big]$; thus if $\psi_{d\mathcal{L}}(x) = 0$ we necessarily have $c_{\underline{i}} = 0$ for all $\underline{i}$ which sport at least one $x_{i_s} \in B_I$. The outcome is that $x \in Ker(\psi_{d\mathcal{L}})$ implies $x = \sum_{\underline{i}\,:\, x_{i_s} \in B_K\, (\forall\, s)} c_{\underline{i}}\, x_{\underline{i}} \in \mathcal{L}(K)$; thus $Ker(\psi_{d\mathcal{L}}) \subseteq \mathcal{L}(K)$, and the converse inclusion is clear because $\psi_{d\mathcal{L}}$ is a derivation. $\square$

Lemma 10.16. *The Lie cobracket δ of $U(\mathcal{L}_\nu)$ preserves τ. That is, for each $\vartheta \in U(\mathcal{L}_\nu)$ in the expansion $\delta_2(\vartheta) = \sum_{\underline{b}_1, \underline{b}_2 \in \mathbb{B}} c_{\underline{b}_1, \underline{b}_2}\, \boldsymbol{\alpha}_{\underline{b}_1} \otimes \boldsymbol{\alpha}_{\underline{b}_2}$ (w.r.t. the basis $\mathbb{B} \otimes \mathbb{B}$, where $\mathbb{B}$ is a PBW basis as in §10.2 w.r.t. some total order of B_ν) we have $\tau\big(\hat{\underline{b}}_1\big) + \tau\big(\hat{\underline{b}}_2\big) = \tau(\vartheta)$ for some $\hat{\underline{b}}_1$, $\hat{\underline{b}}_2$ with $c_{\hat{\underline{b}}_1, \hat{\underline{b}}_2} \neq 0$, so $\tau\big(\delta(\vartheta)\big) := \max\big\{\tau(\underline{b}_1) + \tau(\underline{b}_2) \,\big|\, c_{\underline{b}_1, \underline{b}_2} \neq 0\big\} = \tau(\vartheta)$ if $\delta(\vartheta) \neq 0$.*

Proof. It follows from Proposition 10.13 that $\tau\big(\delta(\vartheta)\big) \leq \tau(\vartheta)$; so $\delta \colon U(\mathcal{L}_\nu) \longrightarrow U(\mathcal{L}_\nu)^{\otimes 2}$ is a morphism of filtered algebras, hence it naturally induces a morphism of graded algebras $\overline{\delta} \colon G_{\underline{\Theta}}\big(U(\mathcal{L}_\nu)\big) \longrightarrow G_{\underline{\Theta}}\big(U(\mathcal{L}_\nu)\big)^{\otimes 2}$ (notation of §§5.3–4). Therefore proving the claim is equivalent to showing that $Ker\big(\overline{\delta}\,\big) = G_{\underline{\Theta} \cap Ker(\delta)}\big(Ker(\delta)\big) =: \overline{Ker(\delta)}$, the latter being thought of as naturally embedded into $G_{\underline{\Theta}}\big(U(\mathcal{L}_\nu)\big)$.

By construction, $\tau(x\,y - y\,x) = \tau\big([x,y]\big) < \tau(x) + \tau(y)$ for $x, y \in U(\mathcal{L}_\nu)$, so $G_{\underline{\Theta}}\big(U(\mathcal{L}_\nu)\big)$ is commutative: indeed, it is clearly isomorphic — as an algebra — to $S(V_\nu)$, the symmetric algebra over V_ν. Moreover, δ acts as a derivation, that is $\delta(x\,y) = \delta(x)\,\Delta(y) + \Delta(x)\,\delta(y)$ (for all x, $y \in U(\mathcal{L}_\nu)$), thus the same holds for $\overline{\delta}$ too. Like in Lemma 10.15, since $G_{\underline{\Theta}}\big(U(\mathcal{L}_\nu)\big)$ is generated by $G_{\underline{\Theta} \cap \mathcal{L}_\nu}(\mathcal{L}_\nu) =: \overline{\mathcal{L}_\nu}$ it follows that $Ker\big(\overline{\delta}\,\big)$ is the free (associative sub)algebra over $Ker\Big(\overline{\delta}\,\big|_{\overline{\mathcal{L}_\nu}}\Big)$, in short $Ker\big(\overline{\delta}\,\big) = \Big\langle Ker\Big(\overline{\delta}\,\big|_{\overline{\mathcal{L}_\nu}}\Big)\Big\rangle$.

Now, by definition $\delta(x_n) = \sum_{\ell=1}^{n-1} (\ell+1)\, x_\ell \wedge x_{n-\ell}$ (cf. Theorem 10.6) is a sum of $\tau-$homogeneous terms of $\tau-$degree equal to $(n-1) = \tau(x_n)$. Since in addition δ enjoys $\delta\big([x,y]\big) = \big[x \otimes 1 + 1 \otimes x, \delta(y)\big] + \big[\delta(x), y \otimes 1 + 1 \otimes y\big]$ (for all x, $y \in \mathcal{L}_\nu$) we have that $\delta\big|_{\mathcal{L}_\nu}$ is even $\tau-$homogeneous, which means that $\delta\big(\tau(z)\big)$ either is zero or can be written as a sum whose summands are all $\tau-$homogeneous terms of τ–degree equal to $\tau(z)$, for any τ–homogeneous $z \in \mathcal{L}_\nu$; this implies that the induced map $\overline{\delta}\,\big|_{\overline{\mathcal{L}_\nu}}$ enjoys $\overline{\delta}\,\big|_{\overline{\mathcal{L}_\nu}}\big(\overline{\vartheta}\,\big) = \overline{0} \iff \delta(\vartheta) = 0$ for any $\vartheta \in \mathcal{L}_\nu$, whence $Ker\big(\,\overline{\delta}\,\big|_{\overline{\mathcal{L}_\nu}}\big) = \overline{Ker\big(\delta\big|_{\mathcal{L}_\nu}\big)}$. On the upshot we get $Ker\big(\overline{\delta}\,\big) = \Big\langle Ker\Big(\overline{\delta}\,\big|_{\overline{\mathcal{L}_\nu}}\Big)\Big\rangle = \Big\langle \overline{Ker\big(\delta\big|_{\mathcal{L}_\nu}\big)}\,\Big\rangle = \overline{Ker(\delta)}$, q.e.d. $\square$

Proposition 10.17. *$\underline{D} = \underline{\Theta}$, that is $D_n = \Theta_n$ for all $n \in \mathbb{N}$, or $\kappa = \tau$. Therefore, given any total order $\preceq$ in B_ν, the set $\mathcal{A}_{\leq n} = \mathcal{A} \cap \Theta_n = \mathcal{A} \cap D_n$ of ordered monomials*

$$\mathcal{A}_{\leq n} \;=\; \Big\{ \boldsymbol{\alpha}_{\underline{b}} = \boldsymbol{\alpha}_{b_1} \cdots \boldsymbol{\alpha}_{b_k} \,\Big|\, k \in \mathbb{N},\; b_1, \dots, b_k \in B_\nu \,,\; b_1 \preceq \cdots \preceq b_k \,,\; \tau(\underline{b}\,) \leq n \Big\}$$

is a $\Bbbk$–basis of D_n, and $\mathcal{A}_n := \big(\mathcal{A}_{\leq n} \mod D_{n-1} \big)$ is a $\Bbbk$–basis of $D_n \big/ D_{n-1}$ $(\forall\, n \in \mathbb{N})$.

Proof. Clearly the claim about the $\mathcal{A}_{\leq n}$'s and the claim about the $\mathcal{A}_n$'s are equivalent, and either of these claims is equivalent to $\underline{D} = \underline{\Theta}$. Note also that $\mathcal{A}_n := \big(\mathcal{A}_{\leq n} \mod D_{n-1} \big) = \big(\mathcal{A}_{\leq n} \setminus \mathcal{A}_{\leq n-1} \mod D_{n-1} \big)$, where clearly $\mathcal{A}_{\leq n} \setminus \mathcal{A}_{\leq n-1} = \big\{ \boldsymbol{\alpha}_{\underline{b}} \in \mathcal{A} \,\big|\, \tau(\underline{b}\,) = n \big\}$.

By Lemma 10.14*(f)* we have $\mathcal{A}_{\leq n} = \mathcal{A} \bigcap \Theta_n \subseteq \mathcal{A} \bigcap D_n \subseteq D_n$; since $\mathcal{A}$ is a basis, $\mathcal{A}_{\leq n}$ is linearly independent and is a $\Bbbk$–basis of Θ_n (by definition): so $\Theta_n \subseteq D_n$ for all $n \in \mathbb{N}$.

