# QUANTUM GROUP DEFORMATIONS AND QUANTUM $R$–(CO)MATRICES VS. QUANTUM DUALITY PRINCIPLE

GASTÓN ANDRÉS GARCÍA$^\flat$ , FABIO GAVARINI$^\sharp$

Abstract. In this paper we describe the effect on quantum groups — namely, both QUEA's and QFSHA's — of deformations by twist and by 2–cocycles, showing how such deformations affect the semiclassical limit.

As a second, more important task, we discuss how these deformation procedures can be extended, via a formal variation of the original recipes, using *quasi-twists* and *quasi-2–cocycles*. These new recipes seemingly should make no sense at all, yet we prove that they do work, thus providing more general deformation procedures. Later on, we explain the underlying motivation: this comes from *Quantum Duality Principle*, through which every "quasi-twist/2–cocycle" for a given quantum group can be seen as a standard twist/2–cocycle for another quantum group, associated to the original one via the appropriate Drinfeld functor.

Finally, we consider standard constructions involving $R$–(co)matrices for Hopf algebras. First we adapt them to quantum groups, then we show that they extend to the case of *quasi–$R$–*(co)matrices, and finally we discuss how these constructions interact with the Quantum Duality Principle. This also yields new symmetries for the underlying pair of dual Poisson (formal) groups that one gets by specialization.

Contents

1. Introduction 2
2. Quantum groups, Quantum Duality Principle, and deformations 4
2.1. Lie bialgebras and Lie deformations 4
2.2. Hopf algebra deformations and $R$–(co)matrices 5
2.3. Quantum groups 8
2.4. The Quantum Duality Principle 11
3. Deformations of quantum groups 12
3.1. Deformations by twist of QUEA's 12
3.2. Deformations by 2-cocycle of QFSHA's 16
3.3. Deformations by quasi-2-cocycle of QUEA's 22
3.4. Deformations by quasi-twist of QFSHA's 33
3.5. Duality issues 42
4. Deformations vs. QDP 43
4.1. Some auxiliary results 43
4.2. Drinfeld's functors and "quasi-deformations" 44
4.3. Drinfeld's functors and (standard) deformations 45
5. Morphisms in the "(co)quasitriangular" case 46
5.1. $R$–matrices and $\varrho$–comatrices w.r.t. QDP: quasi–(co)matrices 46
5.2. Morphisms from $R$–matrices and $\varrho$–comatrices 48
5.3. Morphisms from quasi–$R$–matrices and quasi–$\varrho$–comatrices 53
5.4. Semiclassical morphisms induced by specialization 55
References 57

2020 *MSC:* 17B37, 17B62 — *Keywords:* Quantum Groups, Quantum Enveloping Algebras.

## 1. Introduction

In Hopf algebra theory, there exists a well-established theory of "deformations" that are produced via specific tools, namely *twists* in one case and *2–cocycles* in the other case. Given a Hopf algebra $H$, a twist for it is a suitable element $\mathcal{F} \in H \otimes H$, while (dually) a 2–cocycle is a suitable 2–form $\sigma \in (H \otimes H)^*$. Deformation by $\mathcal{F}$ provides $H$ with a new Hopf algebra structure, by modifying the coproduct (and the antipode) but not the product, while deformation by $\sigma$ endows $H$ with yet another Hopf structure by changing the product (and the antipode) but not the coproduct.

Quantum groups are Hopf algebras of special type, in two versions: QUEAs (= quantized universal enveloping algebras) and QFSHAs (= quantized formal series Hopf algebras). Roughly speaking, a QUEA is a (topological) Hopf algebra $U_\hbar$ over the $\Bbbk$–algebra of formal power series $\Bbbk[[\hbar]]$ such that $U_0 := U_\hbar \big/ \hbar\, U_\hbar$ is isomorphic to $U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$. Then $U(\mathfrak{g})$ inherits from $U_\hbar$ a Poisson cobracket, which makes it into a co-Poisson Hopf algebra, hence $\mathfrak{g}$ bears a Lie cobracket making it into a Lie bialgebra. One then says that $U_\hbar$ is a *quantization* of the co-Poisson Hopf algebra $U(\mathfrak{g})$, or just of the Lie bialgebra $\mathfrak{g}$. Dually, a QFSHA is a (topological) Hopf algebra $F_\hbar$ over $\Bbbk[[\hbar]]$ such that $F_0 := F_\hbar \big/ \hbar\, F_\hbar$ is isomorphic to $F[[G]]$ for some formal algebraic group $G$. Then $F[[G]]$ inherits from $F_\hbar$ a Poisson bracket, which makes it into a Poisson Hopf algebra, thus $G$ bears a Poisson structure which makes it into a formal Poisson (algebraic) group. One says then that $F_\hbar$ is a *quantization* of the Poisson Hopf algebra $F[[G]]$, or just of the (formal) Poisson group $G$.

As a general philosophy, from any Hopf-theoretical notion — at the quantum level — one typically infers a Lie-theoretical counterpart — at the semiclassical level. When dealing with deformations, this leads to devising suitable notions of "twists" and "2–cocycles" for Lie bialgebras as well as "deformations" (of Lie bialgebras) by them. In particular, a deformation by twist yields a new Lie bialgebra structure where only the Lie cobracket is modified, whereas deformation by 2–cocycle defines yet another, similar structure where only the Lie bracket is changed.

Via this recipe, we expect the following: when we deform (as a Hopf algebra) a quantization $U_\hbar$ of $\mathfrak{g}$ by a twist which is trivial modulo $\hbar$, we get a quantization of $\mathfrak{g}'$, the latter being a deformation by twist (as a Lie bialgebra) of $\mathfrak{g}$: moreover, the (Lie) twist working on $\mathfrak{g}$ is "induced" by the (Hopf) twist that works upon $U_\hbar$, namely the former (Lie) twist is the "semiclassical limit" of the latter (Hopf) twist.

Dually, the following also should hold: when we deform (as a Hopf algebra) a quantization $F_\hbar$ of $G$ by a 2–cocycle which is trivial modulo $\hbar$, we get a quantization of $G'$, the latter being a (formal) Poisson group whose cotangent Lie bialgebra is a deformation by 2–cocycle of $\mathfrak{g}^* := \mathit{Lie}(G)^*$: moreover, the (Lie) 2–cocycle acting on $\mathfrak{g}^*$ is "induced" by the (Hopf) 2–cocycle that acts on $F_\hbar$, namely the former (Lie) 2–cocycle is the "semiclassical limit", in some sense, of the latter (Hopf) 2–cocycle.

Nevertheless, neither of the two results mentioned above seems to be published anywhere in literature (to the best of the authors' knowledge, say). Therefore, as a first contribution in this paper we provide a full, complete statement and proof for the above sketched results, turning them into well-established theorems.

As a second step — our main contribution in this paper — we extend the notions of (Hopf) twist and 2–cocycle, as well as the construction of (Hopf) deformations by them, to a wider setup. Namely, we introduce the notions of *quasi–twist* for a QFSHA and of *quasi–2–cocycle* for a QUEA: roughly speaking, a quasi–twist for $F_\hbar$ has the formal Hopf properties of a twist but has the form $\exp\big(\hbar^{-1}\varphi\big)$, while any twist (trivial modulo $\hbar$) looks like $\exp\big(\hbar^{+1}\phi\big)$ — and similarly for the link between quasi–2–cocycles and 2–cocycles. Thus even the very definition of these "quasi–objects", at least in this form, seems to be problematic — as multiplying by $\hbar^{-1}$ is meaningless. In spite of this, we show that the recipe defining deformations still makes sense if we replace "twists" with "quasi–twists", resp. "2–cocycles" with "quasi–2–cocycles". Moreover, we can describe the semiclassical limit of these deformations (by "quasi–objects"), again in terms of deformations of Lie bialgebras by some (Lie) twist, resp. 2–cocycle, that can be explicitly read out as the semiclassical limit of the quantum (Hopf) quasi–twist, resp. quasi–2–cocycle, that we started with. In a nutshell, we find the perfect "quasi–versions" of the results mentioned above for standard quantum group deformations, i.e. those by twist or by 2–cocycle.

The fact that "deformations by *quasi*–objects" do make sense can be explained in light of the *Quantum Duality Principle (=QDP)*. In fact, the latter provides functorial recipes (via *Drinfeld's functors*) which turn any QUEA into a QFSHA and any QFSHA into a QUEA. Then, through the QDP lens, every "quasi-twist" for a QFSHA, resp. every "quasi–2–cocycle" for a QUEA, is just a sheer standard twist, resp. 2–cocycle, for the QUEA, resp. the QFSHA, obtained when applying the appropriate Drinfeld functor. In this way, our deformations "by quasi–objects" turn out to be tightly related with standard ones, but applied to different quantum groups. Nevertheless, one still has to prove that the (standard) deformation applied to the new quantum group can actually be adapted to the original quantum group.

Finally, we consider some constructions of morphisms that, in general Hopf algebra theory, are provided by $R$–matrices or $\varrho$–comatrices. We apply these constructions to quantum groups, showing that their outcome is much finer than expected from the general theory, and bringing to light their geometrical meaning at the semiclassical level. In addition, we improve those results as follows: we introduce the notions of *quasi–$R$–matrices* and *quasi–$\varrho$–comatrices* (much in the same spirit as with quasi–twists and quasi–2–cocycles), and then we extend the construction of the above morphisms to quasi–$R$–matrices and quasi–$\varrho$–comatrices, again involving the QDP.

The paper is organized as follows.

In §2 we quickly recall the material we work with. In §3 we present the bulk of the paper First we study deformations by twist and by 2–cocycles, then we introduce *quasi–2–cocycles* and *quasi–twists* and the procedures of deformations by these. All this material is discussed again in §4, in light of the *Quantum Duality Principle*. Finally, in §5 we study the morphisms associated with $R$–matrices or $\varrho$–comatrices in the case of quantum groups, also explaining their meaning at the semiclassical limit. Moreover, we extend those constructions and results to the newly minted notions of quasi–$R$–matrices and quasi–$\varrho$–comatrices.

N.B.: a longer version of this work, including full-detailed computations, is available on-line as electronic preprint [GaGa3].

ACKNOWLEDGEMENTS

This article was partially supported by: *(1)* the European research network *COST Action CaLISTA CA21109*; *(2)* CONICET, ANPCyT, Secyt (Argentina), *(3)* the MIUR *Excellence Department Project MatMod@TOV (CUP E83C23000330006)* awarded to the Department of Mathematics, University of Rome "Tor Vergata" (Italy), *(4)* INdAM research branch GNSAGA (Italy).

The authors thank Benjamin Enriquez for sparking their interest in the topics of §5.

## 2. Quantum groups, Quantum Duality Principle, and deformations

In this section we recap the basic notions we deal with in this paper: Lie bialgebras, quantum groups, deformations of both, and the Quantum Duality Principle.

### 2.1. Lie bialgebras and Lie deformations.

In this subsection we recall some definitions and basic facts about Lie bialgebras and their deformations. For a more detailed treatment we refer to [CP], [Mj].

Throughout the paper, $\Bbbk$ will be a field of characteristic zero.

**2.1.1. Generalities.** A *Lie bialgebra* is a triple $\big(\mathfrak{g}\,;[\ ,\ ],\delta\big)$ such that $(\mathfrak{g},[\ ,\ ])$ is a Lie algebra over $\Bbbk$ , $(\mathfrak{g},\delta)$ is a *Lie coalgebra* with *Lie cobracket* $\delta:\mathfrak{g}\longrightarrow\mathfrak{g}\wedge\mathfrak{g}$ , i.e. $\delta^*:\mathfrak{g}^*\wedge\mathfrak{g}^*\longrightarrow\mathfrak{g}^*$ is a Lie algebra bracket on $\mathfrak{g}^*$ ), and the two structures are linked by the constraint that $\delta$ is a 1–cocycle for the Chevalley-Eilenberg cohomology of the Lie algebra $\big(\mathfrak{g}\,;[\ ,\ ]\big)$ with coefficients in $\mathfrak{g}\wedge\mathfrak{g}$ :

$$\begin{aligned}\delta([x,y]) \;&=\; \mathrm{ad}_x\big(\delta(y)\big)-\mathrm{ad}_y\big(\delta(x)\big) \;=\\ &=\; \big[x,y_{[1]}\big]\otimes y_{[2]}+y_{[1]}\otimes\big[x,y_{[2]}\big]-\big[y,x_{[1]}\big]\otimes x_{[2]}-x_{[1]}\otimes\big[y,x_{[2]}\big]\end{aligned}\tag{2.1}$$

using Sweedler's-like notation $\delta(x)=x_{[1]}\otimes x_{[2]}$ for any $x\in\mathfrak{g}$ . We write also $x\wedge y:=2^{-1}(x\otimes y-y\otimes x)$ and thus we identify $\mathfrak{g}\wedge\mathfrak{g}$ with the subspace $(\mathfrak{g}\otimes\mathfrak{g})^{\mathbb{Z}_2}$ .

Finite-dimensional Lie bialgebras are *self-dual*, in the sense that $\big(\mathfrak{g}\,;[\ ,\ ],\delta\big)$ is a Lie bialgebra if and only if $\big(\mathfrak{g}^*\,;\delta^*,[\ ,\ ]^*\big)$ is so; the latter is called the *dual* Lie bialgebra to $\big(\mathfrak{g}\,;[\ ,\ ],\delta\big)$ . This also holds in the infinite-dimensional case, up to technicalities. We denote a Lie bialgebra simply by $\mathfrak{g}$ , and by $\mathfrak{g}^*$ its dual.

Given $r=r_1\otimes r_2$ in $\mathfrak{g}\otimes\mathfrak{g}$ , we write $r_{2,1}:=r_2\otimes r_1$ and $r_{1,2}:=r_1\otimes r_2\otimes 1$ , $r_{2,3}:=1\otimes r_1\otimes r_2$ $r_{1,3}:=r_1\otimes 1\otimes r_2\in\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$. For $s=s_1\otimes s_2\in\mathfrak{g}\otimes\mathfrak{g}$ we define

$$\begin{aligned}[[r,s]] &:= [r_{1,2},s_{1,3}]+[r_{1,2},s_{2,3}]+[r_{1,3},s_{2,3}]\\ &= \big[r_1,s_1\big]\otimes r_2\otimes s_2+r_1\otimes\big[r_2,s_1\big]\otimes s_2+r_1\otimes s_1\otimes\big[r_2,s_2\big].\end{aligned}$$

**2.1.2. Deformations of Lie bialgebras.** In this work, we are mainly interested in two kinds of deformations, where either the Lie cobracket or the Lie bracket alone is deformed. A general theory of deformations for Lie bialgebras using cohomology theory exists, see e.g. [CG], [MW], and references therein for more details.

Let $\big(\mathfrak{g}\,;[\ ,\ ],\delta\big)$ be a Lie bialgebra and $c\in\mathfrak{g}\otimes\mathfrak{g}$ be such that

$$\mathrm{ad}_x\big((\delta\otimes\mathrm{id})(c)+\text{c.p.}+[[c\,,c]]\big) \;=\; 0 \quad , \quad \mathrm{ad}_x\big(c+c_{2,1}\big) \;=\; 0 \qquad \forall\; x\in\mathfrak{g} \tag{2.2}$$

where $\mathrm{ad}_x$ denotes the standard adjoint action of $x$ and c.p. means cyclic permutations on the tensor factors of the previous summand. Then

$$\delta^{\,c} \;:=\; \delta-\partial(c) \quad , \qquad \text{i.e.}\quad \delta^{\,c}(x) \;:=\; \delta(x)-\mathrm{ad}_x(c) \qquad \forall\; x\in\mathfrak{g} \tag{2.3}$$

defines a new Lie cobracket $\delta^{\,c}:\mathfrak{g}\longrightarrow\mathfrak{g}\wedge\mathfrak{g}$ on $\big(\mathfrak{g}\,;[\ ,\ ]\big)$ making $\big(\mathfrak{g}\,;[\ ,\ ],\delta^c\big)$ into a new Lie bialgebra (cf. [Mj, Theorem 8.1.7]).

**Definition 2.1.3.** An element $c\in\mathfrak{g}\otimes\mathfrak{g}$ satisfying (2.2) is called a *twist* of the Lie bialgebra $\mathfrak{g}$ , and the corresponding Lie bialgebra $\mathfrak{g}^{\,c}:=\big(\mathfrak{g}\,;[\ ,\ ],\delta^{\,c}\big)$ is called a *deformation by twist* or *twist deformation* of $\mathfrak{g}$ . $\diamondsuit$

**Remark 2.1.4.** We are adopting here conventions that are slightly different from those in [Mj], yet equivalent. Indeed, we choose to *define* the deformed Lie cobracket in (2.3) as $\delta^c:=\delta-\partial(c)$ , whereas Majid's definition is $\delta^c:=\delta+\partial(c)$ .

Now we introduce a deformation of the Lie bracket. Let $\big(\mathfrak{g}\,;[\ ,\ ]\,,\delta\big)$ be a Lie bialgebra and $\gamma \in \mathit{Hom}_{\Bbbk}\big(\mathfrak{g}\otimes\mathfrak{g}\,,\Bbbk\big)$. We identify $\mathit{Hom}_{\Bbbk}\big(\mathfrak{g}\otimes\mathfrak{g}\,,\Bbbk\big) = (\mathfrak{g}\otimes\mathfrak{g})^* = \mathfrak{g}^*\otimes\mathfrak{g}^*$ for finite-dimensional $\mathfrak{g}$; up to technicalities, the outcome is the same in the infinite-dimensional case too. Dualizing the notion of twist for $\mathfrak{g}^*$ we obtain the notion of 2–cocycle: condition (2.2) with $\mathfrak{g}^*$ replacing $\mathfrak{g}$ and $\gamma$ in the role of $c$ yields

$$\mathrm{ad}_\psi\big(\partial_*(\gamma)+[[\gamma\,,\gamma]]_*\big) \,=\, 0\ ,\quad \mathrm{ad}_\psi\big(\gamma+\gamma_{2,1}\big) \,=\, 0 \qquad \forall\ \psi\in\mathfrak{g}^* \tag{2.4}$$

where $\gamma_{2,1} := \gamma^T$ and $\big(\partial_*(\gamma)\big)(a\,,b\,,c) \,=\, \gamma\big([a\,,b\,]\,,c\big) + \text{c.p.}$. Similarly, $[[\ ,\ ]]_*$ has the same meaning as above but with respect to $\mathfrak{g}^*$.

For any $\gamma$ satisfying (2.4), the map $[\ ,\ ]_\gamma : \mathfrak{g}\wedge\mathfrak{g} \longrightarrow \mathfrak{g}$ given by

$$[x,y]_\gamma \,:=\, [x,y] \,+\, \gamma\big(x_{[1]},y\big)\,x_{[2]} \,-\, \gamma\big(y_{[1]},x\big)\,y_{[2]} \qquad \forall\ x,y\in\mathfrak{g} \tag{2.5}$$

defines a new Lie bracket on the Lie coalgebra $\big(\mathfrak{g}\,;\delta\big)$ making $\big(\mathfrak{g}\,;[\ ,\ ]_\gamma\,,\delta\big)$ into a new Lie bialgebra (cf. [Mj, Exercise 8.1.8]).

**Definition 2.1.5.** Every $\gamma\in\mathit{Hom}_{\Bbbk}\big(\mathfrak{g}\wedge\mathfrak{g}\,,\Bbbk\big)$ that obeys (2.4) is called a *2–cocycle* of the Lie bialgebra $\mathfrak{g}$, and the Lie bialgebra $\mathfrak{g}_\gamma := \big(\mathfrak{g}\,;[\ ,\ ]_\gamma\,,\delta\big)$ is called a *deformation by 2–cocycle* or *2–cocycle deformation*) of $\mathfrak{g}$. $\diamondsuit$

**Remark 2.1.6.** Another observation, dual to Remark 2.1.4 applies to our given definition of 2–cocycle and of 2–cocycle deformation. Again, our notion of 2–cocycle is different from, yet equivalent to, Majid's because any $\gamma\in(\mathfrak{g}\otimes\mathfrak{g})^*$ is a 2–cocycle in our sense if and only if its opposite $-\gamma$ is a 2–cocycle in Majid's, and viceversa.

The following result, which is standard, formalizes the fact that the notions of "twist" and of "2–cocycle" for Lie bialgebras are devised to be dual to each other.

**Proposition 2.1.7.** *Let $\mathfrak{g}$ be a Lie bialgebra, and $\mathfrak{g}^*$ the dual Lie bialgebra.*

*(a) Let $c$ be a twist for $\mathfrak{g}$, and $\gamma_c$ the image of $c$ in $(\mathfrak{g}\otimes\mathfrak{g})^*$ for the natural composed embedding $\mathfrak{g}\otimes\mathfrak{g} \lhook\joinrel\longrightarrow \mathfrak{g}^{**}\otimes\mathfrak{g}^{**} \lhook\joinrel\longrightarrow \big(\mathfrak{g}^*\otimes\mathfrak{g}^*\big)^*$. Then $\gamma_c$ is a 2–cocycle for $\mathfrak{g}^*$, and there exists a canonical isomorphism $\big(\mathfrak{g}^*\big)_{\gamma_c} \cong \big(\mathfrak{g}^c\big)^*$.*

*(b) Let $\gamma$ be a 2–cocycle for $\mathfrak{g}$; assume that $\mathfrak{g}$ is finite-dimensional, and let $c_\gamma$ be the image of $\chi$ in the natural identification $(\mathfrak{g}\otimes\mathfrak{g})^* = \mathfrak{g}^*\otimes\mathfrak{g}^*$. Then $c_\gamma$ is a twist for $\mathfrak{g}^*$, and there exists a canonical isomorphism $\big(\mathfrak{g}^*\big)^{c_\gamma} \cong \big(\mathfrak{g}_\gamma\big)^*$.* $\square$

## 2.2. **Hopf algebra deformations and $R$–(co)matrices.**

We recall some notions on deformations for Hopf algebras. We mainly refer to [Ra], and to [Ks], [CP] and [KS] for *topological* Hopf algebras, using standard notation.

There exist two standard methods to deform Hopf algebras, leading to so-called "2–cocycle deformations" and to "twist deformations": hereafter we recall both.

**Definition 2.2.1.** Let $H$ be a bialgebra (possibly topological, over some commutative ground ring), and let $\mathcal{F}\in H\otimes H$. Then:

*(a)* $\mathcal{F}$ is said to be *unitary* if

$$\big(\epsilon\otimes\mathrm{id}\big)(\mathcal{F}) \,=\, 1 \,=\, \big(\mathrm{id}\otimes\epsilon\big)(\mathcal{F}) \tag{2.6}$$

*(b)* $\mathcal{F}$ is called a *twist* if it is *invertible* in $H\otimes H$, it is *unitary*, and

$$\mathcal{F}_{12}\,\big(\Delta\otimes\mathrm{id}\big)(\mathcal{F}) \,=\, \mathcal{F}_{23}\,\big(\mathrm{id}\otimes\Delta\big)(\mathcal{F}) \tag{2.7}$$

*(c)* $\mathcal{F}$ is called a *(quantum) $R$–matrix* if it is *invertible* in $H\otimes H$ and

$$\big(\Delta\otimes id\big)(\mathcal{F}) \,=\, \mathcal{F}_{13}\,\mathcal{F}_{23}\quad ,\qquad \big(id\otimes\Delta\big)(\mathcal{F}) \,=\, \mathcal{F}_{13}\,\mathcal{F}_{12} \tag{2.8}$$

*(d)* $\mathcal{F}$ is called a *(quantum) R–matrix twist* if it complies both *(b)* and *(c)* above

*(e)* $\mathcal{F}$ is said to be a *solution of the quantum Yang-Baxter equation (=QYBE)* if

$$\mathcal{F}_{12}\,\mathcal{F}_{13}\,\mathcal{F}_{23} \;=\; \mathcal{F}_{23}\,\mathcal{F}_{13}\,\mathcal{F}_{12} \tag{2.9}$$

**Remarks 2.2.2.** *(a)* If $H$ is a Hopf algebra (including a topological one) and there exists $\mathcal{F} \in H \otimes H$ which is *invertible* and such that

$$\mathcal{F}\,\Delta(x)\,\mathcal{F}^{-1} = \Delta^{\mathrm{op}}(x) \qquad\qquad \forall\;\, x \in H \tag{2.10}$$

then $H$ is said to be *quasicocommutative*. If in addition $\mathcal{F}$ obeys also (2.8), then $H$ is said to be *quasitriangular*. Indeed, the standard notion of "$R$–matrix" in literature usually demands the constraint (2.10) besides condition (2.8). Any "$R$–matrix" as in Definition 2.2.1*(c)* is called "weak $R$–matrix" in [Ch], Definition 1.1.

*(b)* Every $R$–matrix as in Definition 2.2.1*(c)* above is automatically *unitary*, cf. [Ch], Lemma 1.2. Conversely, if $\mathcal{F}$ is *unitary* and enjoys (2.8), then it is *invertible* too, hence it is an $R$–matrix. In short, the two conditions are equivalent.

*(c)* If $\mathcal{R}$ is an $R$–matrix for $H$, then so is $\big(\mathcal{R}^{-1}\big)_{21} = \big(\mathcal{R}_{21}\big)^{-1}$ ; moreover, $\mathcal{R}_{21}$ and $\mathcal{R}^{-1}$ are $R$–matrices for $H^{\mathrm{op}}$ and $H^{\mathrm{cop}}$ alike — see [Mj], [Ra].

*(d)* Formulas (2.7) and (2.8) jointly imply (2.9), while (2.8) and (2.9) imply (2.7).

**2.2.3. Deformations by twist.** Let $H$ be a bialgebra (over some ring $\Bbbk$ ), and let $\mathcal{F} \in H \otimes H$ be a twist for it — as in Definition 2.2.1*(b)*. Then $H$ bears a second bialgebra structure, denoted $H^{\mathcal{F}}$ and called *twist deformation* of the old one, with the old product, unit and counit, but with a new "twisted" coproduct $\Delta^{\mathcal{F}}$ given by

$$\Delta^{\mathcal{F}}(x) \;:=\; \mathcal{F}\,\Delta(x)\,\mathcal{F}^{-1} \qquad\qquad\qquad \forall\;\, x \in H$$

If in addition $H$ is a Hopf algebra with antipode $\mathcal{S}$ , then this "twisted" bialgebra $H^{\mathcal{F}}$ is again a Hopf algebra with antipode $\mathcal{S}^{\mathcal{F}}$ given by

$$\mathcal{S}^{\mathcal{F}}(x) \;:=\; v\,\mathcal{S}(x)\,v^{-1} \qquad\qquad\qquad \forall\;\, x \in H$$

where $v := \sum_{\mathcal{F}} \mathcal{S}(f'_1)\, f'_2$ — with $\sum_{\mathcal{F}} f'_1 \otimes f'_2 = \mathcal{F}^{-1}$ — is invertible in $H$ (see, [CP, §4.2.$E$], for further details). *When $H$ is in fact a topological bialgebra or Hopf algebra*, then the same notions still make sense, and the related results apply again.

We present now the dual picture:

**Definition 2.2.4.** Let $H$ be a bialgebra, and let $\sigma \in \big(H^{\otimes 2}\big)^*$ . Then:

*(a)* $\sigma$ is said to be *unitary* if

$$\sigma(a,1) \;=\; \epsilon(a) \;=\; \sigma(1,a) \qquad\qquad \forall\, a \in H \tag{2.11}$$

*(b)* $\sigma$ is called a $2$*–cocycle* if it is *(convolution) invertible* in $\big(H^{\otimes 2}\big)^*$ , it is unitary, and such that

$$\sigma\big(a_{(1)},b_{(1)}\big)\,\sigma\big(a_{(2)}b_{(2)},c\big) \;=\; \sigma\big(b_{(1)},c_{(1)}\big)\,\sigma\big(a,b_{(2)}c_{(2)}\big) \tag{2.12}$$

for all $a,b,c \in H$ — where we abuse of notation identifying $\sigma \in \big(H \otimes H\big)^*$ with the corresponding $\Bbbk$–bilinear map $\rho : H \times H \longrightarrow \Bbbk$ , and we adapt notation accordingly;

*(c)* $\sigma$ is called a *(quantum) $\varrho$–comatrix* if it is *(convolution) invertible* in $\big(H^{\otimes 2}\big)^*$ and — for all $a,b,c \in H$ — we have

$$\sigma(a\,b\,,c) \;=\; \sigma\big(a\,,c_{(1)}\big)\,\sigma\big(b\,,c_{(2)}\big) \;\;, \qquad \sigma(a\,,b\,c) \;=\; \sigma\big(a_{(1)},c\big)\,\sigma\big(a_{(2)},b\big) \tag{2.13}$$

*(d)* $\sigma$ is called a *(quantum) $\varrho$–comatrix* $2$*–cocycle* if it complies with *(b)* and *(c)*;

*(e)* $\sigma$ is said to be a *solution of the quantum Yang-Baxter equation (=QYBE)* if

$$\sigma_{12} * \sigma_{13} * \sigma_{23} \;=\; \sigma_{23} * \sigma_{13} * \sigma_{12} \tag{2.14}$$

where hereafter "$*$" denotes the convolution product.

**Remarks 2.2.5.** *(a)* If $H$ is a Hopf algebra and there exists $\sigma \in (H \otimes H)^*$ which is *(convolution) invertible* and such that

$$\sigma * m * \sigma^{-1} \;=\; m_{\mathrm{op}} \tag{2.15}$$

then $H$ is said to be *quasicommutative*. If in addition $\sigma$ obeys also (2.13), then $H$ itself is said to be *coquasitriangular*. Indeed, the standard notion of "$\varrho$–comatrix", or "dual $R$–matrix", in literature usually demands (2.15) besides (2.13). Following [Ch], one might also use terminology "weak $\varrho$–comatrix", or "weak dual $R$–matrix".

*(b)* Every $\varrho$–matrix as in Definition 2.2.4*(c)* above is *unitary*. Conversely, if $\rho$ is *unitary* and enjoys (2.13), then it is *(convolution) invertible* too, hence it is a $\varrho$–comatrix (cf. [Mj], Lemma 2.2.2). In short, the two conditions are equivalent.

*(c)* Much like for $R$–matrices, if $\rho$ is a $\varrho$–comatrix for $H$, then so is $\big(\rho^{-1}\big)_{21} = \big(\rho_{21}\big)^{-1}$; moreover, $\rho_{21}$ and $\rho^{-1}$ are $\varrho$–comatrices for $H^{\mathrm{op}}$ and $H^{\mathrm{cop}}$ alike.

*(d)* Formulas (2.12) and (2.13) imply (2.14), while (2.13) and (2.14) yield (2.12).

**2.2.6. Deformations by 2–cocycles.** Let $H$ be a bialgebra (over some ring $\Bbbk$), and let $\sigma \in (H \otimes H)^*$ be a 2–cocycle. Then $H$ bears a second bialgebra structure, denoted $H_\sigma$ and called 2*–cocycle deformation* of the old one, with the old coproduct, counit and unit, but with new product $m_\sigma = \sigma * m * \sigma^{-1} : H \otimes H \longrightarrow H$ given by

$$m_\sigma(a,b) \;=\; a \cdot_\sigma b \;=\; \sigma(a_{(1)}, b_{(1)})\, a_{(2)}\, b_{(2)}\, \sigma^{-1}(a_{(3)}, b_{(3)}) \qquad \forall\; a, b \in H$$

If in addition $H$ is a Hopf algebra with antipode $\mathcal{S}$, then this "deformed" bialgebra $H_\sigma$ is again a Hopf algebra with antipode $\mathcal{S}_\sigma$, which in detail reads

$$\mathcal{S}_\sigma(a) \;=\; \sigma(a_{(1)}, \mathcal{S}(a_{(2)}))\, \mathcal{S}(a_{(3)})\, \sigma^{-1}(\mathcal{S}(a_{(4)}), a_{(5)}) \qquad \forall\; a \in H$$

(see [Doi] for more details). *If $H$ is a topological bialgebra or Hopf algebra*, all this construction applies again, as well as the related results, up to technicalities.

The two notions of "2–cocycle" and of "twist", as well as the corresponding deformations, are so devised as to be dual to each other with respect to Hopf duality (cf. [Mj]), also in the setup of *topological* Hopf algebras as with QUEA's and QFSHA's. The same holds for the notions of "$\varrho$–comatrix" and of "$R$–matrix". All this is recorded in the following result, whose proof is trivial (an exercise in Hopf theory):

**Proposition 2.2.7.** *Let $H$ be a Hopf algebra (possibly topological), and $H^*$ its dual Hopf algebra (possibly in topological sense).*

*(a) Let $\mathcal{F}$ be a twist, resp. an $R$–matrix, for $H$, and $\sigma_{\mathcal{F}}$ the image of $\mathcal{F}$ in $(H \otimes H)^*$ for the natural embedding $H \otimes H \lhook\joinrel\longrightarrow H^{**} \otimes H^{**} \lhook\joinrel\longrightarrow \big(H^* \otimes H^*\big)^*$. Then $\sigma_{\mathcal{F}}$ is a 2–cocycle, resp. a $\varrho$–comatrix, for $H^*$. Moreover, in the first case there exists a canonical Hopf algebra isomorphism $\big(H^*\big)_{\sigma_{\mathcal{F}}} \cong \big(H^{\mathcal{F}}\big)^*$.*

*(b) Let $\sigma$ be a 2–cocycle, resp. a $\varrho$–comatrix, for $H$; assume that we have a natural identification $(H \otimes H)^* = H^* \otimes H^*$ (e.g., if $H$ is finite-dimensional), and let $\mathcal{F}_\sigma$ be the image of $\sigma$ in $H^* \otimes H^*$ via this identification. Then $\mathcal{F}_\sigma$ is a twist, resp. an $R$–matrix, for $H^*$. Moreover, in the first case there exists a canonical Hopf algebra isomorphism $\big(H^*\big)^{\mathcal{F}_\sigma} \cong \big(H_\sigma\big)^*$.* $\square$

**2.2.8. Hopf morphisms from $R$–matrices and $\varrho$–comatrices.** Let $H$ be a Hopf algebra, possibly in topological sense. We assume that its (possibly topological) finite dual $H^*$ is a Hopf algebra as well (possibly in a topological sense).

Hereafter we recall some well-known constructions, somewhat shortly: further details can be found, e.g., in [CP], [KS] and [Mj].

**Proposition 2.2.9.** *(cf.* [GaGa3] *for a proof)*

*(a) Every $R$–matrix $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ (using Sweedler's-like notation) for $H$ provides two Hopf algebra morphisms*

$$\overleftarrow{\Phi}_{\mathcal{R}} : H^* \longrightarrow H^{\mathrm{cop}} \ \big(\eta \mapsto \eta(\mathcal{R}_1)\,\mathcal{R}_2\big) \quad , \qquad \overrightarrow{\Phi}_{\mathcal{R}} : H^* \longrightarrow H^{\mathrm{op}} \ \big(\eta \mapsto \mathcal{R}_1\,\eta(\mathcal{R}_2)\big)$$

*(b) If $\mathcal{R}$ is an $R$–matrix for $H$, and $\mathcal{R}^{-1}$ is its inverse, then $\overleftarrow{\Phi}_{\mathcal{R}}$, resp. $\overrightarrow{\Phi}_{\mathcal{R}}$, is convolution invertible, with convolution inverse $\overleftarrow{\Phi}_{\mathcal{R}^{-1}}$, resp. $\overrightarrow{\Phi}_{\mathcal{R}^{-1}}$.* $\square$

The previous result has its dual counterpart, whose proof is again straightforward:

**Proposition 2.2.10.**

*(a) Every $\varrho$–comatrix $\rho$ for $H$ provides two Hopf algebra morphisms*

$$\overleftarrow{\Psi}_{\rho} : H \longrightarrow \big(H^*\big)^{\mathrm{cop}} ,\ \ \ell \mapsto \rho(\ell\,, -) \quad , \qquad \overrightarrow{\Psi}_{\rho} : H \longrightarrow \big(H^*\big)^{\mathrm{op}} ,\ \ \ell \mapsto \rho(-\,,\ell)$$

*(b) If $\rho$ is a $\varrho$–comatrix for $H$, and $\rho^{-1}$ is its (convolution) inverse, then $\overleftarrow{\Psi}_{\rho}$, resp. $\overrightarrow{\Psi}_{\rho}$, is convolution invertible, with convolution inverse $\overleftarrow{\Psi}_{\rho^{-1}}$, resp. $\overrightarrow{\Psi}_{\rho^{-1}}$.* $\square$

**Remark 2.2.11.** Inasmuch as any $R$–matrix, resp. any $\varrho$–comatrix, for $H$ is a $\varrho$–comatrix, resp. an $R$–matrix, for the dual Hopf algebra $H^*$ — cf. Proposition 4.1.2 — applying Proposition 2.2.9 to $H^*$ we get Proposition 2.2.10, and, conversely, applying Proposition 2.2.10 to $H^*$ we get Proposition 2.2.9. In the same spirit, the following result about Hopf algebras in duality follows from the very definitions:

**Proposition 2.2.12.** *Let $K$ and $\varGamma$ be two Hopf algebras (over the same ground ring, and possibly topological) that are dual to each other, say $\varGamma = K^*$ and $K = \varGamma^\star$ for suitably defined dual functors $(\ )^*$ and $(\ )^\star$. Let also $\mathcal{R} = \rho$ be an $R$–matrix for $K$ and a $\varrho$–comatrix for $\varGamma$ — applying Proposition 2.2.7. Then for the morphisms in Proposition 2.2.9 and Proposition 2.2.10 we have canonical identifications*

$$\overleftarrow{\Phi}_{\mathcal{R}} = \overleftarrow{\Psi}_{\rho} \ : \ K^* = \varGamma \longrightarrow \big(\varGamma^\star = K\big)^{\mathrm{cop}} , \quad \overrightarrow{\Phi}_{\mathcal{R}} = \overrightarrow{\Psi}_{\rho} \ : \ K^* = \varGamma \longrightarrow \big(\varGamma^\star = K\big)^{\mathrm{op}} \qquad \square$$

## 2.3. **Quantum groups.**

We recall hereafter the basic notions on quantum groups, in the shape of either quantized universal enveloping algebras (=QUEA's) or quantized formal series Hopf algebras (=QFSHA's) — both being Hopf algebras in a topological sense.

**2.3.1. Classical and quantum preliminaries.** Hereafter we fix a base field $\Bbbk$ of characteristic zero. We recall the following from [CP].

For any Lie algebra $\mathfrak{g}$ over $\Bbbk$, its universal enveloping algebra $U(\mathfrak{g})$ has a canonical structure of Hopf algebra, which is cocommutative and connected. If $\mathfrak{g}$ is also a Lie bialgebra, with Lie cobracket $\delta$, then $\delta$ uniquely extends to define a *Poisson cobracket* $\delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$, just by imposing that it fulfill the co-Leibnitz identity $\delta(x\,y) = \delta(x)\,\Delta(y) + \Delta(x)\,\delta(y)$. Conversely, if the Hopf algebra $U(\mathfrak{g})$ is actually even a Hopf *co-Poisson* algebra, then its Poisson co-bracket $\delta$ maps $\mathfrak{g}$ into $\mathfrak{g} \otimes \mathfrak{g}$, thus yielding a Lie cobracket for $\mathfrak{g}$ that makes the latter into a Lie bialgebra.

Dually, let $G$ be any formal algebraic group $G$ over $\Bbbk$ : by this we loosely mean that $G$ is the spectrum of its formal function algebra $F[[G]]$, the latter being a topological Hopf algebra which is commutative and $I$–adically complete, where $I := Ker(\epsilon)$ is the augmentation ideal of $F[[G]]$. Then $G$ is a (formal) Poisson group if and only if its formal function algebra $F[[G]]$ is a *Poisson* (formal) Hopf algebra, with respect to some Poisson bracket $\{\ ,\ \}$. In this case, the cotangent space $I/I^2$ of $G$ has a Lie bracket induced by $\{\ ,\ \}$ via $[x,y] := \{x',y'\} \ \big(\mathrm{mod}\ I^2\big)$ for all $x,y \in I/I^2$ with $x = x' \ \big(\mathrm{mod}\ I^2\big)$, $y = y' \ \big(\mathrm{mod}\ I^2\big)$: this makes $I/I^2$ into a Lie algebra, but its dual $\mathfrak{g} = Lie(G) := \big(I/I^2\big)^*$ is also a Lie algebra (the tangent Lie algebra to $G$) and the two structures are compatible, so that $\mathfrak{g}^\star := I/I^2$ is a *Lie bialgebra* indeed.

We come now to *quantizations* of the previous co-Poisson/Poisson structures.

Let $\mathcal{T}_{\widehat{\otimes}}$ be the category whose objects are all topological $\Bbbk[[h]]$–modules which are topologically free (i.e. isomorphic to $V[[h]]$ for some $\Bbbk$–vector space $V$, with the $h$–adic topology) and with morphisms the $\Bbbk[[h]]$–linear maps (then automatically continuous). This is a tensor category for the product $T_1 \widehat{\otimes} T_2$ which is the separated $h$–adic completion of the algebraic tensor product $T_1 \otimes_{\Bbbk[[h]]} T_2$ (for all $T_1$, $T_2 \in \mathcal{T}_{\widehat{\otimes}}$).

Let $\mathcal{P}_{\widetilde{\otimes}}$ be the category whose objects are all topological $\Bbbk[[h]]$–modules isomorphic to modules of the type $\Bbbk[[h]]^E$ for some set $E$: these are complete w.r.t. to the weak topology and whose morphisms in $\mathcal{P}_{\widetilde{\otimes}}$ are the $\Bbbk[[h]]$–linear continuous maps. This is a tensor category w.r.t. the tensor product $P_1 \widetilde{\otimes} P_2$ defined to be the completion of the algebraic tensor product $P_1 \otimes_{\Bbbk[[h]]} P_2$ w.r.t. the weak topology: therefore $P_i \cong \Bbbk[[h]]^{E_i}$ $(i=1,2)$ yields $P_1 \widetilde{\otimes} P_2 \cong \Bbbk[[h]]^{E_1 \times E_2}$ (for all $P_1$, $P_2 \in \mathcal{P}_{\widetilde{\otimes}}$).

Note that the objects of $\mathcal{T}_{\widehat{\otimes}}$ and of $\mathcal{P}_{\widetilde{\otimes}}$ are complete and separated w.r.t. the $h$–adic topology, so one has $X \cong X_0[[h]]$ for every such object $X$, with $X_0 := X \big/ \hbar X$.

We denote by $\mathcal{HA}_{\widehat{\otimes}}$ the subcategory of $\mathcal{T}_{\widehat{\otimes}}$ whose objects are all the Hopf algebras in $\mathcal{T}_{\widehat{\otimes}}$ and whose morphisms are all the Hopf algebra morphisms in $\mathcal{T}_{\widehat{\otimes}}$. Similarly, we call $\mathcal{HA}_{\widetilde{\otimes}}$ the subcategory of $\mathcal{P}_{\widetilde{\otimes}}$ whose objects are all the Hopf algebras in $\mathcal{P}_{\widetilde{\otimes}}$ and whose morphisms are all the Hopf algebra morphisms in $\mathcal{P}_{\widetilde{\otimes}}$. To simplify notation, we shall usually drop the subscripts "$\widehat{\ }$" and "$\widetilde{\ }$" from the symbol "$\otimes$".

Finally, when dealing with any $\Bbbk[[\hbar]]$–module $M$ by such notation as $\mathcal{O}\big(\hbar^s\big)$ we shall mean any (unspecified) element belonging to $\hbar^s M$, for all $s,n \in \mathbb{N}$; in other words, for any $x \in M$ by writing $x = \mathcal{O}\big(\hbar^s\big)$ we mean that $x \equiv 0 \ \big(\mathrm{mod}\ \hbar^s M\big)$.

We are ready now to define quantum groups, in two different incarnations:

**2.3.2. Quantized Universal Enveloping Algebras (=QUEA's).** Retain notation as in §2.3.1 above. A *quantized universal enveloping algebra* — or QUEA in short — is a (topological) Hopf algebra $U_\hbar$ in $\mathcal{HA}_{\widehat{\otimes}}$ such that $U_0 := U_\hbar \big/ \hbar U_\hbar$ is a connected, cocommutative Hopf algebra over $\Bbbk$ — or, equivalently, $U_0$ is isomorphic to an enveloping algebra $U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$. Then the formula

$$\delta(x) := \frac{\Delta\big(x'\big) - \Delta^{\mathrm{op}}\big(x'\big)}{\hbar} \quad \mathrm{mod}\ \hbar\, U_\hbar^{\widehat{\otimes}\, 2} \tag{2.16}$$

— where $x' \in U_\hbar$ is any lift of $x \in \mathfrak{g}$ — defines a co-Poisson structure on $U_0 = U(\mathfrak{g})$, hence a Lie bialgebra structure on $\mathfrak{g}$. In this case, we say that $U_\hbar$ is a *quantization* of the co-Poisson Hopf algebra $U(\mathfrak{g})$, or (with a slight abuse of language) of the Lie bialgebra $\mathfrak{g}$; conversely, $U(\mathfrak{g})$ — or just $\mathfrak{g}$ alone — is the *semiclassical limit* of $U_\hbar$. We summarize it writing $U_\hbar(\mathfrak{g}) := U_\hbar$. In the following, we denote by $\mathcal{QUEA}$ the full subcategory of $\mathcal{HA}_{\widehat{\otimes}}$ whose objects are all of the QUEAs.

**2.3.3. Quantized Formal Series Hopf Algebras (=QFSHA's).** Retain again notation as in §2.3.1 above. A *quantized formal series Hopf algebra* — or QFSHA in short — is a (topological) Hopf algebra $F_\hbar$ in $\mathcal{HA}_{\widetilde{\otimes}}$ such that $F_0 := F_\hbar \big/ \hbar \, F_\hbar$ is a commutative, $I$–adically complete topological Hopf algebra over $\Bbbk$ , where $I$ is the augmentation ideal — or, equivalently, $F_0$ is isomorphic to the algebra of functions for some formal algebraic group $F[[G]]$ . Then the formula

$$\{x,y\} \, := \, \frac{x' \, y' \, - \, y' \, x'}{\hbar} \quad \mod \hbar \, F_\hbar \tag{2.17}$$

— where $x', y' \in F_\hbar$ are lifts of $x, y \in F[[G]]$ — defines a Poisson bracket in $F[[G]]$ , thus making $G$ into a (formal) Poisson group. In this case, we say that $F_\hbar$ is a *quantization* of the Poisson Hopf algebra $F[[G]]$ , or (stretching a point) of the formal Poisson group $G$ ; conversely, $F[[G]]$ — or just $G$ alone — is the *semiclassical limit* of $F_\hbar$ . We summarize it writing $F_\hbar[[G]] := F_\hbar$ . In the following, we denote by $\mathcal{QFSHA}$ the full subcategory of $\mathcal{HA}_{\widetilde{\otimes}}$ whose objects are all of the QFSHAs.

