MULTIPARAMETER QUANTUM SUPERGROUPS, DEFORMATIONS AND SPECIALIZATIONS

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To the memory of Pierre Cartier, with our deepest admiration.

INTRODUCTION

In the last forty years, quantum groups — mainly in the form of quantized universal enveloping algebras, hereafter shortened into QUEA's — have been thoroughly studied, both in their *formal* version (à la Drinfeld, say), and in their *polynomial* formulation (following Jimbo and Lusztig). The semiclassical limit underlying them is given by Lie bialgebras (infinitesimally) and Poisson-Lie groups (globally): conversely, the "quantization problem" for a Lie bialgebra consists in the quest for some QUEA whose semiclassical limit yields back the given Lie bialgebra.

Quite soon, *multiparameter* quantum groups have been introduced too, and specifically multiparameter QUEA's — or MpQUEA's, in short. A standard method for that uses the technique of *deformation* of Hopf algebras, which comes in two versions: deformation by twist — where only the comultiplication and the antipode are modified — and deformation by 2–cocycle — where one changes only the multiplication and the antipode. The twist deformation was applied by Reshetikhin to Drinfeld's formal QUEA's $U_\hbar(\mathfrak{g})$, thus finding some multiparameter QUEA's in which the new, "additional" parameters show up to describe the new coalgebra structure. The second method instead was applied by several people, all working with Jimbo & Lusztig's polynomial QUEA $U_q(\mathfrak{g})$ — in a suitable "quantum double version" — thus providing some other multiparameter QUEA's whose parameters enter in the description of the (new) algebra structure. A common, *unified* point of view on this matter was presented in [GG3] (see also [GG1]): the key message there is that, although the two families of MpQUEA's arising from deformations by twist or by 2–cocycle seem to be entirely disconnected, they form instead one and the same family. In addition, a second key result from [GG3] is that deformations of either kind of any MpQUEA still provide some new MpQUEA's, in this same family.

Looking at MpQUEA's, it was quickly realized that their many parameters could always be re-arranged so that only one of them kept the role of "quantization parameter", while all the other took instead a "geometrical" meaning. Namely, when one specializes the single "quantum parameter" and thus gets the semiclassical limit, the remaining parameters enter in the description of the Poisson structure (in the form of a Lie cobracket) that is induced onto the underlying Lie algebra. In this way, specialization of MpQUEA's yields *multiparameter* structures onto the semiclassical limit, so one ends up with multiparameter Lie bialgebras (or MpLbA's, in

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short). On the other hand, one can also introduce this family of MpLbA's via an independent construction, and prove that this family is stable under deformations (in the sense of Lie bialgebra theory) by twist or by 2–cocycle; see [GG3] for details.

The present paper extends the previous world-building process to the setup of quantum *super*groups: namely, we introduce *formal multiparameter quantized enveloping superalgebras* (or *formal MpQUESA's*, in short) and we study their deformations, by twist and by 2–cocycle. Lie superalgebras and their quantizations, the quantum universal enveloping superalgebras (QUESA), can be regarded as Lie algebras and quantum universal enveloping algebras in the category of super vector spaces, that is, $\mathbb{Z}_2$-graded vector spaces. As noted by Majid [Mj], both classical and super objects are related via the process of the so-called *bosonization* which yields a functor from one category to the other. Despite this strong connection, Lie superalgebras and QUESA's are interesting on their own, mainly because of their wide range of applications. Simple Lie superalgebras over an algebraically closed field of characteristic zero were classified by Kac in [Ka2]: this classification is closely related to that of simple Lie as it is given in terms of Cartan matrices and (generalized) Dynkin diagrams. The distinguished issue here is that isomorphic Lie superbialgebras may have non-isomorphic Dynkin diagrams. As the study of MpQUEA's (and their semiclassical limit) was mostly bounded to those whose semiclassical limits were Lie algebras of contragredient type, associated with symmetrizable Cartan matrices, we focus in this paper onto MpQUESA's whose semiclassical limit are simple contragredient Lie superalgebras of type A–G (w.r.t. the classification in [Ka2]). These types of QUESA's were first introduced by Yamane, in [Ya1], who defined formal QUESA's of type A–G: for them, relations are not only exhausted by commutation relations and super quantum Serre relations, as one also needs to add higher order relations. Likewise, the MpLSbA's that we construct are, as Lie superalgebras, the contragredient ones of type A–G, again.