$\underline{n = 0}$: By definition $D_0 := Ker(\delta_1) = \Bbbk \cdot 1_{\mathcal{H}} =: \Theta_0$, spanned by $\mathcal{A}_{\leq 0} = \{1_{\mathcal{H}}\}$, q.e.d.

$\underline{n = 1}$: Let $\eta' \in D_1 := Ker(\delta_2)$. Let $\mathbb{B}$ be a PBW-like basis of ${\mathcal{H}_\hbar}^\vee = U(\mathcal{L}_\nu)$ as mentioned in Lemma 10.16; expanding η' w.r.t. the basis $\mathcal{A}$ we have $\eta' = \sum_{\boldsymbol{\alpha}_{\underline{b}} \in \mathcal{A}} c_{\underline{b}}\, \boldsymbol{\alpha}_{\underline{b}} = \sum_{\underline{b} \in \mathbb{B}} c_{\underline{b}}\, \boldsymbol{\alpha}_{\underline{b}}$. Then we have also $\eta := \eta' - \sum_{\tau(\underline{b}\,) \leq 1} c_{\underline{b}}\, \boldsymbol{\alpha}_{\underline{b}} = \sum_{\tau(\underline{b}\,) > 1} c_{\underline{b}}\, \boldsymbol{\alpha}_{\underline{b}} \in D_1$ because $\boldsymbol{\alpha}_{\underline{b}} \in \mathcal{A}_1 \subseteq \Theta_1 \subseteq D_1$ whenever $\tau(\underline{b}\,) \leq 1$.

Now, $\boldsymbol{\alpha}_1 := \mathbf{a}_1$ and $\boldsymbol{\alpha}_s := \mathbf{a}_s - {\mathbf{a}_1}^s = \hbar \big(\mathbf{x}_s + \hbar^{s-1} {\mathbf{x}_1}^s \big)$ for all $s \in \mathbb{N}_\nu \setminus \{1\}$ yield

$$\eta \;=\; \textstyle\sum_{\underline{b} \in \mathbb{B}\,,\, \tau(\underline{b}\,) > 1} c_{\underline{b}}\, \boldsymbol{\alpha}_{\underline{b}} \;=\; \sum_{\underline{b} \in \mathbb{B}\,,\, \tau(\underline{b}\,) > 1} \hbar^{\,g(\underline{b}\,)}\, c_{\underline{b}} \big(\mathbf{x}_{\underline{b}} + \hbar\, \chi_{\underline{b}} \big) \;\in\; {\mathcal{H}_\hbar}^\vee$$

for some $\chi_{\underline{b}} \in {\mathcal{H}_\hbar}^\vee$: hereafter we set $g(\underline{b}\,) := k$ for each $\underline{b} = b_1 \cdots b_k \in \mathbb{B}$ (i.e. $g(\underline{b}\,)$ is the degree of $\underline{b}$ as a monomial in the b_i's). If $\eta \neq 0$, let $g_0 := \min \big\{ g(\underline{b}\,) \,\big|\, \tau(\underline{b}\,) > 1 \,,\, c_{\underline{b}} \neq 0 \big\}$; then $g_0 > 0$, $\eta_+ := \hbar^{-g_0}\, \eta \in {\mathcal{H}_\hbar}^\vee \setminus \hbar\, {\mathcal{H}_\hbar}^\vee$ and

$$0 \;\neq\; \overline{\eta_+} \;=\; \sum_{g(\underline{b}\,) = g_0} c_{\underline{b}}\, \overline{\mathbf{x}_{\underline{b}}} \;=\; \sum_{g(\underline{b}\,) = g_0} c_{\underline{b}}\, x_{\underline{b}} \;\in\; {\mathcal{H}_\hbar}^\vee \Big/ \hbar\, {\mathcal{H}_\hbar}^\vee \;=\; U(\mathcal{L}_\nu)\,.$$

Now $\delta_2(\eta) = 0$ yields $\delta_2\big(\overline{\eta_+}\big) = 0$, thus $\sum_{g(\underline{b}\,) = g_0} c_{\underline{b}}\, x_{\underline{b}} = \overline{\eta_+} \in P\big(U(\mathcal{L}_\nu)\big) = \mathcal{L}_\nu$; therefore all PBW monomials occurring in the last sum do belong to B_ν (and $g_0 = 1$). In addition, $\delta_2(\eta) = 0$ also implies $\delta_2(\eta_+) = 0$ which yields also $\delta\big(\overline{\eta_+}\big) = 0$ for the Lie cobracket δ of $\mathcal{L}_\nu$ arising as semiclassical limit of $\Delta_{{\mathcal{H}_\hbar}^\vee}$ (see Theorem 10.6); therefore $\overline{\eta_+} = \sum_{b \in B_\nu} c_b\, x_b$ is an element of $\mathcal{L}_\nu$ killed by the Lie cobracket δ, i.e. $\overline{\eta_+} \in Ker(\delta)$.

Now we apply Lemma 10.15 to $V = V_\nu$, $\mathcal{L}(V) = \mathcal{L}(V_\nu) =: \mathcal{L}_\nu$ and $\psi = \delta\big|_{V_\nu}$, so that $\psi_{d\mathcal{L}} = \delta$. From the formulas for δ in Theorem 10.6 we see that $K := Ker(\psi) = Ker\Big(\delta\big|_{V_\nu}\Big) = Span\big(\{x_1, x_2\}\big)$, hence $\mathcal{L}(K) = \mathcal{L}\big(Span\big(\{x_1, x_2\}\big)\big)$: by definition the last space is nothing but $Span\Big(\big\{ x_b \,\big|\, b \in B_\nu \,;\, \tau(b) = 1 \big\}\Big)$, thus eventually via Theorem 10.6 we get $Ker(\delta) = \mathcal{L}(K) = Span\big(\big\{ x_b \,\big|\, b \in B_\nu \,;\, \tau(b) = 1 \big\}\big)$.

Since $\overline{\eta_+} \in Ker(\delta) = Span\big(\big\{ x_b \,\big|\, b \in B_\nu \,;\, \tau(b) = 1 \big\}\big)$ we have $\overline{\eta_+} = \sum_{\substack{b \in B_\nu \\ \tau(b) = 1}} c_b\, x_b$; but $c_b = 0$ whenever $\tau(b) \leq 1$, by construction of η: thus $\overline{\eta_+} = 0$, a contradiction. The outcome is $\eta = 0$, whence finally $\eta' \in \Theta_1$, q.e.d.

$\underline{n>1}$: We must show that $D_n = \varTheta_n$, while assuming by induction that $D_m = \varTheta_m$ for all $m<n$. Let $\eta = \sum_{\underline{b}\in\mathbb{B}} c_{\underline{b}}\, \boldsymbol{\alpha}_{\underline{b}} \in D_n$; then $\tau(\eta) = \max\big\{ \tau(\underline{b}) \,\big|\, c_{\underline{b}} \neq 0 \big\}$. If $\delta_2(\eta)=0$ then $\eta \in D_1 = \varTheta_1$ by the previous analysis, and we're done. Otherwise, $\delta_2(\eta) \neq 0$ and $\tau\big(\delta_2(\eta)\big) = \tau(\eta)$ by Lemma 10.16. On the other hand, since $\underline{D}$ is a Hopf algebra filtration we have $\delta_2(\eta) \in \sum_{\substack{r+s=n\\ r,s>0}} D_r \otimes D_s = \sum_{\substack{r+s=n\\ r,s>0}} \varTheta_r \otimes \varTheta_s$, thanks to the induction; but then $\tau\big(\delta_2(\eta)\big) \leq n$, by definition of τ. Thus $\tau(\eta) = \tau\big(\delta_2(\eta)\big) \leq n$, which means $\eta \in \varTheta_n$. $\square$

Theorem 10.18. *For any $b \in B_\nu$ set $\widehat{\boldsymbol{\alpha}}_b := \hbar^{\kappa(\boldsymbol{\alpha}_b)}\, \boldsymbol{\alpha}_b = \hbar^{\tau(b)}\, \boldsymbol{\alpha}_b$.*

(a) The set of ordered monomials

$$\widehat{\mathcal{A}}_{\leq n} \;:=\; \Big\{\, \widehat{\boldsymbol{\alpha}}_{\underline{b}} := \widehat{\boldsymbol{\alpha}}_{b_1} \cdots \widehat{\boldsymbol{\alpha}}_{b_k} \;\Big|\; k \in \mathbb{N}\,,\, b_1, \dots, b_k \in B\,,\, b_1 \preceq \cdots \preceq b_k\,,\, \kappa\big(\boldsymbol{\alpha}_{\underline{b}}\big) = \tau(\underline{b}) \leq n \Big\}$$

is a $\Bbbk[\hbar]$–basis of $D'_n = D_n\big(\mathcal{H}_\hbar{}'\big) = \hbar^n D_n$. So $\widehat{\mathcal{A}} := \bigcup_{n\in\mathbb{N}} \widehat{\mathcal{A}}_{\leq n}$ is a $\Bbbk[\hbar]$–basis of $\mathcal{H}_\hbar{}'$.