**2.3.4. Equivalence and duality between quantizations.** If $H_1$, $H_2$, are two QUEA's, respectively two QFSHA's, we say that $H_1$ *is equivalent to* $H_2$, and we write $H_1 \equiv H_2$ , if there is an isomorphism $\varphi : H_1 \cong H_2$ (in $\mathcal{QUEA}$, resp. in $\mathcal{QFSHA}$) such that $\varphi = id \mod h$ . In particular, in both cases the semiclassical limit of either $H_1$ or $H_2$ is the same.

By their very construction, the categories $\mathcal{QUEA}$ and $\mathcal{QFSHA}$ are dual to each other (w. r. to the natural, topological linear duality functors in both directions). In detail, by *dual* of any $U_\hbar \in \mathcal{QUEA}$ , denoted $U_\hbar^*$, we take the set of all $\Bbbk[[\hbar]]$–linear functions from $U_\hbar$ to $\Bbbk[[\hbar]]$ (which are automatically continuous w. r. to the $\hbar$–adic topology): this is naturally an object in $\mathcal{QFSHA}$ . On the other hand, by *dual* of any $F_\hbar \in \mathcal{QFSHA}$ , denoted $F_\hbar^\star$, we take the set of all maps from $F_\hbar$ to $\Bbbk[[\hbar]]$ that are continuous with respect to the $\hbar$–adic topology on $\Bbbk[[\hbar]]$ and to the $I_\hbar$–adic topology on $F_\hbar$ , with $I_\hbar := \hbar \, F_\hbar + Ker\big(\epsilon_{F_\hbar}\big)$ ; this $F_\hbar^\star$ is an object in $\mathcal{QUEA}$ . Finally, $(\ )^*$ and $(\ )^\star$ are contravariant functors inverse to each other — cf. [Ga1].

We finish this part with a trivial, technical result, that we will use several times:

**Lemma 2.3.5.** *Let $H$ be a Hopf algebra (possibly topological). We denote by $[\ ,\ ]$ the commutator operation in $H$, and write $H^+ := Ker(\epsilon)$ . Then:*

*(a) There exists a splitting into direct sum $H = \Bbbk \oplus H^+$ . With respect to that splitting, every $z \in H$ uniquely splits into $z = \epsilon(z) + z^+$ with $z^+ := z - \epsilon(z) \in H^+$ .*

*(b) For any $x, y \in H$ we have $[x\,, y\,] = \big[x^+\,, y^+\big]$ — see (a) — so $[H\,, H\,] \subseteq H^+$ .*

*(c) Assume that $H = F_\hbar[[G]]$ is a QFSHA, with $J_\hbar := H^+$ . Then we have $[H\,, H\,] = \big[J_\hbar\,, J_\hbar\big] \subseteq \hbar \, J_\hbar$ , and more in general (for all $k, r_1, r_2, r_3, \dots, r_k, s \in \mathbb{N}_+$ )*

$$\Big[J_\hbar^{r_1}, \Big[J_\hbar^{r_2}, \Big[J_\hbar^{r_3}, \cdots \Big[J_\hbar^{r_k}, J_\hbar^{s}\Big] \cdots \Big]\Big]\Big] \;\subseteq\; (1-\delta_{s,0}) \prod_{i=1}^{k} (1-\delta_{r_i,0}) \, \hbar^k \, J_\hbar^{r_1+r_2+r_3+\cdots+r_k+s-k}$$

*(d) We have $\Delta(z) \,=\, \epsilon(z) \cdot 1 \otimes 1 \,+\, z^+ \otimes 1 \,+\, 1 \otimes z^+ \,+\, \big(z_{(1)}\big)^+ \otimes \big(z_{(2)}\big)^+$ for any $z \in H$ , or also $\Delta(z) \,=\, -\epsilon(z) \cdot 1 \otimes 1 \,+\, z \otimes 1 \,+\, 1 \otimes z \,+\, \big(z_{(1)}\big)^+ \otimes \big(z_{(2)}\big)^+$ . In particular, for $\nabla := \Delta - \Delta^{\mathrm{op}}$ this yields*

$$\nabla(z) \;=\; \big(z_{(1)}\big)^+ \otimes \big(z_{(2)}\big)^+ - \big(z_{(2)}\big)^+ \otimes \big(z_{(1)}\big)^+ \;\in\; Ker(H)^{\otimes 2} \qquad \square$$

## 2.4. **The Quantum Duality Principle.**

We recall hereafter the main facets of the so-called "Quantum Duality Principle", which establishes an equivalence between the category of QUEA's and that of QFSHA's (whereas linear duality provides an *anti*equivalence); cf. [Ga1] for details.

**Definition 2.4.1.** *(Drinfeld's functors)* We define Drinfeld's functors from $\mathcal{QUEA}$ to $\mathcal{QFSHA}$ and viceversa as follows:

*(a)* Let $U_\hbar(\mathfrak{g})$ be any QUEA, and assume for simplicity that $\mathfrak{g}$ is finite-dimensional. Let $\iota : \Bbbk[[\hbar]] \longrightarrow U_\hbar(\mathfrak{g})$ and $\epsilon : U_\hbar(\mathfrak{g}) \longrightarrow \Bbbk[[\hbar]]$ be its unit and counit maps; moreover, for every $n \in \mathbb{N}$ set $\delta_n := (\mathrm{id} - \iota \circ \epsilon)^{\otimes n} \circ \Delta^{(n-1)}$ — mapping $U_\hbar(\mathfrak{g})$ to $U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, n}$. Then we define

$$U_\hbar(\mathfrak{g})' \;:=\; \left\{ \eta \in U_\hbar(\mathfrak{g}) \,\middle|\, \delta_n(\eta) \in \hbar^n \, U_\hbar(\mathfrak{g})^{\otimes n} \;\; \forall\, n \in \mathbb{N} \right\}$$

This defines the functor $(\;)'$, from $\mathcal{QUEA}$ to $\mathcal{QFSHA}$, onto objects: then onto morphisms it is clearly defined by taking restriction.

*(b)* Let $F_\hbar[[G]]$ be any QFSHA, and assume for simplicity that $G$ be finite-dimensional. Let $\epsilon_F : F_\hbar[[G]] \longrightarrow \Bbbk[[\hbar]]$ be its counit map, and consider also $I_{F_\hbar[[G]]} := \hbar\, F_\hbar[[G]] + Ker(\epsilon_F)$. Then we define

$$F_\hbar[[G]]^\vee \;:=\; \hbar\text{–adic completion of } \textstyle\sum_{n \geq 0} \hbar^{-n} I_{F_\hbar[[G]]}^{\,n}$$

This defines the functor $(\;)^\vee$, from $\mathcal{QFSHA}$ to $\mathcal{QUEA}$, onto objects: onto morphisms, we define it via scalar extension — from $\Bbbk[[\hbar]]$ to $\Bbbk((\hbar))$ — followed by restriction and completion. $\diamondsuit$

The main result about the above "Drinfeld's functors" is the following:

**Theorem 2.4.2.** *("The quantum duality principle"; cf.* [Dr], [Ga1]*)*

*(a) The assignments $H \mapsto H'$ and $H \mapsto H^\vee$ respectively define functors of tensor categories $\mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$ and $\mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$, that are inverse to each other, thus yielding an equivalence of catefories.*

*(b) For all $U_\hbar(\mathfrak{g}) \in \mathcal{QUEA}$ and all $F_\hbar[[G]] \in \mathcal{QFSHA}$ one has*

$$U_\hbar(\mathfrak{g})' \Big/ h\, U_\hbar(\mathfrak{g})' \;=\; F[[G^*]] \quad , \qquad F_\hbar[[G]]^\vee \Big/ h\, F_\hbar[[G]]^\vee \;=\; U(\mathfrak{g}^*)$$

*that is, if $U_\hbar(\mathfrak{g})$ is a quantization of $U(\mathfrak{g})$ then $U_\hbar(\mathfrak{g})'$ is a quantization of $F[[G^*]]$, and if $F_\hbar[[G]]$ is a quantization of $F[[G]]$ then $F_\hbar[[G]]^\vee$ is a quantization of $U(\mathfrak{g}^*)$.*

*(c) Both Drinfeld's functors preserve equivalence, that is $H_1 \equiv H_2$ implies that ${H_1}' \equiv {H_2}'$ and ${H_1}^\vee \equiv {H_2}^\vee$ in either case.* $\square$

Drinfeld's functors are dual to each other, namely (cf. [Ga1], notation as in §2.3.4)

$$\big(U_\hbar(\mathfrak{g})^*\big)^\vee = \big(U_\hbar(\mathfrak{g})'\,\big)^\star \qquad \text{and} \qquad \big(F_\hbar[[G]]^\vee\,\big)^* = \big(F_\hbar[[G]]^\star\big)' \tag{2.18}$$

On the other hand, it is worth stressing a strong asymmetry between these functors. Indeed, the definition of $F_\hbar[[G]]^\vee$ is pretty *concrete* (through an explicit generating procedure) whereas that of $U_\hbar(\mathfrak{g})'$ is somewhat *implicit* (it is described as the set of solution of a system of countably many equations). However, an alternative description for $U_\hbar(\mathfrak{g})'$ exists, namely the following (cf. [Ga2, Proposition 3.1.2]):

**Proposition 2.4.3.** *For any $\Bbbk$–basis $\{\overline{y}_i\}_{i \in I}$ of $\mathfrak{g}$, there are $y_i \in U_\hbar(\mathfrak{g})$ such that:*

*(a) $\epsilon(y_i) = 0$, $\big(y_i \mod \hbar\, U_\hbar(\mathfrak{g})\big) = \overline{y}_i$ and $y'_i := \hbar\, y_i \in U_\hbar(\mathfrak{g})'$ for all $i \in I$;*

*(b) $U_\hbar(\mathfrak{g})'$ is the completion of the unital $\Bbbk[[\hbar]]$–subalgebra of $U_\hbar(\mathfrak{g})$ generated by all the $x'_i$'s with respect to its $I'_\hbar$–adic topology, where $I'_\hbar$ is the ideal (in that subalgebra) generated by $\hbar$ and all the $x'_i$'s, so that $U_\hbar(\mathfrak{g})' = \Bbbk\big[\big[\{x'_i\}_{i \in I} \cup \{\hbar\}\big]\big]$.* $\square$

## 3. Deformations of quantum groups

This section is dedicated to explore the effect of deformations of quantum groups, either by twist or by 2-cocycle, seeting the cases of QUEA's and QFSHA's apart.

### 3.1. **Deformations by twist of QUEA's.**

In this subsection we consider deformations by twist of QUEA's (the easier case, in a sense). We begin with a technical result, whose proof is left to the reader:

**Lemma 3.1.1.** *(cf.* [GaGa3]*) Let $H$ be an $\hbar$–adically complete Hopf algebra over $\Bbbk[[\hbar]]$, and let consider an element of the form $\mathcal{F} = \exp\!\big(\hbar\,\varphi\big) \in H \otimes H$, with $\varphi = \varphi_1 \otimes \varphi_2 \in H^{\otimes 2}$, such that $(\epsilon \otimes \mathrm{id})(\mathcal{F}) = 1 = (\mathrm{id} \otimes \epsilon)(\mathcal{F})$. Then*

$$\epsilon(\varphi_1) \otimes \varphi_2 \,=\, 0 \;\;, \qquad \varphi_1 \otimes \epsilon(\varphi_2) \,=\, 0 \;\;, \qquad \epsilon(\varphi_1) \otimes \epsilon(\varphi_2) \,=\, 0$$

*As a consequence, one can assume $\varphi_1 = \varphi_1^+$, $\varphi_2 = \varphi_2^+ \in Ker(\epsilon)$, so $\varphi \in Ker(\epsilon)^{\otimes 2}$.*

We are now ready for our first meaningful result:

**Theorem 3.1.2.** *Let $U_\hbar(\mathfrak{g})$ be a QUEA over $\mathfrak{g} = \big(\mathfrak{g}\,;[\ ,\ ]\,,\delta\,\big)$. Let $\mathcal{F} \in U_\hbar(\mathfrak{g})^{\widehat{\otimes}\,2}$ be a twist for $U_\hbar(\mathfrak{g})$ such that $\mathcal{F} \equiv 1 \left(\bmod \hbar\, U_\hbar(\mathfrak{g})^{\widehat{\otimes}\,2}\right)$; then $\kappa := \hbar^{-1} \log(\mathcal{F}\,) \in U_\hbar(\mathfrak{g})^{\widehat{\otimes}\,2}$, and $\mathcal{F} = \exp\big(\hbar\,\kappa\big)$. Last, we set $\kappa_a := \kappa - \kappa_{2,1}$. Then we have:*

*(a) $\kappa$ is antisymmetric, i.e. $-\kappa = \kappa_{2,1}$, iff $\mathcal{F}$ is orthogonal, i.e. $\mathcal{F}^{-1} = \mathcal{F}_{2,1}$;*

*(b) the element $c := \overline{\kappa_a} = \kappa_a \left(\bmod \hbar\, U_\hbar(\mathfrak{g})^{\widehat{\otimes}\,2}\right)$ belongs to $\mathfrak{g} \otimes \mathfrak{g}$, and it is an antisymmetric twist element for the Lie bialgebra $\mathfrak{g}$;*

*(c) the deformation $\big(U_\hbar(\mathfrak{g})\big)^{\mathcal{F}}$ of $U_\hbar(\mathfrak{g})$ by the twist $\mathcal{F}$ is a QUEA for the Lie bialgebra $\mathfrak{g}^c = \big(\mathfrak{g}\,;[\ ,\ ]\,,\delta^{\,c}\big)$ which is the deformation of $\mathfrak{g}$ by the twist $c$; in a nutshell, we have $\big(U_\hbar(\mathfrak{g})\big)^{\mathcal{F}} \cong U_\hbar\big(\mathfrak{g}^c\,\big)$.*

*Proof. (a)* This follows from standard identities for exponentials and for logarithms.

*(b)* We fix hereafter the notation $U_\hbar := U_\hbar(\mathfrak{g})$ and $J_\hbar := Ker\big(\epsilon_{U_\hbar}\big)$, and we write $\kappa \in U_\hbar^{\widehat{\otimes} 2}$ with Sweedler's like $\sigma$–notation $\kappa = \kappa_1 \otimes \kappa_2$. By Lemma 3.1.1 we can assume (as we shall do henceforth) that $\kappa_1, \kappa_2 \in J_\hbar$, hence $\kappa \in J_\hbar^{\widehat{\otimes} 2}$.

Now we consider the identity $\mathcal{F}_{12}\,\big(\Delta \otimes \mathrm{id}\big)(\mathcal{F}\,) \,=\, \mathcal{F}_{23}\,\big(\mathrm{id} \otimes \Delta\big)(\mathcal{F}\,)$. Writing $\mathcal{F} = \exp\big(\hbar\,\kappa_1 \otimes \kappa_2\big)$ and $\Delta(\kappa_s) = \kappa_s^{(1)} \otimes \kappa_s^{(2)}$ $(s = 1,2)$ this reads

$$\exp\!\big(\hbar\,\kappa_1 \otimes \kappa_2 \otimes 1\big)\,\exp\!\Big(\hbar\,\kappa_1^{(1)} \otimes \kappa_1^{(2)} \otimes \kappa_2\Big) \;=\; \exp\!\big(\hbar\,1 \otimes \kappa_1 \otimes \kappa_2\big)\,\exp\!\Big(\hbar\,\kappa_1 \otimes \kappa_2^{(1)} \otimes \kappa_2^{(2)}\Big)$$

Now taking $\hbar$–adic expansion in both sides of this last identity, at order 0 — in $\hbar$ — we get $1 \otimes 1 \otimes 1 = 1 \otimes 1 \otimes 1$, hence from order 1 we get the non-trivial identity

$$\kappa_1 \otimes \kappa_2 \otimes 1 + \kappa_1^{(1)} \otimes \kappa_1^{(2)} \otimes \kappa_2 \;\underset{\hbar}{\equiv}\; 1 \otimes \kappa_1 \otimes \kappa_2 + \kappa_1 \otimes \kappa_2^{(1)} \otimes \kappa_2^{(2)} \tag{3.1}$$

where hereafter any symbol $\underset{\hbar^n}{\equiv}$ means "congruent modulo $\hbar^n\, U_\hbar^{\widehat{\otimes} 3}$" (for any $n \in \mathbb{N}$). Then taking (3.1) modulo $\hbar$ we get

$$\overline{\kappa_1} \otimes \overline{\kappa_2} \otimes 1 \,+\, \overline{\kappa_1}^{\,(1)} \otimes \overline{\kappa_1}^{\,(2)} \otimes \overline{\kappa_2} \;\;=\;\; 1 \otimes \overline{\kappa_1} \otimes \overline{\kappa_2} \,+\, \overline{\kappa_1} \otimes \overline{\kappa_2}^{\,(1)} \otimes \overline{\kappa_2}^{\,(2)} \tag{3.2}$$

where hereafter $\overline{x} := x \,(\mathrm{mod}\,\hbar)$ " and we took into account that $\overline{\kappa_s^{(i)}} = \overline{\kappa_s}^{(i)}$ for all $s, i \in \{1,2\}$. Now, $\overline{\kappa_s^{(1)}} \otimes \overline{\kappa_s^{(2)}} = \Delta(\overline{\kappa_s})$ with $\overline{\kappa_s} \in U_\hbar \big/ \hbar\, U_\hbar = U(\mathfrak{g})$ has the form

$$\overline{\kappa_s}^{(1)} \otimes \overline{\kappa_s}^{(2)} \;=\; \overline{\kappa_s} \otimes 1 \,+\, 1 \otimes \overline{\kappa_s} \,+\, \dot{\overline{\kappa_s}}^{(1)} \otimes \dot{\overline{\kappa_s}}^{(2)} \tag{3.3}$$

for some $\dot{\overline{\kappa_s}}^{(i)} \in \mathit{Ker}\big(\epsilon_{U(\mathfrak{g})}\big)$ — $i \in \{1,2\}$ — having the following property: if we denote by $U(\mathfrak{g})_n$ the $n$–th piece in the canonical filtration of $U(\mathfrak{g})$ and for any $\mathrm{x} \in U(\mathfrak{g})_n \setminus U(\mathfrak{g})_{n-1}$ we set $\partial(\mathrm{x}) := n$, then in (3.3) we have $\partial\Big(\dot{\overline{\kappa_s}}^{(i)}\Big) \lneqq \partial(\overline{\kappa_s})$. Now, using (3.3) to re-write (3.2) we find, after cancelling out three summands on both sides, that $\dot{\overline{\kappa_1}}^{(1)} \otimes \dot{\overline{\kappa_1}}^{(2)} \otimes \overline{\kappa_2} = \overline{\kappa_1} \otimes \dot{\overline{\kappa_2}}^{(1)} \otimes \dot{\overline{\kappa_2}}^{(2)}$, and then the condition $\partial\Big(\dot{\overline{\kappa_s}}^{(i)}\Big) \lneqq \partial(\overline{\kappa_s})$ forces $\dot{\overline{\kappa_1}}^{(1)} \otimes \dot{\overline{\kappa_1}}^{(2)} = 0 = \dot{\overline{\kappa_2}}^{(1)} \otimes \dot{\overline{\kappa_2}}^{(2)}$. Thus (3.3) reads $\Delta(\overline{\kappa_s}) = \overline{\kappa_s}^{(1)} \otimes \overline{\kappa_s}^{(2)} = \overline{\kappa_s} \otimes 1 + 1 \otimes \overline{\kappa_s}$ ; this means $\overline{\kappa_s} \in \mathfrak{g} \in \big(\subseteq U(\mathfrak{g})\big)$ — for $s \in \{1,2\}$ — so $\overline{\kappa} = \overline{\kappa_1} \otimes \overline{\kappa_2} \in \mathfrak{g} \otimes \mathfrak{g}$, hence $c := \overline{\kappa_a} \in \mathfrak{g} \otimes \mathfrak{g}$.

Now we have to prove that $c$ is an *antisymmetric twist* for the Lie bialgebra $\mathfrak{g}$.

Keeping notation from above, since $\overline{\kappa_s} \in \mathfrak{g}$ we have

$$\Delta(\kappa_s) \underset{\hbar^2}{\equiv} \kappa_s \otimes 1 \,+\, 1 \otimes \kappa_s \,+\, \hbar\, \kappa_s^{[1]} \otimes \kappa_s^{[2]} \tag{3.4}$$

with $\overline{\kappa_s}^{[1]} \otimes \overline{\kappa_s}^{[2]} - \overline{\kappa_s}^{[2]} \otimes \overline{\kappa_s}^{[1]} = \delta(\overline{\kappa_s})$ being the Lie cobracket of $\overline{\kappa_s}$, by assumption. When we plug (3.4) in the $\hbar$–adic expansion of the identity

$$\exp\!\big(\hbar\, \kappa_1 \otimes \kappa_2 \otimes 1\big)\, \exp\!\Big(\hbar\, \kappa_1^{(1)} \otimes \kappa_1^{(2)} \kappa_2\Big) \;=\; \exp\!\big(\hbar\, 1 \otimes \kappa_1 \otimes \kappa_2\big)\, \exp\!\Big(\hbar\, \kappa_1 \otimes \kappa_2^{(1)} \otimes \kappa_2^{(2)}\Big)$$

we find that at order 2 — in $\hbar$ — it implies an identity

$$\overline{\kappa_1}^{[1]} \otimes \overline{\kappa_1}^{[2]} \otimes \overline{\kappa_2} + \overline{\kappa_{1,2}} \cdot \overline{\kappa_{1,3}} + \overline{\kappa_{1,2}} \cdot \overline{\kappa_{2,3}} \;=\; \overline{\kappa_1} \otimes \overline{\kappa_2}^{[1]} \otimes \overline{\kappa_2}^{[2]} + \overline{\kappa_{2,3}} \cdot \overline{\kappa_{1,2}} + \overline{\kappa_{2,3}} \cdot \overline{\kappa_{1,3}} \tag{3.5}$$

where each $\overline{\kappa_{i,j}}$, as usual, is the tensor in $\mathfrak{g}^{\otimes 3}$ which sports the $\kappa_1$'s in position $i$, the $\kappa_2$'s in position $j$, and a (repeated) tensor factor 1 in the last remaining position.

Now let us consider $\Bbbk\big[\mathbb{S}_3\big]$, the group algebra over $\Bbbk$ of the symmetric group $\mathbb{S}_3$, the "antisymmetrizer" $\mathit{Alt}_3 := \big(\mathrm{id} - (1\,2) - (2\,3) - (3\,1) + (1\,2\,3) + (3\,2\,1)\big)$ in $\Bbbk\big[\mathbb{S}_3\big]$, and the natural action of $\Bbbk\big[\mathbb{S}_3\big]$ onto $U(\mathfrak{g})^{\otimes 3}$. Let $\mathit{Alt}_3$ act on the identity (3.5): a sheerly straightforward calculation shows that the outcome, using notation $c := \overline{\kappa_a} = \overline{\kappa} - \overline{\kappa}_{2,1}$, eventually is

$$\big(\delta \otimes \mathrm{id}\big)(c) \,+\, \text{c.p.} \,+\, [[c\,,c]] \;=\; 0$$

This means exactly that $c$ is a twist for the Lie bialgebra $\mathfrak{g}$, as in Definition 2.1.3, which is obviously antisymmetric (by construction), q.e.d.

*(c)* Due to the peculiar form of the twist — namely, its being trivial modulo $\hbar$ — it is easy to see that the Hopf algebra $U_\hbar(\mathfrak{g})^{\mathcal{F}}$ is again a QUEA, over some bialgebra $\tilde{\mathfrak{g}}$, i.e. $U_\hbar(\mathfrak{g})^{\mathcal{F}} \Big/ \hbar\, U_\hbar(\mathfrak{g})^{\mathcal{F}} = U(\tilde{\mathfrak{g}})$, and even that one has $\tilde{\mathfrak{g}} = \mathfrak{g}$ as Lie algebras. In fact, since the twist $\mathcal{F}$ is trivial modulo $\hbar$, we have that $U_\hbar(\mathfrak{g}) \big/ \hbar\, U_\hbar(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})^{\mathcal{F}} \big/ \hbar\, U_\hbar(\mathfrak{g})^{\mathcal{F}}$ are *isomorphic as Hopf algebras*; in particular, then, $U_\hbar(\mathfrak{g})^{\mathcal{F}}$ itself is again a QUEA, on the same Lie algebra $\mathfrak{g}$ from $U_\hbar(\mathfrak{g})$ but possibly inducing on $\mathfrak{g}$ a different Lie cobracket. Indeed, what is actually affected, a priori, is the co-Poisson structure on the semiclassical limit — hence the Lie cobracket on $\mathfrak{g}$ — which in general on $U_\hbar(\mathfrak{g})^{\mathcal{F}} \big/ \hbar\, U_\hbar(\mathfrak{g})^{\mathcal{F}}$ will be different from that on $U_\hbar(\mathfrak{g}) \big/ \hbar\, U_\hbar(\mathfrak{g})$.

Let us compute the Lie coalgebra structure of $\tilde{\mathfrak{g}}$ given by (2.16). Given $\mathrm{x} \in \tilde{\mathfrak{g}}$ , let $x \in U_\hbar(\mathfrak{g})^{\mathcal{F}}$ be any lift of x: using obvious notation, its twisted coproduct is

$$\begin{aligned}\Delta^{\mathcal{F}}(x) \;&=\; \mathcal{F}\,\Delta(x)\,\mathcal{F}^{-1} \;=\; e^{\hbar\kappa}\Big(x\otimes 1+1\otimes x+\hbar\textstyle\sum_i x_1^{[i]}\otimes x_2^{[i]}+O\big(\hbar^2\big)\Big)\,e^{-\hbar\kappa} \;=\\ &=\; \big(1\otimes 1+\hbar\,\kappa\big)\Big(x\otimes 1+1\otimes x+\hbar\textstyle\sum_i x_1^{[i]}\otimes x_2^{[i]}\Big)\big(1\otimes 1-\hbar\,\kappa\big)+O\big(\hbar^2\big) \;=\\ &=\; x\otimes 1+1\otimes x+\hbar\Big(\textstyle\sum_i x_1^{[i]}\otimes x_2^{[i]}-\mathrm{ad}_x(\kappa)\Big) \;+\; O\big(\hbar^2\big)\end{aligned}$$

On the other hand, the opposite twisted coproduct is

$$\begin{aligned}\big(\Delta^{\mathcal{F}}\big)^{\mathrm{op}}(x) \;&=\; (\mathcal{F})_{21}\,\Delta^{\mathrm{op}}(x)\,(\mathcal{F})_{21}^{-1} \;=\; e^{\hbar\,\kappa_{2,1}}\,\Delta^{\mathrm{op}}(x)\,e^{-\hbar\,\kappa_{2,1}} \;=\\ &=\; x\otimes 1+1\otimes x+\hbar\Big(\textstyle\sum_i x_2^{[i]}\otimes x_1^{[i]}-\mathrm{ad}_x\big(\kappa_{2,1}\big)\Big) \;+\; O\big(\hbar^2\big)\end{aligned}$$

Thus, by the very definition of the cobracket — as in (2.16) — we have

$$\delta^{\mathcal{F}}(x) \;:=\; \delta(x)+\big(\mathrm{ad}_x\big(\kappa_{2,1}-\kappa\big)\big) \;\;(\bmod\ \hbar\,) \;=\; \delta(x)-\mathrm{ad}_x(c) \;=:\; \delta^c(x)$$

hence $\tilde{\mathfrak{g}}$ is the twist deformation by $c$ of the Lie bialgebra $\mathfrak{g}$ , as claimed. □

**Observation 3.1.3.** Let us point out that the twists $\mathcal{F}$ considered in Theorem 3.1.2 above are those of "trivial type", as they are the identity modulo $\hbar$ . This ensures that twisting $U_\hbar(\mathfrak{g})$ by such an $\mathcal{F}$ does not affect the Hopf structure of the semiclassical limit; in particular, it still is of the form $U\big(\tilde{\mathfrak{g}}\big)$ , with $\tilde{\mathfrak{g}}$ equal to $\mathfrak{g}$ as a Lie algebra but with a different Lie coalgebra structure. A more general twist might be "unfit", i.e. the deformed Hopf algebra $U_\hbar(\mathfrak{g})^{\mathcal{F}}$ might no longer be a QUEA.

We present now a concrete example, taken from [GaGa2], where formal "multiparameter" QUEAs are studied in detail.

**Example 3.1.4.** Let $n \in \mathbb{N}_+$ and $I := \{1,\dots,n\}$ . We fix a free $\Bbbk[[\hbar]]$–module $\mathfrak{h}$ of finite rank $t$ , and we pick subsets $\Pi^\vee := \big\{T_i^+, T_i^-\big\}_{i\in I} \subseteq \mathfrak{h}$ , $\Pi := \big\{\alpha_i\big\}_{i\in I} \subseteq \mathfrak{h}^* := \mathit{Hom}_{\Bbbk[[\hbar]]}\big(\mathfrak{h}\,,\Bbbk[[\hbar]]\big)$ . Let $P \in M_n\big(\Bbbk[[\hbar]]\big)$ be any $(n\times n)$–matrix with entries in $\Bbbk[[\hbar]]$ . A *realization* of $P$ over $\Bbbk[[\hbar]]$ of rank $t$ is a triple $\mathcal{R} := \big(\mathfrak{h}\,,\Pi\,,\Pi^\vee\big)$ where $\alpha_j\big(T_i^+\big) = p_{ij}$ , $\alpha_j\big(T_i^-\big) = p_{ji}$ $(\forall\, i,j \in I)$, and $\overline{\Sigma} := \big\{\,\overline{S}_i := 2^{-1}\big(T_i^+ + T_i^-\big)\ (\bmod\ \hbar\,\mathfrak{h}\,)\big\}_{i\in I}$ is $\Bbbk$–linearly independent as a subset in $\overline{\mathfrak{h}} := \mathfrak{h}\big/\hbar\,\mathfrak{h}$ .

Let $A := \big(a_{ij}\big)_{i,j\in I} \in M_n(\Bbbk)$ be a symmetrisable generalized Cartan matrix, with associated diagonal matrix $D := \big(d_i\,\delta_{ij}\big)_{i,j\in I}$ . We say that a matrix $P \in M_n(\Bbbk[[\hbar]])$ is *of Cartan type* with corresponding Cartan matrix $A$ if $P_s := 2^{-1}\big(P+P^{\,T}\big) = DA$ .

A *formal multiparameter quantum universal enveloping algebra* (=FoMpQUEA) with multiparameter $P$ and realization $\mathcal{R}$ is the unital, associative, topological, $\hbar$–adically complete $\Bbbk[[\hbar]]$–algebra $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$ generated by the $\Bbbk[[\hbar]]$–submodule $\mathfrak{h}$ and all $E_i$ , $F_i$ (for all $i \in I$ ), with relations (for all $T, T', T'' \in \mathfrak{h}$ , $i\,,j \in I$ )

$$\begin{gathered}T\,E_j \,-\, E_j\,T \;=\; +\alpha_j(T)\,E_j \;\;, \qquad T\,F_j \,-\, F_j\,T \;=\; -\alpha_j(T)\,F_j\\ T'\,T'' \;=\; T''\,T' \;\;, \qquad E_i\,F_j \,-\, F_j\,E_i \;=\; \delta_{i,j}\,\frac{e^{+\hbar\,T_i^+} \,-\, e^{-\hbar\,T_i^-}}{q_i^{+1} \,-\, q_i^{-1}}\\ \sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i} q_{ij}^{+k/2}\,q_{ji}^{-k/2}\,E_i^{1-a_{ij}-k}E_jE_i^k \;=\; 0 \qquad (\,i\neq j\,)\\ \sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i} q_{ij}^{+k/2}\,q_{ji}^{-k/2}\,F_i^{1-a_{ij}-k}F_jF_i^k \;=\; 0 \qquad (\,i\neq j\,)\end{gathered} \tag{3.6}$$

By [GaGa2, Theorem 4.3.2], every FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ bears a structure of topological Hopf algebra over $\Bbbk[[\hbar]]$ — with coproduct taking values into the $\hbar$–adically completed tensor product $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \mathop{\widehat{\otimes}}\limits_{\Bbbk[[\hbar]]} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ — given by ( $\forall\, T \in \mathfrak{h}\,,\ \ell \in I$)

$$\Delta\big(E_\ell\big) = E_\ell \otimes 1 + e^{\hbar T_\ell^+} \otimes E_\ell\ ,\quad \Delta\big(T\big) = T \otimes 1 + 1 \otimes T\ ,\quad \Delta\big(F_\ell\big) = F_\ell \otimes e^{-\hbar T_\ell^-} + 1 \otimes F_\ell$$
$$\epsilon\big(E_\ell\big) \,=\, 0 \quad ,\qquad\qquad \epsilon\big(T\big) \,=\, 0 \quad ,\qquad\qquad \epsilon\big(F_\ell\big) \,=\, 0$$
$$\mathcal{S}\big(E_\ell\big) \,=\, -e^{-\hbar T_\ell^+} E_\ell \quad ,\qquad \mathcal{S}\big(T\big) \,=\, -T \quad ,\qquad \mathcal{S}\big(F_\ell\big) \,=\, -F_\ell\, e^{+\hbar T_\ell^-}$$

Furthermore, by [GaGa2, Theorem 6.1.4], $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is a *quantized universal enveloping algebra* whose semiclassical limit is $U\big(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}\big)$ , where $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}$ is a Lie *multiparameter* Lie bialgebra. In particular, writing again $T$, $E_i$ and $F_i$ for the "specialized" images of the generators $T \in \mathfrak{h}$ and $E_i\,,\ F_i$ ( $i \in I$ ), the Lie algebra structure of $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}$ is given by (3.6) with the commutator replaced by the (Lie) bracket and the quantum Serre relations by the adjoint actions $\mathrm{ad}(E_i)^{1-a_{ij}}(E_j) = 0$ and $\mathrm{ad}(F_i)^{1-a_{ij}}(F_j) = 0\,$, whereas the Lie cobracket is given by $\delta\big(T\big) = 0\,$, $\delta\big(E_i\big) = 2\, T_i^+ \wedge E_i\,$, $\delta\big(F_i\big) = 2\, T_i^- \wedge F_i\,$.

For example, if we take $P := DA\,$, $r := \mathrm{rk}\big(DA\big)$ and $\mathcal{R} := \big(\mathfrak{h}\,,\Pi\,,\Pi^\vee\big)$ a realization of $DA\,$, where $\mathrm{rk}(\mathfrak{h}) = 2\,n - r$ and $T_i^+ = T_i^-$ in $\Pi^\vee$ , for all $i \in I$ , one has that $U_{DA,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is the "quantum double version" of the usual Drinfeld's QUEA $U_\hbar\big(\mathfrak{g}_{{}_A}\big)$ for the Kac-Moody algebra $\mathfrak{g}_{{}_A}$ associated with the Cartan matrix $A\,$; in particular, its semiclassical limit is $U\big(\mathfrak{g}_A^{{}_{MD}}\big)$ , where $\mathfrak{g}_A^{{}_{MD}}$ is the "Manin double version" of $\mathfrak{g}_{{}_A}$ .

Now take any $\Bbbk[[\hbar]]$–basis $\big\{H_g\big\}_{g \in \mathcal{G}}$ of $\mathfrak{h}$ where $|\mathcal{G}| = \mathrm{rk}(\mathfrak{h}) = t\,$. Taking

$$\mathfrak{J}_\Phi \;:=\; \textstyle\sum_{i,j=1}^n \phi_{gk}\, H_g \otimes H_k \;\in\; \mathfrak{h} \otimes \mathfrak{h} \;\subseteq\; U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \otimes U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$$

for any antisymmetric matrix $\Phi = \big(\phi_{i,j}\big)_{1 \le i,j \le n} \in \mathfrak{so}_n\big(\Bbbk[[\hbar]]\big)$ , the element

$$\mathcal{F}_\Phi \;\;:=\;\; e^{\hbar\, 2^{-1} \mathfrak{J}_\Phi} \;\;=\;\; \exp\Big(\hbar\, 2^{-1} \textstyle\sum_{g,k=1}^t \phi_{gk}\, H_g \otimes H_k\Big)$$

in $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \mathbin{\widehat{\otimes}} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ is actually a *twist* for $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ . For $i \in I$ , define the elements $\mathcal{L}_{\Phi,i} := e^{+\hbar\, 2^{-1} \sum_{g,k=1}^t \alpha_i(H_g)\, \phi_{gk} H_k}$ and $\mathcal{K}_{\Phi,i} := e^{+\hbar\, 2^{-1} \sum_{g,k=1}^t \alpha_i(H_g)\, \phi_{kg} H_k}$ . Then, the new coproduct in $\big(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})\big)^{\mathcal{F}_\Phi}$ is given by

$$\begin{aligned} \Delta^{\Phi}\big(E_i\big) \;&=\; E_i \otimes \mathcal{L}_{\Phi,i}^{+1} \,+\, e^{+\hbar T_i^+} \mathcal{K}_{\Phi,i}^{+1} \otimes E_i & \qquad \big(\forall\;\, i \in I\,\big) \\ \Delta^{\Phi}\big(T\big) \;&=\; T \otimes 1 \,+\, 1 \otimes T & \qquad \big(\forall\;\, T \in \mathfrak{h}\,\big) \\ \Delta^{\Phi}\big(F_i\big) \;&=\; F_i \otimes \mathcal{L}_{\Phi,i}^{-1}\, e^{-\hbar T_i^-} \,+\, \mathcal{K}_{\Phi,i}^{-1} \otimes F_i & \qquad \big(\forall\;\, i \in I\,\big) \end{aligned}$$

while the "twisted" antipode $\mathcal{S}^{\mathcal{F}_\Phi}$ can be deduced from the twisted coproduct (see [GaGa3]) and the counit $\epsilon^{\Phi} := \epsilon$ is actually invariant.

With respect to the semiclassical limit, $\overline{\mathfrak{J}}_\Phi := \mathfrak{J}_\Phi \pmod{\hbar}$ is actually a (toral) twist for the Lie bialgebra $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}$ . The deformed Lie cobracket is given by the formula

$$\delta^{\overline{\mathfrak{J}}_\Phi}(x) \;:=\; \delta(x) - \mathrm{ad}_x\big(\overline{\mathfrak{J}}_\Phi\big) \;\;=\;\; \delta(x) - \textstyle\sum_{g,k=1}^t \overline{\phi_{gk}}\,\big(\big[x, H_g\big] \otimes H_k + H_g \otimes \big[x, H_k\big]\big)$$

— for all $x \in \mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}$ , with $\overline{\phi_{gk}} := \phi_{gk} \pmod{\hbar}$ — that on generators reads

$$\delta^{\overline{\mathfrak{J}}_\Phi}(E_i) = 2\, T_{\Phi,i}^+ \wedge E_i\ ,\quad \delta^{\overline{\mathfrak{J}}_\Phi}(T) = 0\ ,\quad \delta^{\overline{\mathfrak{J}}_\Phi}(F_i) = 2\, T_{\Phi,i}^- \wedge F_i\ ,\quad \forall\, i \in I\,,\, T \in \mathfrak{h}$$

where $T_{\Phi,i}^{\pm} = T_i^{\pm} \pm \sum_{g,k=1}^t \overline{\phi_{kg}}\, \alpha_i(H_g)\, H_k$ for all $i \in I$ .

In conclusion, one may consider the deformation $\big(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}\big)^{\overline{\mathfrak{J}}_\Phi}$ of $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}$ by the (Lie) twist $\overline{\mathfrak{J}}_\Phi\,$, as well as the deformation $\big(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})\big)^{\mathcal{F}_\Phi}$ of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ by the (Hopf) twist $\mathcal{F}_\Phi$ . By

[GaGa2, Theorem 6.2.2], we know that $\big(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})\big)^{\mathcal{F}_\Phi}$ is a QUEA, with semiclassical limit $U\Big(\big(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}\big)^{\bar{\mathfrak{J}}_\Phi}\Big) = U\big(\mathfrak{g}_{\bar{P}_\Phi}^{\bar{\mathcal{R}}_\Phi}\big)$ : indeed, $\big(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})\big)^{\mathcal{F}_\Phi} \cong U_{P_\Phi,\hbar}^{\mathcal{R}_\Phi}(\mathfrak{g})$ and $\big(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}\big)^{\bar{\mathfrak{J}}_\Phi} \cong \mathfrak{g}_{\bar{P}_\Phi}^{\bar{\mathcal{R}}_\Phi}$ .

### 3.2. **Deformations by 2-cocycle of QFSHA's.**

We consider now deformations by 2-cocycle of QFSHA's. The outcome is, in short, the dual counterpart of Theorem 3.1.2 above.

**Theorem 3.2.1.**
*Let $F_\hbar[[G]]$ be a QFSHA over the Poisson group $G$, with tangent Lie bialgebra $\mathfrak{g} = \big(\mathfrak{g}\,;\,[\ ,\ ]\,,\,\delta\big)$ . Let $\sigma$ be a 2–cocycle for $F_\hbar[[G]]$ s.t. $\sigma \equiv \epsilon^{\otimes 2} \left(\bmod \hbar \left(F_\hbar[[G]]^{\widetilde{\otimes}\, 2}\right)^*\right)$ ; then $\varsigma := \hbar^{-1} \log_*(\sigma) \in \left(F_\hbar[[G]]^{\widetilde{\otimes}\, 2}\right)^*$ , where "$\log_*$" is the logarithm with respect to the convolution product, and $\sigma = \exp_*\big(\hbar\,\varsigma\big)$ . Last, we set $\varsigma_a := \varsigma - \varsigma_{2,1}$ . Then:*

*(a) $\varsigma$ is antisymmetric, i.e. $\varsigma_{2,1} = -\varsigma$ , iff $\sigma$ is orthogonal, i.e. $\sigma_{2,1} = \sigma^{-1}$ ;*

*(b) the element $\overline{\varsigma_a} := \varsigma_a \left(\bmod \hbar \left(F_\hbar[[G]]^{\widetilde{\otimes}\, 2}\right)^*\right)$ provides a well-defined element $\zeta \in \big(\mathfrak{g}^* \otimes \mathfrak{g}^*\big)^* = \mathfrak{g} \otimes \mathfrak{g}$ that is an antisymmetric 2-cocycle for the Lie bialgebra $\mathfrak{g}^*$;*

*(c) letting $\zeta$ be as in claim (b), the deformation $\big(F_\hbar[[G]]\big)_\sigma$ of $F_\hbar[[G]]$ by the 2–cocycle $\sigma$ is a QFSHA for the formal Poisson group $G_\sigma$ with cotangent Lie bialgebra*

$$Lie\,(G_\zeta)^* \;=\; (\mathfrak{g}^*)_\zeta \;=\; \big(\mathfrak{g}^*\,;\,\big([\ ,\ ]_*\big)_\zeta\,,\,\delta_*\big)$$

*which is the deformation of $\mathfrak{g}^*$ by the 2-cocycle $\zeta$ ; in short, $\big(F_\hbar[[G]]\big)_\sigma \cong F_\hbar[[G_\zeta]]$ .*

*Proof. (a)* This is obvious, just by construction.