Specifically, we introduce multiparameter versions of Yamane's formal QUESA's (in short, FoMpQUESA's), see Definition 4.2.2, by mimicking the strategy we developed in [GG3] for the non-super case. The very definition is based upon the connection between the multiparameters and the action of a fixed commutative subsuperalgebra: this is encoded in the notion of *realization* of a multiparameter matrix P, much like for Kac-Moody algebras. This allows us to relate the quantum objects with their semiclassical limit, and also multiparameter objects with Yamane's standard ones: the latter step goes through deformation(s). To do this, we consider deformations by toral twists of Yamane's uniparameter QUESA's, just like Reshetikhin did with Drinfeld's uniparameter QUEA's, thus finding suitable new, deformed QUESA's that are now "multiparameter" ones, as they depend on the many parameters coming from the twist: these parameters, though, only show up in the description of the coalgebra structure, while the algebra structure is independent of them. As a consequence, one may endow these deformed QUESA's with a Hopf superalgebra structure. The second, key idea is to perform on these MpQUESA's a careful "change of presentation", modifying the original generators (from Yamane's presentation) into new ones: the outcome is a new presentation where the parameters appear in the defining relations, whereas the description of the coalgebra structure now is entirely parameter-free. Finally, we take this last presentation as "the" pattern for our definition of MpQUESA, given again by generators and relations, mimicking the last presentation of the deformed MpQUEA's that we found after the "change-the-generators" step. The arguments used in this analysis apply again to study, more in general, the deformations by twist of *any* one of our MpQUESA's: thus we find that any such deformation is yet another MpQUESA — this is proved in Theorem 5.1.3. Moreover, we prove in Proposition 4.2.8 that any FoMpQUESA can be obtained from a Yamane's uniparameter QUESA by a proper choice of the twist. Then we draw our attention to an extension of the dual notion of twist, *polar–2–cocycles*, and show in Theorem 5.2.15 that actually the family of FoMpQUESA is again stable by deformations of the superalgebra structure by a polar–2–cocycle. In conclusion, any FoMpQUESA can be obtained form Yamane's uniparameter QUESA either by a twist deformation or by a polar–2–cocycle deformation. So, for every FoMpQUESA one can decide to focus the dependence on the discrete multiparameters either on the supercoalgebra structure or on the superalgebra structure: in fact, we chose the second option. As in the non-super setting [GG3], everything is encoded in the notion of "realization" of multiparameters.

Quite naturally, the same strategy used in the quantum setup can be adopted as well in the semiclassical framework, even with some simplifications. Therefore, we develop our construction of MpLSbA's, and the study of their deformations, *independently* of the quantum setup — though everything about that might be deduced by specialization from the quantum level (cf. Theorem 6.1.2). We obtain then a self-contained "chapter" of Lie superalgebra theory, that introduces a new family of multiparameter Lie superalgebras (over simple, contragredient Lie superalgebras of type A–G) and their deformations, the key fact being that this family is stable under deformations (by twist and by 2–cocycle), see Theorem 3.3.3 and Theorem 3.4.3. These MpLSbA's come equipped with a presentation "à la Serre", in which the parameters rule the Lie superalgebra structure (cf. §3.2). In addition, every such MpLSbA admits an alternative presentation, in which the Lie algebra structure stands fixed, while the Lie coalgebra structure varies according to the multiparameter matrix P. Like in the quantum setup, the isomorphism between the two presentations is quite meaningful, as it boils down to a well-chosen change of generators. In short, we might say that our multiparameters are encoded in realizations, FoMpQUESA's are quantizations of MpLSbA's, and multiparameter objects are given by deformations of either the (super)algebra or the (super)coalgebra structure of uniparameter objects — see a diagrammatic summary in §6.3.

Finally, we briefly introduce *polynomial* MpQUESA's: mimicking the recipe of Lustig & Jimbo for polynomial QUEA's, these are certain subalgebras — up to modifying the ground ring — of our FoMpQUESA's. We also study both types of deformations for these objects. Now the polar–2–cocycles become actually Hopf 2–cocycles, and the deformation of the algebraic structure is well-understood. On the other hand, the twist deformations need to satisfy an arithmetic condition in order to be meaningful; nevertheless, these types of deformations cannot be seen, *strictu sensu*, as genuine "twist deformations" of our polynomial MpQUESA's.

A last word about the organization of the paper, and its detailed content.

In Section 2 we introduce the "combinatorial data" underlying our constructions of MpLSbA's and FoMpQUESA's alike: the notion of multiparameters, their realizations and deformations. Then in Section 3 we present our MpLSbA's, given by an explicit presentation (involving parameters) and bearing a Lie bialgebra structure which arise as deformation of one arising from Yamane's QUESA's. We also study their deformations, both by (toral) twists and by (toral) 2–cocycles: in particular, this proves that the family of these Lie superbialgebras is *stable w.r.t. deformations*. In addition, before presenting our new results about multiparameter objects and their deformations, we quickly resume the general theory of Lie superbialgebras and their deformations, as the literature on the subject seems to be quite poor.

Section 4 is dedicated to our newly minted notion of FoMpQUESA's. We begin by introducing them as superalgebras with a given presentation (by generators and relations), which closely mimics that of Yamane's QUESA's in [Ya1] but for involving more general parameters. Then we prove that they do bear a structure of Hopf superalgebra, which is obtained from that of a usual Yamane's QUESA through deformation by a suitable (toral) twist followed by a suitable "change of variables". Finally, we prove the functoriality (w.r.t. the underlying, Cartan-like combinatorial data) of our construction of FoMpQUEA's, and their triangular decomposition.

We spend Section 5 to discuss deformations of FoMpQUEA's by (toral) twists and by (toral) polar–2–cocycles. In short, we prove that these deformations turn FoMpQUEA's into new FoMpQUEA's again, hence the family of all FoMpQUESA's is *stable by such deformations*. Moreover, under mild assumptions things go the other way around too, so that any FoMpQUESA's can be realized as a suitable deformation — both by (toral) twist and by (toral) polar–2–cocycle — of one of Yamane's uniparameter QUESA's.

In Section 6 we compare the classical and the quantum setups. Namely, we perform specializations of FoMpQUESA's, showing that their semiclassical limit is always a MpLSbA, with the same underlying "Cartan datum". Conversely, any possible MpLSbA arises as such a limit — in other words, for any MpLSbA there is a suitable FoMpQUESA which is quantization of it. As a second step, we compare deformations before and after specialization/quantization: the outcome is, in a nutshell, that "specialization/quantization and deformation commute with each other".

Finally, in Section 7 we introduce the notion of *polynomial MpQUESA's*, and we study their behavior under the processes of both types of deformations.

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