(b) $\mathcal{H}_\hbar{}' \;=\; \Bbbk[\hbar] \Big\langle \big\{ \widehat{\boldsymbol{\alpha}}_b \big\}_{b\in B_\nu} \Big\rangle \Big/\!\!\Big/ \bigg(\Big\{ \big[\widehat{\boldsymbol{\alpha}}_{b_1}, \widehat{\boldsymbol{\alpha}}_{b_2} \big] - \hbar\, \widehat{\boldsymbol{\alpha}}_{[b_1,b_2]} \;\Big|\; \forall\, b_1, b_2 \in B_\nu \Big\} \bigg)$.

(c) $\mathcal{H}_\hbar{}'$ is a graded Hopf $\Bbbk[\hbar]$–subalgebra of $\mathcal{H}_\hbar$.

(d) $\mathcal{H}_\hbar{}'\Big|_{\hbar=0} := \mathcal{H}_\hbar{}' \Big/ \hbar\, \mathcal{H}_\hbar{}' = \widetilde{\mathcal{H}} = F\big[\varGamma_{\mathcal{L}_\nu}^\star\big]$, where $\varGamma_{\mathcal{L}_\nu}^\star$ is a connected Poisson algebraic group with cotangent Lie bialgebra isomorphic to $\mathcal{L}_\nu$ (as a Lie algebra) with the graded Lie bialgebra structure given by $\delta(x_n) = (n-2)\, x_{n-1} \wedge x_1$ (for all $n \in \mathbb{N}_\nu$). Indeed, $\mathcal{H}_\hbar{}'\Big|_{\hbar=0}$ is the free Poisson (commutative) algebra over $\mathbb{N}_\nu$, generated by all the $\bar{\boldsymbol{\alpha}}_n := \widehat{\boldsymbol{\alpha}}_n\big|_{\hbar=0}$ ($n \in \mathbb{N}_\nu$) with Hopf structure given (for all $n \in \mathbb{N}_\nu$) by

$$\Delta\big(\bar{\boldsymbol{\alpha}}_n\big) \;=\; \bar{\boldsymbol{\alpha}}_n \otimes 1 \,+\, 1 \otimes \bar{\boldsymbol{\alpha}}_n \,+\, \sum_{k=2}^{n-1} \binom{n}{k}\, \bar{\boldsymbol{\alpha}}_k \otimes \bar{\boldsymbol{\alpha}}_1^{\,n-k} \,+\, \sum_{k=1}^{n-1} (k+1)\, \bar{\boldsymbol{\alpha}}_1^{\,k} \otimes \bar{\boldsymbol{\alpha}}_{n-k}$$

$$S\big(\bar{\boldsymbol{\alpha}}_n\big) \;=\; -\,\bar{\boldsymbol{\alpha}}_n \,-\, \sum_{k=2}^{n-1} \binom{n}{k}\, S\big(\bar{\boldsymbol{\alpha}}_k\big)\, \bar{\boldsymbol{\alpha}}_1^{\,n-k} \,-\, \sum_{k=1}^{n-1} (k+1)\, S\big(\bar{\boldsymbol{\alpha}}_1\big)^k\, \bar{\boldsymbol{\alpha}}_{n-k}\,, \qquad \epsilon\big(\bar{\boldsymbol{\alpha}}_n\big) \;=\; 0\,.$$

Thus $\mathcal{H}_\hbar{}'\Big|_{\hbar=0}$ is the polynomial algebra $\Bbbk\big[\{\eta_b\}_{b\in B_\nu}\big]$ generated by a set of indeterminates $\{\eta_b\}_{b\in B_\nu}$ in bijection with B_ν, so $\varGamma_{\mathcal{L}_\nu}^\star \cong \mathbb{A}_\Bbbk^{B_\nu}$ as algebraic varieties.

Finally, $\mathcal{H}_\hbar{}'\Big|_{\hbar=0} = F\big[\varGamma_{\mathcal{L}_\nu}^\star\big] = \Bbbk\big[\{\eta_b\}_{b\in B_\nu}\big]$ is a graded Poisson Hopf algebra w.r.t. the grading $\partial(\bar{\boldsymbol{\alpha}}_n) = n$ (inherited from $\mathcal{H}_\hbar{}'$) and w.r.t. the grading induced from $\kappa = \tau$ (on $\mathcal{H}$), and a graded algebra w.r.t. the (polynomial) grading $d(\bar{\boldsymbol{\alpha}}_n) = 1$ (for all $n \in \mathbb{N}_+$).

(e) The analogues of statements (a)–(d) hold with $\mathcal{K}$ instead of $\mathcal{H}$, with X^+ instead of X for all $X = \mathcal{L}_\nu, B_\nu, \mathbb{N}_\nu$, and with $\varGamma_{\mathcal{L}_\nu^+}^\star$ instead of $\varGamma_{\mathcal{L}_\nu}^\star$.

Proof. *(a)* This follows from Proposition 10.17 and the characterization of $\mathcal{H}_\hbar{}'$ in §10.10.

(b) This is a direct consequence of claim *(a)* and Lemma 10.14*(g)*.

(c) Thanks to claims *(a)* and *(b)*, we can look at $\mathcal{H}_\hbar{}'$ as a Poisson algebra, whose Poisson bracket is given by $\{x,y\} := \hbar^{-1}[x,y] = \hbar^{-1}(x\,y - y\,x)$ (for all $x, y \in \mathcal{H}_\hbar{}'$); then $\mathcal{H}_\hbar{}'$ itself

is the free associative Poisson algebra generated by $\big\{\, \widehat{\boldsymbol{\alpha}}_n \,\big|\, n \in \mathbb{N} \,\big\}$. Clearly Δ is a Poisson map, therefore it is enough to prove that $\Delta\big(\widehat{\boldsymbol{\alpha}}_n\big) \in {\mathcal{H}_\hbar}' \otimes {\mathcal{H}_\hbar}'$ for all $n \in \mathbb{N}_+$. This is clear for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ which are primitive; as for $n > 2$, we have, like in Proposition 10.13,

$$\begin{aligned} \Delta\big(\widehat{\boldsymbol{\alpha}}_n\big) \; &= \; \textstyle\sum_{k=2}^n \hbar^{k-1} \boldsymbol{\alpha}_k \otimes \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) \, + \, \sum_{k=0}^{n-1} \hbar^k {\boldsymbol{\alpha}_1}^k \otimes \hbar^{n-k-1} Z_{n-k}^k(\boldsymbol{\alpha}_*) \; = \\ &= \; \textstyle\sum_{k=2}^n \widehat{\boldsymbol{\alpha}}_k \otimes \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) \, + \, \sum_{k=0}^{n-1} \widehat{\boldsymbol{\alpha}}_1^{\,k} \otimes \hbar^{n-k-1} Z_{n-k}^k(\boldsymbol{\alpha}_*) \; \in \; {\mathcal{H}_\hbar}' \otimes {\mathcal{H}_\hbar}' \end{aligned} \tag{10.8}$$

thanks to Lemma 10.12 (with notations used therein). In addition, $S\big({\mathcal{H}_\hbar}'\big) \subseteq {\mathcal{H}_\hbar}'$ also follows by induction from (10.8) because Hopf algebra axioms along with (10.8) give

$$S\big(\widehat{\boldsymbol{\alpha}}_n\big) \; = \; -\widehat{\boldsymbol{\alpha}}_n \, - \, \textstyle\sum_{k=2}^{n-1} S\big(\widehat{\boldsymbol{\alpha}}_k\big) \, \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) \, - \, \sum_{k=1}^{n-1} S\big(\widehat{\boldsymbol{\alpha}}_1^{\,k}\big) \, \hbar^{n-k-1} Z_{n-k}^k(\boldsymbol{\alpha}_*) \; \in \; {\mathcal{H}_\hbar}'$$

for all $n \in \mathbb{N}_\nu$ (using induction). The claim follows.