*(b)* First, we prove that $\overline{\varsigma_a} := \varsigma_a \left(\bmod \hbar \left(F_\hbar[[G]]^{\widetilde{\otimes}\, 2}\right)^*\right)$ yields a uniquely defined element $\zeta \in \big(\mathfrak{g}^* \otimes \mathfrak{g}^*\big)^* = \mathfrak{g} \otimes \mathfrak{g}$ . We realize $\mathfrak{g}^*$ as $\mathfrak{g}^* = \mathfrak{m}\big/\mathfrak{m}^2$ with $\mathfrak{m} := Ker\big(\epsilon_{F[[G]]}\big)$ , hence $\mathfrak{g}^* \otimes \mathfrak{g}^* = \big(\mathfrak{m}/\mathfrak{m}^2\big) \otimes \big(\mathfrak{m}/\mathfrak{m}^2\big) \cong \big(\mathfrak{m} \otimes \mathfrak{m}\big)\big/\big(\mathfrak{m} \otimes \mathfrak{m}^2 + \mathfrak{m}^2 \otimes \mathfrak{m}\big)$ , thus we have to prove that the function $\overline{\varsigma_a} := \varsigma_a \,\big(\bmod \hbar\big)$ kills $\mathfrak{m} \otimes \mathfrak{m}^2 + \mathfrak{m}^2 \otimes \mathfrak{m}$ , hence induces $\zeta$ defined onto $\big(\mathfrak{g}^* \otimes \mathfrak{g}^*\big)^* = \big(\mathfrak{m} \otimes \mathfrak{m}\big)\Big/\Big(\mathfrak{m} \otimes \mathfrak{m}^2 + \mathfrak{m}^2 \otimes \mathfrak{m}\Big)$ by the recipe $\zeta\,\big(\overline{u \otimes v}\big) := \overline{\varsigma_a}\,(u \otimes v)$ for each $u, v \in \mathfrak{m}$ . In fact, since $\varsigma_a$ is antisymmetric it is enough to prove that $\overline{\varsigma_a}\,\big(\mathfrak{m} \otimes \mathfrak{m}^2\big) = 0$ ; in turn, this amounts to showing that

$$\varsigma_a\big(a\,,b\,c\big) \underset{\hbar}{\equiv} 0 \qquad\qquad \forall\;\; a, b, c \in J_\hbar := Ker\big(\epsilon_{F_\hbar[[G]]}\big) \tag{3.7}$$

For the given $a, b, c \in J_\hbar := Ker\big(\epsilon_{F_\hbar[[G]]}\big)$ , the 2–cocycle nature of $\sigma$ gives

$$\sigma\big(b_{(1)}\,, c_{(1)}\big)\,\sigma\big(a\,, b_{(2)}\,c_{(2)}\big) \;=\; \sigma\big(a_{(1)}\,, b_{(1)}\big)\,\sigma\big(a_{(2)}\,b_{(2)}\,, c\big) \tag{3.8}$$

Now we expand $\sigma \;=\; \epsilon^{\otimes 2} + \hbar \sum_\varsigma \varsigma' \otimes \varsigma'' + \mathcal{O}\big(\hbar^2\big)$ (cf. §2.3.1 for notation "$\mathcal{O}\big(\hbar^2\big)$") using sort of Sweedler's-like notation $\varsigma \;=\; \sum_\varsigma \varsigma' \otimes \varsigma''$ for $\varsigma$ ; this and (3.8) yield

$$\begin{aligned} \epsilon(a)\,\epsilon(b)\,\epsilon(c) \;+\; \hbar\,\Big(\sum_\varsigma \varsigma'(a)\,\varsigma''(b\,c) \;+\; \epsilon(a) \sum_\varsigma \varsigma'(b)\,\varsigma''(c)\Big) \;+\; \mathcal{O}\big(\hbar^2\big) \;\;= \\ =\;\; \epsilon(a)\,\epsilon(b)\,\epsilon(c) \;+\; \hbar\,\Big(\sum_\varsigma \varsigma'(a\,b)\,\varsigma''(c) \;+\; \sum_\varsigma \varsigma'(a)\,\varsigma''(b)\,\epsilon(c)\Big) \;+\; \mathcal{O}\big(\hbar^2\big) \end{aligned}$$

which implies also $\hbar \sum_\varsigma \varsigma'(a)\,\varsigma''(b\,c) + \mathcal{O}(\hbar^2) = \hbar \sum_\varsigma \varsigma'(a\,b)\,\varsigma''(c) + \mathcal{O}(\hbar^2)$ whence

$$\varsigma(a\,,b\,c\,) \;=\; \sum_\varsigma \varsigma'(a)\,\varsigma''(b\,c\,) \;\underset{\hbar}{\equiv}\; \sum_\varsigma \varsigma'(a\,b)\,\varsigma''(c\,) \;=\; \varsigma(a\,b\,,c\,) \tag{3.9}$$

Recall also that $F_\hbar[[G]]$ is commutative modulo $\hbar$, so that $x\,y \underset{\hbar}{\equiv} y\,x$ for all $x, y \in F_\hbar[[G]]$. Using this along with several instances of (3.9) one gets

$$\varsigma(a\,,b\,c\,) \underset{\hbar}{\equiv} \varsigma(a\,b\,,c\,) \underset{\hbar}{\equiv} \varsigma(b\,a\,,c\,) \underset{\hbar}{\equiv} \varsigma(b\,,a\,c\,) \underset{\hbar}{\equiv} \varsigma(b\,,c\,a) \underset{\hbar}{\equiv} \varsigma(b\,c\,,a)$$

from which we eventually conclude that

$$\varsigma_a(a\,,b\,c\,) \;:=\; \varsigma(a\,,b\,c\,) - \varsigma(b\,c,\,a) \;\underset{\hbar}{\equiv}\; \varsigma(b\,c,\,a) - \varsigma(b\,c,\,a) \;=\; 0 \quad , \qquad \text{q.e.d.}$$

As a second step, we note that $\zeta$ is antisymmetric, by construction, since $\varsigma_a$ is.

Third, we need to prove that $\zeta : \mathfrak{g}^* \otimes \mathfrak{g}^* \longrightarrow \Bbbk$ satisfies the remaining condition of (2.4). Expanding $\sigma$ as $\sigma = \exp_*\big(\hbar\,\varsigma\big) = \epsilon^{\otimes 2} + \hbar\,\varsigma + \hbar^2\,\varsigma^{*2}\big/2 + \mathcal{O}\big(\hbar^3\big)$ and plugging this into (3.8), we find, for all $a, b, c \in J_\hbar := Ker\big(\epsilon_{F_\hbar[[G]]}\big)$ again,

$$\begin{aligned}
&\bigg(\epsilon\big(b_{(1)}\big)\,\epsilon\big(c_{(1)}\big) + \hbar\,\varsigma\big(b_{(1)}\,,c_{(1)}\big) + \hbar^2\,\varsigma\Big(b_{(1)_{(1)}}, c_{(1)_{(1)}}\Big)\,\varsigma\Big(b_{(1)_{(2)}}, c_{(1)_{(2)}}\Big)\Big/2 + \mathcal{O}\big(\hbar^3\big)\bigg)\cdot \\
&\qquad\cdot\bigg(\epsilon(a)\,\epsilon\big(b_{(2)}\big)\,\epsilon\big(c_{(2)}\big) + \hbar\,\varsigma\big(a\,,b_{(2)}\,c_{(2)}\big) + \\
&\qquad\qquad + \hbar^2\,\varsigma\Big(a_{(1)}\,,b_{(2)_{(1)}}c_{(2)_{(1)}}\Big)\,\varsigma\Big(a_{(2)}\,,b_{(2)_{(2)}}c_{(2)_{(2)}}\Big)\Big/2 + \mathcal{O}\big(\hbar^3\big)\bigg) \;= \\
&= \bigg(\epsilon\big(a_{(1)}\big)\,\epsilon\big(b_{(1)}\big) + \hbar\,\varsigma\big(a_{(1)}\,,b_{(1)}\big) + \hbar^2\,\varsigma\Big(a_{(1)_{(1)}}, b_{(1)_{(1)}}\Big)\,\varsigma\Big(a_{(1)_{(2)}}, b_{(1)_{(2)}}\Big)\Big/2 + \mathcal{O}\big(\hbar^3\big)\bigg)\cdot \\
&\qquad\cdot\bigg(\epsilon\big(a_{(2)}\big)\,\epsilon\big(b_{(2)}\big)\,\epsilon(c) + \hbar\,\varsigma\big(a_{(2)}\,b_{(2)}\,,c\big) + \\
&\qquad\qquad + \hbar^2\,\varsigma\Big(a_{(2)_{(1)}}b_{(2)_{(1)}}, c\Big)\,\varsigma\Big(a_{(2)_{(2)}}b_{(2)_{(2)}}, c\Big)\Big/2 + \mathcal{O}\big(\hbar^3\big)\bigg)
\end{aligned}$$

then multiplying, truncating at order 3, and recalling that $\epsilon(a) = 0 = \epsilon(c)$, we get

$$\begin{aligned}
&\varsigma(a\,,b\,c\,) - \varsigma(a\,b\,,c\,) + \hbar\,\Big(\varsigma^{*2}(a\,,b\,c\,)/2 - \varsigma^{*2}(a\,b\,,c\,)/2 \;+ \\
&\quad + \varsigma\big(b_{(1)}\,,c_{(1)}\big)\,\varsigma\big(a\,,b_{(2)}\,c_{(2)}\big) - \varsigma\big(a_{(1)}\,,b_{(1)}\big)\,\varsigma\big(a_{(2)}\,b_{(2)}\,,c\,\big)\Big) \;\underset{\hbar^2}{\equiv}\; 0
\end{aligned} \tag{3.10}$$

Now let $\Bbbk\big[\mathbb{S}_3\big]$ act onto $F_\hbar[[G]]^{\otimes 3}$ and consider in particular the action of the antisymmetrizer $Alt_3 := \big(\mathrm{id} - (1\,2) - (2\,3) - (3\,1) + (1\,2\,3) + (3\,2\,1)\big)$ onto the equation in (3.10), which yields a new equation: denoting equation (3.10) by $\circledast = 0$, we will write $Alt_3(\circledast) = 0$ for the newly found equation. To see the latter explicitly, we compute the left-hand member $Alt_3(\circledast)$: a first contribution is

$$\begin{aligned}
&Alt_3\big(\,1^{st}\text{ line in (3.10)}\big) \;= \\
&\;= \varsigma(a\,,b\,c\,) - \varsigma(b\,,a\,c\,) - \varsigma(a\,,c\,b\,) - \varsigma(c\,,b\,a\,) + \varsigma(c\,,a\,b\,) + \varsigma(b\,,c\,a\,) \;- \\
&\;- \varsigma(a\,b\,,c\,) + \varsigma(b\,a\,,c\,) + \varsigma(a\,c\,,b\,) + \varsigma(c\,b\,,a\,) - \varsigma(c\,a\,,b\,) - \varsigma(b\,c\,,a\,) \;= \\
&\;= \varsigma_a\big(a\,,[b\,,c\,]\big) + \varsigma_a\big(b\,,[c\,,a\,]\big) + \varsigma_a\big(c\,,[a\,,b\,]\big) \;=\; \varsigma_a\big(a\,,[b\,,c\,]\big) + \text{c.p.}
\end{aligned} \tag{3.11}$$

where notation $[\,u\,,v\,] := u\,v - v\,u$ is used to denote the usual commutator. Modulo $\hbar$, such a commutator in $F_\hbar[[G]]$ yields the Poisson bracket in $F[[G]]$, hence we can write $[\,u\,,v\,] \;=\; \hbar\,\big\{\,\overline{u}\,,\overline{v}\,\big\}'$ where we write $\overline{z} := \big(z \bmod \hbar\,F_\hbar[[G]]\big)$ for each

$z \in F_\hbar[[G]]$ and $\mathrm{f}' :=$ some lift in $F_\hbar[[G]]$ of any $\mathrm{f} \in F[[G]]$, i.e. $\overline{\mathrm{f}'} = \mathrm{f}$; note that $\mathrm{f}'$ is only defined up to $\hbar^2 F_\hbar[[G]]$, yet that is enough for us. Then (3.11) turns into

$$Alt_3\big(\mathit{1^{st}\ line\ in}\ (3.10)\big) \ = \ \hbar\left(\varsigma_a\Big(a\,,\{\overline{b}\,,\overline{c}\,\}'\Big) \ + \ \text{c.p.}\right) \tag{3.12}$$

Looking at (3.10), this entails that the $\hbar$–adic expansion of $Alt_3(\circledast)$ has zero term at order $0$, while at order 1 it also has a contribution coming from (3.12).

Now we go and compute the contribution to $Alt_3(\circledast)$ issuing from the third line in (3.10). Again, direct calculations give

$$Alt_3\big(\mathit{3^{rd}\ line\ in}\ (3.10)\big) \ = \ \varsigma_a\big(a_{(1)}\,,b_{(1)}\big)\,\varsigma_a\big(c\,,a_{(2)}\,b_{(2)}\big) \ + \ \text{c.p.} \tag{3.13}$$

Since one always has $x = \epsilon(x) + x_+$ with $x_+ := \big(x - \epsilon(x)\big) \in Ker(\epsilon)$, applying this to each element $x \in \{a\,,b\,,c\}$ occurring in (3.13), then expanding everything and taking into account that $\varsigma_a\big(J_\hbar\,,J_\hbar^2\big) \underset{\hbar}{\equiv} 0 \underset{\hbar}{\equiv} \varsigma_a\big(J_\hbar^2,J_\hbar\big)$ — cf. (3.7) — we obtain

$$Alt_3\big(\mathit{3^{rd}\ line\ in}\ (3.10)\big) \ \underset{\hbar}{\equiv} \ \varsigma_a\big(a\,,b^\wedge_{(1)}\big)\,\varsigma_a\big(c\,,b^\wedge_{(2)}\big) \ + \ \text{c.p.}$$

where we make use of short-hand notation $x^\wedge_{(1)} \otimes x^\wedge_{(2)} := x_{(1)} \otimes x_{(2)} - x_{(2)} \otimes x_{(1)}$.

Finally, we go and compute the contribution to $Alt_3(\circledast)$ issuing from the second line in (3.10). Dropping the coefficients $\hbar$ and $1/2$ we find the following:

$$\begin{aligned} Alt_3\big(\mathit{2^{nd}\ line\ in}\ (3.10)\big) \ &= \ Alt_3\big(\varsigma^{*2}(a\,,b\,c) \ - \ \varsigma^{*2}(b\,,a\,c)\big) \ = \\ &= \ \varsigma^{*2}\big(a\,,[b\,,c]\big) \ + \ \varsigma^{*2}\big(b\,,[c\,,a]\big) \ + \ \varsigma^{*2}\big(c\,,[a\,,b]\big) \ - \\ &\quad - \ \varsigma^{*2}\big([b\,,c]\,,a\big) \ - \ \varsigma^{*2}\big([c\,,a]\,,b\big) \ - \ \varsigma^{*2}\big([a\,,b]\,,c\big) \end{aligned}$$

which in turn implies $Alt_3\big(\varsigma^{*2}(a\,,b\,c) \ - \ \varsigma^{*2}(b\,,a\,c)\big) \ = \ \mathcal{O}(\hbar)$ — since $[\,u\,,v\,] = \mathcal{O}(\hbar)$ for all $u, v \in F_\hbar[[G]]$. The outcome then is that the contribution to $Alt_3(\circledast)$ given by the second line in (3.10) is trivial modulo $\hbar^2$.

Summing up, the outcome of the previous analysis is that

$$\varsigma_a\Big(a\,,\{\overline{b}\,,\overline{c}\,\}'\Big) \ + \ \text{c.p.} \ + \ \varsigma_a\big(a\,,b^\wedge_{(1)}\big)\,\varsigma_a\big(c\,,b^\wedge_{(2)}\big) \ + \ \text{c.p.} \ \underset{\hbar}{\equiv} \ 0$$

Taking the latter modulo $\hbar\,F_\hbar[[G]]$ we find, for the elements $\overline{a}\,,\overline{b}\,,\overline{c} \in F[[G]]$,

$$\varsigma_a\big(\overline{a}\,,\{\overline{b}\,,\overline{c}\,\}\big) \ + \ \text{c.p.} \ + \ \varsigma_a\Big(\overline{a}\,,\overline{b}^\wedge_{(1)}\Big)\,\varsigma_a\Big(\overline{c}\,,\overline{b}^\wedge_{(2)}\Big) \ + \ \text{c.p.} \ = \ 0$$

Now recall that for $x \in J_\hbar$ with $\overline{x} := \big(x \bmod \hbar\,F_\hbar[[G]]\big)$ and $\mathrm{x} := \big(\overline{x} \bmod \mathfrak{m}^2\big)$ we have $\delta(\mathrm{x}) := \mathrm{x}^\wedge_{(1)} \otimes \mathrm{x}^\wedge_{(2)}$ for the induced Lie cobracket of $\mathfrak{g}^* = \mathfrak{m}/\mathfrak{m}^2$ computed on $\mathrm{x}$, by definition; this means that, using our previously established notation $\delta(\mathrm{x}) := \mathrm{x}_{[1]} \otimes \mathrm{x}_{[2]}$, the last formula above yields

$$\zeta\big(\mathrm{a}\,,[\,\mathrm{b}\,,\mathrm{c}\,]\big) \ + \ \text{c.p.} \ + \ \zeta\big(\mathrm{a}\,,\mathrm{b}_{[1]}\big)\,\zeta\big(\mathrm{c}\,,\mathrm{b}_{[2]}\big) \ + \ \text{c.p.} \ = \ 0 \tag{3.14}$$

Finally, the antisymmetry of $\zeta$ gives $\zeta\big(\mathrm{a}\,,[\,\mathrm{b}\,,\mathrm{c}\,]\big) + \text{c.p.} \ = \ -\zeta\big([\,\mathrm{a}\,,\mathrm{b}\,]\,,\mathrm{c}\big) + \text{c.p.}$, while a straightforward check shows that $\zeta\big(\mathrm{a}\,,\mathrm{b}_{[1]}\big)\,\zeta\big(\mathrm{c}\,,\mathrm{b}_{[2]}\big) + \text{c.p.} \ = \ -[[\,\zeta\,,\zeta\,]]_*$. Therefore, (3.14) is equivalent to

$$\zeta\big([\,\mathrm{a}\,,\mathrm{b}\,]\,,\mathrm{c}\big) \ + \ \text{c.p.} \ + \ [[\,\zeta\,,\zeta\,]]_* \ = \ 0$$

which means that $\zeta$ is indeed a (strong type of) 2–cocycle for $\mathfrak{g}^*$, q.e.d.

*(c)* Let us consider the deformed algebra $\big(F_\hbar[[G]]\big)_\sigma$, which coincides with $F_\hbar[[G]]$ as a $\Bbbk[[\hbar]]$–module but is endowed with the deformed multiplication " $\underset{\sigma}{\cdot}$ " defined by

$$a \underset{\sigma}{\cdot} b \ := \ \sigma\big(a_{(1)},b_{(1)}\big)\,a_{(2)}\,b_{(2)}\,\sigma^{-1}\big(a_{(3)},b_{(3)}\big) \qquad\qquad \forall\; a,b \in F_\hbar[[G]] \tag{3.15}$$

As $\sigma$ is of the form $\sigma = \exp_*\big(\hbar\,\varsigma\big)$, it follows from (3.15) that the deformed multiplication " $\underset{\sigma}{\cdot}$ " coincides with the old one modulo $\hbar$, i.e. $a \underset{\sigma}{\cdot} b \equiv a\,b \ \big(\mathrm{mod}\ \hbar\, F_\hbar[[G]]\big)$. Thus $\big(F_\hbar[[G]]\big)_\sigma$ is again a QFSHA, say $\big(F_\hbar[[G]]\big)_\sigma = F_\hbar\big[\big[G_{(\sigma)}\big]\big]$. Then, in order to prove that the new group $G_{(\sigma)}$ is indeed $G_\zeta$ it is enough to show that the Lie bracket induced in $\mathfrak{m}_\sigma \big/ \mathfrak{m}_\sigma^2$ — where $\mathfrak{m}_\sigma := \mathit{Ker}\big(\epsilon_{F[[G_{(\sigma)}]]}\big)$ — is $[\ ,\ ]_\zeta$.

Let us take $\mathrm{a}\,,\mathrm{b} \in \mathfrak{m}_\sigma \big/ \mathfrak{m}_\sigma^2$; then we can pick $a, b \in J_\hbar := \mathit{Ker}\big(\epsilon_{F_\hbar[[G]]}\big)$ such that $\mathrm{a} = a\ \Big(\mathrm{mod}\ \big(\hbar\, J_\hbar + J_\hbar^2\big)\Big)$ and $\mathrm{b} = b\ \Big(\mathrm{mod}\ \big(\hbar\, J_\hbar + J_\hbar^2\big)\Big)$. Now, using the expansion $\sigma = \exp_*\big(\hbar\,\varsigma\big) = \epsilon^{\otimes 2} + \hbar\,\varsigma + \mathcal{O}\big(\hbar^2\big)$, formula (3.15) turns into

$$\begin{aligned} a \underset{\sigma}{\cdot} b \ &= \ \big(\epsilon\big(a_{(1)}\big)\epsilon\big(b_{(1)}\big) + \hbar\,\varsigma\big(a_{(1)}, b_{(1)}\big)\big)\, a_{(2)}\, b_{(2)}\, \big(\epsilon\big(a_{(3)}\big)\epsilon\big(b_{(3)}\big) + \hbar\,\varsigma\big(a_{(3)}, b_{(3)}\big)\big) + \\ &\quad + \mathcal{O}\big(\hbar^2\big) \ = \ a\,b \ + \ \hbar\,\Big(\varsigma\big(a_{(1)}\,, b_{(1)}\big)\, a_{(2)}\, b_{(2)} \ - \ a_{(1)}\, b_{(1)}\, \varsigma\big(a_{(2)}\,, b_{(2)}\big)\Big) \ + \ \mathcal{O}\big(\hbar^2\big) \end{aligned}$$

Therefore, using "$[\ ,\ ]_\sigma$" and "$[\ ,\ ]$" to denote the commutator with respect to the new and the old multiplication, we also have (using that $a_{(s)}\, b_{(s)} \underset{\hbar}{\equiv} b_{(s)}\, a_{(s)}$ )

$$\begin{aligned} [a\,,b\,]_\sigma \ &:= \ a \underset{\sigma}{\cdot} b - b \underset{\sigma}{\cdot} a \ = \ a\,b \ + \ \hbar\,\Big(\,\varsigma\big(a_{(1)}\,, b_{(1)}\big)\, a_{(2)}\, b_{(2)} \ - \ a_{(1)}\, b_{(1)}\, \varsigma\big(a_{(2)}\,, b_{(2)}\big)\Big) \ + \\ &\quad + \mathcal{O}\big(\hbar^2\big) \ - \ b\,a \ - \ \hbar\,\Big(\,\varsigma\big(b_{(1)}\,, a_{(1)}\big)\, b_{(2)}\, a_{(2)} \ - \ b_{(1)}\, a_{(1)}\, \varsigma\big(b_{(2)}\,, a_{(2)}\big)\Big) \ + \ \mathcal{O}\big(\hbar^2\big) \ = \\ &= \ [\,a\,,b\,] \ + \ \hbar\,\Big(-\varsigma_a\big(a_{(2)}\,, b_{(2)}\big)\, a_{(1)}\, b_{(1)} \ - \ \varsigma_a\big(b_{(1)}\,, a_{(1)}\big)\, b_{(2)}\, a_{(2)}\Big) \ + \ \mathcal{O}\big(\hbar^2\big) \end{aligned}$$

Recall that $[\,a\,,b\,] = \hbar\,\big\{\overline{a}\,,\overline{b}\,\big\}'$, where hereafter we write $\overline{x} := x \ \big(\mathrm{mod}\ \hbar\, J_\hbar\big)$ and $\mathrm{f}'$ to denote any lift in $J_\hbar$ of some given f in $J_\hbar$, as we did before; similarly, we have $[\,a\,,b\,]_\sigma = \hbar\,\big\{\overline{a}\,,\overline{b}\,\big\}_\sigma'$. Then modulo $\hbar$ our previous computations give

$$\big\{\overline{a}\,,\overline{b}\,\big\}_\sigma \ = \ \big\{\overline{a}\,,\overline{b}\,\big\} \ - \ \overline{\varsigma}_a\big(\overline{a}_{(2)}\,,\overline{b}_{(2)}\big)\,\overline{a}_{(1)}\,\overline{b}_{(1)} \ - \ \overline{\varsigma}_a\big(\overline{b}_{(1)}\,,\overline{a}_{(1)}\big)\,\overline{b}_{(2)}\,\overline{a}_{(2)} \tag{3.16}$$

For each $x \in \big\{\,\overline{a}_{(s)}\,,\overline{b}_{(s)}\,\big|\, s = 1,2\,\big\}$ we have $x = \epsilon(x) + x^+$ with $x^+ := \big(x - \epsilon(x)\big) \in J_\hbar$. Using this in (3.16) along with $\overline{a}_{(s)}^{\,+}\,\overline{b}_{(s)}^{\,+} \underset{\mathfrak{m}^2}{\equiv} 0 \underset{\mathfrak{m}^2}{\equiv} \overline{a}_{(s)}^{\,+}\,\overline{b}_{(s)}^{\,+}$, we get an equivalence modulo $\mathfrak{m}^2 = \mathfrak{m}_\sigma^2$ (noting that $\mathfrak{m} = \mathfrak{m}_\sigma$ as $\Bbbk$–modules), namely

$$\begin{aligned} \big\{\overline{a}\,,\overline{b}\,\big\}_\sigma' \ &= \ \big\{\overline{a}\,,\overline{b}\,\big\} \ - \ \overline{\varsigma}_a\big(\overline{a}_{(2)}\,,\overline{b}_{(2)}\big)\,\overline{a}_{(1)}\,\overline{b}_{(1)} \ - \ \overline{\varsigma}_a\big(\overline{b}_{(1)}\,,\overline{a}_{(1)}\big)\,\overline{b}_{(2)}\,\overline{a}_{(2)} \ \underset{\mathfrak{m}^2}{\equiv} \\ &\underset{\mathfrak{m}^2}{\equiv} \ \big\{\overline{a}\,,\overline{b}\,\big\} \ - \ \overline{\varsigma}_a\big(\overline{a}_{(2)}\,,\overline{b}_{(2)}\big)\,\overline{a}_{(1)}\,\epsilon\big(\overline{b}_{(1)}\big) \ - \ \overline{\varsigma}_a\big(\overline{a}_{(2)}\,,\overline{b}_{(2)}\big)\,\epsilon\big(\overline{a}_{(1)}\big)\,\overline{b}_{(1)} \ - \\ &\qquad - \ \overline{\varsigma}_a\big(\overline{b}_{(1)}\,,\overline{a}_{(1)}\big)\,\overline{b}_{(2)}\,\epsilon\big(\overline{a}_{(2)}\big) \ - \ \overline{\varsigma}_a\big(\overline{b}_{(1)}\,,\overline{a}_{(1)}\big)\,\epsilon\big(\overline{b}_{(2)}\big)\,\overline{a}_{(2)} \ = \\ &= \ \big\{\overline{a}\,,\overline{b}\,\big\} - \big(\,\overline{\varsigma}_a\big(\overline{a}_{(2)},\overline{b}\,\big)\,\overline{a}_{(1)} - \overline{\varsigma}_a\big(\overline{a}_{(1)},\overline{b}\,\big)\,\overline{a}_{(2)}\,\big) - \big(\,\overline{\varsigma}_a\big(\overline{b}_{(1)},\overline{a}\,\big)\,\overline{b}_{(2)} - \overline{\varsigma}_a\big(\overline{b}_{(2)},\overline{a}\,\big)\,\overline{b}_{(1)}\,\big) \end{aligned}$$

where the element in last line actually belongs to $\mathfrak{m} = \mathfrak{m}_\sigma$. When we reduce all this modulo $\mathfrak{m}^2 = \mathfrak{m}_\sigma^2$, we eventually end up with

$$[\,\mathrm{a}\,,\mathrm{b}\,]_{(\sigma)} \ = \ [\,\mathrm{a}\,,\mathrm{b}\,]_* \ - \ \zeta\big(\mathrm{a}_{[2]},\mathrm{b}\big)\,\mathrm{a}_{[1]} \ - \ \zeta\big(\,\mathrm{b}_{[1]},\mathrm{a}\big)\,\mathrm{b}_{[2]} \ =: \ \big([\,\mathrm{a}\,,\mathrm{b}\,]_*\big)_\zeta$$

thus(cf. Definition 2.5) the Lie bracket we were looking for is just $\big([\ ,\ ]_*\big)_\zeta$. $\square$

**Observation 3.2.2.** We would better point out that the 2-cocycles $\sigma$ considered in Theorem 3.2.1 above are those of "trivial-modulo-$\hbar$-type", in that they are the identity modulo $\hbar$. With this assumption, deforming $F_\hbar[[G]]$ by such a $\sigma$ does not affect the Hopf structure of the semiclassical limit; in particular, it still reads as $F\big[\big[\widetilde{G}\big]\big]$, with $\widetilde{G}$ being the same formal group as $G$ but with a different Poisson structure. A more general 2-cocycle might be "unfit", in that the deformed Hopf algebra $\big(F_\hbar[[G]]\big)_\sigma$ may no longer be a QFSHA, in general.

**Example 3.2.3.** Let $G := GL_n(\Bbbk)$ be the general linear group over $\Bbbk$, and $\mathfrak{g} := \mathfrak{gl}_n(\Bbbk)$ its tangent Lie algebra. It is well-known — cf. [Dr], [CP] — that a quantization of $\mathfrak{g}$ is provided by the QUEA $U_\hbar(\mathfrak{g}) = U_\hbar\big(\mathfrak{gl}_n(\Bbbk)\big)$ defined as follows: it is the unital, associative, $\hbar$–adically complete $\Bbbk[[\hbar]]$–algebra with generators

$$F_1\,, F_2\,, \dots, F_{n-1}\,, \varGamma_1\,, \varGamma_2\,, \dots, \varGamma_{n-1}\,, \varGamma_n\,, E_1\,, E_2\,, \dots, E_{n-1}$$

and relations (for all $i,j \in \{1,\dots,n-1\}$, $k,\ell \in \{1,\dots,n\}$ )

$$\begin{aligned} \big[\varGamma_k\,, \varGamma_\ell\big] \,=\, 0 \;\;, \quad \big[\varGamma_k\,, F_j\big] \,=\, -\delta_{k,j}\,F_j \;\;, \quad \big[\varGamma_k\,, E_j\big] \,=\, +\delta_{k,j}\,E_j \\ \big[E_i\,, F_j\big] \,=\, \delta_{i,j}\,\frac{e^{\hbar\,(\varGamma_i-\varGamma_{i+1})} - e^{\hbar\,(\varGamma_{i+1}-\varGamma_i)}}{e^{+\hbar} - e^{-\hbar}} \\ \big[E_i\,, E_j\big] \,=\, 0 \;\;, \qquad \big[F_i\,, F_j\big] \,=\, 0 \qquad\qquad \forall\;\, i\,,j \,:\, |i-j|>1 \\ E_i^2\,E_j - \big(q+q^{-1}\big)\,E_i\,E_j\,E_i + E_j\,E_i^2 \,=\, 0 \qquad\qquad \forall\;\, i\,,j \,:\, |i-j|=1 \\ F_i^2\,F_j - \big(q+q^{-1}\big)\,F_i\,F_j\,F_i + F_j\,F_i^2 \,=\, 0 \qquad\qquad \forall\;\, i\,,j \,:\, |i-j|=1\,. \end{aligned}$$

where $[X\,,Y] := X\,Y - Y\,X$. The (topological) Hopf algebra structure is given by

$$\begin{aligned} \Delta(F_i) \,=\, F_i \otimes e^{\hbar\,(\varGamma_{i+1}-\varGamma_i)} + 1 \otimes F_i\;, \qquad S(F_i) \,=\, -F_i\,e^{\hbar\,(\varGamma_i-\varGamma_{i+1})}\;, \qquad \epsilon(F_i) \,=\, 0 \\ \Delta(\varGamma_k) \,=\, \varGamma_k \otimes 1 + 1 \otimes \varGamma_k\;, \qquad S(\varGamma_k) \,=\, -\varGamma_k\;, \qquad \epsilon(\varGamma_k) \,=\, 0 \\ \Delta(E_i) \,=\, E_i \otimes 1 + e^{\hbar\,(\varGamma_i-\varGamma_{i+1})} \otimes E_i\;, \qquad S(E_i) \,=\, -e^{\hbar\,(\varGamma_{i+1}-\varGamma_i)}\,E_i\;, \qquad \epsilon(E_i) \,=\, 0 \end{aligned}$$

It is also well-known — cf. [Dr], [CP] — that a quantization of $G := GL_n(\Bbbk)$ is provided by the QFSHA $F_\hbar[[G]] = F_\hbar\big[\big[GL_n(\Bbbk)\big]\big]$ defined as follows: it is the unital, associative, $I_\hbar$–adically complete $\Bbbk[[\hbar]]$–algebra generated by the elements of the set $\big\{x_{ij} \,\big|\, i,j = 1,\dots,n+1\big\}$ *arranged in a $q$–matrix, with* $q := \exp(\hbar)$, with $I_\hbar$ being the ideal generated by $\big\{\hbar\,, x_{1,1}\,,\dots, x_{n,n}\big\}$; this is a quick way to say that the given generators obey the relations

$$\begin{aligned} x_{ij}\,x_{ik} \,=\, q\,x_{ik}\,x_{ij}\;, \qquad x_{ik}\,x_{hk} \,=\, q\,x_{hk}\,x_{ik} \qquad\qquad \forall\;\, j<k\,,i<h \\ x_{il}\,x_{jk} \,=\, x_{jk}\,x_{il}\;, \qquad x_{ik}\,x_{jl} - x_{jl}\,x_{ik} \,=\, \big(q-q^{-1}\big)\,x_{il}\,x_{jk} \qquad \forall\;\, i<j\,,k<l \end{aligned}$$

whereas $\Delta$, $\epsilon$ and $S$ are given (in matrix formulation) by

$$\Delta\Big(\big(x_{ij}\big)_{i=1,\dots,n;}^{j=1,\dots,n;}\Big) \,:=\, \big(x_{ij}\big)_{i=1,\dots,n;}^{j=1,\dots,n;} \otimes \big(x_{ij}\big)_{i=1,\dots,n;}^{j=1,\dots,n;}$$

$$\epsilon\Big(\big(x_{ij}\big)_{i=1,\dots,n;}^{j=1,\dots,n;}\Big) \,:=\, \big(\delta_{ij}\big)_{i=1,\dots,n;}^{j=1,\dots,n;} \quad , \qquad S\Big(\big(x_{ij}\big)_{i=1,\dots,n;}^{j=1,\dots,n;}\Big) \,:=\, \Big(\big(x_{ij}\big)_{i=1,\dots,n;}^{j=1,\dots,n;}\Big)^{-1}$$

which in down-to-earth terms read, for all $i,j = 1,\dots,n$,

$$\Delta\big(x_{ij}\big) \,=\, \textstyle\sum_{k=1}^n x_{ik} \otimes x_{kj} \;\;, \qquad \epsilon\big(x_{ij}\big) \,=\, \delta_{ij} \;\;, \qquad S\big(x_{ij}\big) \,=\, (-q)^{j-i} D_q\Big(\big(x_{hk}\big)_{h\neq j}^{k\neq i}\Big)$$

where $D_q$ is the *quantum determinant*, defined on any square $q$–matrix of size $\ell$ by

$$D_q\Big(\big(x_{ij}\big)_{i=1,\dots,\ell;}^{j=1,\dots,\ell;}\Big) \,:=\, \textstyle\sum_{\sigma\in S_\ell} (-q)^{l(\sigma)} x_{1,\sigma(1)}\,x_{2,\sigma(2)} \cdots x_{\ell,\sigma(\ell)}$$

We have also explicit identifications $F_\hbar[[G]] = U_\hbar(\mathfrak{g})^*$ as well as $U_\hbar(\mathfrak{g}) = F_\hbar[[G]]^\star$, which can be described via the Hopf pairing $\langle\;,\;\rangle : F_\hbar[[G]] \times U_\hbar(\mathfrak{g}) \longrightarrow \Bbbk[[\hbar]]$ uniquely given by the following values on generators:

$$\big\langle x_{i,j}\,, \varGamma_k\big\rangle \,=\, \delta_{i,j}\,\delta_{i,k} \;\;, \quad \big\langle x_{i,j}\,, E_t\big\rangle \,=\, \delta_{i+1,j}\,\delta_{i,t} \;\;, \quad \big\langle x_{i,j}\,, F_t\big\rangle \,=\, \delta_{i,j+1}\,\delta_{t,j} \qquad (3.17)$$

Now consider in $U_\hbar(\mathfrak{g})\,\widehat{\otimes}\,U_\hbar(\mathfrak{g})$ the element $\mathcal{F} := \exp\Big(\hbar\,2^{-1}\sum_{k,\ell=1}^n \phi_{t,k}\,\varGamma_t \otimes \varGamma_k\Big)$ that is a twist for $U_\hbar(\mathfrak{g})$ By Proposition 2.2.7*(a)*, we can see this $\mathcal{F}$ as a 2–cocycle

$\sigma_{\mathcal{F}}$ for $U_\hbar(\mathfrak{g})^* = F_\hbar[[G]]$, simply given by *evaluation at* $\mathcal{F}$, namely

$$\sigma_{\mathcal{F}} \,:\, F_\hbar[[G]] \times F_\hbar[[G]] \longrightarrow \Bbbk[[\hbar]] \quad , \qquad (\varphi\,,\psi) \mapsto \big\langle\, \varphi \otimes \psi\,,\, \mathcal{F} \,\big\rangle \tag{3.18}$$

Now, from (3.18) and (3.17), direct calculation gives

$$\sigma_{\mathcal{F}}\big(x_{i,r}\,,x_{\ell,h}\big) \;=\; \sum_{m=0}^{+\infty} \frac{\hbar^m\, 2^{-m}}{m!} \,\left\langle\, \Delta^{(m-1)}\big(x_{i,r} \otimes x_{\ell,h}\big)\,,\, \Big(\textstyle\sum_{t,\,k=1}^n \phi_{t,k}\, \varGamma_t \otimes \varGamma_k\Big)^{\otimes m} \right\rangle$$

Let us consider $\left\langle\, \Delta^{(m-1)}\big(x_{i,r} \otimes x_{\ell,h}\big)\,,\, \Big(\sum_{t,\,k=1}^n \phi_{t,k}\, \varGamma_t \otimes \varGamma_k\Big)^{\otimes m} \right\rangle$ . Definitions give

$$\left\langle\, \Delta^{(m-1)}\big(x_{i,r} \otimes x_{\ell,h}\big)\,,\, \Big(\textstyle\sum_{t,\,k=1}^n \phi_{t,k}\, \varGamma_t \otimes \varGamma_k\Big)^{\otimes m} \right\rangle \;=$$

$$=\; \sum_{\substack{s_1,\dots,\,s_{m-1}=1\\ e_1,\dots,\,e_{m-1}=1}}^{n} \left\langle x_{i,\,s_1} \otimes x_{\ell,\,e_1} \otimes \cdots \otimes x_{s_{m-1},\,r} \otimes x_{e_{m-1},\,h}\,,\, \Big(\textstyle\sum_{t,\,k=1}^n \phi_{t,k}\, \varGamma_t \otimes \varGamma_k\Big)^{\otimes m} \right\rangle \;=$$

$$=\; \sum_{\substack{s_1,\dots,\,s_{m-1}=1\\ e_1,\dots,\,e_{m-1}=1}}^{n} \prod_{c=1}^{m} \textstyle\sum_{t,\,k=1}^n \phi_{t,k} \left\langle x_{s_{c-1},\,s_c}\,,\varGamma_t \right\rangle \left\langle x_{e_{c-1},\,e_c}\,,\varGamma_k \right\rangle$$

where we set $s_0 := i$, $s_m := r$, $e_0 := \ell$, $e_m := h$. Now, the formulas in (3.17) guarantee that $\left\langle x_{s_{c-1},\,s_c}\,,\varGamma_t \right\rangle \left\langle x_{e_{c-1},\,e_c}\,,\varGamma_k \right\rangle = 0$ whenever $s_{c-1} \neq s_c$ or $e_{c-1} \neq e_c$; therefore, from the previous computation one eventually gets

$$\sigma_{\mathcal{F}}\big(x_{i,r}\,,x_{\ell,h}\big) \;=\; \delta_{i,r}\,\delta_{\ell,h} \sum_{m=0}^{+\infty} \frac{\hbar^m\, 2^{-m}}{m!} \Big(\textstyle\sum_{t,\,k=1}^n \phi_{t,k} \left\langle x_{i,\,i}\,,\varGamma_t \right\rangle \left\langle x_{\ell,\,\ell}\,,\varGamma_k \right\rangle\Big)^m \;=$$

$$=\; \delta_{i,r}\,\delta_{\ell,h} \sum_{m=0}^{+\infty} \frac{\hbar^m\, 2^{-m}}{m!} \,(\phi_{i,\ell})^m \;=\; \delta_{i,r}\,\delta_{\ell,h}\, \exp\big(\hbar\, 2^{-1} \phi_{i,\ell}\big) \;=\; \delta_{i,r}\,\delta_{\ell,h}\, e^{\hbar\, \phi_{i,\ell}/2}$$

i.e. $$\sigma_{\mathcal{F}}\big(x_{i,r}\,,x_{\ell,h}\big) \;=\; \delta_{i,r}\,\delta_{\ell,h}\, e^{\hbar\, \phi_{i,\ell}/2} \qquad \forall\;\, i\,,r,\ell,h \in \{1,\dots,n\} \tag{3.19}$$

Using this formula, the deformed product in $F_\hbar[[G]]_{\sigma_{\mathcal{F}}}$ can be described as follows:

$$\begin{aligned} x_{i,j} \,\dot{\sigma}_{\mathcal{F}}\, x_{\ell,t} \;&:=\; \sigma_{\mathcal{F}}\big((x_{i,j})_{(1)}\,,\,(x_{\ell,t})_{(1)}\big)\, (x_{i,j})_{(2)}\, (x_{\ell,t})_{(2)}\, \sigma_{\mathcal{F}}^{-1}\big((x_{i,j})_{(3)}\,,\,(x_{\ell,t})_{(3)}\big) \;= \\ &=\; \sigma_{\mathcal{F}}\big(x_{i,i}\,,\, x_{\ell,\ell}\big)\, x_{i,j}\, x_{\ell,t}\, \sigma_{\mathcal{F}}^{-1}\big(x_{j,j}\,,\, x_{t,t}\big) \;=\; e^{\hbar\,(\phi_{i,\ell}\,-\,\phi_{j,t})/2}\, x_{i,j}\, x_{\ell,t} \end{aligned}$$

i.e. $$x_{i,j} \,\dot{\sigma}_{\mathcal{F}}\, x_{\ell,t} \;=\; e^{\hbar\,(\phi_{i,\ell}\,-\,\phi_{j,t})/2}\, x_{i,j}\, x_{\ell,t} \qquad \forall\;\, i\,,j,\ell,t \in \{1,\dots,n\} \tag{3.20}$$

Note that this formula shows how the new, deformed product is equivalent modulo $\hbar$ to the old one: this happens because we work with 2–cocycles of the form $\exp\big(\hbar\,\varsigma\big)$ where $\varsigma$. By this same reason, *any set of elements which generate, as an algebra, the QFSHA under exam, will also generate it w.r.t. the new, deformed product.* For this reason, (3.20) is enough to describe $F_\hbar[[G]]_{\sigma_{\mathcal{F}}}$ as the latter is generated (w.r.t. the new product) by the $x_{i,j}$'s, just like $F_\hbar[[G]]$ was (with the old product).

Let us now see how (3.20) yields a new Poisson bracket in the semiclassical limit of $F_\hbar[[G]]_{\sigma_{\mathcal{F}}}$. With notation $\overline{x}_{r,s} := x_{r,s} \,\big(\mathrm{mod}\;\, \hbar\, F_\hbar[[G]]_{\sigma_{\mathcal{F}}}\big)$, such a Poisson bracket is given by $\big\{\overline{x}_{i,j}\,,\overline{x}_{\ell,t}\big\}_{\sigma_{\mathcal{F}}} := \dfrac{\big[x_{i,j}\,,x_{\ell,t}\big]_{\sigma_{\mathcal{F}}}}{\hbar} \,\Big(\mathrm{mod}\;\, \hbar\, F_\hbar[[G]]_{\sigma_{\mathcal{F}}}\Big)$ . Now

$$\begin{aligned} \big[x_{i,j}\,,x_{\ell,t}\big]_{\sigma_{\mathcal{F}}} \;&=\; e^{\hbar\,(\phi_{i,\ell}\,-\,\phi_{j,t})/2}\, x_{i,j}\, x_{\ell,t} \;-\; e^{\hbar\,(\phi_{j,t}\,-\,\phi_{i,\ell})/2}\, x_{\ell,t}\, x_{i,j} \;= \\ &=\; e^{\hbar\,(\phi_{i,\ell}\,-\,\phi_{j,t})/2} \big[x_{i,j}\,,x_{\ell,t}\big] \;+\; \big(e^{\hbar\,(\phi_{i,\ell}\,-\,\phi_{j,t})/2} - e^{\hbar\,(\phi_{j,t}\,-\,\phi_{i,\ell})/2}\big)\, x_{\ell,t}\, x_{i,j} \end{aligned}$$

hence expanding the exponentials we get

$$\big[x_{i,j}\,,x_{\ell,t}\big]_{\sigma_{\mathcal{F}}} \;=\; \big(1 + \hbar\,(\phi_{i,\ell} \,-\, \phi_{j,t})/2\big) \big[x_{i,j}\,,x_{\ell,t}\big] \,+\, \hbar\,\big(\phi_{i,\ell} \,-\, \phi_{j,t}\big)\, x_{\ell,t}\, x_{i,j} \,+\, \mathcal{O}\big(\hbar^2\big)$$

from which we eventually get

$$\big\{\overline{x}_{i,j}\,,\overline{x}_{\ell,t}\big\}_{\sigma_{\mathcal{F}}} \;:=\; \big\{\overline{x}_{i,j}\,,\overline{x}_{\ell,t}\big\} \,+\, \Big(\,\overline{\phi}_{i,\ell} \,-\, \overline{\phi}_{j,t}\Big)\,\overline{x}_{\ell,t}\,\overline{x}_{i,j} \tag{3.21}$$

where $\big\{\overline{x}_{i,j}\,,\overline{x}_{\ell,t}\big\}$ denotes the old (undeformed) Poisson bracket and we used that $\big[\overline{x}_{i,j}\,,\overline{x}_{\ell,t}\big] \,=\, 0$ and deformed and undeformed product do coincide modulo $\hbar$.

In addition, the formula (3.21) also induces a concrete description of the modified Lie bracket in the cotangent Lie bialgebra $\mathfrak{g}^* := \mathfrak{m}\big/\mathfrak{m}^2$, where $\mathfrak{m}$ is the augmentation ideal of $F_\hbar[[G]]_{\sigma_{\mathcal{F}}}$. Indeed, the latter has as $\Bbbk$–basis the set of cosets $\big($modulo $\mathfrak{m}^2\,\big)$

$$\Big\{\, \mathrm{x}_{i,j} := \big(\overline{x}_{i,j} - \delta_{i,j}\big) \;\text{ mod }\; \mathfrak{m}^2 \,\Big|\, i\,,j=1,\dots,n \,\Big\}$$

and for these elements from (3.21) we deduce the deformed Lie bracket as given by

$$\begin{aligned}
\big[\mathrm{x}_{i,j}\,,\mathrm{x}_{\ell,t}\big]_{\sigma_{\mathcal{F}}} \,=\, \big[\mathrm{x}_{i,j}\,,\mathrm{x}_{\ell,t}\big] \quad , \quad \big[\mathrm{x}_{i,i}\,,\mathrm{x}_{\ell,\ell}\big]_{\sigma_{\mathcal{F}}} \,=\, \big[\mathrm{x}_{i,i}\,,\mathrm{x}_{\ell,\ell}\big] & \qquad \forall\;\, i\neq j\,,\; \ell\neq t \\
\big[\mathrm{x}_{i,i}\,,\mathrm{x}_{\ell,t}\big]_{\sigma_{\mathcal{F}}} \;=\; \big[\mathrm{x}_{i,i}\,,\mathrm{x}_{\ell,t}\big] \,+\, \Big(\,\overline{\phi}_{i,\ell} \,-\, \overline{\phi}_{i,t}\Big)\,\mathrm{x}_{\ell,t} & \qquad \forall\;\, \ell\neq t \\
\big[\mathrm{x}_{i,j}\,,\mathrm{x}_{\ell,\ell}\big]_{\sigma_{\mathcal{F}}} \;:=\; \big[\mathrm{x}_{i,i}\,,\mathrm{x}_{\ell,t}\big] \,+\, \Big(\,\overline{\phi}_{i,\ell} \,-\, \overline{\phi}_{j,\ell}\Big)\,\mathrm{x}_{i,j} & \qquad \forall\;\, i\neq j
\end{aligned}$$

### 3.3. **Deformations by quasi-2-cocycle of QUEA's.**

This subsection is dedicated to deformations by quasi-2-cocycle of QUEA's. The result we achieve is somewhat surprising, as we are "stretching the standard recipe", as the 2-cocycles that we use to deform our Hopf $\Bbbk[[\hbar]]$–algebras are valued in the field $\Bbbk((\hbar))$ rather than in $\Bbbk[[\hbar]]$. Thus, *a priori* nothing guarantees that the recipe would just work and produce a new Hopf algebra over $\Bbbk[[\hbar]]$; nonetheless, we eventually find quite a meaningful result, which says that *the standard procedure of deformation by twist for QUEA's can be extended somewhat beyond its natural borders.*

We begin with two ancillary results.

