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DUALITY FOR ACTION BIALGEBROIDS

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INTRODUCTION

In the theory of (Lie) groupoids, a first important subclass is that of *action groupoids*, that is, those that express a group action: in fact, one can see this as the first non-trivial example that naturally extends the notion of groups to that of groupoids. The corresponding infinitesimal notion is that of *action Lie algebroid*, that is, a Lie algebroid that arises from a Lie algebra acting on some representation space, see, *e.g.*, [Mac, CaLaPi].

In a purely algebraic language, groups correspond, loosely speaking, to (special subclasses of) Hopf algebras, and groupoids to Hopf algebroids, or, more in general, to bialgebroids; in particular, action groupoids correspond to the subclass of *action bialgebroids*, also known as *smash product bialgebroids*, or still *scalar extension bialgebroids*, see, *e.g.*, [BáSz, St, StŠk].

A special feature of the notion of Hopf algebra is its self-duality, in that the linear dual of a (finitely generated) Hopf algebra is again a Hopf algebra (up to technicalities). This nice behaviour extends to the more general notion of bialgebroids (and even Hopf algebroids) that are also self-dual, up to an important caveat: indeed, the very notion of bialgebroid is two-fold as one defines both *left* and *right* bialgebroids. Moreover, every bialgebroid (either left or right) has two natural duals, a *left* and a *right* one; hence, both a left and a right linear duality functor are defined and starting with left or right bialgebroids, this construction eventually gives rise to four linear duality functors. Finally, these four duality functors switch chirality: the dual of a left bialgebroid (U, A) is a right one, and vice versa.

All this is well known [KadSz] and fully settled under some projectivity and finiteness assumptions for (one of) the underlying A-module structure; if this fails to be, one still has some control by considering notions such as topological bialgebroids, or the like, similar to the Hopf algebra case. In fact, this is what happens with universal enveloping algebras VL and jet bialgebroids JL, see [KoPo], associated to some Lie-Rinehart algebra (L, A): assuming that L is finitely generated projective over the base algebra A, both VL and JL are not finitely generated projective but they can still be seen as dual to each other, *i.e.*, non-degenerately paired via linear duality functors as above, up to some technicalities (including that JL is a bialgebroid only in a suitable topological sense).

A second special type of duality arises when dealing with bialgebroids W_h that are (formal) quantisations of some VL or some JL, for some Lie-Rinehart algebra (L, A) as above: one has two functors $W_h \to W_h^\vee$ and $W_h \to W_h'$ such that

– if W_h is a quantisation of JL, then $W_h^\vee$ is a quantisation of $V(L^*)$,
– if W_h is a quantisation of VL, then W_h' is a quantisation of $J(L^*)$,

where $L^* = \mathrm{Hom}_A(L, A)$ denotes the dual Lie-Rinehart algebra. This phenomenon, known as *quantum duality principle*, sprouts from a key idea of Drinfeld for quantum groups, later extended to quantum groupoids in [ChGa]. Note that, in this case, both functors *preserve* chirality, in that if W_h is a left (resp. right) bialgebroid, then so are $W_h^\vee$ and W_h'.

In this paper, we investigate what happens with linear duality (in general) and Drinfeld duality functors (in the quantum setup) when working with action bialgebroids.

First of all, an action bialgebroid has the form $R\#U$, where (U, A) is a left bialgebroid and R a braided commutative monoid in the category of Yetter-Drinfeld modules over U. Theorem 2.5 presents a couple of criteria in order to recognise those bialgebroids that are action bialgebroids. Starting from a right bialgebroid (V, B) and for a braided commutative monoid S

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in a respective Yetter-Drinfeld category, a similar construction of an action bialgebroid of the form $V\#S$ applies. In Lemma 1.3 and Theorem 2.7, we prove (see the main text for details and notation):

Theorem. *Let (U, A) be a left bialgebroid such that ${}_{\triangleright}U$ is finitely generated A-projective and let $U_* := \mathrm{Hom}_A({}_{\triangleright}U, A)$ be its left dual. Then, if R is a braided commutative monoid in ${}_U\mathbf{YD}^U$, it is also so in ${}^{U_*}\mathbf{YD}_{U_*}$ and vice versa; in particular, there is an isomorphism*

$$\mathrm{Hom}_R(R\#U, R) \simeq U_*\#R,$$

of right bialgebroids, which, if U is a right Hopf algebroid (over a left A-bialgebroid), is an isomorphism of left Hopf algebroids (over right R-bialgebroids).

This expresses the fact that (linear) duality commutes with the action bialgebroid construction. As for Drinfeld duality, we study the effect of the two Drinfeld functors $(-)^\vee$ and $(-)'$ to quantum groupoids that are, in addition, action bialgebroids, say, $F_h\#R_h$ and $R_h\#U_h$. The outcome is formulated in Theorems 3.13 & 3.14, see again the main text for the mentioned technical details and notation:

Theorem.

(i) *Let (F_h, A_h) be left quantum formal series bialgebroid and R_h an h-topologically complete $k[[h]]$-algebra that is a braided commutative monoid in ${}_{F_h}\mathbf{YD}^{F_h}$, and such that R_h/hR_h is commutative. Then, up to a technical assumption, R_h is a braided commutative YD algebra over $F_h^\vee$ as well, and there is a canonical isomorphism*

$$R_h\#F_h^\vee \;\simeq\; (R_h\#F_h)^\vee$$

of topological left bialgebroids over R_h. In particular, $R_h\#F_h^\vee$ is a left quantum universal enveloping bialgebroid over R_h.

(ii) *Let (U_h, A_h) be a left quantum universal enveloping bialgebroid and R_h an h-topologically complete $k[[h]]$-algebra that is a braided commutative monoid in ${}_{U_h}\mathbf{YD}^{U_h}$, and such that R_h/hR_h is commutative. Then, up to a technical assumption, R_h is a braided commutative YD algebra over U_h' as well, and there is a canonical isomorphism*

$$R_h\#U_h' \;\simeq\; (R_h\#U_h)'$$

of left bialgebroids over R_h. In particular, $R_h\#U_h'$ is a left quantum formal series bialgebroid over R_h.

These statements, in turn, express the fact that Drinfeld functors commute with the action bialgebroid construction as well. It is worth stressing that for proving $(R_h\#U_h)' \simeq R_h\#U_h'$ we actually resort to a linear duality trick, relying on the previous results (suitably adapted to the specific situation).

Some final words about the organisation of the paper. Section 1 presents some categorical equivalences regarding Yetter-Drinfeld modules and linear duality in the realm of bialgebroids. Section 2 focuses on action bialgebroids, extending the well-known construction from [BrzMi] from Hopf algebras to (right) Hopf algebroids (over left bialgebroids), formulating a right handed-version of this construction, presenting a characterisation of such action bialgebroids, and finally also the above mentioned results on the dual of action bialgebroids. Section 3 is devoted to action bialgebroids in the quantum framework, namely the one set up in [ChGa]: in particular, building upon the results therein about Drinfeld functors and the *Quantum Duality Principle*, we apply Drinfeld functors to action bialgebroids, thus finding that those functors "commute" with the action bialgebroid construction. Finally, the appendices list essentially all technical tools needed in the paper.

Notation. Let k be a commutative ring, possibly a field or of characteristic zero. As customary, unadorned tensor products or Homs refer to those over k. Sweedler notation subscripts typically refer to left bialgebroids, superscripts to right bialgebroids, while left comodules are indicated by round brackets, right comodules by square brackets. The three appendices give a detailed account on all employed bialgebroid related notation.

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