(d) Thanks to *(a)* and *(b)*, ${\mathcal{H}_\hbar}'\Big|_{\hbar=0}$ is a polynomial $\Bbbk$–algebra as claimed, over the set of indeterminates $\big\{\, \bar{\boldsymbol{\alpha}}_b := \widehat{\boldsymbol{\alpha}}_b\big|_{\hbar=0} \big(\in {\mathcal{H}_\hbar}'\big|_{\hbar=0}\big) \,\big|\, b \in B_\nu \,\big\}$. Furthermore, in the proof of *(c)* we noticed that ${\mathcal{H}_\hbar}'$ is also the free Poisson algebra generated by $\big\{\, \widehat{\boldsymbol{\alpha}}_n \,\big|\, n \in \mathbb{N} \,\big\}$; therefore ${\mathcal{H}_\hbar}'\Big|_{\hbar=0}$ is the free commutative Poisson algebra generated by $\big\{\, \bar{\boldsymbol{\alpha}}_n := \check{\boldsymbol{\alpha}}_{x_n}\big|_{\hbar=0} \,\big|\, n \in \mathbb{N} \,\big\}$. Then formula (10.8) — for all $n \in \mathbb{N}_\nu$ — describes uniquely the Hopf structure of ${\mathcal{H}_\hbar}'$, hence the formula it yields at $\hbar = 0$ will describe the Hopf structure of ${\mathcal{H}_\hbar}'\big|_{\hbar=0}$.

Expanding $\hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*)$ in (10.8) w.r.t. the basis $\widehat{\mathcal{A}}$ in *(a)* we find a sum of terms of τ–degree less or equal than $(n-k)$, and the sole one achieving equality is $\widehat{\boldsymbol{\alpha}}_1^{\,n-k}$, which occurs with coefficient $\binom{n}{k}$: similarly, when expanding $\hbar^{n-k-1} Z_{n-k}^k(\boldsymbol{\alpha}_*)$ in (10.8) w.r.t. $\widehat{\mathcal{A}}$ all summands have τ–degree less or equal than $(n-k-1)$, and equality holds only for $\widehat{\boldsymbol{\alpha}}_{n-k}$, whose coefficient is $(k+1)$. Therefore for some $\boldsymbol{\eta} \in {\mathcal{H}_\hbar}'\big|_{\hbar=0}$ we have

$$\Delta\big(\widehat{\boldsymbol{\alpha}}_n\big) \; = \; \textstyle\sum_{k=2}^n \widehat{\boldsymbol{\alpha}}_k \otimes \displaystyle\binom{n}{k} \widehat{\boldsymbol{\alpha}}_1^{\,n-k} \, + \, \textstyle\sum_{k=0}^{n-1} (k+1) \, \widehat{\boldsymbol{\alpha}}_1^{\,k} \otimes \widehat{\boldsymbol{\alpha}}_{n-k} \, + \, \hbar \, \boldsymbol{\eta} \, ;$$

this yields the formula for Δ, from which the formula for S follows too as usual.

Finally, let $\varGamma := Spec\big({\mathcal{H}_\hbar}'\big|_{\hbar=0}\big)$ be the algebraic Poisson group such that $F\big[\varGamma\big] = {\mathcal{H}_\hbar}'\big|_{\hbar=0}$, and let $\boldsymbol{\gamma}_\nu := co\mathcal{L}ie\,(\varGamma)$ be its cotangent Lie bialgebra. Since ${\mathcal{H}_\hbar}'\big|_{\hbar=0}$ is Poisson free over $\big\{\bar{\boldsymbol{\alpha}}_n\big\}_{n \in \mathbb{N}_\nu}$, as a Lie algebra $\boldsymbol{\gamma}_\nu$ is free over $\big\{\, d_n := \bar{\boldsymbol{\alpha}}_n \mod \mathfrak{m}^2 \,\big\}_{n \in \mathbb{N}_\nu}$ (where $\mathfrak{m} := J_{{\mathcal{H}_\hbar}'|_{\hbar=0}}$), so $\boldsymbol{\gamma}_\nu \cong \mathcal{L}_\nu$, via $d_n \mapsto x_n (n \in \mathbb{N}_+)$ as a Lie algebra. The Lie cobracket is

$$\begin{aligned} \delta_{\boldsymbol{\gamma}_\nu}\big(d_n\big) \; &= \; \big(\Delta - \Delta^{\mathrm{op}}\big)\big(\bar{\boldsymbol{\alpha}}_n\big) \mod \mathfrak{m}_\otimes \; = \\ &= \; \sum_{k=2}^{n-1} \binom{n}{k} \bar{\boldsymbol{\alpha}}_k \wedge \bar{\boldsymbol{\alpha}}_1^{\,n-k} \, + \, \sum_{k=1}^{n-1} (k+1) \, \bar{\boldsymbol{\alpha}}_1^{\,k} \wedge \bar{\boldsymbol{\alpha}}_{n-k} \mod \mathfrak{m}_\otimes \; = \\ &= \; \binom{n}{n-1} \bar{\boldsymbol{\alpha}}_{n-1} \wedge \bar{\boldsymbol{\alpha}}_1 \, + \, 2 \, \bar{\boldsymbol{\alpha}}_1 \wedge \bar{\boldsymbol{\alpha}}_{n-1} \mod \mathfrak{m}_\otimes \; = \\ &= \; (n-2) \, \bar{\boldsymbol{\alpha}}_{n-1} \wedge \bar{\boldsymbol{\alpha}}_1 \mod \mathfrak{m}_\otimes \; = \; (n-2) \, d_{n-1} \wedge d_1 \; \in \; \boldsymbol{\gamma} \otimes \boldsymbol{\gamma} \end{aligned}$$

where $\mathfrak{m}_\otimes := \Big(\mathfrak{m}^2 \otimes {\mathcal{H}_\hbar}'|_{\hbar=0} + \mathfrak{m} \otimes \mathfrak{m} + {\mathcal{H}_\hbar}'|_{\hbar=0} \otimes \mathfrak{m}^2\Big)$, whence $\varGamma = \varGamma_{\mathcal{L}_\nu}^\star$ as claimed in *(d)*.

Finally, the statements about gradings of ${\mathcal{H}_\hbar}'\Big|_{\hbar=0} = F\big[\varGamma_{\mathcal{L}_\nu}^\star\big]$ hold by construction.

(e) This should be clear from the whole discussion, since all arguments apply again — *mutatis mutandis* — when starting with $\mathcal{K}$ instead of $\mathcal{H}$; we leave details to the reader. $\square$

10.19 Drinfeld's algebra $\big(\mathcal{H}_\hbar{}'\big)^\vee$. I look now at the other Drinfeld's functor at $\hbar$, and consider $\big(\mathcal{H}_\hbar{}'\big)^\vee := \sum_{n \in \mathbb{N}} \hbar^{-n} {J'}^n$, where $J' := J_{\mathcal{H}_\hbar{}'}$. Theorem 2.2 tells us that $\big(\mathcal{H}_\hbar{}'\big)^\vee$ *is a Hopf $\Bbbk[\hbar]$–subalgebra of $\mathcal{H}_\hbar$, and the specialization of $\big(\mathcal{H}_\hbar{}'\big)^\vee$ at $\hbar = 0$, i.e. $\big(\mathcal{H}_\hbar{}'\big)^\vee\Big|_{\hbar=0} := \big(\mathcal{H}_\hbar{}'\big)^\vee \Big/ \hbar \, \big(\mathcal{H}_\hbar{}'\big)^\vee$, is the universal enveloping algebra of the cotangent Lie bialgebra of the connected algebraic Poisson group which is the spectrum of $\mathcal{H}_\hbar{}'\Big|_{\hbar=0}$, that is exactly $\varGamma_{\mathcal{L}_\nu}^\star$.* Thanks to Theorem 10.18, this means $\big(\mathcal{H}_\hbar{}'\big)^\vee\Big|_{\hbar=0} = U(\mathcal{L}_\nu)$ as co-Poisson Hopf $\Bbbk$–algebras, the Lie cobracket of $\mathcal{L}_\nu$ being the one given in Theorem 10.18*(d)*.

Therefore we must show that $\big(\mathcal{H}_\hbar{}'\big)^\vee\Big|_{\hbar=0}$ is a cocommutative Hopf $\Bbbk$–algebra, it is generated by its primitive elements, and the latter set inherits a Lie bialgebra structure isomorphic to that of $\gamma_\nu := co\mathcal{L}ie\big(\varGamma_{\mathcal{L}_\nu}^\star\big)$. We prove all this directly, via an explicit description of $\big(\mathcal{H}_\hbar{}'\big)^\vee$ and its specialization at $\hbar = 0$, provided in the following

Theorem 10.20. *For any $b \in B_\nu$ set $\check{\boldsymbol{\alpha}}_b := \hbar^{\,\kappa(\boldsymbol{\alpha}_b)-1} \boldsymbol{\alpha}_b = \hbar^{\,\tau(b)-1} \boldsymbol{\alpha}_b = \hbar^{-1} \widehat{\boldsymbol{\alpha}}_b$.*

(a) $\big(\mathcal{H}_\hbar{}'\big)^\vee = \Bbbk[\hbar] \Big\langle \big\{ \check{\boldsymbol{\alpha}}_b \big\}_{b \in B_\nu} \Big\rangle \bigg/\!\!\bigg/ \bigg(\Big\{ \big[\check{\boldsymbol{\alpha}}_{b_1}, \check{\boldsymbol{\alpha}}_{b_2} \big] - \check{\boldsymbol{\alpha}}_{[b_1,b_2]} \;\Big|\; \forall\, b_1, b_2 \in B_\nu \Big\} \bigg)$.