**Lemma 3.3.1.** *Let $U_\hbar := U_\hbar(\mathfrak{g})$ be any QUEA, and $J_\hbar := Ker\big(\epsilon_{U_\hbar}\big)$. For every $z \in U_\hbar$, there exists $N \in \mathbb{N}$ such that $\delta_n(z) \,\in\, \hbar^{\max(n,N)-N} J_\hbar^{\otimes n}$ for every $n \in \mathbb{N}$.*

*Proof.* By Theorem 2.4.2*(a)*, applying $(\ )^\vee$ after $(\ )'$ to the QUEA $U_\hbar$ we get $U_\hbar = \big(U_\hbar'\big)^\vee$: therefore, letting $I_\hbar' := \hbar\, U_\hbar' + Ker\big(\epsilon_{U_\hbar'}\big)$, this last identity reads

$$U_\hbar \;=\; \hbar\text{-adic completion of } \sum_{n\geq 0} \hbar^{-n}\big(I_\hbar'\big)^n \;=\; \hbar\text{-adic completion of } \bigcup_{n\geq 0} \hbar^{-n}\big(I_\hbar'\big)^n$$

In particular, this implies that for our $z \in U_\hbar(\mathfrak{g})$ there exist some $N \in \mathbb{N}$ and $z' \in \big(I_\hbar'\big)^N$ such that $z \,\equiv\, \hbar^{-N} z' \,\big(\text{mod}\; \hbar\, U_\hbar(\mathfrak{g})\big)$. Now, given $n \in \mathbb{N}$ we have $\delta_n\big(z'\big) \in \hbar^n U_\hbar^{\otimes n}$ because $z' \in \big(I_\hbar'\big)^N \subseteq U_\hbar'$, and also $\delta_n\big(z'\big) \in \sum_{s_1+\cdots+s_n=N} \otimes_{i=1}^n \big(I_\hbar'\big)^{s_i}$ because $I_\hbar'$ is a Hopf ideal; moreover, $Ker\big(\epsilon_{U_\hbar'}\big) \subseteq \hbar\, U_\hbar'$ again by construction, hence $I_\hbar' \subseteq \hbar\, U_\hbar'$. In the end, all this yields $\delta_n\big(z'\big) \in \hbar^{\max(n,N)}\, J_\hbar^{\otimes n}$, therefore $\delta_n(z) \in \hbar^{\max(n,N)-N}\, J_\hbar^{\otimes n}$ as claimed. $\square$

We fix now some more notation: namely, we denote by "$\log_*$" and "$\exp_*$" the logarithm and the exponential w.r.t. the convolution product, whenever defined.

**Lemma 3.3.2.** *Let $U_\hbar(\mathfrak{g})$ be any QUEA, and let $\chi$ be a $\Bbbk[[\hbar]]$–bilinear form on $U_\hbar(\mathfrak{g})$ such that $\chi(z\,,1) = 0 = \chi(1\,,z)$ for any $z \in U_\hbar(\mathfrak{g})$; denote also by the same symbol $\chi$ the scalar extension of $\chi$ to a $\Bbbk((\hbar))$–bilinear form for the $\Bbbk((\hbar))$–vector space $\mathbb{U}_\hbar(\mathfrak{g}) := \Bbbk((\hbar)) \otimes_{\Bbbk[[\hbar]]} U_\hbar(\mathfrak{g})$. Then:*

*(a) the formal expression* $\sigma := \exp_*\big(\hbar^{-1}\chi\big)$ *uniquely provides a well-defined,* $\Bbbk(\!(\hbar)\!)$*–valued bilinear form for* $\mathbb{U}_\hbar(\mathfrak{g})$ *;*

*(b)* $\sigma(z\,,1) = \epsilon(z) = \sigma(1\,,z)$ *for any* $z \in U_\hbar(\mathfrak{g})$ *;*

*(c)* $\sigma$ *is orthogonal, i.e.* $\sigma_{2,1} = \sigma^{-1}$*, iff* $\chi$ *is antisymmetric, i.e.* $\chi_{2,1} = -\chi$ *.*

*Proof. (a)* Fix notation $U_\hbar := U_\hbar(\mathfrak{g})$ and $J_\hbar := Ker\big(\epsilon_{U_\hbar}\big)$. For any $z \in U_\hbar$, set

$$\hat{z} := \epsilon(z)\,, \quad z^+ := z - \epsilon(z) = z - \hat{z} \in J_\hbar\,, \quad \text{hence} \quad z = z^+ + \hat{z} \tag{3.22}$$

The assumption $\chi(z\,,1) = 0 = \chi(1\,,z)$ for $z \in U_\hbar(\mathfrak{g})$ implies (for all $u, v \in U_\hbar$)

$$\chi(u,v) \;=\; \chi\big(u^+ + \hat{u}\,, v^+ + \hat{v}\big) \;=\; \chi\big(u^+, v^+\big) \tag{3.23}$$

Now, for any $a\,, b \in U_\hbar$, the formula $\sigma = \exp_*\big(\hbar^{-1}\chi\big) \;=\; \sum_{n\geq 0} \hbar^{-n}\chi^{*\,n}\big/n!$ gives

$$\sigma(a\,,b\,) \;=\; \textstyle\sum_{n\geq 0} \hbar^{-n} \prod_{i=1}^n \chi\big(a_{(i)}, b_{(i)}\big)\Big/n! \;=\; \sum_{n\geq 0} \hbar^{-n} \prod_{i=1}^n \chi\big(a_{(i)}^+, b_{(i)}^+\big)\Big/n! \tag{3.24}$$

where we took into account that $\chi^{*\,k}(u\,,v\,) \;=\; \prod_{s=1}^k \chi\big(u_{(s)}, v_{(s)}\big) \;=\; \prod_{s=1}^k \chi\big(u_{(s)}^+, v_{(s)}^+\big)$ for each $u, v \in U_\hbar$, $k \in \mathbb{N}$, by definitions along with (3.23). Now we notice that $\otimes_{i=1}^n a_{(i)}^+ = \delta_n(a)$ and $\otimes_{i=1}^n b_{(i)}^+ = \delta_n(b)$, hence Lemma 3.3.1 above guarantees that $h^{-n} \prod\limits_{i=1}^n \chi\big(a_{(i)}^+, b_{(i)}^+\big) \in \hbar^{-n+\max(n,A)-A+\max(n,B)-B}$ $\big(\forall\; n \in \mathbb{N}_+\big)$, whence in particular

$$\begin{aligned} h^{-n} \prod_{i=1}^n \chi\big(a_{(i)}^+, b_{(i)}^+\big) \;&\in\; \hbar^{-\min(A,B)}\, \Bbbk[[\hbar]] \qquad \forall\; n \in \mathbb{N}_+ \\ h^{-n} \prod_{i=1}^n \chi\big(a_{(i)}^+, b_{(i)}^+\big) \;&\in\; \hbar^{n-(A+B)}\, \Bbbk[[\hbar]] \qquad \forall\; n \geq A+B \end{aligned} \tag{3.25}$$

where $A \in \mathbb{N}$, resp. $B \in \mathbb{N}$, plays for $a$, resp. for $b$, the role of $N$ for $z$ in Lemma 3.3.1 above; by this, the formal expansion for $\sigma(a\,,b\,)$ in (3.24) yields a well defined element in $\Bbbk[[\hbar]]$, hence $\sigma$ is a well-defined $\Bbbk(\!(\hbar)\!)$–bilinear form of $\mathbb{U}_\hbar(\mathfrak{g})$ as claimed.

*(b–c)* Both claims are obvious, by construction, as they follow from standard identities for formal exponentials. $\square$

The previous result leads us to introduce the following notion:

**Definition 3.3.3.** Let $U_\hbar(\mathfrak{g})$ be a QUEA, and $\mathbb{U}_\hbar(\mathfrak{g}) := \Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[[\hbar]]} U_\hbar(\mathfrak{g})$. Note that $\mathbb{U}_\hbar(\mathfrak{g})$ has a natural "Hopf algebra structure" of $\mathbb{U}_\hbar(\mathfrak{g})$ induced by scalar extension from $U_\hbar(\mathfrak{g})$ — so that, in particular, the "coproduct" takes values in $\Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[[\hbar]]} \big(U_\hbar(\mathfrak{g})\,\widehat{\otimes}_{\Bbbk[[\hbar]]} U_\hbar(\mathfrak{g})\big)$ rather than in $\mathbb{U}_\hbar(\mathfrak{g}) \otimes_{\Bbbk(\!(\hbar)\!)} \mathbb{U}_\hbar(\mathfrak{g})$.

We call *quasi-2-cocycle of* $U_\hbar(\mathfrak{g})$ any $\Bbbk(\!(\hbar)\!)$–bilinear form $\sigma$ of $\mathbb{U}_\hbar(\mathfrak{g})$ which has the form $\sigma := \exp_*\big(\hbar^{-1}\chi\big)$ for some $\Bbbk[[\hbar]]$–bilinear form $\chi \in \Big(U_\hbar(\mathfrak{g})^{\widehat{\otimes} 2}\Big)^*$ of $U_\hbar(\mathfrak{g})$ such that $\chi(z\,,1) = 0 = \chi(1\,,z)$ for all $z \in U_\hbar(\mathfrak{g})$, and in addition enjoys the 2–cocycle properties with respect to the above "Hopf algebra structure" of $\mathbb{U}_\hbar(\mathfrak{g})$.

**Remark 3.3.4.** The notion of "quasi-2-cocycle" for a QUEA $U_\hbar(\mathfrak{g})$ can also be cast in the following, equivalent shape. Recall that $F_\hbar[[G]] := U_\hbar(\mathfrak{g})^*$ is a QFSHA (cf. §2.3.4), and then $\Big(U_\hbar(\mathfrak{g})^{\widehat{\otimes} 2}\Big)^* \;=\; U_\hbar(\mathfrak{g})^* \,\widetilde{\otimes}\, U_\hbar(\mathfrak{g})^* \;=\; F_\hbar[[G]] \,\widetilde{\otimes}\, F_\hbar[[G]]$. Given $\chi \in \Big(U_\hbar(\mathfrak{g})^{\widehat{\otimes} 2}\Big)^*$ as in Definition 3.3.3 above, the condition $\chi(z\,,1) = 0 = \chi(1\,,z)$ for all $z \in U_\hbar(\mathfrak{g})$ means that $\chi \in J_{F_\hbar[[G]]} \,\widetilde{\otimes}\, J_{F_\hbar[[G]]}$, with $J_{F_\hbar[[G]]} := Ker\big(\epsilon_{F_\hbar[[G]]}\big)$, hence we have $\chi \in \hbar^2 \big(J_{F_\hbar[[G]]}^\vee\big)^{\widehat{\otimes}\, 2} \subseteq \hbar^2 \big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\, 2}$ where $J_{F_\hbar[[G]]}^\vee := \hbar^{-1} J_{F_\hbar[[G]]}$ and

$F_\hbar[[G]]^\vee$ is the QUEA defined in §2.4 out of $F_\hbar[[G]]$. Thus, it follows that $\hbar^{-1}\chi \in \hbar\big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\,2}$, so $\sigma := \exp_*\big(\hbar^{-1}\chi\big)$ is a well-defined element in $\big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\,2}$.

Now, the requirement that $\sigma := \exp_*\big(\hbar^{-1}\chi\big)$ be a quasi-2-cocycle for $U_\hbar(\mathfrak{g})$ in the sense of Definition 3.3.3 above is equivalent to the property of $\sigma$ being a twist element for $F_\hbar[[G]]^\vee$ — which makes perfectly sense in sight of Proposition 2.2.7.

Clearly, every 2-cocycle for $U_\hbar(\mathfrak{g})$ is a quasi-2-cocycle as well; the converse, instead, is not true, in general (counterexamples do exist). However, the key point is that *every quasi-2-cocycle still provides a well-defined deformation by 2-cocycle of* $U_\hbar(\mathfrak{g})$.

**Theorem 3.3.5.** *Let $U_\hbar(\mathfrak{g})$ be a QUEA, and $\sigma = \exp_*\big(\hbar^{-1}\chi\big)$ a quasi-2-cocycle for it, as in Definition 3.3.3. Then the procedure of 2-cocycle deformation by $\sigma$ applied to $\mathbb{U}_\hbar(\mathfrak{g})$ actually restricts to $U_\hbar(\mathfrak{g})$, making the latter into a new QUEA.*

*Proof.* First of all, we explain the statement itself. By definitions and by Lemma 3.3.2, we can perform the deformation by $\sigma$ onto the $\mathbb{U}_\hbar(\mathfrak{g}) := \Bbbk((\hbar))\,\widehat{\otimes}\,U_\hbar(\mathfrak{g})$. Our statement then claims the resulting deformed Hopf structure onto $\mathbb{U}_\hbar(\mathfrak{g})$ "restricts" to a deformation of $U_\hbar(\mathfrak{g})$ itself: in turn, this amounts to claiming that $U_\hbar(\mathfrak{g})$ is closed for the $\sigma$–deformed product in $\big(\mathbb{U}_\hbar(\mathfrak{g})\big)_\sigma$ — so we tackle this last problem.

Fix notation $U_\hbar := U_\hbar(\mathfrak{g})$, $J_\hbar := Ker\big(U_\hbar\big)$, $J'_\hbar := Ker\big(U'_\hbar\big)$ and $\widetilde{J}_\hbar := \hbar^{-1}J'_\hbar$, where $U'_\hbar := U_\hbar(\mathfrak{g})'$ is given in Definition 2.4.1*(a)*. As it was mentioned in the proof of Lemma 3.3.1, Theorem 2.4.2*(a)* implies that $U_\hbar = \big(U'_\hbar\big)^\vee$, that is

$$U_\hbar \;=\; \hbar\text{-adic completion of } \sum_{n\geq 0} \hbar^{-n}\big(I'_\hbar\big)^n$$

where $I'_\hbar := \hbar\,U'_\hbar + Ker\big(\epsilon_{U'_\hbar}\big) = \hbar\,U'_\hbar + J'_\hbar$; then a moment's thought shows that the previous expression of $U_\hbar$ reads also

$$U_\hbar \;=\; \hbar\text{-adic completion of } \sum_{n\geq 0}\hbar^{-n}\big(J'_\hbar\big)^n \;=\; \hbar\text{-adic completion of } \sum_{n\geq 0}\widetilde{J}_\hbar^{\,n} \tag{3.26}$$

Note also that $J'_\hbar$ is a Hopf ideal in $U'_\hbar$, and $J'_\hbar \subseteq \hbar\,J_\hbar$ (by construction); thus for $z' \in {J'_\hbar}^N$ (with $N \in \mathbb{N}$), acting like in the proof of Lemma 3.3.1 one gets

$$\delta_n\big(z'\big) \;\in\; \hbar^n J_\hbar^{\otimes n} \bigcap \Big(\textstyle\sum_{\sum_i N_i = N} \bigotimes_{i=1}^n {J'_\hbar}^{N_i}\Big) \;\subseteq\; \hbar^{\max(n,N)} J_\hbar^{\otimes n} \tag{3.27}$$

Again, for any $z \in U_\hbar$ we retain notation as in (3.22) above, that is

$$\hat{z} := \epsilon(z)\,, \quad z^+ := z - \epsilon(z) = z - \hat{z} \,\in\, J_\hbar\,, \quad \text{hence} \quad z = z^+ + \hat{z} \tag{3.28}$$

and we recall also that for all $u, v \in U_\hbar$ we have

$$\chi(u,v) \;=\; \chi\big(u^+ + \hat{u}\,, v^+ + \hat{v}\big) \;=\; \chi\big(u^+, v^+\big) \tag{3.29}$$

Thanks to (3.26), in order to prove that $U_\hbar(\mathfrak{g}) =: U_\hbar$ is closed for the $\sigma$–deformed product $\underset{\sigma}{\cdot}$ it is enough to show that $\widetilde{J}_\hbar^{\,A} \underset{\sigma}{\cdot} \widetilde{J}_\hbar^{\,B} \subseteq \sum_{n\geq 0}\widetilde{J}_\hbar^{\,n}$ for any $A, B \in \mathbb{N}_+$.

To begin with, we pick $a \in \widetilde{J}_\hbar^{\,A} = \hbar^{-A}{J'_\hbar}^A$ and $b \in \widetilde{J}_\hbar^{\,B} = \hbar^{-B}{J'_\hbar}^B$; by definition,

$$a \underset{\sigma}{\cdot} b \;:=\; \sigma\big(a_{(1)}, b_{(1)}\big)\, a_{(2)}\, b_{(2)}\, \sigma^{-1}\big(a_{(3)}, b_{(3)}\big)$$

whence expanding the formal formula $\sigma = \exp_*\big(\hbar^{-1}\chi\big) = \sum_{n\geq 0}\hbar^{-n}\chi^{*\,n}/n!$ — much like in the proof of Lemma 3.3.2 — we get

$$\begin{aligned} a \underset{\sigma}{\cdot} b \;&=\; \textstyle\sum_{t,\ell\geq 0}\hbar^{-(t+\ell)}\,(-1)^\ell\,(t!\,\ell!)^{-1}\,\chi^{*\,t}\big(a_{(1)}, b_{(1)}\big)\, a_{(2)}\, b_{(2)}\, \chi^{*\,\ell}\big(a_{(3)}, b_{(3)}\big) \;= \\ &=\; a\cdot b + \sum_{t+\ell>0}\hbar^{-(t+\ell)}\,(-1)^\ell\,(t!)^{-1}(\ell!)^{-1}\,\chi^{*\,t}\big(a_{(1)}, b_{(1)}\big)\, a_{(2)}\, b_{(2)}\, \chi^{*\,\ell}\big(a_{(3)}, b_{(3)}\big) \end{aligned} \tag{3.30}$$

where we took into account coassociativity and counitality properties.

Let us analyze each summand in the very last line in (3.30). From the identities $\chi^{*\,k}(u\,,v\,) = \prod_{s=1}^{k}\chi\big(u_{(s)},v_{(s)}\big) = \prod_{s=1}^{k}\chi\big(u^{+}_{(s)},v^{+}_{(s)}\big)$ — cf. (3.29) — we get

$$\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,a_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;=\; = \prod_{i=1}^{t}\chi\big(a^{+}_{(i)},b^{+}_{(i)}\big)\,a_{(t+1)}\,b_{(t+1)}\prod_{j=1}^{\ell}\chi\big(a^{+}_{(t+1+j)},b^{+}_{(t+1+j)}\big)$$

The map $j_{t+1}\colon U_\hbar^{\otimes(t+\ell)}\longrightarrow U_\hbar^{\otimes(t+\ell+1)}$, $\bigotimes_{s=1}^{t+\ell}x_s\mapsto\Big(\bigotimes_{s=1}^{t}x_s\Big)\otimes1\otimes\Big(\bigotimes_{s=t+1}^{t+\ell}x_s\Big)$, together with the expansion in (3.28) gives

$$\Big(\bigotimes_{i=1}^{t}a^{+}_{(i)}\Big)\otimes a^{+}_{(t+1)}\otimes\Big(\bigotimes_{j=1}^{\ell}a^{+}_{(t+1+j)}\Big) \;=\; \delta_{t+\ell+1}(a) \;\in\; \hbar^{\max(t+\ell+1,A)\,-A}\,U_\hbar^{\otimes(t+\ell+1)}$$

$$\Big(\bigotimes_{i=1}^{t}a^{+}_{(i)}\Big)\otimes\widehat{a}_{(t+1)}\otimes\Big(\bigotimes_{j=1}^{\ell}a^{+}_{(t+1+j)}\Big) \;=\; j_{t+1}\big(\delta_{t+\ell}(a)\big) \;\in\; \hbar^{\max(t+\ell,A)\,-A}\,U_\hbar^{\otimes(t+\ell+1)}$$

so that, summing up,

$$\Big(\bigotimes_{i=1}^{t}a^{+}_{(i)}\Big)\otimes a_{\,t+1}\otimes\Big(\bigotimes_{j=1}^{\ell}a^{+}_{(t+1+j)}\Big) = \delta_{t+\ell+1}(a)+j_{t+1}\big(\delta_{t+\ell}(a)\big)\in\hbar^{\max(t+\ell,A)\,-A}\,U_\hbar^{\otimes(t+\ell+1)}$$

— like in the proof of Lemma 3.3.1 — and similarly with $b$, resp. $B$, replacing $a$, resp. $A$. Eventually, for all $t+\ell>0$ this gives

$$\begin{aligned}&\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,a_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;=\\ &\;=\;\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,\widehat{a}_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;+\; \chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,a^{+}_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big)\end{aligned} \tag{3.31}$$

where for the two summands in second line, writing $n:=t+\ell$, we have

$$\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,\widehat{a}_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;= \;=\;\prod_{i=1}^{t}\chi\big(a^{+}_{(i)},b^{+}_{(i)}\big)\,\widehat{a}_{(t+1)}\,b_{(t+1)}\prod_{k=t+2}^{n+1}\chi\big(a^{+}_{(k)},b^{+}_{(k)}\big) \;\in\; \hbar^{\max(n,A)\,-A+\max(n,B)\,-B}\,U_\hbar^{\otimes(n+1)}$$

$$\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,a^{+}_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;= \;=\;\prod_{i=1}^{t}\chi\big(a^{+}_{(i)},b^{+}_{(i)}\big)\,a^{+}_{(t+1)}\,b_{(t+1)}\prod_{k=t+2}^{n+1}\chi\big(a^{+}_{(k)},b^{+}_{(k)}\big) \;\in\; \hbar^{\max(n+1,A)\,-A+\max(n,B)\,-B}\,U_\hbar^{\otimes(n+1)}$$

*Let us now assume that* $A:=1$, so that $n:=t+\ell>0$ implies $n:=t+\ell\geq1=A$. Then the last estimates read

$$\begin{aligned}\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,\widehat{a}_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;&\in\; \hbar^{\,n-1+\max(n,B)\,-B}\,U_\hbar^{\otimes\,(n+1)}\\ \chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,a^{+}_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;&\in\; \hbar^{\,n+\max(n,B)\,-B}\,U_\hbar^{\otimes\,(n+1)}\end{aligned} \tag{3.32}$$

The term in the second line, when plugged in (3.31) and then in (3.30), yields the contribution $\frac{(-1)^\ell}{t!\,\ell!}\,\hbar^{-n}\,\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,a^{+}_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;\in\; \hbar^{\max(n,B)\,-B}\,U_\hbar^{\otimes\,(n+1)}$, so for growing $n$ these elements sum up to a convergent series in $U_\hbar$, and we are done. As to the term in the first line, we split it into

$$\begin{aligned}&\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,\widehat{a}_{(2)}\,b_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;=\\ &\;=\;\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,\widehat{a}_{(2)}\,\widehat{b}_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;+\; \chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,\widehat{a}_{(2)}\,b^{+}_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big)\end{aligned} \tag{3.33}$$

Then for the first summand we have (almost by definition, or acting as before)

$$\chi^{*\,t}\big(a_{(1)},b_{(1)}\big)\,\widehat{a}_{(2)}\,\widehat{b}_{(2)}\,\chi^{*\,\ell}\big(a_{(3)},b_{(3)}\big) \;=\; \chi^{*\,(t+\ell)}(a\,,b\,)$$

so when we plug every such term in (3.31) and then in (3.30), they sum up to $\sum_{t+\ell>0} \hbar^{-(t+\ell)} (-1)^\ell (t!)^{-1} (\ell!)^{-1} \chi^{*t}\big(a_{(1)}, b_{(1)}\big) \, \widehat{a}_{(2)} \, \widehat{b}_{(2)} \, \chi^{*\ell}\big(a_{(3)}, b_{(3)}\big) =$

$$= \sum_{\substack{t+\ell=n \\ n>0}} \frac{(-1)^\ell \hbar^{-n}}{t!\,\ell!} \chi^{*n}(a\,,b\,) = \sum_{n>0} \frac{1}{n!} \hbar^{-n} \bigg( \sum_{t+\ell=n} (-1)^\ell \binom{n}{\ell} \bigg) \chi^{*n}(a\,,b\,) = 0$$

just because of the combinatorial identity $\sum_{t+\ell=n} (-1)^\ell \binom{n}{\ell} = 0$ .

Finally, we have to dispose of the summands of type

$$\chi^{*t}\big(a_{(1)}, b_{(1)}\big) \, \widehat{a}_{(2)} \, b^+_{(2)} \, \chi^{*\ell}\big(a_{(3)}, b_{(3)}\big) \tag{3.34}$$

for which the analogue of the first identity in (3.32) holds true, namely

$$\chi^{*t}\big(a_{(1)}, b_{(1)}\big) \, \widehat{a}_{(2)} \, b^+_{(2)} \, \chi^{*\ell}\big(a_{(3)}, b_{(3)}\big) \in \hbar^{n-1+\max(n+1,B)-B} \, U_\hbar^{\otimes (n+1)} \tag{3.35}$$

where $n := t+\ell$ , taking into account that $\delta_{n+1}(b) \in \hbar^{\max(n+1,B)-B} \, U_\hbar^{\otimes (n+1)}$ .

Then we have to distinguish two cases, depending on $n := t+\ell$ .

First we assume $n := t+\ell \geq B$ . Then $n-1+\max(n+1,B)-B \geq n$ , hence the first identity in (3.32) yields $\chi^{*t}\big(a_{(1)}, b_{(1)}\big) \, \widehat{a}_{(2)} \, b^+_{(2)} \, \chi^{*\ell}\big(a_{(3)}, b_{(3)}\big) \in \hbar^n \, U_\hbar^{\otimes (n+1)}$ , and then, when plugged in (3.33), and subsequently in (3.31) and in (3.30), this provides to the expansion of $a \underset{\sigma}{\cdot} b$ a contribution of the form

$$\hbar^{-n} \frac{(-1)^\ell}{t!\,\ell!} \chi^{*t}\big(a_{(1)}, b_{(1)}\big) \, \widehat{a}_{(2)} \, b^+_{(2)} \, \chi^{*\ell}\big(a_{(3)}, b_{(3)}\big) \in \hbar^{-n} \, \hbar^n \, U_\hbar^{\otimes (n+1)} = U_\hbar^{\otimes (n+1)}$$

— which is fair! — hence we are done with it.

Then we are left with the case $n := t+\ell \leq B-1$ . Tracking backwards our construction, all these case provide to (3.30) a contribution of the form

$$\begin{aligned} &\sum_{t+\ell=1}^{B-1} \hbar^{-(t+\ell)} \frac{(-1)^\ell}{t!\,\ell!} \chi^{*t}\big(a_{(1)}, b_{(1)}\big) \, \widehat{a}_{(2)} \, b^+_{(2)} \, \chi^{*\ell}\big(a_{(3)}, b_{(3)}\big) = \\ &\quad = \sum_{n=1}^{B-1} \frac{1}{n!} \hbar^{-n} \sum_{t+\ell=n} (-1)^\ell \binom{n}{\ell} \chi^{*t}\big(a_{(1)}, b_{(1)}\big) \, \widehat{a}_{(2)} \, b^+_{(2)} \, \chi^{*\ell}\big(a_{(3)}, b_{(3)}\big) \end{aligned} \tag{3.36}$$

With no loss of generality, *we can assume that* $a \not\equiv 0$ , $b \not\equiv 0 \ \big(\mathrm{mod}\ \hbar U_\hbar\big)$ . Then for their corresponding cosets $\overline{a}\,, \overline{b} \in U_\hbar \big/ \hbar U_\hbar \cong U(\mathfrak{g})$ we have $\overline{a} \in U(\mathfrak{g})_1$ and $\overline{b} \in U(\mathfrak{g})_B$ , where $\big\{U(\mathfrak{g})_n\big\}_{n\in\mathbb{N}}$ is the standard, coradical filtration of $U(\mathfrak{g})$ , and also $\delta_1(\,\overline{a}\,) \neq 0$ as well as $\delta_n\big(\,\overline{b}\,\big) \neq 0$ for $1 \leq n \leq B$ — cf. [Ga1], Lemma 3.3. Moreover, we recall that $U_\hbar := U_\hbar(\mathfrak{g})$ *is cocommutative modulo* $\hbar U_\hbar$ , as it is a QUEA: in particular, this implies that $\delta_n\big(\,\overline{b}\,\big)$ *is a symmetric tensor* — for $1 \leq n \leq B$ — hence we can write $\delta_n(b\,)$ in the form

$$\delta_n(b\,) = b^+_{(1)} \otimes \cdots \otimes b^+_{(n)} = \beta_{\langle 1 \rangle} \otimes \cdots \otimes \beta_{\langle n \rangle} + \mathcal{O}_n\big(\hbar^1\big) \tag{3.37}$$

(for $1 \leq n \leq B$ ) where $\beta_{\langle 1 \rangle} \otimes \cdots \otimes \beta_{\langle n \rangle}$ — using some $\sigma$–notation of sort, as usual — is some *symmetric* tensor in $U_\hbar^{\otimes n}$ and *hereafter* $\mathcal{O}_n\big(\hbar^s\big)$ *stands for some element in* $\hbar^s \, U_\hbar^{\otimes n}$ , for every $s, n \in \mathbb{N}$ . Then plugging (3.37) in (3.34) we find

$$\begin{aligned} \chi^{*t}\big(a_{(1)}, b_{(1)}\big) \, \widehat{a}_{(2)} \, b^+_{(2)} \, \chi^{*\ell}\big(a_{(3)}, b_{(3)}\big) &= \prod_{i=1}^{t} \chi\big(a^+_{(i)}, b^+_{(i)}\big) \, \widehat{a}_{(t+1)} \, b^+_{(t+1)} \prod_{k=t+2}^{t+\ell+1} \chi\big(a^+_{(k)}, b^+_{(k)}\big) = \\ &= \prod_{i=1}^{t} \chi\big(a^+_{(i)}, \beta_{\langle i \rangle}\big) \, \widehat{a}_{(t+1)} \, \beta_{\langle t+1 \rangle} \prod_{k=t+2}^{t+\ell+1} \chi\big(a^+_{(k)}, \beta_{\langle k \rangle}\big) + \mathcal{O}_1\big(\hbar^{t+\ell}\big) \end{aligned}$$

for all $t+\ell \leq B-1$, with $\prod_{i=1}^{t} \chi\big(a^+_{(i)}, \beta_{\langle i \rangle}\big)\, \widehat{a}_{(t+1)}\, \beta_{\langle t+1 \rangle} \prod_{k=t+2}^{t+\ell+1} \chi\big(a^+_{(k)}, \beta_{\langle k \rangle}\big) \;\in\; \hbar^{t+\ell-1}\, U_\hbar$. Therefore, the contribution to (3.30) given in (3.36) now reads

$$\sum_{n=1}^{B-1} \frac{1}{n!}\, \hbar^{-n} \sum_{t+\ell=n} (-1)^\ell \binom{n}{\ell} \chi^{*\,t}\big(a_{(1)}, b_{(1)}\big)\, \widehat{a}_{(2)}\, b^+_{(2)}\, \chi^{*\,\ell}\big(a_{(3)}, b_{(3)}\big) \;=$$
$$= \sum_{n=1}^{B-1} \frac{1}{n!}\, \hbar^{-n} \sum_{t+\ell=n} (-1)^\ell \binom{n}{\ell} \prod_{i=1}^{t} \chi\big(a^+_{(i)}, \beta_{\langle i \rangle}\big)\, \widehat{a}_{(t+1)}\, \beta_{\langle t+1 \rangle} \prod_{k=t+2}^{n+1} \chi\big(a^+_{(k)}, \beta_{\langle k \rangle}\big) \;+\; \mathcal{O}_1\big(\hbar^0\big)$$

where in the last formula we have

$$\sum_{n=1}^{B-1} \frac{1}{n!}\, \hbar^{-n} \sum_{t+\ell=n} (-1)^\ell \binom{n}{\ell} \prod_{i=1}^{t} \chi\big(a^+_{(i)}, \beta_{\langle i \rangle}\big)\, \widehat{a}_{(t+1)}\, \beta_{\langle t+1 \rangle} \prod_{k=t+2}^{n+1} \chi\big(a^+_{(k)}, \beta_{\langle k \rangle}\big) \;\in\; \hbar^{-1}\, U_\hbar$$

Now, observe that, setting $n := t + \ell$, we can re-write

$$\prod_{i=1}^{t} \chi\big(a^+_{(i)}, \beta_{\langle i \rangle}\big)\, \widehat{a}_{(t+1)}\, \beta_{\langle t+1 \rangle} \prod_{k=t+2}^{n+1} \chi\big(a^+_{(k)}, \beta_{\langle k \rangle}\big) \;=\; \varPhi\big(\delta_n(a) \otimes \beta_{\langle 1 \rangle} \otimes \cdots \otimes \beta_{\langle n+1 \rangle}\big) \tag{3.38}$$

with $\varPhi := \mu \circ \big(\chi^{\otimes t} \otimes \mathrm{id}_{U_\hbar}^{\otimes 2} \otimes \chi^{\otimes (n-t)}\big) \circ \varsigma_{n+1}$ mapping $U_\hbar^{\otimes 2(n+1)}$ to $U_\hbar$, where

*(1)* $\varsigma_{n+1} : U_\hbar^{\otimes (2n+1)} \longrightarrow U_\hbar^{\otimes (2n+1)}$ is the "shuffle" map

$$x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n \otimes y_{n+1} \;\mapsto\; x_1 \otimes y_1 \otimes x_2 \otimes y_2 \otimes \cdots \otimes x_n \otimes y_n \otimes y_{n+1}$$

and, considering $\Bbbk[[\hbar]]$ as embedded into $U_\hbar$ via the unit map,

*(2)* $\mu : U_\hbar^{\otimes n} \longrightarrow U_\hbar$ is the obvious ($n$–fold iterated) multiplication by scalars.

Now recall that $\beta_{\langle 1 \rangle} \otimes \cdots \otimes \beta_{\langle n+1 \rangle}$ represents a tensor in $\sigma$–notation, so more explicitly we might write $\beta_{\langle 1 \rangle} \otimes \cdots \otimes \beta_{\langle n+1 \rangle} = \sum_{s=1}^{N} \beta_{s,1} \otimes \beta_{s,\,n+1}$; so in the formula we are dealing with what is written as a product $\prod_{i=1}^{t} \chi\big(a^+_{(i)}, \beta_{\langle i \rangle}\big)\, \widehat{a}_{(t+1)}\, \beta_{\langle t+1 \rangle} \prod_{k=t+2}^{n+1} \chi\big(a^+_{(k)}, \beta_{\langle k \rangle}\big)$ is actually a sum of several products as $\prod_{i=1}^{t} \chi\big(a^+_{(i)}, \beta_{s,\,i}\big)\, \widehat{a}_{(t+1)}\, \beta_{s,\,t+1} \prod_{k=t+2}^{n+1} \chi\big(a^+_{(k)}, \beta_{s,\,k}\big)$. But then recall that this tensor $\beta_{\langle 1 \rangle} \otimes \cdots \otimes \beta_{\langle n+1 \rangle} = \sum_{s=1}^{N} \beta_{s,1} \otimes \beta_{s,\,n+1}$ is *symmetric*, therefore, the various products $\prod_{i=1}^{t} \chi\big(a^+_{(i)}, \beta_{s,\,i}\big)\, \widehat{a}_{(t+1)}\, \beta_{s,\,t+1} \prod_{k=t+2}^{n+1} \chi\big(a^+_{(k)}, \beta_{s,\,k}\big)$ actually all *coincide*: letting $C_n$ be their "common value", we deduce that

$$\sum_{n=1}^{B-1} \frac{1}{n!}\, \hbar^{-n} \sum_{t+\ell=n} (-1)^\ell \binom{n}{\ell} \prod_{i=1}^{t} \chi\big(a^+_{(i)}, \beta_{\langle i \rangle}\big)\, \widehat{a}_{(t+1)}\, \beta_{\langle t+1 \rangle} \prod_{k=t+2}^{n+1} \chi\big(a^+_{(k)}, \beta_{\langle k \rangle}\big) \;=$$
$$= \sum_{n=1}^{B-1} \frac{1}{n!}\, \hbar^{-n} \Bigg( \sum_{t+\ell=n} (-1)^\ell \binom{n}{\ell} \Bigg) C_n \;=\; 0$$

again because of the identity $\sum_{t+\ell=n} (-1)^\ell \binom{n}{\ell} \;=\; 0$.

Thus, also the last contributions to (3.30) given in (3.36) actually belong to $U_\hbar$.

To sum up, *we have proved that (for all* $a \in \widetilde{J}_\hbar^{\,1} = \hbar^{-1} J'_\hbar$, $b \in \widetilde{J}_\hbar^{\,B} = \hbar^{-B} {J'_\hbar}^{B}$ *)*

$$a \underset{\sigma}{\cdot} b \;=\; a \cdot b \,+\, z \qquad \textit{with} \quad z \in J_\hbar \tag{3.39}$$

*and similarly (for all* $a \in \widetilde{J}_\hbar^{\,1} = \hbar^{-1} J'_\hbar$, $b \in \widetilde{J}_\hbar^{\,B} = \hbar^{-B} {J'_\hbar}^{B}$ *) also*

$$b \underset{\sigma}{\cdot} a \;=\; b \cdot a \,+\, x \qquad \textit{with} \quad x \in J_\hbar \tag{3.40}$$

Let $\big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]}^{\sigma}$ the unital $\Bbbk[[\hbar]]$–subalgebra of $\big(\mathbb{U}_\hbar(\mathfrak{g})\big)_\sigma := \big(\Bbbk((\hbar)) \otimes_{\Bbbk[[\hbar]]} U_\hbar\big)_\sigma$ generated by $\widetilde{J}_\hbar$. Now recall that $\widetilde{J}_\hbar := \hbar^{-1} J_\hbar'$ with $J_\hbar' := \mathit{Ker}\big(U_\hbar'\big)$; then $U_\hbar' = J_\hbar' \oplus \Bbbk[[\hbar]] \cdot 1$, which implies $\Delta\big(J_\hbar'\big) \subseteq J_\hbar' \otimes 1 + J_\hbar' \otimes J_\hbar' + 1 \otimes J_\hbar'$. Then also $\Delta\big(\widetilde{J}_\hbar\big) \subseteq \widetilde{J}_\hbar \otimes 1 + \widetilde{J}_\hbar \otimes \widetilde{J}_\hbar + 1 \otimes \widetilde{J}_\hbar$. As the coalgebra structure is the same in $\mathbb{U}_\hbar(\mathfrak{g})$ and $\big(\mathbb{U}_\hbar(\mathfrak{g})\big)_\sigma$, it follows that $\big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]}^{\sigma}$ *is a Hopf* $\Bbbk[[\hbar]]$*–subalgebra inside* $\big(\mathbb{U}_\hbar(\mathfrak{g})\big)_\sigma$.

By repeated use of (3.39) or (3.40) alike, we find that $\big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]}^{\sigma} \subseteq U_\hbar = \big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]}$.

Now observe that the original product "$\cdot$" in $U_\hbar := U_\hbar(\mathfrak{g})$ and $\mathbb{U}_\hbar(\mathfrak{g})$ can be obtained from "$\underset{\sigma}{\cdot}$" through deformation via the inverse 2–cocycle $\sigma^{-1}$. Thanks to this, we can reverse the roles of $U_\hbar = \big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]}$ and $\big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]}^{\sigma}$ in the previous construction, thus achieving $\big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]} \subseteq \big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]}^{\sigma}$. Therefore $\big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]} \subseteq \big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]}^{\sigma}$, which in particular implies that $U_\hbar = \big\langle \widetilde{J}_\hbar \big\rangle_{\Bbbk[[\hbar]]}$ is closed for the $\sigma$–product. $\square$

**Definition 3.3.6.** With assumptions as in Theorem 3.3.5, the new QUEA obtained from $U_\hbar(\mathfrak{g})$ via 2–cocycle deformation by $\sigma$ of $\mathbb{U}_\hbar(\mathfrak{g})$ followed by restriction will be called *quasi-2-cocycle deformation of* $U_\hbar(\mathfrak{g})$ *by* $\sigma$, and it will be denoted by $U_\hbar(\mathfrak{g})_\sigma$.

To complete our analysis, next result sheds light onto the new, quasi-2-cocycle deformed QUEA $U_\hbar(\mathfrak{g})_\sigma$, describing in detail its semiclassical limit:

**Theorem 3.3.7.** *Let* $U_\hbar(\mathfrak{g})$ *be a QUEA over the Lie bialgebra* $\mathfrak{g} = \big(\mathfrak{g}\,;\,[\ ,\ ]\,,\,\delta\big)$. *Let* $\sigma$ *be a quasi-2-cocycle for* $U_\hbar(\mathfrak{g})$, *so* $\sigma = \exp_*\big(\hbar^{-1}\chi\big)$ *for some* $\chi \in \Big(U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}\Big)^*$ *with* $\chi(z\,,1) = 0 = \chi(1\,,z)$ *for* $z \in U_\hbar(\mathfrak{g})$. *Set also* $\chi_a := \chi - \chi_{2,1}$. *Then:*

*(a)* $\chi$ *is antisymmetric, i.e.* $\chi_{2,1} = -\chi$, *iff* $\sigma$ *is orthogonal, i.e.* $\sigma_{2,1} = \sigma^{-1}$;

*(b) the* $\Bbbk$*–linear map* $\gamma := \chi_a \left(\mathrm{mod}\ \hbar\,\big(U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}\big)^*\right)\Big|_{\mathfrak{g}\otimes\mathfrak{g}}$ *from* $\mathfrak{g}\otimes\mathfrak{g}$ *to* $\Bbbk$ *is antisymmetric 2–cocycle for the Lie bialgebra* $\mathfrak{g}$;

*(c) the quasi-2-cocycle deformation* $\big(U_\hbar(\mathfrak{g})\big)_\sigma$ *of* $U_\hbar(\mathfrak{g})$ *is a QUEA for the Lie bialgebra* $\mathfrak{g}_\gamma = \big(\mathfrak{g}\,;\,[\ ,\ ]_\gamma\,,\,\delta\big)$ *which is the deformation of* $\mathfrak{g}$ *by the 2–cocycle* $\gamma$; *in a nutshell, we have* $\big(U_\hbar(\mathfrak{g})\big)_\sigma \cong U_\hbar\big(\mathfrak{g}_\gamma\big)$.

*In particular, if* $\sigma$ *is* $\Bbbk[[\hbar]]$*–valued — i.e., it is an ordinary 2–cocycle for the Hopf* $\Bbbk[[\hbar]]$*–algebra* $U_\hbar(\mathfrak{g})$ *— or equivalently* $\chi \in \hbar\,\big(U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}\big)^*$, *then we have just* $\gamma = 0$ *and* $\big(U_\hbar(\mathfrak{g})\big)_\sigma \cong U_\hbar\big(\mathfrak{g}_\gamma\big) = U_\hbar(\mathfrak{g})$.

*Proof.* *(a)* This follows from claim *(c)* in Lemma 3.3.2.

*(b)* We are interested in the restriction to $\mathfrak{g}\otimes\mathfrak{g}$ of the specialization of $\sigma$ modulo $\hbar$. So we start with $\mathrm{a}\,,\mathrm{b}\,,\mathrm{c} \in \mathfrak{g}$, that we realize as $\mathrm{a} = a\,\big(\mathrm{mod}\ \hbar\,U_\hbar\big)$, $\mathrm{b} = b\,\big(\mathrm{mod}\ \hbar\,U_\hbar\big)$ and $\mathrm{c} = c\,\big(\mathrm{mod}\ \hbar\,U_\hbar\big)$ for some "lifts" $a, b, c \in U_\hbar$. By the identity $U_\hbar = \big(U_\hbar'\big)^\vee$ and by Lemma 3.3 in [Ga1], *we can choose the lifts* $a$, $b$ *and* $c$ *belong to* $\widetilde{J}_\hbar := \hbar^{-1} J_\hbar'$, *so that* $a' := \hbar\,a$, $b' := \hbar\,b$ *and* $c' := \hbar\,c$ *belong to* $J_\hbar'$.

