MULTIPARAMETER QUANTUM GENERAL LINEAR SUPERGROUP

FABIO GAVARINI ♯ , MARGHERITA PAOLINI ♭

1. Introduction

The so-called "quantum groups" appear in literature in two forms: either quantised universal enveloping algebras (in short, QUEA's) over some Lie algebra $\mathfrak{g}$, or quantised function algebras over some Lie or algebraic group G. In both cases, "quantised" is meant in two possible ways, namely:

— a *formal* one, where we consider *topological* Hopf algebras, say $U_\hbar(\mathfrak{g})$ or $F_\hbar[[G]]$, over the ring $\Bbbk[[\hbar]]$ of formal power series in the deformation parameter $\hbar$,

— a *polynomial* one, where we consider *standard* Hopf algebras, $U_q(\mathfrak{g})$ or $F_q[G]$, over a base ring where an element q takes the role of "deformation parameter", e.g. $\Bbbk(q)$ or $\Bbbk\big[\,q\,,q^{-1}\big]$ (the ring of rational functions or Laurent polynomials in q).

In either case, the quantisation canonically defines, as "semiclassical limit", an additional Poisson structure on the underlying geometrical object, namely a Lie cobracket on $\mathfrak{g}$ — turning the latter into a Lie bialgebra — and a Poisson bracket on G — making it into a Poisson (Lie or algebraic) group; see [Dr] or [CP] for details.

One can also consider *multiparametric* quantisations, involving several parameters: nevertheless, only one of them "rules" the quantisation, whereas the others are responsible for the induced Poisson structure at the semiclassical limit. A typical procedure to produce such quantisations involve Hopf-theoretical deformations, either via twists or via 2–cocycles: one starts with a uniparametric quantisation, and then applying a deformation (by twist or by 2–cocycle) ends up with a multiparametr quantisation, whose "extra parameters" come from the twist or the 2–cocycle involved in the process (cf. e.g., [GG3], [GG4] and references therein for more details).

All the above applies as well, up to technicalities, to the context of "quantum supergroups", i.e. quantisations of Lie superalgebras and supergroups.

In the present paper, we deal with the general linear Lie superalgebra and supergroup, that is $\mathfrak{gl}_n$ and GL_n endowed with some "parity". In this setup, *uniparametric* quantised universal enveloping superalgebras (in short, QUESA's) have been introduced in [Ya1] (in great generality) in a *formal* version, and then taken up again in [Zha] in *polynomial* form. Starting from that, *multiparametric* QUESA's have been constructed in [GGP] via the process of deformation by twist explained above.

Our goal is to work out the dual side, i.e. to introduce suitable dual objects to the QUESA's for $\mathfrak{gl}_n^p$ mentioned above, in all their variants — uniparametric or multiparametric, formal or polynomial.

In the uniparametric setting, we start from Yamane's QUESA $U_\hbar\big(\mathfrak{gl}_n^p\big)$, where the superscript "p" accounts for the underlying parity. Through a direct approach, we construct its full linear dual, which is concretely realised as a topological Hopf superalgebra $F_\hbar\big[\big[GL_n^p\big]\big]$ with a non-degenerate Hopf pairing with $U_\hbar\big(\mathfrak{gl}_n^p\big)$. In particular, we find that this $F_\hbar\big[\big[GL_n^p\big]\big]$ is indeed a quantum formal series Hopf superalgebra

2020 *MSC:* 20G42, 16T05, 81R50 — *Keywords:* Quantum Groups, Quantum Function Algebras.

(in short, QFSHA) as we were looking for: we provide for it an explicit presentation by generators and relations and a suitable PBW-like theorem.

For the multiparametric side of the story, we rely on the uniparametric one and resort to a deformation procedure. Indeed, as every multiparametric QUESA $U_\hbar^\Phi(\mathfrak{gl}_n^p)$ from [GGP] is obtained from Yamane's QUESA $U_\hbar(\mathfrak{gl}_n^p)$ via deformation by some twist $\mathcal{F}_\Phi$, one can get the dual $\big(U_\hbar^\Phi(\mathfrak{gl}_n^p)\big)^*$ as deformation of $\big(U_\hbar(\mathfrak{gl}_n^p)\big)^*$ by the 2–cocycle σ_Φ corresponding to $\mathcal{F}_\Phi$. But $\big(U_\hbar^\Phi(\mathfrak{gl}_n^p)\big)^* = F_\hbar\big[\big[GL_n^p\big]\big]$, the uniparametric QFSHA that we just constructed; so we only have to compute the deformation $\big(F_\hbar\big[\big[GL_n^p\big]\big]\big)_{\sigma_\Phi}$. The final outcome is a multiparametric QFSHA $F_\hbar^\Phi\big[\big[GL_n^p\big]\big]$ for which we find a presentation by generators and relations and a PBW-like theorem.