(b) $\big(\mathcal{H}_\hbar{}'\big)^\vee$ *is a graded Hopf $\Bbbk[\hbar]$–subalgebra of $\mathcal{H}_\hbar$.*

(c) $\big(\mathcal{H}_\hbar{}'\big)^\vee\Big|_{\hbar=0} := \big(\mathcal{H}_\hbar{}'\big)^\vee \Big/ \hbar \, \big(\mathcal{H}_\hbar{}'\big)^\vee \cong U\big(\mathcal{L}_\nu\big)$ *as co-Poisson Hopf algebra, where $\mathcal{L}_\nu$ bears the Lie bialgebra structure given by $\delta(x_n) = (n-2)\, x_{n-1} \wedge x_1$ (for all $n \in \mathbb{N}_\nu$).*

Finally, the grading d given by $d(x_n) := 1$ $(n \in \mathbb{N}_+)$ makes $\big(\mathcal{H}_\hbar{}'\big)^\vee\Big|_{\hbar=0} = U(\mathcal{L}_\nu)$ into a graded co-Poisson Hopf algebra, and the grading ∂ given by $\partial(x_n) := n$ $(n \in \mathbb{N}_+)$ makes $\big(\mathcal{H}_\hbar{}'\big)^\vee\Big|_{\hbar=0} = U(\mathcal{L}_\nu)$ into a graded Hopf algebra and $\mathcal{L}_\nu$ into a graded Lie bialgebra.

(d) The analogues of statements (a)–(c) hold with $\mathcal{K}$, $\mathcal{L}_\nu^+$, B_ν^+ and $\mathbb{N}_\nu^+$ respectively instead of $\mathcal{H}$, $\mathcal{L}_\nu^+$, B_ν and $\mathbb{N}_\nu^+$.

Proof. *(a)* This follows from Theorem 10.18*(b)* and the very definition of $\big(\mathcal{H}_\hbar{}'\big)^\vee$ in §10.19.

(b) This is a direct consequence of claim *(a)* and Theorem 10.18*(c)*.

(c) It follows from claim *(a)* that mapping $\check{\boldsymbol{\alpha}}_b\big|_{\hbar=0} \mapsto b$ $(\forall\, b \in B_\nu)$ yields a well-defined algebra isomorphism $\Phi \colon \big(\mathcal{H}_\hbar{}'\big)^\vee\Big|_{\hbar=0} \xhookrightarrow{\;\cong\;} U\big(\mathcal{L}_\nu\big)$. In addition, when expanding $\hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*)$ in (10.8) w.r.t. the basis $\mathcal{A}$ (see Proposition 10.17) we find a sum of terms of τ–degree less than or equal to $(n-k)$, and equality is achieved only for $\boldsymbol{\alpha}_1^{\,n-k}$, which occurs with coefficient $\binom{n}{k}$: similarly, the expansion of $\hbar^{n-k-1} Z_{n-k}^k(\boldsymbol{\alpha}_*)$ in (10.8) yields a sum of terms whose τ–degree is less or equal than $(n-k-1)$, with equality only for

$\boldsymbol{\alpha}_{n-k}$, whose coefficient is $(k+1)$. Thus using the relation $\widehat{\boldsymbol{\alpha}}_s = \hbar\,\check{\boldsymbol{\alpha}}_s$ ($s \in \mathbb{N}_+$) we get

$$\Delta\big(\check{\boldsymbol{\alpha}}_n\big) = \check{\boldsymbol{\alpha}}_n \otimes 1 + 1 \otimes \check{\boldsymbol{\alpha}}_n + \textstyle\sum_{k=2}^{n-1} \check{\boldsymbol{\alpha}}_k \otimes \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) + \sum_{k=1}^{n-1} \check{\boldsymbol{\alpha}}_1^{\,k} \otimes \hbar^{n-1} Z_{n-k}^k(\boldsymbol{\alpha}_*) =$$

$$= \check{\boldsymbol{\alpha}}_n \otimes 1 + 1 \otimes \check{\boldsymbol{\alpha}}_n + \textstyle\sum_{k=2}^{n-1} \hbar^{n-k}\, \check{\boldsymbol{\alpha}}_k \otimes \displaystyle\binom{n}{k} \check{\boldsymbol{\alpha}}_1^{\,n-k} + \textstyle\sum_{k=1}^{n-1} \hbar^k\, (k+1)\, \check{\boldsymbol{\alpha}}_1^{\,k} \otimes \check{\boldsymbol{\alpha}}_{n-k} + \hbar^2\, \boldsymbol{\eta} =$$

$$= \check{\boldsymbol{\alpha}}_n \otimes 1 + 1 \otimes \check{\boldsymbol{\alpha}}_n + \hbar \left(n\, \check{\boldsymbol{\alpha}}_{n-1} \otimes \check{\boldsymbol{\alpha}}_1 + 2\, \check{\boldsymbol{\alpha}}_1 \otimes \check{\boldsymbol{\alpha}}_{n-1}\right) + \hbar^2\, \boldsymbol{\chi}$$

for some $\boldsymbol{\eta}, \boldsymbol{\chi} \in \big({\mathcal{H}_\hbar}'\big)^\vee \otimes \big({\mathcal{H}_\hbar}'\big)^\vee$. It follows that $\Delta\big(\check{\boldsymbol{\alpha}}_n\big|_{\hbar=0}\big) = \check{\boldsymbol{\alpha}}_n\big|_{\hbar=0} \otimes 1 + 1 \otimes \check{\boldsymbol{\alpha}}_n\big|_{\hbar=0}$ for all $n \in \mathbb{N}_\nu$. Similarly we have $S\big(\check{\boldsymbol{\alpha}}_n\big|_{\hbar=0}\big) = -\check{\boldsymbol{\alpha}}_n\big|_{\hbar=0}$ and $\epsilon\big(\check{\boldsymbol{\alpha}}_n\big|_{\hbar=0}\big) = 0$ for all $n \in \mathbb{N}_\nu$, thus Φ *is an isomorphism of Hopf algebras* too. In addition, the Poisson cobracket of $\big({\mathcal{H}_\hbar}'\big)^\vee\Big|_{\hbar=0}$ inherited from $\big({\mathcal{H}_\hbar}'\big)^\vee$ is given by

$$\delta\big(\check{\boldsymbol{\alpha}}_n\big|_{\hbar=0}\big) = \Big(\hbar^{-1}(\Delta - \Delta^{\mathrm{op}})\big(\check{\boldsymbol{\alpha}}_n\big)\Big) \mod \hbar\, \big({\mathcal{H}_\hbar}'\big)^\vee \otimes \big({\mathcal{H}_\hbar}'\big)^\vee =$$

$$= \big(n\, \check{\boldsymbol{\alpha}}_{n-1} \wedge \check{\boldsymbol{\alpha}}_1 + 2\, \check{\boldsymbol{\alpha}}_1 \wedge \check{\boldsymbol{\alpha}}_{n-1}\big) \mod \hbar\, \big({\mathcal{H}_\hbar}'\big)^\vee \otimes \big({\mathcal{H}_\hbar}'\big)^\vee = (n-2)\, \check{\boldsymbol{\alpha}}_{n-1}\big|_{\hbar=0} \wedge \check{\boldsymbol{\alpha}}_1\big|_{\hbar=0}$$

hence Φ is also an isomorphism of *co-Poisson* Hopf algebras, as claimed.

The statements on gradings of $\big({\mathcal{H}_\hbar}'\big)^\vee\Big|_{\hbar=0} = U(\mathcal{L}_\nu)$ should be clear by construction.