As $\sigma$ is a normalized Hopf 2–cocycle for $U_\hbar$, it must obey the equality

$$\sigma\big(b'_{(1)}, c'_{(1)}\big)\,\sigma\big(a', b'_{(2)}c'_{(2)}\big) \;=\; \sigma\big(a'_{(1)}, b'_{(1)}\big)\,\sigma\big(a'_{(2)}b'_{(2)}, c'\big) \tag{3.41}$$

Let us focus on the left hand side of (3.41). Expanding the exponential we get

$$\sigma\big(b'_{(1)}, c'_{(1)}\big)\,\sigma\big(a', b'_{(2)}c'_{(2)}\big) \;=\; \sum_{n,m\geq 0} \frac{\hbar^{-(n+m)}}{n!\,m!}\,\chi^{*\,n}\big(b'_{(1)}, c'_{(1)}\big)\,\chi^{*\,m}\big(a', b'_{(2)}c'_{(2)}\big) \;=$$

$$\begin{aligned}
&= \epsilon\big(a'\big)\,\epsilon\big(b'\big)\,\epsilon\big(c'\big) \,+\, \hbar^{-1}\,\chi\big(b',c'\big)\,\epsilon\big(a'\big) \,+\, \hbar^{-1}\,\chi\big(a',b'c'\big) \,+ \\
&+\, \hbar^{-2}\,\chi\big(b'_{(1)},c'_{(1)}\big)\,\chi\big(a',b'_{(2)}c'_{(2)}\big) \,+\, \hbar^{-2}\,2^{-1}\,\chi^{*\,2}\big(b',c'\big)\,\epsilon\big(a'\big) \,+ \\
&+\, \hbar^{-2}\,2^{-1}\,\chi^{*\,2}\big(a',b'c'\big) \,+ \sum_{n+m\geq 3} \frac{\hbar^{-(n+m)}}{n!\,m!}\,\chi^{*\,n}\big(b'_{(1)},c'_{(1)}\big)\,\chi^{*\,m}\big(a',b'_{(2)}c'_{(2)}\big)
\end{aligned}$$

Then, noting that $\epsilon\big(a'\big) = \epsilon\big(b'\big) = \epsilon\big(c'\big) = 0$ by construction, and analyzing all other summands as in the proof of claim Theorem 3.3.5, we obtain

$$\begin{aligned}
\sigma\big(b'_{(1)},c'_{(1)}\big)\,\sigma\big(a',b'_{(2)}c'_{(2)}\big) \;&=\; \hbar^{-1}\chi\big(a',b'c'\big) + \hbar^{-2}\chi\big(b'_{(1)},c'_{(1)}\big)\,\chi\big(a',b'_{(2)}c'_{(2)}\big) \,+ \\
&+\, \hbar^{-2}\,2^{-1}\,\chi^{*\,2}\big(a',b'c'\big) \,+\, \big(\textit{sum of all terms with } n+m\geq 3\big)
\end{aligned} \tag{3.42}$$

Writing $z' = z'^{+} + \epsilon\big(z'\big)$ and using that $\chi(z,1) = 0 = \chi(1,z)$ and

$$\chi\big(x'_{(1)},y'_{(1)}z'_{(1)}\big)\,\chi\big(x'_{(2)},y'_{(2)}\,\epsilon\big(z'_{(2)}\big)\big) \;=\; \chi\big(x'_{(1)},y'_{(1)}z'\big)\,\chi\big(x'_{(2)},y'_{(2)}\big)$$

we have that

$$\begin{aligned}
\chi^{*\,2}\big(a',b'c'\big) \;&=\; \chi\big(a'_{(1)},b'_{(1)}c'_{(1)}\big)\,\chi\big(a'_{(2)},b'_{(2)}c'_{(2)}\big) \;= \\
&=\; \chi\big(a'^{+}_{(1)},b'^{+}_{(1)}c'^{+}_{(1)}\big)\,\chi\big(a'^{+}_{(2)},b'^{+}_{(2)}c'^{+}_{(2)}\big) \,+\, \chi\big(a'^{+}_{(1)},b'^{+}_{(1)}\big)\,\chi\big(a'^{+}_{(2)},b'^{+}_{(2)}c'\big) \,+ \\
&\quad + \chi\big(a'^{+}_{(1)},c'^{+}_{(1)}\big)\,\chi\big(a'^{+}_{(2)},b'c'^{+}_{(2)}\big) \,+\, \chi\big(a'^{+}_{(1)},b'c'^{+}_{(1)}\big)\,\chi\big(a'^{+}_{(2)},c'^{+}_{(2)}\big) \,+ \\
&\quad + \chi\big(a'^{+}_{(1)},b'\big)\,\chi\big(a'^{+}_{(2)},c'\big) \,+\, \chi\big(a'^{+}_{(1)},b'^{+}_{(1)}c'\big)\,\chi\big(a'^{+}_{(2)},b'^{+}_{(2)}\big) \,+\, \chi\big(a'^{+}_{(1)},c'\big)\,\chi\big(a'^{+}_{(2)},b'\big)
\end{aligned}$$

Since $z' = \hbar\,z$ and $z'^{\,+}_{(i)} \in \hbar\,U_\hbar$ , we may re-write the expression above as

$$\chi^{*\,2}\big(a',b'c'\big) \;=\; \hbar^4\,\chi\big(a_{(1)},b\big)\,\chi\big(a_{(2)},c\big) \,+\, \hbar^4\,\chi\big(a_{(1)},c\big)\,\chi\big(a_{(2)},b\big) \,+\, \mathcal{O}\big(\hbar^5\big)$$

Performing a similar analysis on the term $\chi\big(b'_{(1)},c'_{(1)}\big)\,\chi\big(a',b'_{(2)}c'_{(2)}\big)$ we get

$$\chi\big(b'_{(1)},c'_{(1)}\big)\,\chi\big(a',b'_{(2)}c'_{(2)}\big) \;=\; \hbar^4\,\chi\big(b,c_{(1)}\big)\,\chi\big(a,c_{(2)}\big) + \hbar^4\,\chi\big(b_{(1)},c\big)\,\chi\big(a,b_{(2)}\big) + \mathcal{O}\big(\hbar^5\big)$$

Moreover, with a similar (yet easier) analysis one finds also that

$$\chi^{*n}\big(b'_{(1)},c'_{(1)}\big)\,\chi^{*m}\big(a',b'_{(2)}c'_{(2)}\big) \;\in\; \hbar^{+2(n+m)}\,\Bbbk[[\hbar]]$$

for all $n+m\geq 3$, so that the (last) summand "$\big(\textit{sum of all terms with } n+m\geq 3\big)$" in (3.42) is of type $\mathcal{O}\big(\hbar^{n+m}\big) = \mathcal{O}\big(\hbar^3\big)$. Putting all together in (3.42) we find

$$\begin{aligned}
&\hbar^3\,\sigma\big(b_{(1)},c_{(1)}\big)\,\sigma\big(a,b_{(2)}c_{(2)}\big) \;= \\
&\quad= \hbar^2\,\chi(a,bc) + \hbar^2\,\chi\big(b,c_{(1)}\big)\,\chi\big(a,c_{(2)}\big) + \hbar^2\,\chi\big(b_{(1)},c\big)\,\chi\big(a,b_{(2)}\big) \,+ \\
&\qquad+\, \hbar^2\,2^{-1}\,\chi\big(a_{(1)},b\big)\,\chi\big(a_{(2)},c\big) \,+\, \hbar^2\,2^{-1}\,\chi\big(a_{(1)},c\big)\,\chi\big(a_{(2)},b\big) \,+\, \mathcal{O}\big(\hbar^3\big)
\end{aligned}$$

An analogous treatment of the right hand side of (3.41) yields

$$\begin{aligned}
&\hbar^3\,\sigma\big(a_{(1)},b_{(1)}\big)\,\sigma\big(a_{(2)}b_{(2)},c\,\big) \;= \\
&\quad= \hbar^2\,\chi(ab,c\,) + \hbar^2\,\chi\big(a,b_{(1)}\big)\,\chi\big(b_{(2)},c\,\big) + \hbar^2\,\chi\big(a_{(1)},b\,\big)\,\chi\big(a_{(2)},c\,\big) \,+ \\
&\qquad+\, \hbar^2\,2^{-1}\,\chi\big(a,c_{(1)}\big)\,\chi\big(b,c_{(2)}\big) \,+\, \hbar^2\,2^{-1}\,\chi\big(b,c_{(1)}\big)\,\chi\big(a,c_{(2)}\big) \,+\, \mathcal{O}\big(\hbar^3\big)
\end{aligned}$$

Altogether, this implies that

$$\begin{aligned}
&\chi(a,bc\,) \,+\, \chi\big(b,c_{(1)}\big)\,\chi\big(a,c_{(2)}\big) \,+\, \chi\big(b_{(1)},c\,\big)\,\chi\big(a,b_{(2)}\big) \,+ \\
&\quad+\, 2^{-1}\,\chi\big(a_{(1)},b\big)\,\chi\big(a_{(2)},c\,\big) \,+\, 2^{-1}\,\chi\big(a_{(1)},c\,\big)\,\chi\big(a_{(2)},b\,\big) \;\underset{\hbar}{\equiv} \\
&\qquad\underset{\hbar}{\equiv}\; \chi(ab,c\,) \,+\, \chi\big(a,b_{(1)}\big)\,\chi\big(b_{(2)},c\,\big) \,+\, \chi\big(a_{(1)},b\,\big)\,\chi\big(a_{(2)},c\,\big) \,+ \\
&\qquad\quad+\, 2^{-1}\,\chi\big(a,c_{(1)}\big)\,\chi\big(b,c_{(2)}\big) \,+\, 2^{-1}\,\chi\big(b,c_{(1)}\big)\,\chi\big(a,c_{(2)}\big)
\end{aligned}$$

where (again) $\underset{\hbar}{\equiv}$ stands for "congruent modulo $\hbar\,\Bbbk[[\hbar]]$", that we re-write as

$$\begin{aligned} &\chi(a,bc) + \chi\big(b,c_{(1)}\big)\,\chi\big(a,c_{(2)}\big) + \chi\big(b_{(1)},c\big)\,\chi\big(a,b_{(2)}\big) + \\ &\quad + 2^{-1}\chi\big(a_{(1)},b\big)\,\chi\big(a_{(2)},c\big) + 2^{-1}\chi\big(a_{(1)},c\big)\,\chi\big(a_{(2)},b\big) - \\ &\qquad - \chi(ab,c) - \chi\big(a,b_{(1)}\big)\,\chi\big(b_{(2)},c\big) - \chi\big(a_{(1)},b\big)\,\chi\big(a_{(2)},c\big) - \\ &\qquad\quad - 2^{-1}\chi\big(a,c_{(1)}\big)\,\chi\big(b,c_{(2)}\big) - 2^{-1}\chi\big(b,c_{(1)}\big)\,\chi\big(a,c_{(2)}\big) \underset{\hbar}{\equiv} 0 \end{aligned} \tag{3.43}$$

Consider now the action of the group algebra $\Bbbk\big[\mathbb{S}_3\big]$ of the symmetric group $\mathbb{S}_3$ on $\big(U_\hbar^{\widehat{\otimes}3}\big)^*$ given by $\sigma\,.\,\varphi(a,b,c) := \varphi\big(\sigma^{-1}\,.\,(a,b,c)\big)$, where the action of $\Bbbk\big[\mathbb{S}_3\big]$ on $U_\hbar^{\widehat{\otimes}3}$ is the natural one that permutes the tensor factors. Then we let the antisymmetrizer $Alt_3$ act on both sides of (3.43): using that $\gamma := \chi_a \Big(\mathrm{mod}\ \hbar\,\big(U_\hbar(\mathfrak{g})^{\widehat{\otimes}\,2}\big)^*\Big)\Big|_{\mathfrak{g}\otimes\mathfrak{g}}$ and that $a\,\big(\mathrm{mod}\ \hbar\,U_\hbar\big) = \mathrm{a}$, $b\,\big(\mathrm{mod}\ \hbar\,U_\hbar\big) = \mathrm{b}$ and $c\,\big(\mathrm{mod}\ \hbar\,U_\hbar\big) = \mathrm{c}$, a straightforward calculation eventually yields

$$\partial_*(\gamma) + \text{c.p.} + [[\gamma\,,\gamma]]_* = 0$$

This means exactly that $\gamma$ is a 2-cocycle for the Lie bialgebra $\mathfrak{g}$ — according to Definition 2.1.5 — that is obviously antisymmetric (by construction), q.e.d.

*(c)* First of all, we start by noting that $U_\hbar(\mathfrak{g})_\sigma := \big(U_\hbar(\mathfrak{g})\big)_\sigma$ is equal to $U_\hbar(\mathfrak{g})$ as a counital $\Bbbk[[\hbar]]$–coalgebra (by construction), but with the new product defined by

$$m_\sigma(a,b) = a \underset{\sigma}{\cdot} b = \sigma\big(a_{(1)},b_{(1)}\big)\,a_{(2)}\,b_{(2)}\,\sigma^{-1}\big(a_{(3)},b_{(3)}\big) \qquad \forall\ a,b \in U_\hbar(\mathfrak{g})$$

In particular, the $\Bbbk[[\hbar]]$–module $U_\hbar(\mathfrak{g})_\sigma = U_\hbar(\mathfrak{g})$ is still topologically free, so that $U_\hbar(\mathfrak{g})_\sigma$ is again a Hopf algebra in $\mathcal{T}_{\widehat{\otimes}}$, cf. §2.3.1. Moreover, its semiclassical limit $\overline{U_\hbar(\mathfrak{g})_\sigma} := U_\hbar(\mathfrak{g})_\sigma \Big/ \hbar\,U_\hbar(\mathfrak{g})_\sigma$ as a coalgebra is the same as that of $U_\hbar(\mathfrak{g})$; hence it is again cocommutative connected. Thus by Milnor-Moore Theorem we have $\overline{U_\hbar(\mathfrak{g})_\sigma} = U\big(\widehat{\mathfrak{g}}\big)$, where $\widehat{\mathfrak{g}} = \mathrm{Prim}\big(\overline{U_\hbar(\mathfrak{g})_\sigma}\big)$ is the space of primitive elements in $\overline{U_\hbar(\mathfrak{g})_\sigma}$, and as such it coincides with $\mathrm{Prim}\big(\overline{U_\hbar(\mathfrak{g})}\big) = \mathrm{Prim}\big(U(\mathfrak{g})\big) = \mathfrak{g}$ as a Lie coalgebra; its Lie algebra structure, on the other hand, does depend on $\sigma$. Altogether, this shows that $U_\hbar(\mathfrak{g})_\sigma$ is a QUEA, whose semiclassical limit is $U\big(\widehat{\mathfrak{g}}\big)$; then we are only left to prove that the Lie bracket on $\widehat{\mathfrak{g}}$ coincides with that of $\mathfrak{g}_\gamma$, while also proving that $\gamma$ is an antisymmetric 2–cocycle for the Lie bialgebra $\mathfrak{g}$.

The Lie bracket in $\widehat{\mathfrak{g}}$ is given by the commutator inside $U\big(\widehat{\mathfrak{g}}\big) = \overline{U_\hbar(\mathfrak{g})_\sigma}$, so we denote it by $[\mathrm{a}\,,\mathrm{b}]_\sigma = \mathrm{a} \underset{\sigma}{\bar{\cdot}} \mathrm{b} - \mathrm{b} \underset{\sigma}{\bar{\cdot}} \mathrm{a}$ (for all $\mathrm{a}\,,\mathrm{b} \in \mathfrak{g}$), where $\underset{\sigma}{\bar{\cdot}}$ is the product in $U\big(\widehat{\mathfrak{g}}\big) = \overline{U_\hbar(\mathfrak{g})_\sigma}$ induced by the ($\sigma$–deformed) product in $U_\hbar(\mathfrak{g})_\sigma$. Therefore, we will compute such a commutator as the coset modulo $\hbar\,U_\hbar$ of a commutator in $U_\hbar$, namely $[\mathrm{a}\,,\mathrm{b}]_\sigma = \mathrm{a} \underset{\sigma}{\bar{\cdot}} \mathrm{b} - \mathrm{b} \underset{\sigma}{\bar{\cdot}} \mathrm{a} = a \underset{\sigma}{\cdot} b - b \underset{\sigma}{\cdot} a \ \big(\mathrm{mod}\ \hbar\,U_\hbar\big)$, where *$a$ and $b$, like in the proof of claim (c), are lifts of* a *and* b *— i.e.,* $a\,\big(\mathrm{mod}\ \hbar\,U_\hbar\big) = \mathrm{a}$ *and* $b\,\big(\mathrm{mod}\,\hbar\,U_\hbar\big) = \mathrm{b}$ *— such that* $a' := \hbar\,a \in J'_\hbar$ *and* $b' := \hbar\,b \in J'_\hbar$.

We re-start back from (3.30), which now gives (taking into account all the analysis carried out there, with $A = 1 = B$)

$$\begin{aligned} &a \underset{\sigma}{\cdot} b - b \underset{\sigma}{\cdot} a \underset{\hbar}{\equiv} a \cdot b - b \cdot a + \\ &\quad + \hbar^{-3}\Big(\chi\big(a'^{+}_{(1)},b'^{+}_{(1)}\big)\big(\widehat{a}'_{(2)}\,b'^{+}_{(2)} + a'^{+}_{(2)}\,\widehat{b}'_{(2)}\big) - \chi\big(b'^{+}_{(1)},a'^{+}_{(1)}\big)\big(\widehat{b}'_{(2)}\,a'^{+}_{(2)} + b'^{+}_{(2)}\,\widehat{a}'_{(2)}\big)\Big) - \\ &\quad - \hbar^{-3}\Big(\big(\widehat{a}'_{(1)}\,b'^{+}_{(1)} + a'^{+}_{(1)}\,\widehat{b}'_{(1)}\big)\,\chi\big(a'^{+}_{(2)},b'^{+}_{(2)}\big) - \big(\widehat{b}'_{(1)}\,a'^{+}_{(1)} + b'^{+}_{(1)}\,\widehat{a}'_{(1)}\big)\,\chi\big(b'^{+}_{(2)},a'^{+}_{(2)}\big)\Big) \end{aligned}$$

Second, letting $\chi_a := \chi - \chi_{2,1}$ , the previous formula greatly simplifies into

$$a \mathop{\cdot}_{\sigma} b - b \mathop{\cdot}_{\sigma} a \;=\; a \cdot b \,-\, b \cdot a \,+\, \hbar^{-3} \Big( \chi_a\big(a'^{+}_{(1)}, b'\big)\, a'^{+}_{(2)} \,-\, \chi_a\big(a'^{+}_{(2)}, b'\big)\, a'^{+}_{(1)} \Big) \,- \\ -\, \hbar^{-3} \Big( \chi_a\big(b'^{+}_{(1)}, a'\big)\, b'^{+}_{(2)} \,-\, \chi_a\big(b'^{+}_{(2)}, a'\big)\, b'^{+}_{(1)} \Big) \,+\, \mathcal{O}_1(\hbar)$$

Now, let us write $z' = \hbar\, z$ for all $z \in \{a\,,b\}$ : then the last formula turns into

$$a \mathop{\cdot}_{\sigma} b - b \mathop{\cdot}_{\sigma} a \;=\; a \cdot b \,-\, b \cdot a \,+\, \hbar^{-1} \Big( \chi_a\big(a^{+}_{(1)}, b\big)\, a^{+}_{(2)} \,-\, \chi_a\big(a^{+}_{(2)}, b\big)\, a^{+}_{(1)} \Big) \,- \\ -\, \hbar^{-1} \Big( \chi_a\big(b^{+}_{(1)}, a\big)\, b^{+}_{(2)} \,-\, \chi_a\big(b^{+}_{(2)}, a\big)\, b^{+}_{(1)} \Big) \,+\, \mathcal{O}_1(\hbar) \qquad (3.44)$$

Here we recall that, working with a QUEA, for $c \in \{a\,,b\}$ we have

$$\Delta(c) \;=\; c \otimes 1 \,+\, 1 \otimes c \,+\, c^{+}_{(1)} \otimes c^{+}_{(2)} \,+\, \mathcal{O}_2\big(\hbar^2\big) \;\;, \qquad c^{+}_{(1)} \otimes c^{+}_{(2)} \,\in\, \hbar\, U_h^{\widehat{\otimes} 2}$$

and moreover — for every $c \in \{a\,,b\}$ and $\mathrm{c} \in \{\mathrm{a}\,,\mathrm{b}\}$ , so that $c$ is a lift of $\mathrm{c}$ —

$$\overline{\hbar^{-1}\big(c^{+}_{(1)} \otimes c^{+}_{(2)} - c^{+}_{(2)} \otimes c^{+}_{(1)}\big)} \;=\; \delta(\mathrm{c}) \;=:\; \mathrm{c}_{[1]} \otimes \mathrm{c}_{[2]} \qquad (3.45)$$

where hereafter any "overlined" object stands for "its coset modulo $\hbar$"; in addition, we recall also that $\chi_a$ is antisymmetric. Then (3.44) and (3.45) altogether yield

$$[\mathrm{a}\,,\mathrm{b}]_\sigma = \mathrm{a} \mathop{\bar{\cdot}}_{\sigma} \mathrm{b} - \mathrm{b} \mathop{\bar{\cdot}}_{\sigma} \mathrm{a} \;=\; \overline{a \mathop{\cdot}_{\sigma} b - b \mathop{\cdot}_{\sigma} a} \;= \\ =\; [\mathrm{a}\,,\mathrm{b}] \,+\, \gamma\big(\mathrm{a}_{[1]}, \mathrm{b}\big)\, \mathrm{a}_{[2]} \,-\, \gamma\big(\mathrm{b}_{[1]}, \mathrm{a}\big)\, \mathrm{b}_{[2]} \;= \\ =\; [\mathrm{a}\,,\mathrm{b}] \,-\, \gamma\big(\mathrm{a}_{[2]}, \mathrm{b}\big)\, \mathrm{a}_{[1]} \,-\, \gamma\big(\mathrm{b}_{[1]}, \mathrm{a}\big)\, \mathrm{b}_{[2]} \;=:\; [\mathrm{a}\,,\mathrm{b}]_\gamma$$

hence $[\mathrm{a}\,,\mathrm{b}]_\sigma = [\mathrm{a}\,,\mathrm{b}]_\gamma$ for all $\mathrm{a}\,,\mathrm{b} \in \mathfrak{g}$ , in the sense of (2.5), and we are done.

Finally, if in particular $\sigma$ is $\Bbbk[[\hbar]]$–valued — i.e., it is an ordinary 2–cocycle for the Hopf $\Bbbk[[\hbar]]$–algebra $U_\hbar(\mathfrak{g})$ — then we have $\chi \,=\, \hbar\, \log_*(\sigma) \,\in\, \hbar\, \big(U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}\big)^*$ , hence we have just $\gamma = 0$ and $\big(U_\hbar(\mathfrak{g})\big)_\sigma \cong U_\hbar\big(\mathfrak{g}_\gamma\big) = U_\hbar(\mathfrak{g})$ . □

**Example 3.3.8.** Some examples of deformation by quasi-2-cocycles are treated in [GaGa2, Section 5.2], for the wide family of *formal multiparameter QUEAs* that treated in Example 3.1.4; we then resume notations and formulas from there.

Fix $n \in \mathbb{N}_+$ and $I := \{1, \dots, n\}$ . We fix $P := \big(p_{i,j}\big)_{i,j \in I} \in M_n\big(\Bbbk[[\hbar]]\big)$ of Cartan type, with associated Cartan matrix $A$ , a realization $\mathcal{R} := \big(\mathfrak{h}\,, \Pi\,, \Pi^\vee\big)$ of it and the Hopf algebra $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$ . Let $\big\{H_g\big\}_{g \in \mathcal{G}}$ be a $\Bbbk[[\hbar]]$–basis in $\mathfrak{h}$, with $|\mathcal{G}| = \mathrm{rk}(\mathfrak{h}) = t$ .

Fix an antisymmetric, $\Bbbk[[\hbar]]$–bilinear map $\chi : \mathfrak{h} \times \mathfrak{h} \longrightarrow \Bbbk[[\hbar]]$ , that corresponds to $X \,=\, \big(\chi_{g\gamma} = \chi(H_g\,, H_\gamma)\big)_{g,\gamma \in \mathcal{G}} \in \mathfrak{so}_t\big(\Bbbk[[\hbar]]\big)$ . Any such $\chi$ also induces uniquely an antisymmetric, $\Bbbk[[\hbar]]$–bilinear map $\tilde{\chi}_U \,:\, U^{\mathcal{R}}_{P,\hbar}(\mathfrak{h}) \times U^{\mathcal{R}}_{P,\hbar}(\mathfrak{h}) \longrightarrow \Bbbk[[\hbar]]$ as follows. By definition, $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{h})$ is an $\hbar$–adically complete topologically free Hopf algebra isomorphic to $\widehat{S}_{\Bbbk[[\hbar]]}(\mathfrak{h}) := \widehat{\bigoplus_{n \in \mathbb{N}} S^n_{\Bbbk[[\hbar]]}(\mathfrak{h})}$ , the $\hbar$–adic completion of the symmetric algebra $S_{\Bbbk[[\hbar]]}(\mathfrak{h}) = \bigoplus_{n \in \mathbb{N}} S^n_{\Bbbk[[\hbar]]}(\mathfrak{h})$ . Then, $\tilde{\chi}_U$ is defined as the unique $\Bbbk[[\hbar]]$–linear (hence $\hbar$–adically continuous) map $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{h}) \otimes U^{\mathcal{R}}_{P,\hbar}(\mathfrak{h}) \xrightarrow{\;\tilde{\chi}_U\;} \Bbbk[[\hbar]]$ such that

$$\begin{aligned} &\tilde{\chi}_U(z,1) := \epsilon(z) =: \tilde{\chi}_U(1,z) && \forall\; z \in \widehat{S}_{\Bbbk[[\hbar]]}(\mathfrak{h}) \\ &\tilde{\chi}_U(x,y) := \chi(x,y) && \forall\; x,y \in S^1_{\Bbbk[[\hbar]]}(\mathfrak{h}) \\ &\tilde{\chi}_U(x,y) := 0 && \forall\; x \in S^r_{\Bbbk}(\mathfrak{h})\,,\; y \in S^s_{\Bbbk}(\mathfrak{h}) \,:\, r,s \geq 1\,,\; r+s > 2 \end{aligned} \qquad (3.46)$$

By construction, $\tilde{\chi}_U$ *is a normalized Hochschild 2–cocycle on* $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{h})$ , that is

$$\epsilon(x)\, \tilde{\chi}_U(y,z) \,-\, \tilde{\chi}_U(xy,z) \,+\, \tilde{\chi}_U(x,yz) \,-\, \tilde{\chi}_U(x,y)\, \epsilon(z) \;=\; 0 \qquad \forall\; x,y,z \in U^{\mathcal{R}}_{P,\hbar}(\mathfrak{h})$$

By [GaGa2, Lemma 5.2.3], the convolution powers of $\tilde{\chi}_U$ satisfy the following property: for all $H_+, H_- \in \mathfrak{h}$ and $k, \ell, m \in \mathbb{N}_+$, we have

$$\tilde{\chi}_U^{\,*\,m}\big(H_+^k, H_-^\ell\big) \;=\; \begin{cases} \delta_{k,m}\,\delta_{\ell,m}\,\big(m!\big)^2\,\chi(H_+,H_-)^m & \text{for} \quad m \geq 1\;, \\ \delta_{k,0}\,\delta_{\ell,0} & \text{for} \quad m = 0\;. \end{cases}$$

This allows us to define a "toral" quasi-2-cocycle $\chi_U$ as the unique $\Bbbk[[\hbar]]$–linear map from $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h}) \underset{\Bbbk[[\hbar]]}{\otimes} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$ to $\Bbbk((\hbar))$ given by the exponentiation of $\hbar^{-1}\,2^{-1}\,\tilde{\chi}_U$, i.e.

$$\chi_U \;:=\; e^{\hbar^{-1}2^{-1}\tilde{\chi}_U} \;=\; \textstyle\sum_{m \geq 0} \hbar^{-m}\,\tilde{\chi}_U^{\,*\,m}\Big/ 2^m\,m!$$

By [GaGa2, Lemma 5.2.2], this $\chi_U$ is a *quasi–2–cocycle* for $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$, in the sense of Definition 3.3.3. Moreover, one has, for all $H_+\,, H_- \in \mathfrak{h}$, and setting $K_\pm := e^{\hbar\,H_\pm}$,

$$\chi_U^{\pm 1}(H_+,H_-) \;=\; \pm\hbar^{-1}\,2^{-1}\chi(H_+,H_-) \quad , \qquad \chi_U(K_+,K_-) \;=\; e^{\hbar\,2^{-1}\chi(H_+,H_-)}$$

Assume that $\chi$ satisfies the additional requirement $\chi(S_i\,,-\,) \;=\; 0 \;=\; \chi(\,-\,,S_i)$ for $S_i \;:=\; 2^{-1}\big(T_i^+ + T_i^-\big)\,(\forall\; i \in I)$. Then $\chi\big(T_i^+\,,T\big) \;=\; \chi\big(-T_i^-\,,T\big)$ and $\chi\big(T\,,T_i^+\big) \;=\; \chi\big(T\,,-T_i^-\big)$ for all $i \in I$, $T \in \mathfrak{h}$; so $\chi$ induces a $\Bbbk[[\hbar]]$–bilinear map $\overline{\chi} : \overline{\mathfrak{h}} \times \overline{\mathfrak{h}} \longrightarrow \Bbbk[[\hbar]]$, where $\overline{\mathfrak{h}} := \mathfrak{h}\big/\mathfrak{s}$ with $\mathfrak{s} := \mathit{Span}_{\Bbbk[[\hbar]]}\big(\{\,S_i\,\}_{i \in I}\big)$, given by

$$\overline{\chi}\,\big(\,T'\!+\mathfrak{s}\,,T''\!+\mathfrak{s}\,\big) \;:=\; \chi\big(T',T''\big) \qquad \forall\;\, T',T'' \in \mathfrak{h}$$

Now, replaying the construction above with $\overline{\mathfrak{h}}$ and $\overline{\chi}$ replacing $\mathfrak{h}$ and $\chi$, we can construct a quasi-2–cocycle $\overline{\chi}_U : U_{P,\hbar}^{\mathcal{R}}\big(\,\overline{\mathfrak{h}}\,\big) \times U_{P,\hbar}^{\mathcal{R}}\big(\,\overline{\mathfrak{h}}\,\big) \longrightarrow \Bbbk((\hbar))$. Since $U_{P,\hbar}^{\mathcal{R}}\big(\,\overline{\mathfrak{h}}\,\big) \cong \widehat{S}_{\Bbbk[[\hbar]]}\big(\,\overline{\mathfrak{h}}\,\big)$, there exists a Hopf algebra epimorphism $\pi : U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \twoheadrightarrow U_{P,\hbar}^{\mathcal{R}}\big(\,\overline{\mathfrak{h}}\,\big)$ given by $\pi(E_i) := 0$, $\pi(F_i) := 0$ — for $i \in I$ — and $\pi(T) := (T + \mathfrak{s}) \in \overline{\mathfrak{h}} \subseteq U_{P,\hbar}^{\mathcal{R}}\big(\,\overline{\mathfrak{h}}\,\big)$ — for $T \in \mathfrak{h}$. Then we consider $\sigma_\chi \;:=\; \overline{\chi}_U \circ (\pi \times \pi) \;:\; U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \times U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) \twoheadrightarrow \Bbbk((\hbar))$ which is *automatically* a normalized, $\Bbbk((\hbar))$–valued Hopf quasi-2–cocycle on $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$.

By Theorem 3.3.7, one may define a "deformed product" on $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ using $\sigma_\chi$ hereafter denoted by $\dot{\sigma}_\chi$. Write $X^{(n)_{\sigma_\chi}} = X \,\dot{\sigma}_\chi \cdots \dot{\sigma}_\chi X$ for the $n$–th power of any $X \in U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ with respect to this deformed product.

Directly from definitions, sheer computation yields the following formulas, relating the deformed product with the old one (for all $T', T'', T \in \mathfrak{h}$, $i\,,\, j \in I$):

$$T' \,\dot{\sigma}_\chi T'' \;=\; T'\,T'' \quad , \qquad E_i \,\dot{\sigma}_\chi F_j \;=\; E_i\,F_j \quad , \qquad F_j \,\dot{\sigma}_\chi E_i \;=\; F_j\,E_i$$

$$T \,\dot{\sigma}_\chi E_j \;=\; T\,E_j \,+\, 2^{-1}\chi\big(T,T_j^+\big)\,E_j \quad , \quad E_j \,\dot{\sigma}_\chi T \;=\; E_j\,T \,+\, 2^{-1}\chi\big(T_j^+,T)\big)\,E_j$$

$$T \,\dot{\sigma}_\chi F_j \;=\; T\,F_j \,+\, 2^{-1}\chi\big(T,T_j^-\big)\,F_j \quad , \quad F_j \,\dot{\sigma}_\chi T \;=\; F_j\,T \,+\, 2^{-1}\chi\big(T_j^-,T\big)\,F_j$$

$$E_i^{(m)_{\sigma_\chi}} \;=\; \textstyle\prod_{\ell=1}^{m-1} \sigma_\chi\Big(e^{+\hbar\,\ell\,T_i^+},\, e^{+\hbar\,T_i^+}\Big)\,E_i^m \;=\; E_i^m$$

$$E_i^m \,\dot{\sigma}_\chi E_j^n \;=\; \sigma_\chi\Big(e^{+\hbar\,m\,T_i^+},\, e^{+\hbar\,n\,T_j^+}\Big)\,E_i^m\,E_j^n \;=\; e^{+\hbar\,m\,n\,2^{-1}\mathring{\chi}_{ij}}\,E_i^m\,E_j^n$$

$$\begin{aligned} E_i^{(m)_{\sigma_\chi}} \,\dot{\sigma}_\chi\, E_j \,\dot{\sigma}_\chi\, E_k^{(n)_{\sigma_\chi}} \;=\; & \Big(\textstyle\prod_{\ell=1}^{m-1} \sigma_\chi\Big(e^{+\hbar\,\ell\,T_i^+},\, e^{+\hbar\,T_i^+}\Big)\Big)\Big(\textstyle\prod_{t=1}^{n-1} \sigma_\chi\Big(e^{+\hbar\,t\,T_k^+},\, e^{+\hbar\,T_k^+}\Big)\Big) \cdot \\ & \cdot\, \sigma_\chi\Big(e^{+\hbar\,m\,T_i^+},\, e^{+\hbar\,T_j^+}\Big)\,\sigma_\chi\Big(e^{+\hbar\,(m\,T_i^+ + T_j^+)},\, e^{+\hbar\,n\,T_k^+}\Big)\,E_i^m\,E_j\,E_k^n \end{aligned}$$

$$F_i^{(m)_{\sigma_\chi}} \;=\; \textstyle\prod_{\ell=1}^{m-1} \sigma_\chi^{-1}\Big(e^{-\hbar\,\ell\,T_i^-},\, e^{-\hbar\,T_i^-}\Big)\,F_i^m \;=\; F_i^m$$

$$F_i^m \,\dot{\sigma}_\chi F_j^n \;=\; \sigma_\chi^{-1}\Big(e^{-\hbar\,m\,T_i^-},\, e^{-\hbar\,n\,T_j^-}\Big)\,F_i^m\,F_j^n \;=\; e^{-\hbar\,m\,n\,2^{-1}\mathring{\chi}_{ij}}\,F_i^m\,F_j^n$$

$$\begin{aligned} F_i^{(m)_{\sigma_\chi}} \,\dot{\sigma}_\chi\, F_j \,\dot{\sigma}_\chi\, F_k^{(n)_{\sigma_\chi}} =\; & \Big(\textstyle\prod_{\ell=1}^{m-1} \sigma_\chi^{-1}\Big(e^{-\hbar\,\ell\,T_i^-},\, e^{-\hbar\,T_i^-}\Big)\Big)\Big(\textstyle\prod_{t=1}^{n-1} \sigma_\chi^{-1}\Big(e^{-\hbar\,t\,T_k^-},\, e^{-\hbar\,T_k^-}\Big)\Big) \cdot \\ & \cdot\, \sigma_\chi^{-1}\Big(e^{-\hbar\,m\,T_i^-},\, e^{-\hbar\,T_j^-}\Big)\,\sigma_\chi^{-1}\Big(e^{-\hbar\,(m\,T_i^- + T_j^-)},\, e^{-\hbar\,n\,T_k^-}\Big)\,F_i^m\,F_j\,F_k^n \end{aligned}$$

Fix now $\mathring{X} := \Big( \mathring{\chi}_{ij} = \chi\big(T_i^+, T_j^+\big) \Big)_{i,j \in I}$ and define the multiparameter matrix

$$P_{(\chi)} := P + \mathring{X} = \Big( p_{ij}^{(\chi)} := p_{ij} + \mathring{\chi}_{ij} \Big)_{i,j \in I} \;, \quad \Pi_{(\chi)} := \Big\{ \alpha_i^{(\chi)} := \alpha_i \pm \chi\big(-, T_i^{\pm}\big) \Big\}_{i \in I}$$

It turns out that $P_{(\chi)}$ is a matrix of Cartan type — the same of $P$ indeed — and $\mathcal{R}_{(\chi)} = \big(\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee\big)$ is a realization of it. Moreover, by [GaGa2, Theorem 5.2.12], there exists an isomorphism of topological Hopf algebras $\big(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})\big)_{\sigma_\chi} \cong U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)}}(\mathfrak{g})$ which is the identity on generators. In short, every toral quasi–2–cocycle deformation of a FoMpQUEA is another FoMpQUEA. Moreover, under mild restrictions, one proves that the FoMpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is isomorphic to a toral quasi–2–cocycle deformation of the Drinfeld's standard double QUEA, cf. [GaGa2, Theorem 5.2.14].

As to the semiclassical limit, taking everything modulo $\hbar$, the map $\chi$ defines a similar antisymmetric, $\Bbbk$–bilinear map $\gamma := \big(\chi \bmod \hbar\big) : \mathfrak{h}_0 \times \mathfrak{h}_0 \longrightarrow \Bbbk$ — where $\mathfrak{h}_0 := \mathfrak{h} \big/ \hbar\, \mathfrak{h} = \overline{\mathfrak{h}}$. Out of $\gamma$ one constructs a toral 2–cocycle $\gamma_{\mathfrak{g}}$ for the Lie bialgebra $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}$, and out of it the 2–cocycle deformed Lie bialgebra $\big(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}\big)_{\gamma_{\mathfrak{g}}}$. Similarly as above, out of $\gamma$ we get the multiparameter matrix $P_{(\gamma)}$ and its realization $\mathcal{R}_{(\gamma)}$: then $P_{(\gamma)} = \bar{P}_{(\chi)}$ and $\mathcal{R}_{(\gamma)} = \bar{\mathcal{R}}_{(\chi)}$. Attached to these we have $U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)}}(\mathfrak{g})$ and $\mathfrak{g}_{P_{(\gamma)}}^{\mathcal{R}_{(\gamma)}} = \mathfrak{g}_{\bar{P}_{(\chi)}}^{\bar{\mathcal{R}}_{(\chi)}}$, again connected via quantization/specialization, and $\mathfrak{g}_{P_{(\gamma)}}^{\mathcal{R}_{(\gamma)}} \cong \big(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}\big)_{\gamma_{\mathfrak{g}}}$ as Lie bialgebras. In fact, *"deformation by (quasi–)2–cocycle commutes with specialization"*, see [GaGa2, Theorem 6.2.4]: with assumptions as above, we have that $\big(U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})\big)_{\sigma_\chi}$ is a QUEA, with semiclassical limit $U\Big(\big(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}}\big)_{\gamma_{\mathfrak{g}}}\Big) \cong U\Big(\mathfrak{g}_{P_{(\gamma)}}^{\mathcal{R}_{(\gamma)}}\Big)$.

**Remark 3.3.9.** It is important to stress that *our notion of quasi–2–cocycle did* not *come out of the blue*, but rather was suggested by the previous example. Indeed, the authors first "met" these objects when studying polynomial-type QUEAs $U_q(\mathfrak{g})$ "à la Jimbo-Lusztig": these are standard Hopf algebras (no topology is involved), to which one can apply deformation by 2–cocycles and then obtain some "multiparameter QUEAs" — cf. [GaGa1], §4.2. Every such *polynomial* $U_q(\mathfrak{g})$ can be realized as a Hopf subalgebra of a *formal* $U_\hbar(\mathfrak{g})$, hence it makes sense to try and extend the 2–cocycle and the associated deformation procedure for $U_q(\mathfrak{g})$ to the larger object $U_\hbar(\mathfrak{g})$. When we fulfilled this task, in [GaGa2], what we actually found was that the unique extension of the 2–cocycle of $U_q(\mathfrak{g})$ to $U_\hbar(\mathfrak{g})$ actually is a *quasi*–2–cocycle (and not a 2–cocycle!), yet despite this the deformation procedure does extend from $U_q(\mathfrak{g})$ to the whole $U_\hbar(\mathfrak{g})$. Thus the very notion of "quasi–2–cocycle" and the associated deformation procedure showed up as something real from this concrete example.

## 3.4. **Deformations by quasi-twist of QFSHA's.**

In this subsection we consider deformations by twist of QFSHA's, but again "stretching the standard recipe", much like in §3.3: in fact, rather than twists in the usual sense we consider some special twist elements belonging to the scalar extension from $\Bbbk[[\hbar]]$ to $\Bbbk((\hbar))$ of our QFSHA. For these elements — that we call "quasi-twists" — nothing ensures *a priori* that the deformation recipe would properly work on the given QFSHA; nevertheless, we eventually find that this is indeed the case. In short, we prove that *the standard procedure of deformation by twist for QFSHA's can be extended (beyond its natural borders) to the case of quasi-twist elements.*

We begin with a couple of technical lemmas:

**Lemma 3.4.1.** *Let $F_\hbar[[G]]$ be a QFSHA, and $F_\hbar[[G]]^\vee$ the associated QUEA defined in §2.4. Let $\varphi \in J_\hbar^2$ , with $J_\hbar := \mathrm{Ker}\big(\epsilon_{F_\hbar[[G]]}\big)$ . Then:*

*(a)* $F := \exp\big(\hbar^{-1}\varphi\big)$ *is a well-defined element in* $F_\hbar[[G]]^\vee$ *;*

*(b)* $\mathrm{Ad}(F)\big(f\big) := F \cdot f \cdot F^{-1} \in F_\hbar[[G]]$ *for all* $f \in F_\hbar[[G]]$ *, so that the adjoint action of* $F$ *onto* $F_\hbar[[G]]^\vee$ *actually restricts to* $F_\hbar[[G]]$ *.*

*Proof.* *(a)* The assumption $\varphi \in J_\hbar^2$ implies $\varphi \in \hbar^2 \big(J_\hbar^\vee\big)^2 \subseteq \hbar^2 F_\hbar[[G]]^\vee$ , where $J_\hbar^\vee := \hbar^{-1} J_\hbar \subseteq F_\hbar[[G]]^\vee$ . Therefore $\hbar^{-1}\varphi \in \hbar\, F_\hbar[[G]]^\vee$ , hence $F := \exp\big(\hbar^{-1}\varphi\big)$ is indeed a well-defined element in $F_\hbar[[G]]^\vee$ , q.e.d.

*(b)* We compute $\mathrm{Ad}(F)\big(f\big)$ , $f \in F_\hbar[[G]]$ : using the identity $\mathrm{Ad}\big(\exp(X)\big)(Y) = \exp\big(\mathrm{ad}(X)\big)(Y)$ and expanding the exponential into a power series we get

$$\mathrm{Ad}(F)\big(f\big) \;=\; \mathrm{Ad}\big(\exp\big(\hbar^{-1}\varphi\big)\big)(f) \;=\; \exp\big(\,\mathrm{ad}\big(\hbar^{-1}\varphi\big)\big)(f) \;=\; \sum_{n=0}^{+\infty} \frac{1}{n!}\, \mathrm{ad}\big(\hbar^{-1}\varphi\big)^n(f)$$

Now Lemma 2.3.5*(c)* and the assumption $\varphi \in J_{F_\hbar}^2$ together guarantee that

$$\mathrm{ad}\big(\hbar^{-1}\varphi\big)^n(f) \;=\; \mathrm{ad}\big(\hbar^{-1}\varphi\big)^n\big(f_+\big) \;\in\; (1-\delta_{s,0})\, J_{F_\hbar}^{n+s} \qquad\qquad \forall\, n \in \mathbb{N}_+$$

with $s \in \mathbb{N}$ such that $f \in J_{F_\hbar}^s$ , hence $\mathrm{Ad}(F)\big(f\big) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, \mathrm{ad}\big(\hbar^{-1}\varphi\big)^n(f)$ is indeed a well-defined element — a *convergent* series! — of $F_\hbar[[G]]$ . □

**Lemma 3.4.2.** *Let $F_\hbar[[G]]$ be a QFSHA, and $F_\hbar[[G]]^\vee$ the associated QUEA defined in §2.4. Let $\phi \in F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$ be such that $(\mathrm{id} \otimes \epsilon)(\phi) = 0 = (\epsilon \otimes \mathrm{id})(\phi)$ . Then:*

*(a)* $\mathcal{F} = \exp\big(\hbar^{-1}\phi\big)$ *is a well-defined element in* $\big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\, 2}$ *;*

*(b)* $\mathcal{F} \cdot (x \otimes y) \cdot \mathcal{F}^{-1} \in F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$ *for all* $x, y \in F_\hbar[[G]]$ *, so that the adjoint action of* $\mathcal{F}$ *onto* $\big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\, 2}$ *actually restricts to* $F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$ *;*

*(c)* $(\mathrm{id} \otimes \epsilon)(\mathcal{F}\,) = 1 = (\epsilon \otimes \mathrm{id})(\mathcal{F}\,)$ *;*

*(d)* $\mathcal{F}$ *is orthogonal, i.e.* $\mathcal{F}_{2,1} = \mathcal{F}^{-1}$ *, iff* $\phi$ *is antisymmetric, i.e.* $\phi_{2,1} = -\phi$ *;*

*Proof.* *(a)–(b)* These follow from Lemma 3.4.1 applied to $F_\hbar[[G \times G]]$ and to $\varphi := \phi$ .

*(c)–(d)* These follow from definitions and by $(\mathrm{id} \otimes \epsilon)(\phi) = 0 = (\epsilon \otimes \mathrm{id})(\phi)$ . □

The previous result leads us to introduce the notion of "quasi-twist", as follows:

**Definition 3.4.3.** Let $F_\hbar[[G]]$ be a QFSHA, and $F_\hbar[[G]]^\vee$ as in 2.4.1*(b)*. We call *quasi-twist (element) of* $F_\hbar[[G]]$ any element in $\big(F_\hbar[[G]]^{\widetilde{\otimes}\, 2}\big)^\vee = \big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\, 2}$ of the form $\mathcal{F} := \exp\big(\hbar^{-1}\phi\big)$ — for some $\phi \in F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$ such that $(\mathrm{id} \otimes \epsilon)(\phi) = 0 = (\epsilon \otimes \mathrm{id})(\phi)$ — which have the property of a twist element for the QUEA $F_\hbar[[G]]^\vee$ .