After achieving our goal for *formal* quantisations, we obtain the parallel result for *polynomial* ones in a very simple way — roughly, selecting suitable subalgebras inside $F_\hbar\big[\big[GL_n^p\big]\big]$ (for the uniparametric case) and $F_\hbar^\Phi\big[\big[GL_n^p\big]\big]$ (for the multiparametric one). In particular, for the latter we discuss a bit its direct comparison with Manin's multiparametric QFSA's introduced in [Ma].

ACKNOWLEDGEMENTS

The authors thankfully acknowledge support by: *(1)* the MIUR *Excellence Department Project MatMod@TOV (CUP E83C23000330006)* awarded to the Department of Mathematics, University of Rome "Tor Vergata" (Italy), *(2)* the research branch *GNSAGA* of INdAM (Italy), *(3)* the European research network *COST Action CaLISTA CA21109*.

References

[An] N. Andruskiewitsch, *Lie superbialgebras and Poisson-Lie supergroups*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 147–163.

[CG] N. Ciccoli, L. Guerra, *The Variety of Lie Bialgebras*, J. Lie Theory **13** (2003), no. 2, 577–588.

[CP] V. Chari, A. Pressley, *A guide to quantum group*, Cambridge University Press, Cambridge, 1995.

[Dr] V. G. Drinfeld, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. **1**, **2** (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[EG] B. Enriquez, N. Geer, *Compatibility of quantization functors of Lie bialgebras with duality and doubling operations*, Selecta Math. (N.S.) **15** (2009), no. 1, 1–38.

[Ga] F. Gavarini, *The quantum duality principle*, Annales de l'Institut Fourier **52** (2002), no. 3, 809–834.

[GG1] G. A. García, F. Gavarini, *Twisted deformations vs. cocycle deformations for quantum groups*, Communications in Contemporary Mathematics **23** (2021), no. 8 — 2050084 (56 pages).

[GG2] ______, ______, *Multiparameter quantum groups at roots of unity*, J. Noncommut. Geom. **16** (2022), no. 3, 839–926.

[GG3] ______, ______, *Formal multiparameter quantum groups, deformations and specializations*, Annales de l'Institut Fourier, 117 pages (to appear); printed on-line on February 5th, 2024, **DOI:** `10.5802/aif.3675` — preprint `arXiv:2203.11023 [math.QA]` (2022).

[GG4] ______ , ______ , *Quantum Group Deformations and Quantum R–(co)matrices vs. Quantum Duality Principle*, 72 pages — preprint `arXiv:2403.15096v2 [math.QA]` (2024).

[GG5] ______ , ______ , *GaGa algebras*, in progress.

[GG6] ______ , ______ , *Quantum GaGa algebras*, in progress.

[GGP] G. A. García, F. Gavarini, M. Paolini, *Multiparameter Quantum Supergroups, Deformations and Specializations*, 65 pages — preprint `arXiv:2410.22549 [math.QA]` (2024).

[Ge1] N. Geer, *Etingof-Kazhdan quantization of Lie superbialgebras*, Adv. Math. **207** (2006), no. 1, 1–38.

[Ge2] ______ , *Some remarks on quantized Lie superalgebras of classical type*, J. Algebra **314** (2007), no. 2, 565–580.

[Ka1] V. G. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge, 1990.

[Ka2] ______ , *Lie superalgebras*, Advances in Math. **26** (1977), no. 1, 8–96.

[Ko] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, in: *Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975)*, pp. 177–306, Lecture Notes in Math., Vol. **570**, Springer, Berlin–New York, 1977.

[KS] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997, xx+552 .

[Le] D. A. Leites, *Cohomologies of Lie superalgebras*, Functional Analysis and Its Applications 9(4) (1975), 340–341.

[Ma] Y. Manin, *Multiparametric Quantum Deformation of the General Linear Supergroup*, Communications in Mathematical Physics **123** (1989), 163–175.

[Mj] S. Majid, *Foundations of quantum groups*, Cambridge University Press, Cambridge, 1995.

[MW] S. Merkulov, T. Willwacher, *Deformation theory of Lie bialgebra properads*, Geometry and physics, Vol. I, 219–247, Oxford Univ. Press, Oxford, 2018.

[Ya1] H. Yamane, *Quantized enveloping algebras associated to simple Lie superalgebras and their universal R–matrices*, Publ. Res. Inst. Math. Sci. **30** (1994), no. 1, 15–87.

[Ya2] ______ , *On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras*, Publ. Res. Inst. Math. Sci. **35** (1999), no. 3, 321–390.

[Ya3] ______ , *Errata to: "On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras"*, Publ. Res. Inst. Math. Sci. **37** (2001), no. 4, 615–619.

[Zha] Zhang R. B., *Structure and Representations of the Quantum General Linear Supergroup*, Commun. Math. Phys. **195** (1998), 525–547.

♯ Dipartimento di Matematica,
Università degli Studi di Roma "Tor Vergata" — INdAM / GNSAGA
Via della ricerca scientifica 1, I-00133 Roma, ITALY — `gavarini@mat.uniroma2.it`

♭ Independent Researcher — `margherita.paolini.mp@gmail.com`