(d) This should be clear from the whole discussion, as all arguments apply again — *mutatis mutandis* — when starting with $\mathcal{K}$ instead of $\mathcal{H}$; details are left to the reader. $\square$

10.21 Specialization limits. So far, Theorem 10.18*(d)* and Theorem 10.20*(c)* prove the following specialization results for ${\mathcal{H}_\hbar}'$ and $\big({\mathcal{H}_\hbar}'\big)^\vee$ respectively:

$${\mathcal{H}_\hbar}' \xrightarrow{\ \hbar \to 0\ } \widetilde{\mathcal{H}} \cong F\big[\mathit{\Gamma}_{\mathcal{L}_\nu}^{\,\star}\big]\,, \qquad \big({\mathcal{H}_\hbar}'\big)^\vee \xrightarrow{\ \hbar \to 0\ } U(\mathcal{L}_\nu)$$

as graded Poisson or co-Poisson Hopf $\Bbbk$–algebras. In addition, Theorem 10.18*(b)* implies that ${\mathcal{H}_\hbar}' \xrightarrow{\ \hbar \to 1\ } \mathcal{H}' = \mathcal{H}$ as graded Hopf $\Bbbk$–algebras. Indeed, by Theorem 10.18*(b)* $\mathcal{H}$ (or even $\mathcal{H}_\hbar$) embeds as an algebra into ${\mathcal{H}_\hbar}'$, via $\boldsymbol{\alpha}_n \mapsto \widehat{\boldsymbol{\alpha}}_n$ (for all $n \in \mathbb{N}_\nu$): then

$$[\boldsymbol{\alpha}_n, \boldsymbol{\alpha}_m] \mapsto \big[\widehat{\boldsymbol{\alpha}}_n, \widehat{\boldsymbol{\alpha}}_m\big] = \hbar\, \widehat{\boldsymbol{\alpha}}_{[x_n, x_m]} \equiv \widehat{\boldsymbol{\alpha}}_{[x_n, x_m]} \mod (\hbar - 1)\, {\mathcal{H}_\hbar}' \qquad (\forall\, n, m \in \mathbb{N}_\nu)$$

thus, thanks to the presentation of ${\mathcal{H}_\hbar}'$ by generators and relations in Theorem 10.18*(b)*, $\mathcal{H}$ is isomorphic to ${\mathcal{H}_\hbar}'\Big|_{\hbar=1} := {\mathcal{H}_\hbar}' \Big/ (\hbar - 1)\, {\mathcal{H}_\hbar}' = \Bbbk\big\langle \widehat{\boldsymbol{\alpha}}_1\big|_{\hbar=1}, \widehat{\boldsymbol{\alpha}}_2\big|_{\hbar=1}, \dots, \widehat{\boldsymbol{\alpha}}_n\big|_{\hbar=1}, \dots \big\rangle$, as a $\Bbbk$–algebra, via $\boldsymbol{\alpha}_n \mapsto \widehat{\boldsymbol{\alpha}}_n\big|_{\hbar=1}$. Moreover, the Hopf structure of ${\mathcal{H}_\hbar}'\Big|_{\hbar=1}$ is given by

$$\Delta\big(\widehat{\boldsymbol{\alpha}}_n\big|_{\hbar=1}\big) = \textstyle\sum_{k=2}^{n} \widehat{\boldsymbol{\alpha}}_k \otimes \hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) + \sum_{k=0}^{n-1} \widehat{\boldsymbol{\alpha}}_1^{\,k} \otimes \hbar^{n-1} Z_{n-k}^k(\boldsymbol{\alpha}_*) \mod (\hbar - 1) {\mathcal{H}_\hbar}' \otimes {\mathcal{H}_\hbar}'\,.$$

Now, $Q_{n-k}^k(\mathbf{a}_*) = Q_{n-k}^k(\boldsymbol{\alpha}_* + {\boldsymbol{\alpha}_1}^*) = \mathcal{Q}_{n-k}^k(\boldsymbol{\alpha}_*)$ for some polynomial $\mathcal{Q}_{n-k}^k(\boldsymbol{\alpha}_*)$ in the $\boldsymbol{\alpha}_i$'s; let $\mathcal{Q}_{n-k}^k(\boldsymbol{\alpha}_*) = \sum_s \mathcal{T}_{n-k}^{s,k}(\boldsymbol{\alpha}_*)$ be the splitting of $\mathcal{Q}_{n-k}^k$ into τ–homogeneous summands (i.e., each $\mathcal{T}_{n-k}^{s,k}(\boldsymbol{\alpha}_*)$ is a homogeneous polynomial of τ–degree s): then

$$\hbar^{n-k} Q_{n-k}^k(\mathbf{a}_*) = \hbar^{n-k} \mathcal{Q}_{n-k}^k(\boldsymbol{\alpha}_*) = \hbar^{n-k} \textstyle\sum_s \mathcal{T}_{n-k}^{s,k}(\boldsymbol{\alpha}_*) = \sum_s \hbar^{n-k-s} \mathcal{T}_{n-k}^{s,k}(\widehat{\boldsymbol{\alpha}}_*)$$

with $n-k-s>0$ for all s (by construction). Since clearly $\hbar^{n-k-s}\mathcal{T}^{s,k}_{n-k}(\widehat{\boldsymbol{\alpha}}_*)\equiv\mathcal{T}^{s,k}_{n-k}(\widehat{\boldsymbol{\alpha}}_*)$ mod $(\hbar-1)\,{\mathcal{H}_\hbar}'$, we find $\hbar^{n-k}\,Q^k_{n-k}(\mathbf{a}_*)\;=\;\hbar^{n-k}\,\mathcal{Q}^k_{n-k}(\boldsymbol{\alpha}_*)\;=\;\sum_s\hbar^{n-k-s}\,\mathcal{T}^{s,k}_{n-k}(\widehat{\boldsymbol{\alpha}}_*)\;\equiv\;\sum_s\mathcal{T}^{s,k}_{n-k}(\widehat{\boldsymbol{\alpha}}_*)$ mod $(\hbar-1)\,{\mathcal{H}_\hbar}'\;=\;\mathcal{Q}^k_{n-k}(\widehat{\boldsymbol{\alpha}}_*)$, for all k and n. Similarly we deduce that $\hbar^{n-1}Z^k_{n-k}(\boldsymbol{\alpha}_*)\equiv Z^k_{n-k}(\widehat{\boldsymbol{\alpha}}_*)$ mod $(\hbar-1)\,{\mathcal{H}_\hbar}'$, for all k and n. The outcome is that

$$\Delta\big(\widehat{\boldsymbol{\alpha}}_n\big|_{\hbar=1}\big)=\sum_{k=2}^n\widehat{\boldsymbol{\alpha}}_k\otimes\hbar^{n-k}\mathcal{Q}^k_{n-k}(\boldsymbol{\alpha}_*)+\sum_{k=0}^{n-1}{\widehat{\boldsymbol{\alpha}}_1}^k\otimes\hbar^{n-1}Z^k_{n-k}(\boldsymbol{\alpha}_*)\ \mathrm{mod}\,(\hbar-1)\,{\mathcal{H}_\hbar}'\otimes{\mathcal{H}_\hbar}'=$$
$$=\sum_{k=2}^n\widehat{\boldsymbol{\alpha}}_k\otimes\mathcal{Q}^k_{n-k}(\widehat{\boldsymbol{\alpha}}_*)+\sum_{k=0}^{n-1}{\widehat{\boldsymbol{\alpha}}_1}^k\otimes Z^k_{n-k}(\widehat{\boldsymbol{\alpha}}_*)\ \ \mathrm{mod}\ (\hbar-1)\,{\mathcal{H}_\hbar}'\otimes{\mathcal{H}_\hbar}'\,.$$

On the other hand, we have $\Delta(\boldsymbol{\alpha}_n)\;=\;\sum_{k=2}^n\boldsymbol{\alpha}_k\otimes\mathcal{Q}^k_{n-k}(\boldsymbol{\alpha}_*)\;+\;\sum_{k=0}^{n-1}{\boldsymbol{\alpha}_1}^k\otimes Z^k_{n-k}(\boldsymbol{\alpha}_*)$ in $\mathcal{H}$. Thus the *graded algebra* isomorphism $\Psi\colon\mathcal{H}\xhookrightarrow{\cong}{\mathcal{H}_\hbar}'\Big|_{\hbar=1}$ given by $\boldsymbol{\alpha}_n\mapsto\widehat{\boldsymbol{\alpha}}_n\big|_{\hbar=1}$ preserves the coproduct too. Similarly, Ψ respects the antipode and the counit, hence it is a graded Hopf algebra isomorphism. In a nutshell, we have (as graded Hopf $\Bbbk$–algebras)

$${\mathcal{H}_\hbar}'\xrightarrow{\ \hbar\to1\ }\mathcal{H}'=\mathcal{H}\;.$$

Similarly, Theorem 10.20 implies that $\big({\mathcal{H}_\hbar}'\big)^\vee\xrightarrow{\ \hbar\to1\ }\mathcal{H}$ as graded Hopf $\Bbbk$–algebras. Indeed, Theorem 10.20*(a)* shows that $\big({\mathcal{H}_\hbar}'\big)^\vee\;\cong\;\Bbbk[\hbar]\otimes_\Bbbk U(\mathcal{L}_\nu)$ *as graded associative algebras*, via $\check{\boldsymbol{\alpha}}_n\mapsto x_n$ $(\,n\in\mathbb{N}_\nu\,)$, in particular $\big({\mathcal{H}_\hbar}'\big)^\vee$ is the free associative $\Bbbk[\hbar]$–algebra over $\big\{\check{\boldsymbol{\alpha}}_n\big\}_{n\in\mathbb{N}_\nu}$; then specialization yields a graded algebra isomorphism

$$\Omega\colon\big({\mathcal{H}_\hbar}'\big)^\vee\Big|_{\hbar=1}:=\;\big({\mathcal{H}_\hbar}'\big)^\vee\Big/(\hbar-1)\,\big({\mathcal{H}_\hbar}'\big)^\vee\xhookrightarrow{\cong}\mathcal{H}\;,\qquad\check{\boldsymbol{\alpha}}_n\big|_{\hbar=1}\mapsto\boldsymbol{\alpha}_n\;\;.$$