Of course, every twist for $F_\hbar[[G]]$ is a quasi-twist too; the converse, in general, is false. However, *every quasi-twist still provides a well-defined deformation of* $F_\hbar[[G]]$ :

**Theorem 3.4.4.** *Let $F_\hbar[[G]]$ be a QFSHA, and $\mathcal{F} = \exp\big(\hbar^{-1}\phi\big)$ a quasi-twist for it, as in Definition 3.4.3. Then the procedure of twist deformation by $\mathcal{F}$ applied to the QUEA $F_\hbar[[G]]^\vee$ restricts to $F_\hbar[[G]]$ , making the latter into a new QFSHA.*

*Proof.* When deforming $F_\hbar[[G]]^\vee$ by the twist $\mathcal{F}$ one introduces the new coproduct $\Delta^{\mathcal{F}}$ given by $\Delta^{\mathcal{F}} := \mathrm{Ad}(\mathcal{F}\,) \circ \Delta$ ; then Lemma 3.4.2*(b)* ensures that $\Delta^{\mathcal{F}}$ restricts to

$F_\hbar[[G]]$, as it maps the latter into $F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$. The antipode is dealt with similarly, so the (deformed) Hopf structure of $\big(F_\hbar[[G]]^\vee\big)^{\mathcal{F}}$ does restrict to $F_\hbar[[G]]$, q.e.d. $\square$

**Definition 3.4.5.** With assumptions as in Theorem 3.4.4, the new QFSHA obtained from $F_\hbar[[G]]$ via twist deformation (by $\mathcal{F}$) of $F_\hbar[[G]]^\vee$ followed by restriction will be called *the quasi-twist deformation of* $F_\hbar[[G]]$ *by* $\mathcal{F}$, and denoted by $F_\hbar[[G]]^{\mathcal{F}}$.

Finally, the next result describes in detail what exactly is the nature of the new, quasi-twist deformed QFSHA $F_\hbar[[G]]^{\mathcal{F}}$, shedding light onto its semiclassical limit:

**Theorem 3.4.6.** *Let $F_\hbar[[G]]$ be a QFSHA over the Lie bialgebra $\mathfrak{g} = \big(\mathfrak{g}\,;\,[\ ,\ ]\,,\,\delta\big)$. Set $\mathfrak{m} := Ker\big(\epsilon_{F[[G]]}\big)$, so $\mathfrak{m}\big/\mathfrak{m}^2 \cong \mathfrak{g}^*$ and $\big(\mathfrak{m}\otimes\mathfrak{m}\big)\big/\big(\mathfrak{m}^2\otimes\mathfrak{m}+\mathfrak{m}\otimes\mathfrak{m}^2\big) \cong \mathfrak{g}^*\otimes\mathfrak{g}^*$ as Lie bialgebras. Let $\mathcal{F}$ be a quasi-twist for $F_\hbar[[G]]$, of the form $\mathcal{F} = \exp\big(\hbar^{-1}\phi\big)$ for some $\phi \in F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$, and set also $\phi_a := \phi - \phi_{2,1}$. Then:*

*(a) $\phi$ is antisymmetric, i.e. $\phi_{2,1} = -\phi$, iff $\mathcal{F}$ is orthogonal, i.e. $\mathcal{F}_{2,1} = \mathcal{F}^{-1}$;*

*(b) the element* $c := \Big(\phi_a\,\Big(\mathrm{mod}\ \hbar\, F_\hbar[[G]]^{\widetilde{\otimes}\, 2}\Big)\ \mathrm{mod}\ \big(\mathfrak{m}^2\otimes\mathfrak{m}+\mathfrak{m}\otimes\mathfrak{m}^2\big)\Big)$ *in* $\big(\mathfrak{m}\otimes\mathfrak{m}\big)\big/\big(\mathfrak{m}^2\otimes\mathfrak{m}+\mathfrak{m}\otimes\mathfrak{m}^2\big) \cong \mathfrak{g}^*\otimes\mathfrak{g}^*$ *is an antisymmetric twist for $\mathfrak{g}^*$;*

*(c) the quasi-twist deformation $\big(F_\hbar[[G]]\big)^{\mathcal{F}}$ of $F_\hbar[[G]]$ is a QFSHA for the Poisson group $G^c$ whose cotangent Lie bialgebra is $Lie\big(G^c\big)^* = \big(\mathfrak{g}^*\big)^c = \big(\mathfrak{g}^*\,;\,[\ ,\ ]_*\,,\,\delta_*^c\big)$ that is the deformation of $\mathfrak{g}^*$ by the twist $c$; in short, $\big(F_\hbar[[G]]\big)^{\mathcal{F}} \cong F_\hbar\big[\big[G^c\big]\big]$.*

*In particular, if $\mathcal{F}$ is $\Bbbk[[\hbar]]$–valued — i.e., it is an ordinary twist for $F_\hbar[[G]]$ — or equivalently $\phi \in \hbar\, F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$, then $c = 0$ and $\big(F_\hbar[[G]]\big)^c \cong F_\hbar\big[\big[G^c\big]\big] = F_\hbar[[G]]$.*

*Proof. (a)* This is a special case of Lemma 3.4.2*(d)*.

*(b)* We start from the twist identity $\mathcal{F}_{12}\,\big(\Delta\otimes\mathrm{id}\big)(\mathcal{F}) = \mathcal{F}_{23}\,\big(\mathrm{id}\otimes\Delta\big)(\mathcal{F})$ that we re-write in the equivalent form

$$\big(\Delta\otimes\mathrm{id}\big)(\mathcal{F})\,\big(\mathrm{id}\otimes\Delta\big)(\mathcal{F})^{-1} \;=\; \mathcal{F}_{12}^{-1}\,\mathcal{F}_{23} \tag{3.47}$$

Replacing $\mathcal{F} = \exp\big(\hbar^{-1}\phi\big)$, we find

$$\big(\Delta\otimes\mathrm{id}\big)(\mathcal{F})\big)\cdot\big(\mathrm{id}\otimes\Delta\big)(\mathcal{F})^{-1} \;=\; \exp\big(\hbar^{-1}\big(\Delta\otimes\mathrm{id}\big)(\phi)\big)\cdot\exp\big(-\hbar^{-1}\big(\mathrm{id}\otimes\Delta\big)(\phi)\big)$$

Now we recall the *Baker-Campbell-Hausdorff's formula*, that is the formal identity

$$\exp(X)\cdot\exp(Y) \;=\; \exp\big(\mathcal{BCH}(X,Y)\big) \tag{3.48}$$

which allows to express the product of two exponential as a single exponential: in it, $\mathcal{BCH}(X,Y) := \log\big(\exp(X)\,\exp(Y)\big)$ is an explicit formal series given by

$$\mathcal{BCH}(X,Y) \;=\; \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}\sum_{\substack{r_i+s_i>0\\ 1\leq i\leq n}}\frac{\big[X^{\bullet r_1}Y^{\bullet s_1}X^{\bullet r_2}Y^{\bullet s_2}\cdots X^{\bullet r_n}Y^{\bullet s_n}\big]}{\big(\sum_{i=1}^n(r_i+s_i)\big)\cdot\prod_{j=1}^n r_i!\,s_i!} \tag{3.49}$$

where we use notation

$$\big[X^{\bullet r_1}Y^{\bullet s_1}\cdots X^{\bullet r_n}Y^{\bullet s_n}\big] \;:=$$
$$=\;\big[\underbrace{X,\big[X,\cdots\big[X,}_{r_1}\big[\underbrace{Y,\big[Y,\cdots\big[Y,}_{s_1}\cdots,\big[\underbrace{X,\big[X,\cdots\big[X,}_{r_n}\big[\underbrace{Y,\big[Y,\cdots,Y}_{s_n}\big]\big]\big]\cdots\big]\big]\cdots\big]\big]$$

with the silent assumption that the Lie monomial $\big[X^{\bullet r_1}Y^{\bullet s_1}\cdots X^{\bullet r_n}Y^{\bullet s_n}\big]$ is just $X$, respectively $Y$, when $n=1$ and $s_1=0$, respectively $r_1=0$, while it is zero whenever $s_n>1$ or $s_n=0$ and $r_n>1$. In words, when $S := \sum_{i=1}^n(r_i+s_i) > 1$

the Lie monomial $\big[\, X^{\bullet r_1} Y^{\bullet s_1} \cdots X^{\bullet r_n} Y^{\bullet s_n} \big]$ is the composition of several operators $\mathrm{ad}(X)^{r_i}$ or $\mathrm{ad}(Y)^{s_i}$ to $Y$ — when $s_n = 1$ — or to $X$ — when $s_n = 0$ and $r_n = 1$. Looking up to second order, (3.49) reads

$$\mathcal{BCH}(X,Y\,) \; := \; X \,+\, Y \,+\, \frac{1}{2}\,[X,Y\,] \,+\, \mathcal{O}_{\mathcal{L}}(3) \tag{3.50}$$

where $\mathcal{O}_{\mathcal{L}}(3)$ denotes a (formal) infinite linear combination of Lie monomials in $X$ and $Y$ of degree at least 3. Setting now $X := \hbar^{-1}\big(\Delta \otimes \mathrm{id}\big)(\phi)$ and $Y := -\hbar^{-1}\big(\mathrm{id} \otimes \Delta\big)(\phi)$, the above analysis yields, rewriting (3.48),

$$\begin{aligned} &\big(\Delta \otimes \mathrm{id}\big)(\mathcal{F}\,) \cdot \big(\mathrm{id} \otimes \Delta\big)(\mathcal{F}\,)^{-1} \;= \\ &\qquad = \; \exp\Big(\, \mathcal{BCH}\Big(\, \hbar^{-1}\big(\Delta \otimes \mathrm{id}\big)(\phi)\,,\, -\hbar^{-1}\big(\mathrm{id} \otimes \Delta\big)(\phi)\Big)\Big) \end{aligned} \tag{3.51}$$

where the BCH series has to be expanded as in (3.49). To this end, writing $\phi = \phi_1 \otimes \phi_2$ (a sum being tacitly intended) with $\phi_1, \phi_2 \in J_\hbar$ (by Lemma 3.1.1) we have

$$\big(\Delta \otimes \mathrm{id}\big)(\phi) \;=\; \Delta\big(\phi_1\big) \otimes \phi_2 \;=\; \phi_{1,3} \,+\, \phi_{2,3} \,+\, \big((\phi_1)_{(1)}\big)^+ \otimes \big((\phi_1)_{(2)}\big)^+ \otimes \phi_2$$

where we expanded $\Delta\big(\phi_1\big)$ as in Lemma 2.3.5*(d)* and we used that $\epsilon\big(\phi_1\big) = 0$. Note that in the expansion of $\big(\Delta \otimes \mathrm{id}\big)(\phi)$ we have

$$\big(\phi_{1,3} \,+\, \phi_{2,3}\big) \;\in\; J_\hbar^{(\otimes 3|2)} \quad , \qquad \big((\phi_1)_{(1)}\big)^+ \otimes \big((\phi_1)_{(2)}\big)^+ \otimes \phi_2 \;\in\; J_\hbar^{\otimes 3} \tag{3.52}$$

where we introduced the notation $J_\hbar^{(\otimes 3|N)} := \sum_{\substack{a,b,c \geqslant 0 \\ a+b+c \geqslant N}} J_\hbar^a \,\widetilde{\otimes}\, J_\hbar^b \,\widetilde{\otimes}\, J_\hbar^c$ (for $N \in \mathbb{N}$).

A similar analysis for $\big(\mathrm{id} \otimes \Delta\big)(\phi)$, just switching the roles of $\phi_1$ and $\phi_2$, yields

$$\big(\mathrm{id} \otimes \Delta\big)(\phi) \;=\; \phi_{1,2} \,+\, \phi_{1,3} \,+\, \phi_1 \otimes \big((\phi_2)_{(1)}\big)^+ \otimes \big((\phi_2)_{(2)}\big)^+$$

$$\text{with} \qquad \big(\phi_{1,2} \,+\, \phi_{1,3}\big) \;\in\; J_\hbar^{(\otimes 3|2)} \,, \qquad \phi_1 \otimes \big((\phi_2)_{(1)}\big)^+ \otimes \big((\phi_2)_{(2)}\big)^+ \;\in\; J_\hbar^{\otimes 3} \tag{3.53}$$

Now, thanks to Lemma 2.3.5*(c)*, from (3.52) and (3.53) together we get the identity $\big[\hbar\, X, \hbar\, Y\big] \;=\; \big[\,\phi_{1,3}\,,\,\phi_{1,2}\big] + \big[\,\phi_{2,3}\,,\,\phi_{1,2}\,\big] + \big[\,\phi_{2,3}\,,\,\phi_{1,3}\,\big] \,+\, \hbar \cdot \mathcal{O}\Big(J_\hbar^{[\otimes 3|4]}\Big)$, hence

$$\big[X,Y\big] \;=\; \hbar^{-1}\Big(\hbar^{-1}\big[\phi_{1,3},\phi_{1,2}\big] + \hbar^{-1}\big[\phi_{2,3},\phi_{1,2}\big] + \hbar^{-1}\big[\phi_{2,3},\phi_{1,3}\big] \,+\, \mathcal{O}\Big(J_\hbar^{[\otimes 3|4]}\Big)\Big) \tag{3.54}$$

for some element $\mathcal{O}\Big(J_\hbar^{[\otimes 3|4]}\Big) \in J_\hbar^{[\otimes 3|4]}$, where hereafter we use notation

$$J_\hbar^{[\otimes 3|N]} \;:=\; \sum_{\substack{a,b,c \geqslant 1 \\ a+b+c \geqslant N}} J_\hbar^{\otimes a} \otimes J_\hbar^{\otimes b} \otimes J_\hbar^{\otimes c} \;\subseteq\; \mathit{Ker}\Big(\epsilon_{F_\hbar[[G\,]]^{\otimes 3}}\Big)^N \qquad \forall\;\, N \in \mathbb{N}_+ \tag{3.55}$$

Pushing the analysis further on, we find easily that

$$\big[\, X^{\bullet r_1} Y^{\bullet s_1} \cdots X^{\bullet r_n} Y^{\bullet s_n} \big] \;\in\; \hbar^{-1}\, J_\hbar^{[\otimes 3|S+1]} \qquad \forall\;\, S := \sum_{i=1}^n (r_i + s_i) > 1 \tag{3.56}$$

looking at (3.49), this tells us that the expansion of the BCH series occurring in (3.51), when expanded as in (3.49), is actually given by $\hbar^{-1}$ multiplied by *a truly convergent series inside* $F_\hbar[[G\,]]^{\otimes 3}$. In other words, tiding everything up we find that there exists some $\mathcal{Z} \in J_\hbar^{\otimes 3} \subseteq F_\hbar[[G]]^{\otimes 3}$ such that $\big(\Delta \otimes \mathrm{id}\big)(\mathcal{F}\,) \cdot \big(\mathrm{id} \otimes \Delta\big)(\mathcal{F}\,)^{-1} = \exp\big(\,\hbar^{-1}\mathcal{Z}\,\big)$ Even more, by (3.49) and (3.50) and the previous analysis we do know the expansion of this $\mathcal{Z}$ up to second order, whence we find

$$\begin{aligned} &\big(\Delta \otimes \mathrm{id}\big)(\mathcal{F}\,) \cdot \big(\mathrm{id} \otimes \Delta\big)(\mathcal{F}\,)^{-1} \;=\; \exp\Big(\,\hbar^{-1}\,\Big(\big(\Delta \otimes \mathrm{id}\big)(\phi) \,-\, \big(\mathrm{id} \otimes \Delta\big)(\phi) \,- \\ &\; -\, \hbar^{-1}2^{-1}\big[\phi_{1,3},\phi_{1,2}\big] - \hbar^{-1}2^{-1}\big[\phi_{2,3},\phi_{1,2}\big] - \hbar^{-1}2^{-1}\big[\phi_{2,3},\phi_{1,3}\big] \,+\, \mathcal{O}\Big(J_\hbar^{[\otimes\, 3|4]}\Big)\Big)\Big) \end{aligned} \tag{3.57}$$

Now we go and work instead on the right-hand side of (3.47). Again, replacing $\mathcal{F} = \exp\big(\hbar^{-1}\phi\big)$, we find

$$\mathcal{F}_{12}^{-1} \cdot \mathcal{F}_{23} \;=\; \exp\big(-\hbar^{-1}\phi_{1,2}\big) \cdot \exp\big(\hbar^{-1}\phi_{2,3}\big) \;=\; \exp\Big(\mathcal{BCH}\big(-\hbar^{-1}\,\phi_{1,2}\,,\,\hbar^{-1}\,\phi_{2,3}\big)\Big)$$

Now for the computation of $\mathcal{BCH}\big(-\hbar^{-1}\,\phi_{1,2}\,,\hbar^{-1}\,\phi_{2,3}\big)$ ; to avoid possible confusion, we denote the second, right-hand instance of $\phi$ by $\phi'$. We begin noting that

$$\big[\,\phi_{1,2}\,,\phi'_{2,3}\,\big] \;=\; \big[\,\phi_1 \otimes \phi_2 \otimes 1\,,1 \otimes \phi'_1 \otimes \phi'_2\,\big] \;=\; \phi_1 \otimes \big[\,\phi_2\,,\phi'_1\,\big] \otimes \phi'_2 \;\in\; \hbar\,J_\hbar^{[\otimes 3|3]}$$

so that for $X := -\hbar^{-1}\,\phi_{1,2}$ and $Y := \hbar^{-1}\,\phi_{2,3}$ we get, using Lemma 2.3.5*(c)*,

$$\big[X,Y\big] \;=\; \big[-\hbar^{-1}\,\phi_{1,2}\,,\hbar^{-1}\,\phi'_{2,3}\,\big] \;=\; -\hbar^{-2}\big[\,\phi_{1,2}\,,\phi'_{2,3}\,\big] \;\in\; \hbar^{-2}\,\hbar\,J_\hbar^{\otimes 3} \;=\; \hbar^{-1}\,J_\hbar^{[\otimes 3|3]}$$

A second, similar step gives (with obvious notation $\phi$, $\phi'$ and $\phi''$)

$$\big[\,\phi_{1,2}\,,\big[\,\phi'_{1,2}\,,\phi''_{2,3}\,\big]\big] \;=\; \big[\,\phi_1 \otimes \phi_2 \otimes 1\,,\phi'_1 \otimes \big[\,\phi'_2\,,\phi''_1\,\big] \otimes \phi''_2\,\big] \;=$$
$$=\; \big[\,\phi_1\,,\phi'_1\,\big] \otimes \phi_2 \cdot \big[\,\phi'_2\,,\phi''_1\,\big] \otimes \phi''_2 \,+\, \phi_1 \cdot \phi'_1 \otimes \big[\,\phi_2\,,\big[\,\phi'_2\,,\phi''_1\,\big]\big] \otimes \phi''_2 \;\in\; \hbar^2\,J_\hbar^{[\otimes 3|4]}$$

hence $\big[X,\big[X,Y\,\big]\big] \in \hbar^{-3}\,\hbar^2\,J_\hbar^{[\otimes 3|4]} \;=\; \hbar^{-1}\,J_\hbar^{[\otimes 3|4]}$. More in general, iteration yields

$$\big[\,X^{\bullet r_1}Y^{\bullet s_1}\cdots X^{\bullet r_n}Y^{\bullet s_n}\big] \;\in\; \hbar^{-S}\,\hbar^{S-1}\,J_\hbar^{[\otimes 3|S+1]} \;=\; \hbar^{-1}\,J_\hbar^{[\otimes 3|S+1]} \tag{3.58}$$

with notation as before, still using Lemma 2.3.5*(c)*. Tiding everything up we find that there exists $\mathcal{W} \in J_\hbar^{\otimes 3} \subseteq F_\hbar[[G]]^{\widetilde{\otimes}\,3}$ such that $\mathcal{F}_{12}^{-1} \cdot \mathcal{F}_{23} \;=\; \exp\big(\hbar^{-1}\mathcal{W}\big)$ ; moreover, by (3.49) and (3.50) along with the previous analysis we can write

$$\mathcal{F}_{1,2}^{-1} \cdot \mathcal{F}_{2,3} \;=\; \exp\Big(\hbar^{-1}\Big(-\,\phi_{1,2} \,+\, \phi_{2,3} \,-\, \hbar^{-1}\,2^{-1}\big[\,\phi_{1,2}\,,\,\phi_{2,3}\,\big] \,+\, \mathcal{O}\Big(J_\hbar^{[\otimes\,3|4]}\Big)\Big) \tag{3.59}$$

Finally, comparing (3.57), (3.59) and (3.47) we get the identity in $F_\hbar[[G]]^{\widetilde{\otimes}\,3}$

$$\big(\Delta \otimes \mathrm{id}\big)(\phi) \,-\, \big(\mathrm{id} \otimes \Delta\big)(\phi) \,-$$
$$-\,\hbar^{-1}2^{-1}\big[\phi_{1,3}\,,\,\phi_{1,2}\big] - \hbar^{-1}2^{-1}\big[\phi_{2,3}\,,\,\phi_{1,2}\big] - \hbar^{-1}2^{-1}\big[\phi_{2,3}\,,\,\phi_{1,3}\big] \,+\, \mathcal{O}\Big(J_\hbar^{[\otimes\,3|4]}\Big) \;=$$
$$=\; -\phi_{1,2} \,+\, \phi_{2,3} \,-\, \hbar^{-1}\,2^{-1}\big[\,\phi_{1,2}\,,\,\phi_{2,3}\,\big] \,+\, \mathcal{O}\Big(J_\hbar^{[\otimes\,3|4]}\Big)$$

that in turn, through simplification and reduction modulo $\hbar\,F_\hbar[[G]]^{\widetilde{\otimes}\,3}$, yields the following identity inside $F[[G]]^{\widetilde{\otimes}\,3}$

$$\begin{aligned} &\big(\Delta \otimes \mathrm{id}\big)\big(\,\overline{\phi}\,\big) \,-\, \big(\mathrm{id} \otimes \Delta\big)\big(\,\overline{\phi}\,\big) \,+\, \overline{\phi}_{1,2} \,-\, \overline{\phi}_{2,3} \,+ \\ &\qquad +\; 2^{-1}\big\{\,\overline{\phi}_{1,2}\,,\,,\overline{\phi}_{1,3}\,\big\} \,+\, \big\{\,\overline{\phi}_{1,2}\,,\,\overline{\phi}_{2,3}\,\big\} \,+\, 2^{-1}\big\{\,\overline{\phi}_{1,3}\,,\,\overline{\phi}_{2,3}\,\big\} \underset{\mathfrak{m}^{[\otimes\,3\,|4]}}{\equiv} 0 \end{aligned} \tag{3.60}$$

where hereafter we adopt the notation for which $\overline{\varphi}$ denotes the coset modulo $\hbar$ of any element $\varphi \in F_\hbar[[G]]^{\widetilde{\otimes}\,3}$ with $n \in \mathbb{N}_+$.

Now let $\Bbbk\big[\mathbb{S}_3\big]$ act onto $F_\hbar[[G]]^{\widetilde{\otimes}\,3}$ and consider the action of the antisymmetrizer $Alt_3 := \big(\,\mathrm{id} - (1\,2) - (2\,3) - (3\,1) + (1\,2\,3) + (3\,2\,1)\big)$ onto (3.60): this in turn yields a new identity. Within the latter, we have a first contribution of the form

$$Alt_3.\Big(\big(\Delta \otimes \mathrm{id}\big)\big(\,\overline{\phi}\,\big) \,-\, \big(\mathrm{id} \otimes \Delta\big)\big(\,\overline{\phi}\,\big)\Big) \;==\; \big(\,\nabla \otimes \mathrm{id}\big)\big(\,\overline{\phi_a}\,\big) \,+\, \mathrm{c.p}$$

and a second contribution $Alt_3.\big(\,\overline{\phi}_{1,2} \,-\, \overline{\phi}_{2,3}\,\big) \;=\; 0$ . The last contribution is

$$Alt_3.\Big(2^{-1}\big\{\,\overline{\phi}_{1,2}\,,\,\overline{\phi}_{1,3}\,\big\}\Big) \,+\, Alt_3.\Big(\big\{\,\overline{\phi}_{1,2}\,,\,\overline{\phi}_{2,3}\,\big\}\Big) \,+\, Alt_3.\Big(2^{-1}\big\{\,\overline{\phi}_{1,3}\,,\,\overline{\phi}_{2,3}\,\big\}\Big)$$

We go and compute the first summand, as follows:

$$Alt_3.\Big(2^{-1}\big\{\,\overline{\phi}_{1,2}\,,\,\overline{\phi}_{1,3}\,\big\}\Big) \;=\; \big\{\,\overline{\phi}_{1,2}\,,\,\overline{\phi}_{1,3}\,\big\} \,+\, \mathrm{c.p.}$$

A similar analysis applies to the third summand, which yields

$$Alt_3 . \Big( 2^{-1} \big\{ \overline{\phi}_{1,3} \, , \, \overline{\phi}_{2,3} \big\} \Big) \;=\; \big\{ \overline{\phi}_{1,3} \, , \, \overline{\phi}_{2,3} \big\} \;+\; \text{c.p.}$$

whereas for the second summand instead we get

$$Alt_3 . \Big( \big\{ \overline{\phi}_{1,2} \, , \, \overline{\phi}_{2,3} \big\} \Big) \;=\; \big\{ \overline{\phi}_{1,2} \, , \, \overline{\phi}_{2,3} \big\} \;-\; \big\{ \overline{\phi}_{3,2} \, , \, \overline{\phi}_{2,1} \big\} \;+\; \text{c.p.}$$

Putting all these together we find

$$Alt_3 . \Big( 2^{-1} \big\{ \overline{\phi}_{1,2} \, , \, \overline{\phi}_{1,3} \big\} \Big) + Alt_3 . \Big( \big\{ \overline{\phi}_{1,2} \, , \, \overline{\phi}_{2,3} \big\} \Big) + Alt_3 . \Big( 2^{-1} \big\{ \overline{\phi}_{1,3} \, , \, \overline{\phi}_{2,3} \big\} \Big) \;=$$

$$= \big\{ \overline{\phi}_{1,2} \, , \overline{\phi}_{1,3} \big\} + \big\{ \overline{\phi}_{1,2} \, , \overline{\phi}_{2,3} \big\} - \big\{ \overline{\phi}_{3,2} \, , \overline{\phi}_{2,1} \big\} + \big\{ \overline{\phi}_{1,3} \, , \overline{\phi}_{2,3} \big\} + \text{c.p.} \;=\; \big\{\big\{ \overline{\phi_a} \, , \overline{\phi_a} \big\}\big\}$$

where the very last identity follows from a routine calculation. Joint with the previously found identities, the latter gives yet the following, last one:

$$\Big( \big( \nabla \otimes \mathrm{id} \, \big) \big( \overline{\phi_a} \, \big) \;+\; \text{c.p.} \Big) \;+\; \big\{\big\{ \overline{\phi_a} \, , \overline{\phi_a} \big\}\big\} \;\underset{\mathfrak{m}^{[\otimes 4]}}{\equiv}\; 0$$

At last, recalling that $c := \overline{\phi_a} \, \big( \mathrm{mod} \; \mathfrak{m}^2 \big)$ in $\mathfrak{m} \big/ \mathfrak{m}^2 = \mathfrak{g}^*$ , and that in the latter Lie bialgebra the Lie cobracket, resp. the Lie bracket, is given by $\nabla$ , resp. by $[ \; , \; ]$ , reduced modulo $\mathfrak{m}^2$ , the last formula above — in $\mathfrak{m}^{\widetilde{\otimes}\, 3}$ — implies

$$\Big( \big( \delta \otimes \mathrm{id} \, \big) ( c \, ) \;+\; \text{c.p.} \Big) \;+\; [[ \, c \, , c \, ]] \;=\; 0$$

within $\big( \mathfrak{g}^* \big)^{\otimes 3}$, which implies exactly that $c$ — which is antisymmetric by construction — is an antisymmetric twist for the Lie bialgebra $\mathfrak{g}^* = \mathfrak{m} \big/ \mathfrak{m}^2$ , q.e.d.

*(c)* We adopt the following notational convention: any element in $F_\hbar[[G]]$ will be denoted by an italic letter, say $f \in F_\hbar[[G]]$ ; then its coset modulo $\hbar \, F_\hbar[[G]]$ will be denoted with a line over that letter, say $\overline{f} := \big( f \; \mathrm{mod} \; \hbar \, F_\hbar[[G]] \big)$ , and finally the coset of the latter modulo $\mathfrak{m}^2$ will be denoted by the corresponding letter in roman font, say $\mathrm{f} := \big( \overline{f} \; \mathrm{mod} \; \mathfrak{m}^2 \big)$ . Note also that every element in $\mathfrak{g}^* = \mathfrak{m} \big/ \mathfrak{m}^2$ can be written as such an $\mathrm{f} = \big( \overline{f} \; \mathrm{mod} \; \mathfrak{m}^2 \big)$ for some $f \in J_\hbar \in \mathit{Ker}\big( \epsilon_{F_\hbar[[G]]} \big)$ .

Similar notation will be used for elements in $F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$ and their coset modulo $\hbar$ and (further on) modulo $\mathfrak{m}^{[\otimes 2 \,|\, 3]} := \mathfrak{m} \otimes \mathfrak{m}^2 + \mathfrak{m}^2 \otimes \mathfrak{m}$ .

Recall that the Lie cobracket induced on $\mathfrak{m} \big/ \mathfrak{m}^2 = \mathfrak{g}^*$ by the deformed quantization is defined by

$$\delta^{\mathcal{F}}(\mathrm{f}\,) \;:=\; \Big( \Delta^{\mathcal{F}} - \big(\Delta^{\mathcal{F}}\big)^{21} \Big) \big( \, \overline{f} \, \big) \;\; \mathrm{mod} \; \mathfrak{m}^{[\otimes 2 \,|\, 3]} \;=\; \overline{\Delta^{\mathcal{F}}(f)} - \overline{\big(\Delta^{\mathcal{F}}\big)^{\mathrm{op}}(f)} \;\; \mathrm{mod} \; \mathfrak{m}^{[\otimes 2 \,|\, 3]}$$

so we start computing $\Delta^{\mathcal{F}}(f)$ . Definitions give

$$\Delta^{\mathcal{F}}(f) \;=\; \mathrm{Ad}(\mathcal{F}\,) \Big( \big( f_{(1)} \otimes 1 \big) \cdot \big( 1 \otimes f_{(2)} \big) \Big) \;=\; \mathrm{Ad}(\mathcal{F}\,) \big( f_{(1)} \otimes 1 \big) \cdot \mathrm{Ad}(\mathcal{F}\,) \big( 1 \otimes f_{(2)} \big)$$

In the last product, we focus on the first factor: thus we get

$$\mathrm{Ad}(\mathcal{F}\,) \big( f_{(1)} \otimes 1 \big) \;=\; \mathrm{Ad} \, \big( \exp\!\big( \hbar^{-1} \phi \big) \big) \big( f_{(1)} \otimes 1 \big) \;=\; \exp \, \big( \mathrm{ad}\big( \hbar^{-1} \phi \big) \big) \big( f_{(1)} \otimes 1 \big) \;=$$

$$=\; \sum_{n=0}^{+\infty} \frac{1}{n!} \, \mathrm{ad}\big( \hbar^{-1} \phi_1 \big)^n \big( f_{(1)} \big) \otimes \phi_2^{\,n} \;=\; f_{(1)} \otimes 1 \;+\; \big[ \hbar^{-1} \phi_1 \, , f_{(1)} \big] \otimes \phi_2 \;+\; \mathcal{O}(2)$$

that is in short $\mathrm{Ad}(\mathcal{F}\,) \big( f_{(1)} \otimes 1 \big) \;=\; f_{(1)} \otimes 1 \;+\; \big[ \hbar^{-1} \phi_1 \, , f_{(1)} \big] \otimes \phi_2 \;+\; \mathcal{O}(2)$ where hereafter $\mathcal{O}(2)$ denotes any element in $J_\hbar^{[\otimes 2 \,|\, 3]}$ . A similar calculation yields

$\mathrm{Ad}(\mathcal{F})\big(1 \otimes f_{(2)}\big) \,=\, 1 \otimes f_{(2)} \,+\, \phi_1 \otimes \big[\,\hbar^{-1}\phi_2\,,f_{(2)}\,\big] \,+\, \mathcal{O}(2)$ . Eventually, this gives

$$\begin{aligned}\Delta^{\mathcal{F}}(f) \;&=\; \mathrm{Ad}(\mathcal{F})\big(\,f_{(1)} \otimes f_{(2)}\big) \;=\; \mathrm{Ad}(\mathcal{F})\big(\,f_{(1)} \otimes 1\,\big) \cdot \mathrm{Ad}(\mathcal{F})\big(1 \otimes f_{(2)}\big) \;= \\ &= \big(\,f_{(1)} \otimes 1 + \big[\,\hbar^{-1}\phi_1\,,f_{(1)}\,\big] \otimes \phi_2 + \mathcal{O}(2)\big) \cdot \big(\,1 \otimes f_{(2)} + \phi_1 \otimes \big[\,\hbar^{-1}\phi_2\,,f_{(2)}\,\big] + \mathcal{O}(2)\big) \;= \\ &= \; f_{(1)} \otimes f_{(2)} + \epsilon\big(f_{(1)}\big)\,\phi_1 \otimes \big[\,\hbar^{-1}\phi_2\,,f_{(2)}\,\big] + \big[\,\hbar^{-1}\phi_1\,,f_{(1)}\,\big] \otimes \phi_2\,\epsilon\big(f_{(2)}\big) + \mathcal{O}(2) \;= \\ &= \; \Delta(f) \,+\, \phi_1 \otimes \big[\,\hbar^{-1}\phi_2\,,f\,\big] \,+\, \big[\,\hbar^{-1}\phi_1\,,f\,\big] \otimes \phi_2 \,+\, \mathcal{O}(2)\end{aligned}$$

so that, in short, $\;\Delta^{\mathcal{F}}(f) \underset{J_\hbar^{[\otimes 2\,|3]}}{\equiv} \Delta(f) \,+\, \phi_1 \otimes \big[\,\hbar^{-1}\phi_2\,,f\,\big] \,+\, \big[\,\hbar^{-1}\phi_1\,,f\,\big] \otimes \phi_2$ .
Therefore, for $\nabla_{\mathcal{F}} := \Delta^{\mathcal{F}} - \big(\Delta^{\mathcal{F}}\big)^{21}$ we get

$$\begin{aligned}\nabla_{\mathcal{F}}(f) \;&\underset{J_\hbar^{[\otimes 2\,|3]}}{\equiv}\; \Delta(f) \,+\, \phi_1 \otimes \big[\,\hbar^{-1}\phi_2\,,f\,\big] \,+\, \big[\,\hbar^{-1}\phi_1\,,f\,\big] \otimes \phi_2 \,- \\ &\qquad -\, \Delta^{\mathrm{op}}(f) \,-\, \big[\,\hbar^{-1}\phi_2\,,f\,\big] \otimes \phi_1 \,-\, \phi_2 \otimes \Big[\,\hbar^{-1}\phi_1\,,f\,\Big] \;= \\ &\qquad =\; \nabla(f) \,-\, \phi_1^{(a)} \otimes \hbar^{-1}\Big[\,f\,,\,\phi_2^{(a)}\Big] \,-\, \hbar^{-1}\Big[\,f\,,\,\phi_1^{(a)}\Big] \otimes \phi_2^{(a)}\end{aligned} \tag{3.61}$$

where we set $\phi_a := \phi - \phi_{21} = \phi_1^{(a)} \otimes \phi_2^{(a)}$ . Reducing (3.61) modulo $\hbar\, J_\hbar^{\otimes 2}$ yields

$$\nabla_{\mathcal{F}}\big(\,\overline{f}\,\big) \underset{\mathfrak{m}^{[\otimes 2\,|3]}}{\equiv} \nabla\big(\,\overline{f}\,\big) \,-\, \overline{\phi_1^{(a)}} \otimes \Big\{\,\overline{f}\,,\,\overline{\phi_2^{(a)}}\,\Big\} \,-\, \Big\{\,\overline{f}\,,\,\overline{\phi_1^{(a)}}\,\Big\} \otimes \overline{\phi_2^{(a)}}$$

hence reducing the latter modulo $\mathfrak{m}^{[\otimes 2\,|3]}$ we find in $\mathfrak{m}^{\otimes 2}\big/\mathfrak{m}^{[\otimes 3]} = \mathfrak{g}^* \otimes \mathfrak{g}^*$ the identity

$$\Big(\,\nabla_{\mathcal{F}} \bmod \mathfrak{m}^{[\otimes 2\,|3]}\Big)(\mathrm{f}) \;=\; \delta(\mathrm{f}) \,-\, \big(\,\mathrm{ad}(\mathrm{f})\big)(c) \;=\; \big(\,\delta - \partial_c\big)(\mathrm{f}) \;=\; \delta^{\,c}(\mathrm{f})$$

which means that the induced Lie cobracket on $\mathfrak{m}\big/\mathfrak{m}^2 = \mathfrak{g}^*$ is just $\delta^{\,c}$, q.e.d. $\square$

**Example 3.4.7.** Let $G := GL_n(\Bbbk)$ and $\mathfrak{g} := \mathfrak{gl}_n(\Bbbk)$ . We consider the QUEA $U_\hbar(\mathfrak{g}) = U_\hbar\big(\mathfrak{gl}_n(\Bbbk)\big)$ and the QFSHA $F_\hbar[[G]] = F_\hbar\big[\big[GL_n(\Bbbk)\big]\big]$ introduced in Example 3.2.3. Letting $\mathfrak{b}^-$ and $\mathfrak{b}^+$ be the Borel Lie subalgebras in $\mathfrak{g}$ of lower triangular and upper triangular matrices, respectively, the subalgebra $U_\hbar(\mathfrak{b}^-)$ of $U_\hbar(\mathfrak{g})$ generated by the $F_i$'s and the $\varGamma_k$'s is a QUEA for $\mathfrak{b}^-$ , while the subalgebra $U_\hbar(\mathfrak{b}^+)$ generated by the $E_i$'s and the $\varGamma_k$'s is a QUEA for $\mathfrak{b}^+$ — both being also Hopf subalgebras of $U_\hbar(\mathfrak{g})$ . Dually, the QFSHA $F_\hbar[[B^-]] = U_\hbar(\mathfrak{b}^-)^*$ identifies with the Hopf quotient of $F_\hbar[[G]]$ obtained by modding out the ideal generated by the $x_{i,j}$'s with $i < j$ ; similarly, $F_\hbar[[B^+]] = U_\hbar(\mathfrak{b}^+)^*$ identifies with the Hopf quotient of $F_\hbar[[G]]$ obtained by modding out the ideal generated by the $x_{i,j}$'s with $i > j$ . Therefore, from the presentation of $F_\hbar[[G]]$ in Example 3.2.3 one deduces the following presentations for these quotient algebras: $F_\hbar[[B^-]]$ is generated by the entries of the "lower triangular $q$–matrix" $\big(x_{i,j}^-\big)_{i=1,\dots,n;}^{j=1,\dots,n;}$ with $x_{i,j}^- := x_{i,j}$ for all $i \geq j$ and $x_{i,j}^- := 0$ for all $i < j$ , and similarly $F_\hbar[[B^+]]$ is generated by the entries of the "upper triangular $q$–matrix" $\big(x_{i,j}^+\big)_{i=1,\dots,n;}^{j=1,\dots,n;}$ with $x_{i,j}^+ := x_{i,j}$ for $i \leq j$ and $x_{i,j}^+ := 0$ for $i > j$ .

Now we consider a new group $G$ which is "double version" of $GL_n(\Bbbk)$ , in that it is a Manin double of $B^-$ and $B^+$ ; its tangent Lie algebra $\mathfrak{g}$ then is the Manin double of $\mathfrak{b}^-$ and $\mathfrak{b}^+$ ; in particular, $G = B^- \times B^+$ as algebraic varieties (not as groups), with $B^-$ and $B^+$ being embedded as subgroups, whereas $\mathfrak{g} = \mathfrak{b}^- \oplus \mathfrak{b}^+$ as vector spaces, with $\mathfrak{b}^-$ and $\mathfrak{b}^+$ being embedded as Lie subalgebras (this case is explained in detail in [GaGa2] for $SL_n(\Bbbk)$ , and $GL_n(\Bbbk)$ is just a very slight variation).

For these new $G$ and $\mathfrak{g}$, a QUEA $U_\hbar(\mathfrak{g})$ is defined as follows: it is the unital, associative, $\hbar$–adically complete $\Bbbk[[\hbar]]$–algebra with generators

$$F_1\,,F_2\,,\dots,F_{n-1}\,,\varGamma_1^-\,,\varGamma_2^-\,,\dots,\varGamma_{n-1}^-\,,\varGamma_n^-\,,\varGamma_1^+\,,\varGamma_2^+\,,\dots,\varGamma_{n-1}^+\,,\varGamma_n^+\,,E_1\,,E_2\,,\dots,E_{n-1}$$

and relations

$$\begin{gathered}
\big[\varGamma_k^\pm\,,\varGamma_\ell^\pm\big]=0\;,\quad \big[\varGamma_k^\pm\,,F_j\big]=-\delta_{k,j}\,F_j\;,\quad \big[\varGamma_k^\pm\,,E_j\big]=+\delta_{k,j}\,E_j\;,\quad \big[\varGamma_k^\pm\,,\varGamma_\ell^\mp\big]=0\\
\big[E_i\,,F_j\big]\;=\;\delta_{i,j}\,\frac{e^{\hbar\,(\varGamma_i^+-\varGamma_{i+1}^+)}-e^{\hbar\,(\varGamma_{i+1}^--\varGamma_i^-)}}{e^{+\hbar}-e^{-\hbar}}\\
\big[E_i\,,E_j\big]\;=\;0\;,\qquad \big[F_i\,,F_j\big]\;=\;0 \qquad\qquad \forall\;i\,,j\,:\,|i-j|>1\\
E_i^2\,E_j-\big(q+q^{-1}\big)\,E_i\,E_j\,E_i+E_j\,E_i^2\;=\;0 \qquad\qquad \forall\;i\,,j\,:\,|i-j|=1\\
F_i^2\,F_j-\big(q+q^{-1}\big)\,F_i\,F_j\,F_i+F_j\,F_i^2\;=\;0 \qquad\qquad \forall\;i\,,j\,:\,|i-j|=1\,.
\end{gathered}$$

where $[X\,,Y]:=X\,Y-Y\,X$ again. The Hopf structure then is given by

$$\begin{aligned}
\Delta(F_i)\;&=\;F_i\otimes e^{\hbar\,(\varGamma_{i+1}^--\varGamma_i^-)}+1\otimes F_i\;, & S(F_i)\;&=\;-F_i\,e^{\hbar\,(\varGamma_i^--\varGamma_{i+1}^-)}\;, & \epsilon(F_i)\;&=\;0\\
\Delta\big(\varGamma_k^\pm\big)\;&=\;\varGamma_k^\pm\otimes 1+1\otimes\varGamma_k^\pm\;, & S\big(\varGamma_k^\pm\big)\;&=\;-\varGamma_k^\pm\;, & \epsilon\big(\varGamma_k^\pm\big)\;&=\;0\\
\Delta(E_i)\;&=\;E_i\otimes 1+e^{\hbar\,(\varGamma_i^+-\varGamma_{i+1}^+)}\otimes E_i\;, & S(E_i)\;&=\;-e^{\hbar\,(\varGamma_{i+1}^+-\varGamma_i^+)}\,E_i\;, & \epsilon(E_i)\;&=\;0
\end{aligned}$$

In fact, this $U_\hbar(\mathfrak{g})$ can be realized as a *quantum double* of $U_\hbar(\mathfrak{b}^-)$ and $U_\hbar(\mathfrak{b}^+)$ : in particular, $U_\hbar(\mathfrak{g})=U_\hbar(\mathfrak{b}^-)\,\widehat{\otimes}\,U_\hbar(\mathfrak{b}^+)$ as coalgebras. Dually, the latter implies that for the QFSHA $F_\hbar[[G]]:=U_\hbar(\mathfrak{g})^*$ we have an identification *as algebras*

$$F_\hbar[[G]]\;=\;\big(U_\hbar(\mathfrak{b}^-)\,\widehat{\otimes}\,U_\hbar(\mathfrak{b}^+)\big)^*\;=\;U_\hbar(\mathfrak{b}^-)^*\,\widetilde{\otimes}\,U_\hbar(\mathfrak{b}^+)^*\;=\;F_\hbar[[B^-]]\,\widetilde{\otimes}\,F_\hbar[[B^+]]$$

Exploiting the presentations above for $F_\hbar[[B^-]]$ and $F_\hbar[[B^+]]$, we find a presentation for $F_\hbar[[G]]$ as the algebra generated by the entries of the "$q$–matrix in blocks"

$$\begin{pmatrix} X^+ & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & X^- \end{pmatrix} \qquad \text{with}\quad X^\pm:=\big(x_{i,j}^\pm\big)_{i=1,\dots,n;}^{j=1,\dots,n;} \quad \text{as defined above (triangular).}$$

Moreover, explicit identifications $F_\hbar[[G]]=U_\hbar(\mathfrak{g})^*$ and $U_\hbar(\mathfrak{g})=F_\hbar[[G]]^\star$ can be encoded in the Hopf pairing $\langle\;,\;\rangle:F_\hbar[[G]]\times U_\hbar(\mathfrak{g})\longrightarrow\Bbbk[[\hbar]]$ given on generators by

$$\begin{gathered}
\Big\langle x_{i,j}^-\,,\textstyle\prod_{k=1}^n\big(\varGamma_k^+\big)^{g_k}\Big\rangle=0=\big\langle x_{i,j}^-\,,E_t\big\rangle\quad,\quad \big\langle x_{i,j}^+\,,F_t\big\rangle=0=\Big\langle x_{i,j}^+\,,\textstyle\prod_{k=1}^n\big(\varGamma_k^-\big)^{g_k}\Big\rangle\\
\big\langle x_{i,j}^-\,,F_t\big\rangle=\delta_{i,j+1}\,\delta_{t,j}\qquad,\qquad \big\langle x_{i,j}^+\,,E_t\big\rangle=\delta_{i+1,j}\,\delta_{i,t}\\
\Big\langle x_{i,j}^+\,,\textstyle\prod_{k=1}^n\big(\varGamma_k^+\big)^{g_k}\Big\rangle\;=\;\delta_{i,j}\,(1-\delta_{g_i,0})\prod_{k\neq i}\delta_{g_k,0}\;=\;\Big\langle x_{i,j}^-\,,\textstyle\prod_{k=1}^n\big(\varGamma_k^-\big)^{g_k}\Big\rangle
\end{gathered}\tag{3.62}$$

In particular, from the first line in (3.62) note that if $\underline{\varGamma}_1$ and $\underline{\varGamma}_2$ are two monomials in the $\varGamma_k^\pm$'s, then for all $i=1,\dots,n$ we have

$$\Big\langle x_{i,i}^\pm\,,\underline{\varGamma}_1\cdot\underline{\varGamma}_2\Big\rangle\;=\;\Big\langle x_{i,i}^\pm\,,\underline{\varGamma}_1\Big\rangle\cdot\Big\langle x_{i,i}^\pm\,,\underline{\varGamma}_2\Big\rangle \tag{3.63}$$

Thanks to Proposition 4.1.2 later on, any quasi–twist for $F_\hbar[[G]]$ can be seen as a quasi–2–cocycle for $U_\hbar(\mathfrak{g})=F_\hbar[[G]]^\star$. Now, some examples of the latter were introduced in Example 3.3.8 above for a large class of QUEA, including that for $\mathfrak{g}=\mathfrak{sl}_n(\Bbbk)$. The same procedure can be applied to the present case, which is a slight variation of that case applied to $\mathfrak{gl}_n(\Bbbk)$ instead of $\mathfrak{sl}_n(\Bbbk)$, as follows.

Let $\mathfrak{h}$ be the $\Bbbk[[\hbar]]$–span of $B_\varGamma:=\big\{\,\varGamma_k^+,\varGamma_k^-\,\big|\,k=1,\dots,n\,\big\}$. Then fix an antisymmetric, $\Bbbk[[\hbar]]$–bilinear map $\chi:\mathfrak{h}\times\mathfrak{h}\longrightarrow\Bbbk[[\hbar]]$ whose matrix of values on pairs of elements in the $\Bbbk[[\hbar]]$–basis $B_\varGamma$ is $X=\Big(\chi_{k,t}^{\varepsilon,\eta}=\chi\big(\varGamma_k^\varepsilon\,,\varGamma_t^\eta\big)\Big)_{k,t=1,\dots,n;}^{\varepsilon,\,\eta\in\{+,-\}}\in\mathfrak{so}_{2n}\big(\Bbbk[[\hbar]]\big)$.

Any such map $\chi$ also induces uniquely an antisymmetric, $\Bbbk[[\hbar]]$–bilinear map $\tilde{\chi}_U$ on $U_\hbar(\mathfrak{h}) = \widehat{S}_{\Bbbk[[\hbar]]}(\mathfrak{h}) := \bigoplus_{n\in\mathbb{N}} \widehat{S^n_{\Bbbk[[\hbar]]}(\mathfrak{h})}$ with values in $\Bbbk[[\hbar]]$, by setting

$$\begin{aligned} &\tilde{\chi}_U(z,1) := \epsilon(z) =: \tilde{\chi}_U(1,z) && \forall\ z \in \widehat{S}_{\Bbbk[[\hbar]]}(\mathfrak{h}) \\ &\tilde{\chi}_U(x,y) := \chi(x,y) && \forall\ x,y \in S^1_{\Bbbk[[\hbar]]}(\mathfrak{h}) \\ &\tilde{\chi}_U(x,y) := 0 \qquad \forall\ x \in S^r_{\Bbbk}(\mathfrak{h})\,,\ y \in S^s_{\Bbbk}(\mathfrak{h}) && :\ r,s \geq 1\,,\ r+s > 2 \end{aligned} \tag{3.64}$$

Then we define the map $\chi_U := e^{\hbar^{-1}2^{-1}\tilde{\chi}_U} = \sum_{m\geq 0} \hbar^{-m}\, \tilde{\chi}_U^{\,*\,m} \Big/ 2^m\, m!$ from $U_\hbar(\mathfrak{h}) \widehat{\otimes} U_\hbar(\mathfrak{h})$ to $\Bbbk((\hbar))$, which, like in [GaGa2, Lemma 5.2.2], is a *quasi–2–cocycle* for $U_\hbar(\mathfrak{h})$.

Assume now that $\chi$ satisfies the additional constraint $\chi(S_i\,, -\,) = 0 = \chi(\,-\,, S_i)$ for all $i \in I := \{1,\dots,n-1\}$, where $S_i := 2^{-1}\big(\varGamma_i^+ - \varGamma_{i+1}^+ + \varGamma_i^- - \varGamma_{i+1}^-\big)$ for all $i \in I$; note that this is equivalent to requiring $\chi^{+,\eta}_{i,t} - \chi^{+,\eta}_{i+1,t} + \chi^{-,\eta}_{i,t} - \chi^{-,\eta}_{i+1,t} = 0$ for all $i \in I$, all $t = 1,\dots,n-1$ and all $\eta \in \{+,-\}$. Then $\chi$ induces a unique $\Bbbk[[\hbar]]$–bilinear map $\overline{\chi} : \overline{\mathfrak{h}} \times \overline{\mathfrak{h}} \longrightarrow \Bbbk[[\hbar]]$, where $\overline{\mathfrak{h}} := \mathfrak{h}\big/\mathfrak{s}$ with $\mathfrak{s} := Span_{\Bbbk[[\hbar]]}\big(\{\, S_i \,\}_{i=1,\dots,n-1;}\big)$, given by $\overline{\chi}\,\big(H' + \mathfrak{s}\,, H'' + \mathfrak{s}\,\big) := \chi\big(H', H''\big)$ for all $H', H'' \in \mathfrak{h}$.

Now repeat the above construction with $\overline{\mathfrak{h}}$ and $\overline{\chi}$ replacing $\mathfrak{h}$ and $\chi$: this yields a quasi–2–cocycle $\overline{\chi}_U$ for $U_\hbar\big(\,\overline{\mathfrak{h}}\,\big)$. But now the additional assumption on $\chi$ implies that there exists a *Hopf algebra epimorphism* $\pi : U_\hbar(\mathfrak{g}) \relbar\joinrel\twoheadrightarrow \widehat{S}_{\Bbbk[[\hbar]]}\big(\,\overline{\mathfrak{h}}\,\big) \cong U_\hbar\big(\,\overline{\mathfrak{h}}\,\big)$ given by $\pi(E_i) := 0$, $\pi(F_i) := 0$ — for $i = 1,\dots,n-1$ — and $\pi(T) := (T + \mathfrak{s}) \in \overline{\mathfrak{h}} \subseteq U_\hbar\big(\,\overline{\mathfrak{h}}\,\big)$ — for $T \in \mathfrak{h}$. Finally, we set $\sigma_\chi := \overline{\chi}_U \circ (\pi \times \pi)$, which is a well-defined quasi–2–cocycle for $U_\hbar(\mathfrak{g})$, again in the sense of Definition 3.3.3. Note that

$$\sigma_\chi := \overline{\chi}_U \circ (\pi \times \pi) = \exp\big(\hbar^{-1}\, 2^{-1}\, \widetilde{\overline{\chi}}_U\big) \circ (\pi \times \pi) = \exp\big(\hbar^{-1}\, 2^{-1}\, \widetilde{\overline{\chi}}_U \circ (\pi \times \pi)\big)$$

Now let us re-think the quasi–2–cocycle $\sigma_\chi$ for $U_\hbar(\mathfrak{g})$ as a quasi-twist for $F_\hbar[[G]]$. First of all, comparing (3.63) and (3.64) we deduce that the form $\widetilde{\overline{\chi}}_U \circ (\pi \times \pi)$ in $\Big(U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}\Big)^*$ identifies with $\varPhi_\chi := \sum_{k,\,t=1}^{n} \sum_{\varepsilon,\,\eta\in\{+,-\}} \chi^{\varepsilon,\eta}_{k,t}\, y^{\varepsilon}_{k,k} \otimes y^{\eta}_{t,t}$ in $F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$, where $y^{\varsigma}_{\ell,\ell} := \log\big(x^{\varsigma}_{\ell,\ell}\big)$ is a well-defined element in $F_\hbar[[G]]$. Then, exponentiating yields

$$\mathcal{F}_\chi \;:=\; \sigma_\chi \;=\; \exp\big(\hbar^{-1}\, 2^{-1}\, \widetilde{\overline{\chi}}_U \circ (\pi \times \pi)\big) \;=\; \exp\big(\hbar^{-1}\, 2^{-1}\, \varPhi_\chi\big)$$

which is exactly the quasi-twist of $F_\hbar[[G]]$ we were looking for.

We can also check directly that this $\mathcal{F}_\chi$ is a quasi-twist. We see this in the simplest case, when $n = 2$; the other cases are quite similar, but require more calculations.

We need to compute the coproduct of the $x^{\varepsilon}_{t,t}$'s in $F_\hbar[[G]]$, which is defined (by construction) by the condition $\big\langle\, \Delta\big(x^{\varepsilon}_{t,t}\big), A \otimes Z \,\big\rangle = \big\langle\, x^{\varepsilon}_{t,t}\,, A \cdot Z \,\big\rangle$ for all $A\,, Z \in U_\hbar(\mathfrak{g})$; since $U_\hbar(\mathfrak{g})$ admits the PBW-type basis

$$\mathcal{B} \;:=\; \Big\{\, F^f\, \big(\varGamma_1^-\big)^{g_1^-}\, \big(\varGamma_2^-\big)^{g_2^-}\, \big(\varGamma_1^+\big)^{g_1^+}\, \big(\varGamma_2^+\big)^{g_2^+}\, E^e \;\Big|\; f, g_1^-, g_2^-, g_1^+, g_2^+, e \in \mathbb{N} \,\Big\}$$

we can replace $A$ and $Z$ with any two PBW monomials from $\mathcal{B}$. Now, let us say that a PBW monomial of the form $\mathcal{M} = F^f\, \big(\varGamma_1^-\big)^{g_1^-}\, \big(\varGamma_2^-\big)^{g_2^-}\, \big(\varGamma_1^+\big)^{g_1^+}\, \big(\varGamma_2^-\big)^{g_2^+}\, E^e$ belongs to the root space $(e-f)\,\alpha$. Then root/weight considerations easily show that $\big\langle\, x^{\varepsilon}_{t,t}\,, \mathcal{M} \,\big\rangle \neq 0$ only for PBW monomials $\mathcal{M}$ in the root space 0, i.e. such that $e = f$. A straightforward computation gives

$$E^e \cdot F^f \;=\; \sum_{s=0}^{e\wedge f} \big([s]_q!\big)^2 \begin{bmatrix} e \\ s \end{bmatrix}_q \begin{bmatrix} f \\ s \end{bmatrix}_q F^{f-s}\, K_{e,f}(s)\, E^{e-s}$$

where $q := \exp(\hbar)$, $[r]_q := \dfrac{q^r - q^{-r}}{q - q^{-1}}$, $[m]_q! := \prod_{r=1}^{m} [r]_q$, $\begin{bmatrix} \ell \\ s \end{bmatrix}_q := \dfrac{[\ell]_q!}{[s]_q!\,[\ell - s]_q!}$ and

$$K_{e,f}(s) := \prod\nolimits_{r=1}^{s} \frac{q^{2s-e-f+1-r} K_+^{+1} - q^{r-1-2s+e+f} K_-^{-1}}{q^r - q^{-r}}$$

with $K_+^{+1} := 1 \otimes \exp\big(+\hbar\,\big(\Gamma_i^+ - \Gamma_{i+1}^+\big)\big)^{\pm 1}$ and $K_-^{-1} := \exp\big(-\hbar\,\big(\Gamma_i^- - \Gamma_{i+1}^-\big)\big)^{\pm 1} \otimes 1$. Then the product of two PBW monomials expands into

$$\mathcal{M}' \cdot \mathcal{M}'' \;=\; \sum\nolimits_{s=0}^{e' \wedge f''} \big([s]_q!\big)^2 \begin{bmatrix} e \\ s \end{bmatrix}_q \begin{bmatrix} f \\ s \end{bmatrix}_q F^{f'} \underline{\Gamma}^{\,\dot{g}} F^{f''-s} K_{e',f''}(s)\, E^{e'-s} \underline{\Gamma}^{\,\ddot{g}} E^{e''} \;=$$

$$=\; \sum\nolimits_{s=0}^{e' \wedge f''} q^{\mathcal{E}(\dot{g},\, f'',\, s\,, e',\, \ddot{g})} \big([s]_q!\big)^2 \begin{bmatrix} e' \\ s \end{bmatrix}_q \begin{bmatrix} f'' \\ s \end{bmatrix}_q F^{f'+f''-s} \underline{\Gamma}^{\,\dot{g}} K_{e',f''}(s)\, \underline{\Gamma}^{\,\ddot{g}} E^{e'-s+e''}$$

where $\underline{\Gamma}^{\,\dot{g}} := \big(\Gamma_1^-\big)^{\dot{g}_1^-} \big(\Gamma_2^-\big)^{\dot{g}_2^-} \big(\Gamma_1^+\big)^{\dot{g}_1^+} \big(\Gamma_2^+\big)^{\dot{g}_2^+}$, $\underline{\Gamma}^{\,\ddot{g}} := \big(\Gamma_1^-\big)^{\ddot{g}_1^-} \big(\Gamma_2^-\big)^{\ddot{g}_2^-} \big(\Gamma_1^+\big)^{\ddot{g}_1^+} \big(\Gamma_2^+\big)^{\ddot{g}_2^+}$ while $\mathcal{E}\big(\dot{g}\,, f''\!, s\,, e'\!, \ddot{g}\big) \in \mathbb{Z}$ is a suitable exponent. When we expand w.r.t. $\mathcal{B}$, the part given by a linear combination of PBW monomials in the root space 0 is

$$\big(\mathcal{M}' \cdot \mathcal{M}''\big)_0 \;=\; \delta_{f',0}\, \delta_{f'',e'}\, \delta_{e'',0}\, q^{\mathcal{E}(\dot{g},\, e',\, e',\, e',\, \ddot{g})} \big([e']_q!\big)^2\, \underline{\Gamma}^{\,\dot{g}} K_{e',e'}\big(e'\big)\, \underline{\Gamma}^{\,\ddot{g}}$$

Eventually, tiding everything up we find

$$\Big\langle \Delta\big(x_{t,t}^{\varepsilon}\big), \mathcal{M}' \otimes \mathcal{M}'' \Big\rangle \;=\; \delta_{f',0}\, \delta_{f'',e'}\, \delta_{e'',0}\, q^{\mathcal{E}(\dot{g},\, e',\, e',\, e',\, \ddot{g})}\, [e']_q!^2 \Big\langle x_{t,t}^{\varepsilon}\,, \underline{\Gamma}^{\,\dot{g}} K_{e',e'}\big(e'\big)\, \underline{\Gamma}^{\,\ddot{g}} \Big\rangle$$

Now, a similar analysis yields (notation being obvious, hopefully)

$$\Big\langle\, x_{t,t}^{\varepsilon}\,, \underline{\Gamma}^{\,\dot{g}} K_{e',e'}\big(e'\big)\, \underline{\Gamma}^{\,\ddot{g}} \Big\rangle \;=\; \Big\langle\, x_{t,t}^{\varepsilon}\,, \underline{\Gamma} \Big\rangle^{\dot{g}} \cdot \Big\langle\, x_{t,t}^{\varepsilon}\,, K_{e',e'}\big(e'\big) \Big\rangle \cdot \Big\langle\, x_{t,t}^{\varepsilon}\,, \underline{\Gamma} \Big\rangle^{\ddot{g}}$$

and finally direct computation gives $\Big\langle\, x_{t,t}^{\varepsilon}\,, K_{e',e'}\big(e'\big) \Big\rangle \;=\; \delta_{e',0}$, which in turn follows from $\chi(S_i\,, -\,) \;=\; 0 \;=\; \chi(\,-\,, S_i)$ for $i = 1, \dots, n-1$. Therefore, we get $\Big\langle \Delta\big(x_{t,t}^{\varepsilon}\big)\,, \mathcal{M}' \otimes \mathcal{M}'' \Big\rangle \neq 0$ only when $\big(f', e'\big) = (0,0) = \big(f'', e''\big)$, and then

$$\Big\langle \Delta\big(x_{t,t}^{\varepsilon}\big)\,, \mathcal{M}' \otimes \mathcal{M}'' \Big\rangle \;=\; \Big\langle \Delta\big(x_{t,t}^{\varepsilon}\big)\,, \underline{\Gamma}^{\,\dot{g}} \otimes \underline{\Gamma}^{\,\ddot{g}} \Big\rangle \;=\; \Big\langle x_{t,t}^{\varepsilon}\,, \mathcal{M}' \Big\rangle \cdot \Big\langle x_{t,t}^{\varepsilon}\,, \mathcal{M}'' \Big\rangle$$

which in short means $\Delta\big(x_{t,t}^{\varepsilon}\big) \;=\; x_{t,t}^{\varepsilon} \otimes x_{t,t}^{\varepsilon}$, i.e. $x_{t,t}^{\varepsilon}$ is group-like. Therefore, for $y_{t,t}^{\varepsilon} := \log\big(x_{t,t}^{\varepsilon}\big)$ instead we have $\Delta\big(y_{t,t}^{\varepsilon}\big) \;=\; y_{t,t}^{\varepsilon} \otimes 1 + 1 \otimes y_{t,t}^{\varepsilon}$, i.e. $y_{t,t}^{\varepsilon}$ is primitive.

Eventually, as all the $y_{t,t}^{\varepsilon}$'s are primitive, a trivial computation shows that $\mathcal{F}_\chi$ does obey condition (2.7), hence it is indeed a quasi-twist, as claimed.

### 3.5. **Duality issues.**

Deformations by twist or by 2–cocycle, both for Lie bialgebras and for Hopf algebras, are dual to each other, see Proposition 2.1.7 and Proposition 2.2.7. This prompts us to compare these two procedures *before* and *after* specialization.

A first result is the following, whose proof is trivial — just track the whole construction of both $c_{\mathcal{F}}$ and $\zeta_\sigma$, and compare the outcomes.

**Proposition 3.5.1.** *Let $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$ be a QUEA and a QFSHA which are dual to each other, that is $F_\hbar[[G]] = U_\hbar(\mathfrak{g})^*$ and $U_\hbar(\mathfrak{g}) = F_\hbar[[G]]^\star$. Then let $\mathcal{F}$ be a twist for $U_\hbar(\mathfrak{g})$, and $\sigma$ be a 2–cocycle for $F_\hbar[[G]]$. Assume that both $\mathcal{F}$ and $\sigma$ are trivial modulo $\hbar$, so that there exists a corresponding twist $c_{\mathcal{F}}$ for $\mathfrak{g}$ (induced by $\mathcal{F}$ via Theorem 3.1.2) and a corresponding 2–cocycle $\zeta_\sigma$ for $\mathfrak{g}^*$ (induced by $\sigma$ via Theorem 3.2.1). Finally, we identify twists for $U_\hbar(\mathfrak{g})$ and 2–cocycles for $F_\hbar[[G]]$ via Proposition 2.2.7, and similarly twists for $\mathfrak{g}$ and 2–cocycles for $\mathfrak{g}^*$ via Proposition 2.1.7.*

*Then the following holds: if $\mathcal{F} = \sigma$, then $c_{\mathcal{F}} = \zeta_\sigma$.* □

A similar result holds for deformations by quasi–2–cocycles and by quasi-twists. Indeed, let us first notice that, if $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$ are a QUEA and a QFSHA in duality — i.e., $F_\hbar[[G]] = U_\hbar(\mathfrak{g})^*$ and $U_\hbar(\mathfrak{g}) = F_\hbar[[G]]^\star$ — then *any quasi–2–cocycle for $U_\hbar(\mathfrak{g})$ is a quasi-twist for $F_\hbar[[G]]$, and viceversa*, cf. Proposition 4.1.2 later on.

Once this is settled, next result (which mirrors Proposition 3.5.1 above) holds too, whose proof again follows by direct comparison of the two deformation procedures (just tracking the whole construction of $\gamma_\sigma$ and $c_{\mathcal{F}}$, and comparing the outcomes):

**Proposition 3.5.2.** *Let $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$ be a QUEA and a QFSHA which are in duality, that is $F_\hbar[[G]] = U_\hbar(\mathfrak{g})^*$ and $U_\hbar(\mathfrak{g}) = F_\hbar[[G]]^\star$. Then let $\sigma$ be a quasi–2–cocycle for $U_\hbar(\mathfrak{g})$, and $\mathcal{F}$ be a quasi-twist for $F_\hbar[[G]]$. Let $\gamma_\sigma$ be the 2–cocycle for $\mathfrak{g}$ induced by $\sigma$ via Theorem 3.3.7, and let $c_{\mathcal{F}}$ be the twist for $\mathfrak{g}^*$ induced by $\mathcal{F}$ via Theorem 3.4.6. Finally, we identify quasi–2–cocycles for $U_\hbar(\mathfrak{g})$ and quasi-twists for $F_\hbar[[G]]$ as mentioned above, and similarly we identify twists for $\mathfrak{g}$ and 2–cocycles for $\mathfrak{g}^*$ via Proposition 2.1.7. Then the following holds: if $\sigma = \mathcal{F}$, then $\gamma_\sigma = c_{\mathcal{F}}$.* $\square$

## 4. Deformations vs. QDP

In this section we investigate how the deformation procedures interact when we interchange QUEA's and QFSHA's via Drinfeld's functors, as in Theorem 2.4.2.

### 4.1. **Some auxiliary results.**

We begin with a key observation: our "quasi-2–cocycles" for any QUEA and "quasi-twists" for any QFSHA are actually *standard* 2–cocycles and twists, respectively, for the QFSHA and for the QUEA that are associated with the original quantum group via Drinfeld's functors from the QDP. Here is the precise result:

**Lemma 4.1.1.**

*(a) Let $U_\hbar(\mathfrak{g})$ be a QUEA, and let $U_\hbar(\mathfrak{g})'$ be its associated QFSHA following Theorem 2.4.2. Let $\sigma := \exp_*\big(\hbar^{-1}\chi\big)$ be a quasi-2-cocycle for $U_\hbar(\mathfrak{g})$ as in Definition 3.3.3. Then the restriction $\sigma\big|_{U_\hbar(\mathfrak{g})'\times U_\hbar(\mathfrak{g})'}$ of $\sigma$ to $U_\hbar(\mathfrak{g})' \times U_\hbar(\mathfrak{g})'$ is a well-defined, $\Bbbk[[\hbar]]$–valued bilinear form on $U_\hbar(\mathfrak{g})'$, of the form $\sigma' = \exp_*\big(\hbar^{+1}\chi'\big)$ with $\chi' := \big(\hbar^{-2}\chi\big)\big|_{U_\hbar(\mathfrak{g})'\times U_\hbar(\mathfrak{g})'}$, and this $\sigma' := \exp_*\big(\hbar^{+1}\chi'\big)$ is a 2-cocycle for $U_\hbar(\mathfrak{g})'$.*

*(b) Let $F_\hbar[[G]]$ be a QFSHA, and let $F_\hbar[[G]]^\vee$ be its associated QUEA following Theorem 2.4.2. Let $\mathcal{F} := \exp\big(\hbar^{-1}\phi\big)$ be a quasi-twist for $F_\hbar[[G]]$ as in Definition 3.4.3. Then $\mathcal{F} := \exp\big(\hbar^{-1}\phi\big) = \exp\big(\hbar^{+1}\phi^\vee\big)$ with $\phi^\vee := \hbar^{-2}\phi \in \big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\, 2}$, and, in these terms, $\mathcal{F}^\vee := \exp\big(\hbar^{+1}\phi^\vee\big)$ is a twist for $F_\hbar[[G]]^\vee$.*

*Proof. (a)* We retain notation from the proof of Lemma 3.3.2, and we proceed along the same lines. Thus we set $U_\hbar := U_\hbar(\mathfrak{g})$ and $J_\hbar := Ker\big(U_\hbar\big)$, and we write

$$\hat{z} := \epsilon(z)\,, \quad z^+ := z - \epsilon(z) = z - \hat{z} \,\in\, J_\hbar\,, \quad \text{hence} \quad z = z^+ + \hat{z} \qquad \forall\; z \in U_\hbar$$

We already saw that $\chi(u,v) \,=\, \chi\big(u^+,v^+\big)$ for all $u, v \in U_\hbar$, and then we have

$$\sigma(a\,,b\,) \;=\; \textstyle\sum_{n\geq 0} \hbar^{-n} \prod_{i=1}^n \chi\big(a_{(i)}^+, b_{(i)}^+\big) \Big/ n! \tag{4.1}$$

for any $a\,,b \in U_\hbar$, where $\otimes_{i=1}^n a_{(i)}^+ = \delta_n(a)$ and $\otimes_{i=1}^n b_{(i)}^+ = \delta_n(b)$.

Now, restricting to $U_\hbar'$ we get that $a', b' \in U_\hbar'$ yields $\delta_n\big(a'\big), \delta_n\big(b'\big) \in \hbar^n\, U_\hbar^{\widehat{\otimes}\, n}$; also, in the sequel we can clearly assume $\epsilon\big(a'\big) = 0 = \epsilon\big(b'\big)$. Then we get

$$\textstyle\prod_{i=1}^n \chi'\big(a'^{+}_{(i)}, b'^{+}_{(i)}\big) \;=\; \prod_{i=1}^n \hbar^{-2} \chi\big(a'^{+}_{(i)}, b'^{+}_{(i)}\big) \;\in\; \hbar^{-2n}\, \hbar^{2n}\, \Bbbk[[\hbar]] \;=\; \Bbbk[[\hbar]] \qquad \forall\; n \in \mathbb{N}_+$$

whence, like in (4.1), we get $\sigma'\big(a', b'\big) = \sum_{n \geq 0} \hbar^{+n} \prod_{i=1}^{n} \chi'\big(a'^{+}_{(i)}, b'^{+}_{(i)}\big) \Big/ n! \in \Bbbk[[\hbar]]$ for all $a', b' \in U'_\hbar$ , which proves the claim.

*(b)* This follows directly from Definition 3.4.3. $\square$

As a direct consequence, we have the following significant result:

**Proposition 4.1.2.** *Let $U_\hbar$ be a QUEA and $F_\hbar$ be a QFSHA that are dual to each other, i.e. such that $F_\hbar = (U_\hbar)^*$ and $U_\hbar = (F_\hbar)^\star$ . Then:*

*(a) $\sigma$ is a quasi-2-cocycle for $U_\hbar \iff \sigma$ is a quasi-twist for $F_\hbar$ ;*

*(b) $\mathcal{F}$ is a quasi-twist for $F_\hbar \iff \mathcal{F}$ is a quasi-2-cocycle for $U_\hbar$ .*

*Proof.* The proof follows directly from the very definitions of quasi-2–cocycle and quasi-twist, along with the observation that $F_\hbar = (U_\hbar)^*$ and $U_\hbar = (F_\hbar)^\star$ imply $F_\hbar^\vee = (U'_\hbar)^\star$ and $U'_\hbar = \big(F_\hbar^\vee\big)^*$ , by (2.18). $\square$

## 4.2. **Drinfeld's functors and "quasi-deformations".**

In this subsection we analyze the interaction between the process of "quasi-deformation" and the action of a Drinfeld's functor on some quantum group.

**4.2.1. Deformations by quasi-twist under $F_\hbar[[G]] \mapsto F_\hbar[[G]]^\vee$ .** We look now what happens with deformations by quasi-twist for a QFSHA when the latter is acted upon by the functor $(\ )^\vee$ which associates with it a QUEA. Here is our result:

**Theorem 4.2.2.**
*Let $F_\hbar[[G]]$ be a QFSHA. Let $\mathcal{F} = \exp\big(\hbar^{-1}\phi\big)$ be a quasi-twist for $F_\hbar[[G]]$ , with $\phi \in F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$ (cf. Definition 3.4.3). Set $\phi^\vee := \hbar^{-1} \log(\mathcal{F}) = \hbar^{-2}\phi$ , $\phi_a := \phi - \phi_{2,1}$ and $\phi^\vee_a := \phi^\vee - \phi^\vee_{2,1}$ . Then we have:*

*(a) $\phi$ is antisymmetric, i.e. $\phi_{2,1} = -\phi$ , iff $\mathcal{F}$ is orthogonal, i.e. $\mathcal{F}_{2,1} = \mathcal{F}^{-1}$ , iff $\phi^\vee$ is antisymmetric, i.e. $\phi^\vee_{2,1} = -\phi^\vee$ ;*

*(b) $\mathcal{F} = \exp\big(\hbar\,\phi^\vee\big)$ is a twist element for the QUEA $U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee$ .*

*(c) Let $c$ be the antisymmetric twist of $\mathfrak{g}^*$ corresponding to $\mathcal{F}$ as provided by Theorem 3.4.6, and let $c^\vee$ be the similar twist provided by Theorem 3.1.2 along with claim (b) above. Then $c = c^\vee$ .*

*Proof.* *(a)* This follows directly by construction.

*(b)* This is granted by Lemma 4.1.1*(b)*.

*(c)* This follows by a careful — yet entirely straightforward — check, just tracking both constructions involved (of $c$ and of $c^\vee$ alike). $\square$

**4.2.3. Deformations by quasi-2-cocycle under $U_\hbar(\mathfrak{g}) \mapsto U_\hbar(\mathfrak{g})'$ .** Given a QUEA, we can apply on it Drinfeld's functor $(\ )'$ ; we now see what happens when a deformation by quasi-2-cocycle is performed. Our result reads as follows:

**Theorem 4.2.4.** *Let $U_\hbar(\mathfrak{g})$ be a QUEA. Let $\sigma = \exp_*\big(\hbar^{-1}\chi\big)$ be a quasi-2-cocycle for $U_\hbar(\mathfrak{g})$ , with $\chi \in \big(U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}\big)^*$ (cf. Definition 3.3.3). Set $\chi' := \hbar^{-1} \log_*(\sigma) = \hbar^{-2}\chi$ , $\chi_a := \chi - \chi_{2,1}$ and $\chi'_a := \chi' - \chi'_{2,1}$ . Then we have:*

*(a) $\chi$ is antisymmetric, i.e. $\chi_{2,1} = -\chi$ , iff $\sigma$ is orthogonal, i.e. $\sigma_{2,1} = \sigma^{-1}$ , iff $\chi'$ is antisymmetric, i.e. $\chi'_{2,1} = -\chi'$ ;*

*(b) $\sigma = \exp_*\big(\hbar\,\chi'\big)$ is a 2-cocycle for the QFSHA $F_\hbar\big[[G^*]\big] := U_\hbar(\mathfrak{g})'$ .*

*(c) Let $\gamma$ be the antisymmetric 2-cocycle of $\mathfrak{g}$ corresponding to $\sigma$ as provided by Theorem 3.3.5, and let $\gamma'$ be the similar 2-cocycle provided by Theorem 3.2.1 along with claim (b) above — using the identification $\mathfrak{g}^{**} = \mathfrak{g}$. Then $\gamma = \gamma'$.*

*Proof. (a)* This follows directly by construction.

*(b)* This is true because of Lemma 4.1.1*(a)*.

*(c)* Much like for Theorem 4.2.2*(c)*, this follows again by a straightforward check, just carefully tracking both constructions involved (of $\gamma$ and of $\gamma$ alike). □

**Remark 4.2.5.** Recall that the notions of twist and that of 2-cocycle are dual to each other (cf. Proposition 2.2.7), and the same holds for those of quasi-twist and quasi-2-cocycle (cf. §3.5). Moreover, Drinfeld's functors are also dual to each other, in the sense of (2.18). Taking all this into account, it turns out easily that Theorem 4.2.2 and Theorem 4.2.4 above are also "dual to each other", in that either one of these two statements can be deduced from the other by a duality argument.

## 4.3. Drinfeld's functors and (standard) deformations.

We analyze now the interaction between the process of deformation — in the standard sense — and the action of a Drinfeld's functor on some quantum group.

**4.3.1. Deformations by twist under $U_\hbar(\mathfrak{g}) \mapsto U_\hbar(\mathfrak{g})'$.** As a first step, we look what happens for deformations by twist of a QUEA when the latter is acted upon by the functor $(\ )'$ which associates with it a QFSHA. It turns out that we find a relevant result when we make the stronger assumption that the given twist is in fact a *(quantum) $R$–matrix twist*, as in Definition 2.2.1*(d)*. Here is our result:

**Theorem 4.3.2.** *Let $U_\hbar(\mathfrak{g})$ be a QUEA, and let $\mathcal{F}$ be a twist for $U_\hbar(\mathfrak{g})$ s.t. $\phi \equiv 1$ $\big(\mathrm{mod}\ \hbar\, U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}\big)$; then $\phi := \hbar^{-1}\log(\mathcal{F}) \in U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}$, and $\mathcal{F} = \exp(\hbar\,\phi)$. Set also $\phi' := \hbar^{-2}\log(\mathcal{F}) = \hbar^{-2}\phi$, $\phi_a := \phi - \phi_{2,1}$ and $\phi'_a := \phi' - \phi'_{2,1}$. Assume in addition that $\mathcal{F}$ is indeed a (quantum) $R$–matrix twist, as in Definition 2.2.1(d). Then:*

*(a) $\phi$ is antisymmetric, i.e. $\phi_{2,1} = -\phi$, iff $\mathcal{F}$ is orthogonal, i.e. $\mathcal{F}_{2,1} = \mathcal{F}^{-1}$, iff $\phi'$ is antisymmetric, i.e. $\phi'_{2,1} = -\phi'$;*

*(b) $\mathcal{F} = \exp\big(\hbar^{-1}\phi'\big)$ is a quasi-twist for the QFSHA $F_\hbar\big[\big[G^*\big]\big] := U_\hbar(\mathfrak{g})'$.*

*(c) Let $c$ be the antisymmetric twist of $\mathfrak{g}$ corresponding to $\mathcal{F}$ as provided by Theorem 3.1.2, and let $c'$ be the similar twist provided by Theorem 3.4.6 along with claim (b) above — using the identification $\mathfrak{g}^{**} = \mathfrak{g}$. Then $c = c'$.*

*Proof. (a)* This follows directly by construction.

*(b)* This is proved in [EH, Theorem 0.1]. Note that the overall assumption there is that $\mathcal{R}$ be an $R$–matrix, in the standard sense — so that $U_\hbar(\mathfrak{g})$ is quasitriangular. *Nevertheless, all the arguments used there to prove the main result only apply the defining properties of an "$R$–matrix" in the sense of Definition 2.2.1, namely* (2.8) *and the right-hand side of* (2.7)*; the assumption* (2.10)*, instead, is never used.* Therefore, the same arguments, and the whole proof, used in [EH] to prove Theorem 0.1 actually do prove also the present statement, that is actually stronger.

*(c)* Here again, the proof follows from a straightforward, careful checking procedure, keeping track of both constructions involved (of $c$ and of $c'$ alike), much like for Theorem 4.2.2*(c)* and for Theorem 4.2.4*(c)*. □

**4.3.3. Deformations by 2-cocycle under $F_\hbar[[G]] \mapsto F_\hbar[[G]]^\vee$.** As a second step, we look what happens to deformations of a QFSHA by 2-cocycle when we apply

Drinfeld's functor $(\ )^\vee$ . Here again, we get a relevant result under the stronger assumption that the given 2-cocycle is in fact a *(quantum) $\varrho$–comatrix 2-cocycle*, as in Definition 2.2.4*(d)*. Our result reads as follows:

**Theorem 4.3.4.** *Let $F_\hbar[[G]]$ be any QFSHA, and let $\sigma$ be a 2-cocycle for $F_\hbar[[G]]$ such that $\sigma \equiv 1 \left( \bmod \hbar \left( F_\hbar[[G]]^{\widetilde{\otimes}\, 2} \right)^\star \right)$ ; then $\varsigma := \hbar^{-1} \log(\sigma) \in \left( F_\hbar[[G]]^{\widetilde{\otimes}\, 2} \right)^\star$ , and $\sigma = \exp\!\big(\hbar\, \varsigma\big)$ . Set also $\varsigma^\vee := \hbar \log(\sigma) = \hbar^2 \varsigma$ , $\varsigma_a := \varsigma - \varsigma_{2,1}$ and $\varsigma_a^\vee := \varsigma^\vee - \varsigma_{2,1}^\vee$ . Assume in addition that $\sigma$ is a (quantum) $\varrho$–comatrix 2-cocycle, as in Definition 2.2.4(d). Then the following holds true:*

*(a) $\varsigma$ is antisymmetric, i.e. $\varsigma_{2,1} = -\varsigma$ , iff $\sigma$ is orthogonal, i.e. $\sigma_{2,1} = \sigma^{-1}$ , iff $\varsigma^\vee$ is antisymmetric, i.e. $\varsigma_{2,1}^\vee = -\varsigma^\vee$ ;*

*(b) $\sigma = \exp\!\big(\hbar^{-1} \varsigma^\vee\big)$ is a quasi-2-cocycle for the QUEA $U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee$ ;*

*(c) Let $\gamma$ be the antisymmetric 2-cocycle of $\mathfrak{g}^*$ corresponding to $\sigma$ as provided by Theorem 3.2.1, and let $\gamma^\vee$ be the similar 2-cocycle provided by 3.3.5 along with claim (b) above. Then $\gamma = \gamma^\vee$ .*

*Proof. (a)* This is obvious, by standard identities for formal exponentials.

*(b)* This claim is the dual to Theorem 4.3.2*(b)*, so it follows from that one via a duality argument — involving the results in §4.1, in particular Proposition 4.1.2.

*(c)* Once more, as in previous cases, the claim follows from direct checking, keeping track of the two involved 2-cocycles — $\gamma$ and of $\gamma^\vee$ — were constructed. □

**Remark 4.3.5.** Much like as we did in Remark 4.2.5, we notice here as well that — by the same reasons as before — Theorem 4.3.2 and Theorem 4.3.4 above are once more "dual to each other", in that either one of these two statements can be deduced from the other by a duality argument.

## 5. Morphisms in the "(co)quasitriangular" case

In this section we focus onto *$R$–matrices* and *$\varrho$–comatrices*. We investigate what happens with *$R$–matrices* and *$\varrho$–comatrices* w.r.t. the QDP, and then we consider the standard constructions of morphisms between a Hopf algebra $H$ and its dual coming from an $R$–matrix or a $\varrho$–comatrix.

### 5.1. $R$–matrices and $\varrho$–comatrices w.r.t. QDP: quasi–(co)matrices.

In next two results, we explain how $R$–matrices and $\rho$–comatrices "behave well" with respect to Drinfeld's functors and the Quantum Duality Principle. In fact, this leads us to introduce the notions of *"quasi–$R$–matrix"* and of *"quasi–$\rho$–comatrix"*, which are straight analogue of the notions of "quasi–twist"and of "quasi–2–cocycle".

We begin introducing some more bare definitions:

**Definition 5.1.1.**

*(a)* Let $F_\hbar[[G]]$ be a QFSHA. We call *"quasi–$R$–matrix"* of $F_\hbar[[G]]$ any $R$–matrix $\mathcal{R}$ for $U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee$ such that $\mathcal{R} \equiv 1^{\otimes 2} \bmod \Big( \hbar\, F_\hbar[[G]]^\vee \,\widehat{\otimes}\, F_\hbar[[G]]^\vee \Big)$ .

*(b)* Let $U_\hbar(\mathfrak{g})$ be a QUEA. We call *"quasi–$\rho$–comatrix"* of $U_\hbar(\mathfrak{g})$ any $\varrho$–comatrix $\rho$ for $F_\hbar[[G^*]] := U_\hbar(\mathfrak{g})'$ such that $\rho \equiv \epsilon^{\otimes 2} \bmod \Big( \hbar \left( U_\hbar(\mathfrak{g})' \,\widetilde{\otimes}\, U_\hbar(\mathfrak{g})' \right)^\star \Big)$ .

**Remark 5.1.2.** In the same spirit of Proposition 2.2.7 and (2.18), it is clear that the notions of "quasi–$R$–matrix" and of "quasi–$\varrho$–comatrix" are dual to each other.

**Observations 5.1.3.** *(a)* With assumptions as in Definition 5.1.1*(a)* above, let $\mathcal{R}$ be any quasi–$R$–matrix for a QFSHA $F_\hbar[[G]]$: since $\mathcal{R} \equiv 1^{\otimes 2} \mod \hbar$, we can write $\mathcal{R}$ in the form $\mathcal{R} = \exp\big(\hbar^{+1}\,\theta\big)$ for some $\theta \in F_\hbar[[G]]^\vee \,\widehat{\otimes}\, F_\hbar[[G]]^\vee$.

Similarly, if $\rho$ is any quasi–$\varrho$–comatrix for a QUEA $U_\hbar(\mathfrak{g})$, then we can write it in the form $\rho = \exp_*\big(\hbar^{+1}\,\varsigma\big)$ for some $\varsigma \in \big(U_\hbar(\mathfrak{g})' \,\widetilde{\otimes}\, U_\hbar(\mathfrak{g})'\big)^\star$.

*(b)* Note that in the very definition of "quasi–$R$–matrix", resp. of "quasi–$\varrho$–comatrix", we assume a condition which is quite close, yet *weaker*, than the one demanded for the definition of "quasi-twist", resp. of "quasi-2–cocycle", in Definition 3.3.3, resp. in Definition 3.4.3. In fact, our choice for these definitions about $R$–matrices and $\varrho$–comatrices is motivated by Proposition 5.1.4 below, which eventually implies that *when the two setups overlap, the stronger condition for twists/2–cocycles actually holds true* — cf. Theorem 4.3.2 and Theorem 4.3.4.

The key result about quasi–$R$–matrices and quasi–$\varrho$–comatrices is the following:

**Proposition 5.1.4.**

*(a) Let $U_\hbar(\mathfrak{g})$ be a QUEA, and let $U_\hbar(\mathfrak{g})'$ be the QFSHA associated to it by the Quantum Duality Principle, as in Theorem 2.4.2. Then for any $R$–matrix of $U_\hbar(\mathfrak{g})$ of the form $\mathcal{R} = \exp\big(\hbar\,\theta\big)$, with $\theta = \hbar^{-1}\log(\mathcal{R}) \in U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}$, we have*

$$\vartheta \;:=\; \hbar^2\,\theta \;=\; \hbar^{+1}\log(\mathcal{R}) \;\in\; \big(U_\hbar(\mathfrak{g})'\big)^{\widetilde{\otimes}\, 2}$$

*(b) Let $F_\hbar[[G]]$ be a QFSHA, and let $F_\hbar[[G]]^\vee$ be the QUEA associated to it by the Quantum Duality Principle, as in Theorem 2.4.2. Then for any $\varrho$–comatrix of $F_\hbar[[G]]$ of the form $\rho = \exp_*\big(\hbar\,\varsigma\big)$, with $\varsigma = \hbar^{-1}\log_*(\rho) \in \Big(F_\hbar[[G]]^{\widehat{\otimes}\, 2}\Big)^*$, we have*

$$\zeta \;:=\; \hbar^2\,\varsigma \;=\; \hbar^{+1}\log_*(\rho) \;\in\; \Big(\big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\, 2}\Big)^*$$

*Proof.* *(a)* This is proved in [EH, Theorem 0.1]. Indeed, the overall assumption there is that $\mathcal{R}$ be an $R$–matrix, in the standard sense — so that $U_\hbar(\mathfrak{g})$ is quasitriangular. *Nevertheless, all the arguments used there to prove the main result only apply the defining properties of an "$R$–matrix" in the sense of Definition 2.2.1, namely* (2.8) *and the right-hand side of* (2.7)*; the assumption* (2.10)*, instead, is never used.* Therefore, the same arguments, and the whole proof, used in [EH] to prove Theorem 0.1 actually do prove also the present, stronger statement.

*(b)* This follows from claim *(a)*, by duality, using the duality relation (2.18), the fact that $F_\hbar[[G]]^\vee$ is a QUEA when $F_\hbar[[G]]$ is a QFSHA, and Proposition 2.2.7. $\square$

The previous result has the following, important consequence:

**Theorem 5.1.5.** *Let $U_\hbar(\mathfrak{g})$ be a QUEA and $F_\hbar[[G]]$ be a QFSHA; let $F_\hbar[[G^*]] := U_\hbar(\mathfrak{g})'$ be the QFSHA and $U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee$ be the QUEA provided by the Quantum Duality Principle, as in Theorem 2.4.2. Then the following holds:*

*(a) every $R$–matrix for $U_\hbar(\mathfrak{g})$ which is congruent to $1^{\otimes 2}$ modulo $\hbar$ is a quasi–$R$–matrix for $U_\hbar(\mathfrak{g})'$;*

*(b) every $\varrho$–comatrix for $F_\hbar[[G]]$ which is congruent to $\epsilon^{\otimes 2}$ modulo $\hbar$ is a quasi–$\varrho$–comatrix for $F_\hbar[[G]]^\vee$;*

*(c) every $\varrho$–comatrix for $U_\hbar(\mathfrak{g})'$ which is congruent to $\epsilon^{\otimes 2}$ modulo $\hbar$ is a quasi–$\varrho$–comatrix for $U_\hbar(\mathfrak{g})$ itself;*

*(d) every $R$–matrix for $F_\hbar[[G]]^\vee$ which is congruent to $1^{\otimes 2}$ modulo $\hbar$ is a quasi–$R$–matrix for $F_\hbar[[G]]$ itself.*

*Proof.* Claims *(a)* and *(b)* follow directly from Proposition 5.1.4, claims *(a)* and *(b)*, respectively. Now claim *(c)* follows from claim *(b)* applied to the QFSHA $F_\hbar\big[[G^*]\big] := U_\hbar(\mathfrak{g})'$, since $F_\hbar\big[[G^*]\big]^\vee = \big(U_\hbar(\mathfrak{g})'\big)^\vee = U_\hbar(\mathfrak{g})$ by Theorem 2.4.2*(a)*. Similarly, claim *(d)* follows from *(a)* applied to the QUEA $U_\hbar\big(\mathfrak{g}^*\big) := F_\hbar[[G]]^\vee$, since $U_\hbar\big(\mathfrak{g}^*\big)' = \big(F_\hbar[[G]]^\vee\big)' = F_\hbar[[G]]$ by Theorem 2.4.2*(a)* again. $\square$

## 5.2. **Morphisms from $R$–matrices and $\varrho$–comatrices.**

We shall now explore what happens if we apply the general constructions leading to Proposition 2.2.9, resp. to Proposition 2.2.10, is (tentatively) applied to a QUEA, resp. a QFSHA, as the Hopf algebra $H$ to start with. To begin with, we check that Proposition 2.2.9 still makes sense when $H := U_\hbar(\mathfrak{g})$ is a QUEA:

**Proposition 5.2.1.** *Let $U_\hbar(\mathfrak{g})$ be a QUEA, and $F_\hbar[[G]] := U_\hbar(\mathfrak{g})^*$ be its dual QFSHA, as in §2.3.4. Let $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ (in Sweedler's notation) be an R–matrix for $U_\hbar(\mathfrak{g})$. Then there exist two morphisms of topological Hopf algebras*

$$\overleftarrow{\Phi}_{\mathcal{R}} : F_\hbar[[G]] := U_\hbar(\mathfrak{g})^* \longrightarrow U_\hbar(\mathfrak{g})^{\mathrm{cop}} \;, \qquad \eta \mapsto \overleftarrow{\Phi}_{\mathcal{R}}(\eta) := \eta(\mathcal{R}_1)\,\mathcal{R}_2$$
$$\overrightarrow{\Phi}_{\mathcal{R}} : F_\hbar[[G]] := U_\hbar(\mathfrak{g})^* \longrightarrow U_\hbar(\mathfrak{g})^{\mathrm{op}} \;, \qquad \eta \mapsto \overrightarrow{\Phi}_{\mathcal{R}}(\eta) := \mathcal{R}_1\,\eta(\mathcal{R}_2)$$

*Proof.* This is straightforward (see [GaGa3] for details). $\square$

Dually, Proposition 2.2.10 still makes sense when $H := F_\hbar[[G]]$ is a QFSHA:

**Proposition 5.2.2.** *Let $F_\hbar[[G]]$ be a QFSHA, and $U_\hbar(\mathfrak{g}) := F_\hbar[[G]]^\star$ be its dual QUEA, as in §2.3.4. Let $\rho$ be a $\varrho$–comatrix for $F_\hbar[[G]]$. Then there exist two morphisms of topological Hopf algebras*

$$\overleftarrow{\Psi}_\rho : F_\hbar[[G]] \longrightarrow \big(F_\hbar[[G]]^\star\big)^{\mathrm{cop}} = U_\hbar(\mathfrak{g})^{\mathrm{cop}} \;, \qquad \ell \mapsto \overleftarrow{\Psi}_\rho(\ell) := \rho(\ell\,,-)$$
$$\overrightarrow{\Psi}_\rho : F_\hbar[[G]] \longrightarrow \big(F_\hbar[[G]]^\star\big)^{\mathrm{op}} = U_\hbar(\mathfrak{g})^{\mathrm{op}} \;, \qquad \ell \mapsto \overrightarrow{\Psi}_\rho(\ell) := \rho(-,\ell\,)$$

*Proof.* This is straightforward again (cf. [GaGa3] for details). $\square$

We shall now show that both previous results can be refined, eventually yielding morphisms that connect quantum groups of the *same* nature, namely both QFSHA'a in one case and both QUEA's in the other case.