As for the Hopf structure, in $\big({\mathcal{H}_\hbar}'\big)^\vee\Big|_{\hbar=1}$ it is given by

$$\Delta\big(\check{\boldsymbol{\alpha}}_n\big|_{\hbar=1}\big)=\sum_{k=2}^n\check{\boldsymbol{\alpha}}_k\big|_{\hbar=1}\otimes\hbar^{n-k}\mathcal{Q}^k_{n-k}(\boldsymbol{\alpha}_*)\big|_{\hbar=1}+\sum_{k=0}^{n-1}{\check{\boldsymbol{\alpha}}_1}^k\big|_{\hbar=1}\otimes\hbar^{n-2}Z^k_{n-k}(\boldsymbol{\alpha}_*)\big|_{\hbar=1}\,.$$

As before, split $\mathcal{Q}^k_{n-k}(\mathbf{a}_*)$ as $\mathcal{Q}^k_{n-k}(\mathbf{a}_*)=\sum_s\mathcal{T}^{s,k}_{n-k}(\boldsymbol{\alpha}_*)$, and split each $\mathcal{T}^{s,k}_{n-k}(\widehat{\boldsymbol{\alpha}}_*)$ into homogeneous components w.r.t. the total degree in the $\widehat{\boldsymbol{\alpha}}_i$'s, say $\mathcal{T}^{s,k}_{n-k}(\widehat{\boldsymbol{\alpha}}_*)=\sum_r\mathcal{Y}^{s,k}_{r,n}(\widehat{\boldsymbol{\alpha}}_*)$: then $\hbar^{n-k-s}\mathcal{T}^{s,k}_{n-k}(\widehat{\boldsymbol{\alpha}}_*)\;=\;\hbar^{n-k-s}\sum_r\mathcal{Y}^{s,k}_{r,n}(\widehat{\boldsymbol{\alpha}}_*)\;=\;\sum_r\hbar^{n-k-s+r}\mathcal{Y}^{s,k}_{r,n}(\check{\boldsymbol{\alpha}}_*)$, because $\widehat{\boldsymbol{\alpha}}_*=\hbar\,\check{\boldsymbol{\alpha}}_*$. As $\hbar^{n-k-s+r}\mathcal{Y}^{s,k}_{r,n}(\check{\boldsymbol{\alpha}}_*)\equiv\mathcal{Y}^{s,k}_{r,n}(\check{\boldsymbol{\alpha}}_*)$ mod $(\hbar-1)\,\big({\mathcal{H}_\hbar}'\big)^\vee$, we eventually get

$$\hbar^{n-k}\mathcal{Q}^k_{n-k}(\mathbf{a}_*)=\sum_{s,r}\hbar^{n-k-s+r}\mathcal{Y}^{s,k}_{r,n}(\check{\boldsymbol{\alpha}}_*)\equiv\sum_{s,r}\mathcal{Y}^{s,k}_{r,n}(\check{\boldsymbol{\alpha}}_*)\ \mathrm{mod}\,(\hbar-1)\,\big({\mathcal{H}_\hbar}'\big)^\vee=Q^k_{n-k}(\mathbf{a}_*)\,.$$

for all k and n. Similarly $\hbar^{n-1}Z^k_{n-k}(\boldsymbol{\alpha}_*)\equiv Z^k_{n-k}(\boldsymbol{\alpha}_*)$ mod $(\hbar-1)\,\big({\mathcal{H}_\hbar}'\big)^\vee$ $(\forall\,k\,,\,n)$. Thus

$$\Delta\big(\check{\boldsymbol{\alpha}}_n\big|_{\hbar=1}\big)=\sum_{k=2}^n\check{\boldsymbol{\alpha}}_k\big|_{\hbar=1}\otimes\hbar^{n-k}\mathcal{Q}^k_{n-k}(\boldsymbol{\alpha}_*)\big|_{\hbar=1}+\sum_{k=0}^{n-1}{\check{\boldsymbol{\alpha}}_1}^k\big|_{\hbar=1}\otimes\hbar^{n-2}Z^k_{n-k}(\boldsymbol{\alpha}_*)\big|_{\hbar=1}=$$
$$=\;\sum_{k=2}^n\check{\boldsymbol{\alpha}}_k\big|_{\hbar=1}\otimes\mathcal{Q}^k_{n-k}(\boldsymbol{\alpha}_*)\big|_{\hbar=1}\;+\;\sum_{k=0}^{n-1}{\check{\boldsymbol{\alpha}}_1}^k\big|_{\hbar=1}\otimes Z^k_{n-k}(\boldsymbol{\alpha}_*)\big|_{\hbar=1}\,.$$

On the other hand, one has $\Delta(\boldsymbol{\alpha}_n)\;=\;\sum_{k=2}^n\boldsymbol{\alpha}_k\otimes\mathcal{Q}^k_{n-k}(\boldsymbol{\alpha}_*)\;+\;\sum_{k=0}^{n-1}{\boldsymbol{\alpha}_1}^k\otimes Z^k_{n-k}(\boldsymbol{\alpha}_*)$ in $\mathcal{H}$, thus the algebra isomorphism $\Omega\colon\big({\mathcal{H}_\hbar}'\big)^\vee\Big|_{\hbar=1}\xhookrightarrow{\cong}\mathcal{H}$ given by $\widehat{\boldsymbol{\alpha}}_n\big|_{\hbar=1}\mapsto\boldsymbol{\alpha}_n$ also preserves the coproduct; similarly, it also respects the antipode and the counit, hence it is a graded Hopf algebra isomorphism. In a nutshell, we have (as graded Hopf $\Bbbk$–algebras)

$$\big({\mathcal{H}_\hbar}'\big)^\vee\xrightarrow{\ \hbar\to1\ }\mathcal{H}\;.$$

Therefore we have filled in the top part of the diagram of deformations (5.5), corresponding to (5.4), for $H = \mathcal{H}$ $(:= \mathcal{H}_\nu)$: it reads

$$F\big[\varGamma_{\mathcal{L}_\nu}^\star\big] = {\mathcal{H}_\hbar}'\Big|_{\hbar=0} \xleftarrow[\;{\mathcal{H}_\hbar}'\;]{0\leftarrow\hbar\rightarrow 1} {\mathcal{H}_\hbar}'\Big|_{\hbar=1} = \mathcal{H} = \big({\mathcal{H}_\hbar}'\big)^\vee\Big|_{\hbar=1} \xleftarrow[\;({\mathcal{H}_\hbar}')^\vee\;]{1\leftarrow\hbar\rightarrow 0} \big({\mathcal{H}_\hbar}'\big)^\vee\Big|_{\hbar=0} = U(\mathcal{L}_\nu)$$

or simply $F\big[\varGamma_{\mathcal{L}_\nu}^\star\big] \xleftarrow[\;{\mathcal{H}_\hbar}'\;]{0\leftarrow\hbar\rightarrow 1} \mathcal{H} \xleftarrow[\;({\mathcal{H}_\hbar}')^\vee\;]{1\leftarrow\hbar\rightarrow 0} U(\mathcal{L}_\nu)$, where $\mathcal{L}_\nu$ is given the Lie bialgebra structure of Theorem 10.18/20 and $\varGamma_{\mathcal{L}_\nu}^\star$ is the corresponding dual Poisson group mentioned in Theorem 10.18. Thus $\mathcal{H}$ *is intermediate between the (Poisson-type) "geometrical symmetries"* $F\big[\varGamma_{\mathcal{L}_\nu}^\star\big]$ *and* $U(\mathcal{L}_\nu)$, so their geometrical meaning should shed some light on it; conversely, the physical meaning of $\mathcal{H}$ should have some reflect on the physical meaning of both $F\big[\varGamma_{\mathcal{L}_\nu}^\star\big]$ and $U(\mathcal{L}_\nu)$.

Remark: The analysis in §10.9 and §10.21 yields a complete description of the deformation features of $\mathcal{H}$ via $\mathcal{H}_\hbar$ and Drinfeld's functors drawn in (5.5). In particular

$$G_+ = \varGamma_{\mathcal{L}_\nu}^\star\,, \quad \mathfrak{k}_+ \equiv \mathfrak{g}_+^\times = \big(\mathcal{L}_\nu, \delta_*\big)\,, \qquad \mathfrak{g}_- = \big(\mathcal{L}_\nu, \delta_\bullet\big)\,, \quad K_- \equiv G_-^\star = G_{\mathcal{L}_\nu}^\star \tag{10.9}$$

(as $Char(\Bbbk) = 0$) where δ_* and $\delta_\bullet$ denote the Lie cobracket on $\mathcal{L}_\nu$ defined respectively in Theorems 10.18/20 and in Theorem 10.6. Next result shows that the four objects in (10.9) are really different, though they share some common feature.

Theorem 10.22.

(a) $\big({\mathcal{H}_\hbar}^\vee\big)' \cong {\mathcal{H}_\hbar}'$ *as Poisson* $\Bbbk[\hbar]$*–algebras, but* $\big({\mathcal{H}_\hbar}^\vee\big)' \not\cong {\mathcal{H}_\hbar}'$ *as Hopf* $\Bbbk[\hbar]$*–algebras.*

(b) $\big(\mathcal{L}_\nu, \delta_*\big) \cong \big(\mathcal{L}_\nu, \delta_\bullet\big)$ *as Lie algebras, but* $\big(\mathcal{L}_\nu, \delta_*\big) \not\cong \big(\mathcal{L}_\nu, \delta_\bullet\big)$ *as Lie bialgebras.*

(c) $G_{\mathcal{L}_\nu}^\star \cong \varGamma_{\mathcal{L}_\nu}^\star$ *as (algebraic) Poisson varieties, but* $G_{\mathcal{L}_\nu}^\star \not\cong \varGamma_{\mathcal{L}_\nu}^\star$ *as (algebraic) groups.*

(d) The analogues of statements (a)–(c) hold with $\mathcal{K}$ *and* $\mathcal{L}_\nu^+$ *instead of* $\mathcal{H}$ *and* $\mathcal{L}_\nu$.

Proof. It follows from Theorem 10.8*(a)* that $\big({\mathcal{H}_\hbar}^\vee\big)'$ can be seen as a Poisson Hopf algebra, with Poisson bracket given by $\{x,y\} := \hbar^{-1}[x,y] = \hbar^{-1}(x\,y - y\,x)$ (for all $x,\, y \in \big({\mathcal{H}_\hbar}^\vee\big)'$); then $\big({\mathcal{H}_\hbar}^\vee\big)'$ is the free Poisson algebra generated by $\Big\{\, \widetilde{\mathbf{b}}_{x_n} = \widetilde{\mathbf{x}}_n = \mathbf{a}_n \,\Big|\, n \in \mathbb{N} \Big\}$; since $\mathbf{a}_n = \boldsymbol{\alpha}_n + (1-\delta_{1,n})\,{\boldsymbol{\alpha}_1}^n$ and $\boldsymbol{\alpha}_n = \mathbf{a}_n - (1-\delta_{1,n})\,{\mathbf{a}_1}^n$ $(n \in \mathbb{N}_+)$ it is also (freely) Poisson-generated by $\big\{\, \boldsymbol{\alpha}_n \,\big|\, n \in \mathbb{N} \big\}$. We also saw that ${\mathcal{H}_\hbar}'$ is the free Poisson algebra over $\big\{\, \widehat{\boldsymbol{\alpha}}_n \,\big|\, n \in \mathbb{N} \big\}$; thus mapping $\boldsymbol{\alpha}_n \mapsto \widehat{\boldsymbol{\alpha}}_n$ $(\forall\, n \in \mathbb{N})$ does define a unique Poisson algebra isomorphism $\Phi\colon \big({\mathcal{H}_\hbar}^\vee\big)' \xrightarrow{\;\cong\;} {\mathcal{H}_\hbar}'$, given by $\widetilde{\boldsymbol{\alpha}}_b := \hbar^{-d(b)}\boldsymbol{\alpha}_b \mapsto \widehat{\boldsymbol{\alpha}}_b$, for all $b \in B_\nu$. This proves the first half of *(a)*, and then also (taking semiclassical limits and spectra) of *(c)*.

The group structure of either $G_{\mathcal{L}_\nu}^\star$ or $\varGamma_{\mathcal{L}_\nu}^\star$ yields a Lie cobracket onto the cotangent space at the unit point of the above, isomorphic Poisson varieties: this cotangent space identifies with $\mathcal{L}_\nu$, and the two cobrackets are given respectively by $\delta_\bullet(x_n) = \sum_{\ell=1}^{n-1}(\ell+1)\,x_\ell \wedge x_{n-\ell}$ for $G_{\mathcal{L}_\nu}^\star$ (by Theorem 10.8) and by $\delta_*(x_n) = (n-2)\,x_{n-1} \wedge x_1$ for $\varGamma_{\mathcal{L}_\nu}^\star$ (by Theorem 10.18), for all $n \in \mathbb{N}_\nu$. It follows that $Ker(\delta_\bullet) = \{0\} \neq Ker(\delta_*)$, which implies that the two

Lie coalgebra structures on $\mathcal{L}_\nu$ are not isomorphic. This proves *(b)*, and also means that $G_{\mathcal{L}_\nu}^\star \not\cong \varGamma_{\mathcal{L}_\nu}^\star$ as (algebraic) groups, hence $F\big[G_{\mathcal{L}_\nu}^\star\big] \not\cong F\big[\varGamma_{\mathcal{L}_\nu}^\star\big]$ as Hopf $\Bbbk$–algebras, and so $\big({\mathcal{H}_\hbar}^\vee\big)' \not\cong {\mathcal{H}_\hbar}'$ as Hopf $\Bbbk[\hbar]$–algebras, which ends the proof of *(c)* and *(a)* too.

Finally, claim *(d)* should be clear: one applies the like arguments *mutatis mutandis*, and everything follows as before. $\square$

10.23 Generalizations. Plenty of features of $\mathcal{H} = \mathcal{H}^{\mathrm{dif}}$ are shared by a whole bunch of *graded* Hopf algebras, which usually arose in connection with some physical problem or some (co)homological construction, and all bear a nice combinatorial content; essentially, most of them can be described as "formal series" over indexing sets — replacing $\mathbb{N}$ — of various (combinatorial) nature: planar trees (with or without labels), forests, graphs, Feynman diagrams, etc. Besides the ice-breaking examples given by Connes and Kreimer (cf. [CK1–3]), which are all commutative or cocommutative Hopf algebras, other non-commutative non-cocommutative examples (like the one of $\mathcal{H}^{\mathrm{dif}}$) are introduced in [BF1–2], roughly through a "disabelianization process" applied to the commutative Hopf algebras of Connes and Kreimer. The most general analysis and wealth of examples in this context is due to Foissy (see [Fo1–3]), who also makes — in other terms — an interesting (although less deep than ours) study of the operators δ_n's and of the functor $H \mapsto H'$ $(H \in \mathcal{HA}_\Bbbk)$. Other examples, issued out of topological motivations, can be found in the works of Loday et al.: see e.g. [LR], and references therein.

When performing the like analysis, as we did for $\mathcal{H}$, for a graded Hopf algebra H of the afore mentioned type, *the arguments used for $\mathcal{H}$ apply essentially the same, up to minor changes, and give much the same results.* To give an example, the Hopf algebras considered by Foissy are non-commutative polynomial, say $H = \Bbbk\big\langle \{x_i\}_{i \in \mathcal{I}} \big\rangle$ for some index set $\mathcal{I}$: then one finds $\widehat{H} = {\mathcal{H}_\hbar}^\vee\big|_{\hbar=0} = U(\mathfrak{g}_-) = U(\mathcal{L}_\mathcal{I})$ where $\mathcal{L}_\mathcal{I}$ is the free Lie algebra over $\mathcal{I}$.

This opens the way to apply the crystal duality principle to all these graded Hopf algebras of great interest for their applications in mathematical physics or in topology (or whatever), with the simplest case of $\mathcal{H}^{\mathrm{dif}}$ playing the role of a toy model which realizes a clear and faithful pattern for many common features of all Hopf algebras of this kind.

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