**Theorem 5.2.3.** *Let $U_\hbar(\mathfrak{g})$ be a QUEA, let $F_\hbar[[G]] := U_\hbar(\mathfrak{g})^*$ be its dual QFSHA, as in §2.3.4, and let $F_\hbar[[G^*]] := U_\hbar(\mathfrak{g})'$, resp. $U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee$, be the QFSHA, resp. the QUEA, introduced in §2.4. Let $\mathcal{R} = \mathcal{R}^i \otimes \mathcal{R}_i$ (sum, possibly infinite, over repeated indices) be an R–matrix for $U_\hbar(\mathfrak{g})$, which is congruent to $1^{\otimes 2}$ modulo $\hbar$.*

*Then, for the two morphisms $F_\hbar[[G]] \xrightarrow{\overleftarrow{\Phi}_{\mathcal{R}}} U_\hbar(\mathfrak{g})^{\mathrm{cop}}$ and $F_\hbar[[G]] \xrightarrow{\overrightarrow{\Phi}_{\mathcal{R}}} U_\hbar(\mathfrak{g})^{\mathrm{op}}$ in Proposition 5.2.1, the following holds:*

*(a) they take values inside $U_\hbar(\mathfrak{g})'$, and so they corestrict to morphisms*

$$\overleftarrow{\Phi}'_{\mathcal{R}} : F_\hbar[[G]] \longrightarrow \big(U_\hbar(\mathfrak{g})'\big)^{\mathrm{cop}} = F_\hbar[[G^*]]^{\mathrm{cop}}$$

*and*

$$\overrightarrow{\Phi}'_{\mathcal{R}} : F_\hbar[[G]] \longrightarrow \big(U_\hbar(\mathfrak{g})'\big)^{\mathrm{op}} = F_\hbar[[G^*]]^{\mathrm{op}}$$

*between QFSHA's for mutually dual (formal) Poisson groups;*

*(b) they uniquely extend to* $F_\hbar[[G]]^\vee = \big(U_\hbar(\mathfrak{g})^*\big)^\vee$ *, i.e. they extend to morphisms*

$$\overleftarrow{\Phi}_{\mathcal{R}}^\vee : U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee \longrightarrow U_\hbar(\mathfrak{g})^{\mathrm{cop}}$$

*and*

$$\overrightarrow{\Phi}_{\mathcal{R}}^\vee : U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee \longrightarrow U_\hbar(\mathfrak{g})^{\mathrm{op}}$$

*between QUEA's for mutually dual Lie bialgebras.*

*Proof. (a)* Recall that $Ker\big(\epsilon_{U_\hbar(\mathfrak{g})'}\big) =: J_{U_\hbar(\mathfrak{g})'} \subseteq \big(J_{U_\hbar(\mathfrak{g})'} + \Bbbk[[\hbar]]\, 1_{U_\hbar(\mathfrak{g})'}\big) =: I_{U_\hbar(\mathfrak{g})'}$ , and $U_\hbar(\mathfrak{g})$ is a topological Hopf algebra with respect to the $I_{U_\hbar(\mathfrak{g})'}$–adic topology.

Recall also that, by Proposition 5.1.4*(a)*, we can write $\mathcal{R} := \exp\big(\hbar^{-1}\vartheta\big)$ with $\vartheta \in U_\hbar(\mathfrak{g})' \,\widetilde{\otimes}\, U_\hbar(\mathfrak{g})'$ ; we write the latter as $\vartheta = \vartheta^i \otimes \vartheta_i$ (sum over repeated indices), where $\vartheta^i, \vartheta_i \in J_{U_\hbar(\mathfrak{g})'}$ . Then $\hbar^{-1}\vartheta = \hbar^{-1}\vartheta^i \otimes \vartheta_i = \big(\hbar^{-1}\vartheta^i\big) \otimes \vartheta_i = \theta^i \otimes \vartheta_i$ with $\theta^i := \hbar^{-1}\vartheta^i \in \hbar^{-1} J_{U_\hbar(\mathfrak{g})'} \subseteq J_{U_\hbar(\mathfrak{g})} := Ker\big(\epsilon_{U_\hbar(\mathfrak{g})}\big)$ , where the latter inclusion follows by the basic properties of $U_\hbar(\mathfrak{g})'$, cf. [Ga1]. Now writing $\big(\theta^i \otimes \vartheta_i\big)^n = \theta^j_{[n]} \otimes \vartheta_{[n],j}$ for each $n \in \mathbb{N}$ , we have in particular $\theta^j_{[n]} \in J^n_{U_\hbar(\mathfrak{g})}$ and $\vartheta_{[n],j} \in J^n_{U_\hbar(\mathfrak{g})'}$ , for every $n \in \mathbb{N}$ . When we expand $\mathcal{R}$ , by all this we find

$$\mathcal{R} \;=\; \exp\big(\hbar^{-1}\vartheta\big) \;=\; \exp\big(\theta^i \otimes \vartheta_i\big) \;=\; \sum_{n \geq 0} \frac{1}{n!}\, \big(\theta^i \otimes \vartheta_i\big)^n \;=\; \sum_{n \geq 0} \frac{1}{n!}\, \big(\theta^j_{[n]} \otimes \vartheta_{[n],j}\big)$$

Thus for $\eta \in F_\hbar[[G]] := U_\hbar(\mathfrak{g})^*$ we get $\overleftarrow{\Phi}_{\mathcal{R}}(\eta) \;=\; \eta\big(\mathcal{R}^s\big)\, \mathcal{R}_s \;=\; \sum_{n \geq 0} \frac{1}{n!}\, \eta\big(\theta^j_{[n]}\big)\, \vartheta_{[n],j}$ , which describes a well-defined element (a convergent series, in the relevant topology!) of $\big(U_\hbar(\mathfrak{g})'\big)^{\mathrm{cop}}$ — equal to $U_\hbar(\mathfrak{g})'$ as a $\Bbbk[[\hbar]]$–module — exactly because $\vartheta_{[n],j} \in J^n_{U_\hbar(\mathfrak{g})'}$ for each $n \in \mathbb{N}$ . So $\Phi^{\leftarrow}_{\mathcal{R}}$ corestricts to $\big(U_\hbar(\mathfrak{g})'\big)^{\mathrm{cop}} = F_\hbar[[G^*]]^{\mathrm{cop}}$ as claimed, q.e.d.

The proof for $\overrightarrow{\Phi}_{\mathcal{R}}$ goes exactly the same, just switching left and right.

*(b)* We begin acting as in the proof of *(a)* above, but switching the roles of left and right hand sides. Namely, we write $\hbar\,\vartheta = \hbar\,\big(\vartheta^i \otimes \vartheta_i\big) = \theta^i \otimes \vartheta_i$ where $\theta_i := \hbar^{-1}\vartheta_i \in \hbar^{-1} J_{U_\hbar(\mathfrak{g})'} \subseteq J_{U_\hbar(\mathfrak{g})} := Ker\big(\epsilon_{U_\hbar(\mathfrak{g})}\big)$ , and also $\big(\vartheta^i \otimes \theta_i\big)^n = \vartheta^j_{[n]} \otimes \theta_{[n],j}$ , with $\vartheta^j_{[n]} \in J^n_{U_\hbar(\mathfrak{g})'}$ and $\theta_{[n],j} \in J^n_{U_\hbar(\mathfrak{g})}$ , for all $n \in \mathbb{N}$ . Then expanding $\mathcal{R}$ yields

$$\mathcal{R} \;=\; \exp\big(\hbar^{-1}\vartheta\big) \;=\; \exp\big(\vartheta^i \otimes \theta_i\big) \;=\; \sum_{n \geq 0} \frac{1}{n!}\, \big(\vartheta^i \otimes \theta_i\big)^n \;=\; \sum_{n \geq 0} \frac{1}{n!}\, \big(\vartheta^j_{[n]} \otimes \theta_{[n],j}\big)$$

hence for every $\mu \in U_\hbar(\mathfrak{g})^*$ we have

$$\overleftarrow{\Phi}_{\mathcal{R}}(\mu) \;:=\; \mu\big(\mathcal{R}^s\big)\, \mathcal{R}_s \;=\; \sum_{n \geq 0} \frac{1}{n!}\, \mu\big(\vartheta^j_{[n]}\big)\, \theta_{[n],j} \tag{5.1}$$

Now, recall that $\big(U_\hbar(\mathfrak{g})^*\big)^\vee = \big(U_\hbar(\mathfrak{g})'\big)^\star$ , by (2.18). Then we consider the formula (5.1) for any $\mu \in \big(U_\hbar(\mathfrak{g})^*\big)^\vee = \big(U_\hbar(\mathfrak{g})'\big)^\star$ — which contains $U_\hbar(\mathfrak{g})^*$ . As all coefficients $\mu\big(\vartheta'_n\big)$ belong to $\Bbbk[[\hbar]]$ , every partial sum in the right-hand side formal series is a well-defined element in $U_\hbar(\mathfrak{g})^{\mathrm{cop}}$ — equal to $U_\hbar(\mathfrak{g})$ as a $\Bbbk[[\hbar]]$–module. In addition, since $\vartheta^j_{[n]} \in J^n_{U_\hbar(\mathfrak{g})'} \subseteq I^n_{U_\hbar(\mathfrak{g})'}$ — for each $n \in \mathbb{N}$ — and $\mu : U_\hbar(\mathfrak{g})' \longrightarrow \Bbbk[[\hbar]]$ is *continuous* (with respect to the $I^n_{U_\hbar(\mathfrak{g})'}$–adic topology on the left and the $\hbar$–adic topology on the right), for every $s \in \mathbb{N}$ there exist $n_s$ such that $\mu\big(\vartheta^j_{[n]}\big) \in \hbar^{n_s}\Bbbk[[\hbar]]$ for all $n \geq n_s$ . This ensures that the formal series in (5.1) is actually convergent in the $\hbar$–adic topology of $U_\hbar(\mathfrak{g})$ , thus describing a well-defined element in $U_\hbar(\mathfrak{g})$ . Letting $\mu$ range freely inside $\big(U_\hbar(\mathfrak{g})^*\big)^\vee$ , this proves that $\overleftarrow{\Phi}_{\mathcal{R}}$ does indeed extend from $U_\hbar(\mathfrak{g})^*$ to $\big(U_\hbar(\mathfrak{g})^*\big)^\vee = F_\hbar[[G^*]]^\vee =: U_\hbar(\mathfrak{g}^*)$ , q.e.d.

Switching left and right in the arguments above we get the proof for $\overrightarrow{\Phi}_{\mathcal{R}}$ too. $\square$

**Remark 5.2.4.** Claim *(a)* of Theorem 5.2.3 above appears also in [EK], §4.5.

In the dual framework, the parallel result holds true as well:

**Theorem 5.2.5.** *Let $F_\hbar[[G]]$ be a QFSHA, let $U_\hbar(\mathfrak{g}) := F_\hbar[[G]]^\star$ be its dual QUEA, as in §2.3.4, and let also $U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee$, resp. $F_\hbar[[G^*]] := U_\hbar(\mathfrak{g})'$, be the QUEA, resp. the QFSHA, introduced in §2.4. Let $\rho$ be a $\varrho$–comatrix for $F_\hbar[[G]]$, which is congruent to $\epsilon^{\otimes 2}$ modulo $\hbar$.*

*Then, for the two morphisms $F_\hbar[[G]] \xrightarrow{\overleftarrow{\Psi}_\rho} U_\hbar(\mathfrak{g})^{\mathrm{cop}}$ and $F_\hbar[[G]] \xrightarrow{\overrightarrow{\Psi}_\rho} U_\hbar(\mathfrak{g})^{\mathrm{op}}$ in Proposition 5.2.2, the following holds:*

*(a) they take values inside $U_\hbar(\mathfrak{g})'$, so they corestrict to morphisms*

$$\overleftarrow{\Psi}'_\rho : F_\hbar[[G]] \longrightarrow \big(U_\hbar(\mathfrak{g})'\big)^{\mathrm{cop}} = F_\hbar[[G^*]]^{cop}$$

*and*

$$\overrightarrow{\Psi}'_\rho : F_\hbar[[G]] \longrightarrow \big(U_\hbar(\mathfrak{g})'\big)^{\mathrm{op}} = F_\hbar[[G^*]]^{op}$$

*between QFSHA's for mutually dual (formal) Poisson groups.*

*(b) they uniquely extend to $U_\hbar(\mathfrak{g}^*) = F_\hbar[[G]]^\vee$, i.e. they extend to morphisms*

$$\overleftarrow{\Psi}^\vee_\rho : U_\hbar(\mathfrak{g}^*) = F_\hbar[[G]]^\vee \longrightarrow U_\hbar(\mathfrak{g})^{\mathrm{cop}}$$

*and*

$$\overrightarrow{\Psi}^\vee_\rho : U_\hbar(\mathfrak{g}^*) = F_\hbar[[G]]^\vee \longrightarrow U_\hbar(\mathfrak{g})^{\mathrm{op}}$$

*between QUEA's for mutually dual Lie bialgebras.*

*Proof. (a)* By the assumption $\rho \equiv \epsilon^{\otimes 2} \pmod{\hbar}$ and by Proposition 5.1.4*(b)*, we can write $\rho$ in the form $\rho = \exp_*\big(\hbar^{-1}\zeta\big)$ for some $\zeta \in \Big(\big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\, 2}\Big)^*$. Then

$$\zeta \in \Big(\big(F_\hbar[[G]]^\vee\big)^{\widehat{\otimes}\, 2}\Big)^* = \big(F_\hbar[[G]]^\vee \,\widehat{\otimes}\, F_\hbar[[G]]^\vee\big)^* = \big(F_\hbar[[G]]^\vee\big)^* \,\widetilde{\otimes}\, \big(F_\hbar[[G]]^\vee\big)^* =$$
$$= \big(F_\hbar[[G]]^\star\big)' \,\widetilde{\otimes}\, \big(F_\hbar[[G]]^\star\big)' = U_\hbar(\mathfrak{g})' \,\widetilde{\otimes}\, U_\hbar(\mathfrak{g})'$$

— thanks to (2.18) — hence $\hbar^{-1}\zeta \in \hbar^{-1}\, U_\hbar(\mathfrak{g}^*)' \,\widetilde{\otimes}\, U_\hbar(\mathfrak{g}^*)'$. Now, the right-hand side of (2.12) for $\sigma := \rho$ implies $\zeta(1, a) = 0$ for all $a \in F_\hbar[[G]]$, hence $\zeta(-, a) \in \mathit{Ker}\big(\epsilon_{U_\hbar(\mathfrak{g})'}\big)$ and so $\hbar^{-1}\zeta(-, a) \in \big(U_\hbar(\mathfrak{g})'\big)^\vee = U_\hbar(\mathfrak{g}) = F_\hbar[[G]]^\star$ for all $a \in F_\hbar[[G]]$. This implies that

$$\hbar^{-1}\zeta(-\,,-) \in \big(U_\hbar(\mathfrak{g})'\big)^\vee \otimes U_\hbar(\mathfrak{g})' = F_\hbar[[G]]^\star \otimes U_\hbar(\mathfrak{g})' \tag{5.2}$$

where hereafter we are being temporarily sloppy with the tensor product — we fix this later on. Clearly, (5.2) implies $\rho = \exp_*\big(\hbar^{-1}\zeta\big) \in F_\hbar[[G]]^\star \otimes U_\hbar(\mathfrak{g})'$ as well. Therefore we get at once $\Psi_{\overleftarrow{\rho}}(\ell) := \rho(\ell\,,-) \in U_\hbar(\mathfrak{g})'$ for all $\ell \in F_\hbar[[G]]$, q.e.d. This proves the claim about $\overleftarrow{\Psi}_\rho$, and that concerning $\overrightarrow{\Psi}_\rho$ is entirely similar.

It remains to "dot your $i$'s" about the tensor product in (5.2). In fact, *a priori* we have $\rho \in F_\hbar[[G]]^\star \,\widehat{\otimes}\, F_\hbar[[G]]^\star = U_\hbar(\mathfrak{g}) \,\widehat{\otimes}\, U_\hbar(\mathfrak{g})$, hence also

$$\hbar^{-1}\zeta \in F_\hbar[[G]]^\star \,\widehat{\otimes}\, F_\hbar[[G]]^\star = U_\hbar(\mathfrak{g}) \,\widehat{\otimes}\, U_\hbar(\mathfrak{g})$$

— where the (completed, topological) tensor product "$\widehat{\otimes}$" is considered. On the other hand, we have found that

$$\zeta \in \big(F_\hbar[[G]]^\star\big)' \,\widetilde{\otimes}\, \big(F_\hbar[[G]]^\star\big)' = U_\hbar(\mathfrak{g})' \,\widetilde{\otimes}\, U_\hbar(\mathfrak{g})'$$

— where the (completed, topological) tensor product "$\widetilde{\otimes}$" is used. Then the critical point is: what kind of tensor product "$\otimes$" is taken in (5.2)?

Instead of giving a direct answer to this question, we point out the following. First observe that $U_\hbar(\mathfrak{g})' \,\widetilde{\otimes}\, U_\hbar(\mathfrak{g})' = \big(U_\hbar(\mathfrak{g}) \,\widehat{\otimes}\, U_\hbar(\mathfrak{g})\big)'$ embeds into $U_\hbar(\mathfrak{g}) \,\widehat{\otimes}\, U_\hbar(\mathfrak{g})$.

Then, when $\hbar^{-1}\zeta \in U_\hbar(\mathfrak{g})\,\widehat{\otimes}\,U_\hbar(\mathfrak{g})$ is expanded into some series $\hbar^{-1}\zeta = \beta^i \otimes \beta_i$ (summing over repeated indices) with $\beta^i, \beta_i \in U_\hbar(\mathfrak{g})$ for all $i$, what we proved above is that we actually have $\beta_i \in U_\hbar(\mathfrak{g})' \; \big(\subseteq U_\hbar(\mathfrak{g})\big)$ for all indices $i$. This is what we loosely wrote as $\hbar^{-1}\zeta \in U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})' = F_\hbar[[G]]^\star \otimes U_\hbar(\mathfrak{g})'$ in (5.2) above.

*(b)* Acting as in part *(a)*, we find $\rho = \exp_*\big(\hbar^{-1}\zeta\big)$ with $\zeta \in \big(U_\hbar(\mathfrak{g})'\big)^{\widetilde{\otimes}\,2}$ and $\zeta \in \big(Ker\big(\epsilon_{U_\hbar(\mathfrak{g})}\big)\big)^{\widehat{\otimes}\,2}$ too, so $\zeta \in \big(Ker\big(\epsilon_{U_\hbar(\mathfrak{g})'}\big)\big)^{\widetilde{\otimes}\,2}$. Since $Ker\big(\epsilon_{U_\hbar(\mathfrak{g})'}\big) \subseteq \hbar\, Ker\big(\epsilon_{U_\hbar(\mathfrak{g})}\big)$, this implies that, expanding $\hbar^{-1}\zeta$ as a (convergent) series $\hbar^{-1}\zeta = \beta^i \otimes \beta_i$, we can assume $\beta^i \in U_\hbar(\mathfrak{g})'$. As $U_\hbar(\mathfrak{g})' = \big(F_\hbar[[G]]^\star\big)' = \big(F_\hbar[[G]]^\vee\big)^*$, we end up with

$$\hbar^{-1}\zeta = \beta^i \otimes \beta_i \in \big(F_\hbar[[G]]^\vee\big)^* \otimes U_\hbar(\mathfrak{g}) \tag{5.3}$$

where again the meaning of the tensor product "$\otimes$" considered in this formula (along with the corresponding convergence issues) is handled just as in part *(a)*. Finally, from (5.3) it follows at once that $\overleftarrow{\Psi}_\rho$ extends from $F_\hbar[[G]]$ to $F_\hbar[[G]]^\vee$ as claimed. This proves our statement for $\overleftarrow{\Psi}_\rho$, and the case of $\overrightarrow{\Psi}_\rho$ is entirely similar. $\square$

**5.2.6. Duality properties.** When we deal with a QUEA and a QFSHA which are dual to each other, it makes sense to compare the previous results. The outcome is that Proposition 2.2.12 turns to an enhanced version (with trivial proof), as follows:

**Theorem 5.2.7.** *Let $U_\hbar(\mathfrak{g})$ be a QUEA, $F_\hbar[[G]]$ a QFSHA, which are dual to each other, i.e. $F_\hbar[[G]] = U_\hbar(\mathfrak{g})^*$ and $U_\hbar(\mathfrak{g}) = F_\hbar[[G]]^\star$. Let $\mathcal{R} = \rho$ be an $R$–matrix for $U_\hbar(\mathfrak{g})$ and a $\varrho$–comatrix for $F_\hbar[[G]]$, which is trivial modulo $\hbar$, i.e. congruent to $1^{\otimes 2}$, resp. to $\epsilon^{\otimes 2}$, modulo $\hbar$. Then, for the morphisms in Proposition 5.2.1, Proposition 5.2.2, Theorem 5.2.3 and Theorem 5.2.5 we have the following identifications*

$$\overleftarrow{\Phi}_{\mathcal{R}} = \overleftarrow{\Psi}_\rho\,,\;\; \overleftarrow{\Phi}'_{\mathcal{R}} = \overleftarrow{\Psi}'_\rho\,,\;\; \overleftarrow{\Phi}^\vee_{\mathcal{R}} = \overleftarrow{\Psi}^\vee_\rho \quad \textit{and} \quad \overrightarrow{\Phi}_{\mathcal{R}} = \overrightarrow{\Psi}_\rho\,,\;\; \overrightarrow{\Phi}'_{\mathcal{R}} = \overrightarrow{\Psi}'_\rho\,,\;\; \overrightarrow{\Phi}^\vee_{\mathcal{R}} = \overrightarrow{\Psi}^\vee_\rho \quad \square$$

**5.2.8. Comparing morphisms (1).** Let us fix assumptions as in Theorem 5.2.3: $U_\hbar(\mathfrak{g})$ is a given QUEA, $F_\hbar[[G]]$ its dual QFSHA, and $\mathcal{R} = \mathcal{R}^s \otimes \mathcal{R}_s$ is a (quantum) $R$–matrix of $U_\hbar(\mathfrak{g})$. Then from Theorem 5.2.3 we have Hopf algebra morphisms

$$F_\hbar[[G]] \xrightarrow{\;\overleftarrow{\Phi}'_{\mathcal{R}}\;} F_\hbar[[G^*]]^{\mathrm{cop}} \quad , \qquad F_\hbar[[G]] \xrightarrow{\;\overrightarrow{\Phi}'_{\mathcal{R}}\;} F_\hbar[[G^*]]^{\mathrm{op}} \tag{5.4}$$

between QFSHA's for mutually dual (formal) Poisson groups, and

$$U_\hbar(\mathfrak{g}^*) \xrightarrow{\;\overleftarrow{\Phi}^\vee_{\mathcal{R}}\;} U_\hbar(\mathfrak{g})^{\mathrm{cop}} \quad , \qquad U_\hbar(\mathfrak{g}^*) \xrightarrow{\;\overrightarrow{\Phi}^\vee_{\mathcal{R}}\;} U_\hbar(\mathfrak{g})^{\mathrm{op}} \tag{5.5}$$

between QUEA's for mutually dual Lie bialgebras, which we re-write in the form

$$U_\hbar(\mathfrak{g}^*)^{\mathrm{cop}} \xrightarrow{\;\overleftarrow{\Phi}^\vee_{\mathcal{R}}\;} U_\hbar(\mathfrak{g}) \quad , \qquad U_\hbar(\mathfrak{g}^*)^{\mathrm{op}} \xrightarrow{\;\overrightarrow{\Phi}^\vee_{\mathcal{R}}\;} U_\hbar(\mathfrak{g}) \tag{5.6}$$

that is entirely equivalent. We now go and compare (5.4) and (5.6).

Recall that $F_\hbar[[G^*]] := U_\hbar(\mathfrak{g})'$ and $U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee$, which are in duality because $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$ are in duality (by construction) and we can apply (2.18). Then also $F_\hbar[[G^*]]^{\mathrm{cop}}$ and $U_\hbar(\mathfrak{g}^*)^{\mathrm{op}}$ are in duality, as well as $F_\hbar[[G^*]]^{\mathrm{op}}$ and $U_\hbar(\mathfrak{g}^*)^{\mathrm{cop}}$.

We are now ready to compare the morphisms in (5.4) with those in (5.6). Namely,

$$\begin{tikzcd} F_\hbar[[G]] \arrow[r, "\overleftarrow{\Phi}'_{\mathcal{R}}"] \arrow[d, squiggly, no head] & F_\hbar[[G^*]]^{\mathrm{cop}} \arrow[d, squiggly, no head] \\ U_\hbar(\mathfrak{g}) & U_\hbar(\mathfrak{g}^*)^{\mathrm{op}} \arrow[l, "\overrightarrow{\Phi}^\vee_{\mathcal{R}}"'] \end{tikzcd} \qquad \begin{tikzcd} F_\hbar[[G]] \arrow[r, "\overrightarrow{\Phi}'_{\mathcal{R}}"] \arrow[d, squiggly, no head] & F_\hbar[[G^*]]^{\mathrm{op}} \arrow[d, squiggly, no head] \\ U_\hbar(\mathfrak{g}) & U_\hbar(\mathfrak{g}^*)^{\mathrm{cop}} \arrow[l, "\overleftarrow{\Phi}^\vee_{\mathcal{R}}"'] \end{tikzcd} \tag{5.7}$$

are diagrams where the vertical, twisting lines denote a relationship of mutual (Hopf) duality. Next result tells us that the link between the morphisms on top row and those underneath is indeed "the best possible one":

**Theorem 5.2.9.** *The two morphisms in left-hand side, resp. in right-hand side, of* (5.7) *are adjoint to each other, that is for all* $\eta \in U_\hbar(\mathfrak{g}^*)$ *and* $f \in F_\hbar[[G]]$ *we have*

$$\Big\langle \overrightarrow{\Phi}_{\mathcal{R}}^{\vee}(\eta)\,, f \Big\rangle \;=\; \Big\langle \eta\,, \overleftarrow{\Phi}_{\mathcal{R}}'(f) \Big\rangle \qquad \textit{and} \qquad \Big\langle \overleftarrow{\Phi}_{\mathcal{R}}^{\vee}(\eta)\,, f \Big\rangle \;=\; \Big\langle \eta\,, \overrightarrow{\Phi}_{\mathcal{R}}'(f) \Big\rangle$$

*where by "*$\langle\ ,\ \rangle$*" we denote the pairing between any two Hopf algebras in duality.*

*Proof.* It is enough to prove half of the claim — the other one being entirely similar — say the right-hand side. Direct computation yields

$$\Big\langle \overleftarrow{\Phi}_{\mathcal{R}}^{\vee}(\eta)\,, f \Big\rangle \;=\; \Big\langle \big\langle \eta\,, \mathcal{R}^s \big\rangle\, \mathcal{R}_s\,, f \Big\rangle \;=\; \Big\langle \eta\,, \mathcal{R}^s \big\langle \mathcal{R}_s\,, f \big\rangle \Big\rangle \;=\; \Big\langle \eta\,, \overrightarrow{\Phi}_{\mathcal{R}}'(f) \Big\rangle$$

for all $\eta \in U_\hbar(\mathfrak{g}^*)$ and $f \in F_\hbar[[G]]$, hence we are done. □

As a second step, let now start with assumptions as in Theorem 5.2.5: $F_\hbar[[G]]$ is a given QFSHA, $U_\hbar(\mathfrak{g})$ its dual QUEA, and $\rho$ is a (quantum) $\varrho$–comatrix of $F_\hbar[[G]]$. Then Theorem 5.2.5 provides Hopf algebra morphisms

$$F_\hbar[[G]] \xrightarrow{\ \overleftarrow{\Psi}_\rho'\ } F_\hbar[[G^*]]^{\mathrm{cop}} \quad , \qquad F_\hbar[[G]] \xrightarrow{\ \overrightarrow{\Psi}_\rho'\ } F_\hbar[[G^*]]^{\mathrm{op}} \tag{5.8}$$

between QFSHA's for mutually dual (formal) Poisson groups, and

$$U_\hbar(\mathfrak{g}^*) \xrightarrow{\ \overleftarrow{\Psi}_\rho^{\vee}\ } U_\hbar(\mathfrak{g})^{\mathrm{cop}} \quad , \qquad U_\hbar(\mathfrak{g}^*) \xrightarrow{\ \overrightarrow{\Psi}_\rho^{\vee}\ } U_\hbar(\mathfrak{g})^{\mathrm{op}} \tag{5.9}$$

between QUEA's for mutually dual Lie bialgebras; we re-write the latter as

$$U_\hbar(\mathfrak{g}^*)^{\mathrm{cop}} \xrightarrow{\ \overleftarrow{\Psi}_\rho^{\vee}\ } U_\hbar(\mathfrak{g}) \quad , \qquad U_\hbar(\mathfrak{g}^*)^{\mathrm{op}} \xrightarrow{\ \overrightarrow{\Psi}_\rho^{\vee}\ } U_\hbar(\mathfrak{g}) \tag{5.10}$$

that is entirely equivalent. We now go and compare (5.8) and (5.10).

Acting as before (for the morphisms induced by an $R$–matrix), we find diagrams

$$\begin{tikzcd} F_\hbar[[G]] \arrow[r, "\overleftarrow{\Psi}_\rho'"] \arrow[d, squiggly, dash] & F_\hbar[[G^*]]^{\mathrm{cop}} \arrow[d, squiggly, dash] \\ U_\hbar(\mathfrak{g}) & U_\hbar(\mathfrak{g}^*)^{\mathrm{op}} \arrow[l, "\overrightarrow{\Psi}_\rho^{\vee}"] \end{tikzcd} \qquad \begin{tikzcd} F_\hbar[[G]] \arrow[r, "\overrightarrow{\Psi}_\rho'"] \arrow[d, squiggly, dash] & F_\hbar[[G^*]]^{\mathrm{op}} \arrow[d, squiggly, dash] \\ U_\hbar(\mathfrak{g}) & U_\hbar(\mathfrak{g}^*)^{\mathrm{cop}} \arrow[l, "\overleftarrow{\Psi}_\rho^{\vee}"] \end{tikzcd} \tag{5.11}$$

where the vertical, twisting lines denote a relationship of mutual (Hopf) duality. Again, the link between the morphisms on top row and those underneath turns out to be "the best possible one", as the following result claims:

**Theorem 5.2.10.** *The two morphisms in left-hand side, resp. in right-hand side, of* (5.7) *are adjoint to each other, that is for all* $\eta \in U_\hbar(\mathfrak{g}^*)$ *and* $f \in F_\hbar[[G]]$ *we have*

$$\Big\langle \overrightarrow{\Psi}_\rho^{\vee}(\eta)\,, f \Big\rangle \;=\; \Big\langle \eta\,, \overleftarrow{\Psi}_\rho'(f) \Big\rangle \qquad \textit{and} \qquad \Big\langle \overleftarrow{\Psi}_\rho^{\vee}(\eta)\,, f \Big\rangle \;=\; \Big\langle \eta\,, \overrightarrow{\Psi}_\rho'(f) \Big\rangle$$

*where by "*$\langle\ ,\ \rangle$*" we denote the pairing between any two Hopf algebras in duality.*

*Proof.* We prove just the left-hand side of the claim. Direct computation gives

$$\Big\langle \overrightarrow{\Psi}_\rho^{\vee}(\eta)\,, f \Big\rangle \;=\; \big\langle\, \rho\,(-,\eta)\,, f\, \big\rangle \;=\; \rho\big(f,\eta\big) \;=\; \big\langle\, \eta\,, \rho\,(\,f,-)\, \big\rangle \;=\; \Big\langle \eta\,, \overleftarrow{\Psi}_\rho'(f) \Big\rangle$$

for all $\eta \in U_\hbar(\mathfrak{g}^*)$ and $f \in F_\hbar[[G]]$, as requested. □

### 5.3. **Morphisms from quasi–$R$–matrices and quasi–$\varrho$–comatrices.**

We shall now explore what happens when the constructions leading to Proposition 2.2.9 or Proposition 2.2.10, respectively, is (tentatively) applied to a QFSHA and a quasi–$R$–comatrix for it, or to a QUEA and a quasi–$\varrho$–comatrix for it.

As a first result, we find that the construction of Hopf morphisms as in Proposition 2.2.9 can be applied again when the Hopf algebra under scrutiny is a QFSHA and its $R$–matrix is replaced by only (!) a quasi–$R$–matrix.

**Proposition 5.3.1.** *Let $F_\hbar[[G]]$ be a QFSHA, and $\mathcal{R}$ a quasi–$R$–matrix for it. Then the recipes in Proposition 2.2.9 provide two well-defined morphisms*

$$\overleftarrow{\underline{\Phi}}_{\mathcal{R}} : F_\hbar[[G^*]] := \big(U_\hbar(\mathfrak{g}^*)\big)^* = \big(F_\hbar[[G]]^\vee\big)^* \longrightarrow \big(F_\hbar[[G]]^\vee\big)^{\mathrm{cop}} = U_\hbar(\mathfrak{g}^*)^{\mathrm{cop}}$$

$$\overrightarrow{\underline{\Phi}}_{\mathcal{R}} : F_\hbar[[G^*]] := \big(U_\hbar(\mathfrak{g}^*)\big)^* = \big(F_\hbar[[G]]^\vee\big)^* \longrightarrow \big(F_\hbar[[G]]^\vee\big)^{\mathrm{op}} = U_\hbar(\mathfrak{g}^*)^{\mathrm{op}}$$

*Proof.* This follows from a direct application of Proposition 5.2.1 to the QUEA $U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee$ and its $R$–matrix $\mathcal{R}$. □

The previous result provide morphisms from a QFSHA to a QUEA. We shall now improve such a result — much like we did in §5.2 — finding a couple of morphisms between QFSHA's and another couple between QUEA's.

**Theorem 5.3.2.** *Assume that $\mathcal{R}$ is a quasi–$R$–matrix for the QFSHA $F_\hbar[[G]]$, i.e. an $R$–matrix for the QUEA $F_\hbar[[G]]^\vee =: U_\hbar(\mathfrak{g}^*)$, of the form $\mathcal{R} = \exp\big(\hbar^{-1} r\big)$ for some $r \in F_\hbar[[G]]^{\widetilde{\otimes}\, 2}$. Then, for the two morphisms $\overleftarrow{\underline{\Phi}}_{\mathcal{R}}$ and $\overrightarrow{\underline{\Phi}}_{\mathcal{R}}$ in Proposition 5.3.1 above, the following holds:*

*(a) they corestrict to morphisms*

$$\overleftarrow{\underline{\Phi}}'_{\mathcal{R}} : F_\hbar[[G^*]] = \big(F_\hbar[[G]]^\vee\big)^* \longrightarrow \Big(\big(F_\hbar[[G]]^\vee\big)^{\mathrm{cop}}\Big)' = \Big(\big(F_\hbar[[G]]^\vee\big)'\Big)^{\mathrm{cop}} = F_\hbar[[G]]^{\mathrm{cop}}$$

*and*

$$\overrightarrow{\underline{\Phi}}'_{\mathcal{R}} : F_\hbar[[G^*]] = \big(F_\hbar[[G]]^\vee\big)^* \longrightarrow \Big(\big(F_\hbar[[G]]^\vee\big)^{\mathrm{op}}\Big)' = \Big(\big(F_\hbar[[G]]^\vee\big)'\Big)^{\mathrm{op}} = F_\hbar[[G]]^{\mathrm{op}}$$

*between QFSHA's for mutually dual (formal) Poisson groups;*

*(b) they extend to morphisms*

$$\overleftarrow{\underline{\Phi}}^\vee_{\mathcal{R}} : U_\hbar(\mathfrak{g}) = F_\hbar[[G]]^\star = \Big(\big(F_\hbar[[G]]^\vee\big)^*\Big)^\vee \longrightarrow \big(F_\hbar[[G]]^\vee\big)^{\mathrm{cop}} = U_\hbar(\mathfrak{g}^*)^{\mathrm{cop}}$$

*and*

$$\overrightarrow{\underline{\Phi}}^\vee_{\mathcal{R}} : U_\hbar(\mathfrak{g}) = F_\hbar[[G]]^\star = \Big(\big(F_\hbar[[G]]^\vee\big)^*\Big)^\vee \longrightarrow \big(F_\hbar[[G]]^\vee\big)^{\mathrm{op}} = U_\hbar(\mathfrak{g}^*)^{\mathrm{op}}$$

*between QUEA's for mutually dual Lie bialgebras.*

*Proof.* First of all, note that the chain of identities

$$\Big(\big(F_\hbar[[G]]^\vee\big)^{\mathrm{cop}}\Big)' = \Big(\big(F_\hbar[[G]]^\vee\big)'\Big)^{\mathrm{cop}} = F_\hbar[[G]]^{\mathrm{cop}}$$

— and similarly with superscript "op" instead of "cop" throughout — is obvious from definitions along with the fact that Drinfeld's functors $(\ )'$ and $(\ )^\vee$ are inverse to each other. Similarly, it is also obviously true the following chain of identities

$$\Big(\big(F_\hbar[[G]]^\vee\big)^*\Big)^\vee = \Big(\big(F_\hbar[[G]]^\vee\big)'\Big)^\star = F_\hbar[[G]]^\star = U_\hbar(\mathfrak{g})$$

As to the rest of the claim, everything follows from Theorem 5.2.3 applied to the QUEA $U_\hbar(\mathfrak{g}^*) := F_\hbar[[G]]^\vee$ along with its $R$–matrix $\mathcal{R}$. □

Now we go for the dual constructions, concerning quasi–$\rho$–comatrices for a QUEA:

**Proposition 5.3.3.** *Assume that $\rho$ is a quasi–$\varrho$–comatrix for the QUEA $U_\hbar(\mathfrak{g})$, i.e. an element of the form $\rho = \exp_*\big(\hbar^{-1}\varrho\big)$ for some $\varrho \in \Big(U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}\Big)^*$ — taking into account Lemma 3.3.2 — which obeys* (2.13).

*Then the recipes in Proposition 2.2.10 provide two well-defined morphisms*

$$\overleftarrow{\underline{\Psi}}_\rho \; : \; F_\hbar[[G^*]] := U_\hbar(\mathfrak{g})' \longrightarrow \Big(\big(U_\hbar(\mathfrak{g})'\big)^\star\Big)^{\mathrm{cop}} = \big(F_\hbar[[G^*]]^\star\big)^{\mathrm{cop}} = U_\hbar(\mathfrak{g}^*)^{\mathrm{cop}}$$

*and*

$$\overrightarrow{\underline{\Psi}}_\rho \; : \; F_\hbar[[G^*]] := U_\hbar(\mathfrak{g})' \longrightarrow \Big(\big(U_\hbar(\mathfrak{g})'\big)^\star\Big)^{\mathrm{op}} = \big(F_\hbar[[G^*]]^\star\big)^{\mathrm{op}} = U_\hbar(\mathfrak{g}^*)^{\mathrm{op}}$$

*Proof.* Everything follows from definitions when applying Proposition 5.2.2 to the QFSHA $F_\hbar[[G^*]] := U_\hbar(\mathfrak{g})'$ and noting that $\big(U_\hbar(\mathfrak{g})'\big)^\star = F[[G^*]]^\star = U_\hbar(\mathfrak{g}^*)$ . □

Once again, the previous result provides morphisms from a QFSHA to a QUEA, and now we "enhance" it — like we did with Theorem 5.3.2 — finding morphisms between QFSHA's and morphisms between QUEA's:

**Theorem 5.3.4.** *Assume that $\rho$ is a quasi–$\varrho$–comatrix for the QUEA $U_\hbar(\mathfrak{g})$, i.e. an element of the form $\rho = \exp_*\big(\hbar^{-1}\varrho\big)$ for some $\varrho \in \Big(U_\hbar(\mathfrak{g})^{\widehat{\otimes}\, 2}\Big)^*$ — taking into account Lemma 3.3.2 — which obeys* (2.13)*. Then, for the two morphisms $\overleftarrow{\underline{\Psi}}_\rho$ and $\overrightarrow{\underline{\Psi}}_\rho$ in Proposition 5.3.3 above, the following holds:*

*(a) they corestrict to morphisms*

$$\overleftarrow{\underline{\Psi}}'_\rho \; : \; F_\hbar[[G^*]] = U_\hbar(\mathfrak{g})' \longrightarrow \bigg(\Big(\big(U_\hbar(\mathfrak{g})'\big)^\star\Big)^{\mathrm{cop}}\bigg)' = \big(U_\hbar(\mathfrak{g})^*\big)^{\mathrm{cop}} =: F_\hbar[[G]]^{\mathrm{cop}}$$

*and*

$$\overrightarrow{\underline{\Psi}}'_\rho \; : \; F_\hbar[[G^*]] = U_\hbar(\mathfrak{g})' \longrightarrow \bigg(\Big(\big(U_\hbar(\mathfrak{g})'\big)^\star\Big)^{\mathrm{op}}\bigg)' = \big(U_\hbar(\mathfrak{g})^*\big)^{\mathrm{op}} =: F_\hbar[[G]]^{\mathrm{op}}$$

*between QFSHA's for mutually dual (formal) Poisson groups;*

*(b) they extend to morphisms*

$$\overleftarrow{\underline{\Psi}}^\vee_\rho \; : \; U_\hbar(\mathfrak{g}) = \big(U_\hbar(\mathfrak{g})'\big)^\vee \longrightarrow \Big(\big(U_\hbar(\mathfrak{g})'\big)^\star\Big)^{\mathrm{cop}} = \big(F_\hbar[[G^*]]^\star\big)^{\mathrm{cop}} = U_\hbar(\mathfrak{g}^*)^{\mathrm{cop}}$$

*and*

$$\overrightarrow{\underline{\Psi}}^\vee_\rho \; : \; U_\hbar(\mathfrak{g}) = \big(U_\hbar(\mathfrak{g})'\big)^\vee \longrightarrow \Big(\big(U_\hbar(\mathfrak{g})'\big)^\star\Big)^{\mathrm{op}} = \big(F_\hbar[[G^*]]^\star\big)^{\mathrm{op}} = U_\hbar(\mathfrak{g}^*)^{\mathrm{op}}$$

*between QUEA's for mutually dual Lie bialgebras.*

*Proof.* As the functors $(\ )'$ and $(\ )^\vee$ are inverse to each other, definitions yield

$$\bigg(\Big(\big(U_\hbar(\mathfrak{g})'\big)^\star\Big)^{\mathrm{cop}}\bigg)' = \bigg(\Big(\big(U_\hbar(\mathfrak{g})'\big)^\star\Big)'\bigg)^{\mathrm{cop}} = \bigg(\Big(\big(U_\hbar(\mathfrak{g})'\big)^\vee\Big)^*\bigg)^{\mathrm{cop}} = \big(U_\hbar(\mathfrak{g})^*\big)^{\mathrm{cop}} =: F_\hbar[[G]]^{\mathrm{cop}}$$

— and similarly with superscript "op" instead of "cop" throughout — also thanks to $(\ )' \circ (\ )^\star = (\ )^* \circ (\ )^\vee$ . Basing on this, the entire claim follows at once from Theorem 5.2.5 applied to the QFSHA $F_\hbar[[G^*]] := U_\hbar(\mathfrak{g})'$ and to its $\varrho$–comatrix $\rho$. □

**5.3.5. Duality properties.** If we consider a QUEA and a QFSHA which are dual to each other, we can compare the previous results: thus we find the following "quasi-analogue" — whose proof is trivial again — of Theorem 5.2.7:

**Theorem 5.3.6.** *Let $U_\hbar(\mathfrak{g})$ be a QUEA, $F_\hbar[[G]]$ a QFSHA, which are dual to each other, i.e. $F_\hbar[[G]] = U_\hbar(\mathfrak{g})^*$ and $U_\hbar(\mathfrak{g}) = F_\hbar[[G]]^\star$ . Let $\rho = \mathcal{R}$ be a quasi–$\rho$–comatrix for $U_\hbar(\mathfrak{g})$ and a quasi–R–matrix for $F_\hbar[[G]]$ .*

*Then, for the morphisms in Proposition 5.3.1, Proposition 5.3.3, Theorem 5.3.2 and Theorem 5.3.4 we have the following identifications*

$$\overleftarrow{\underline{\Phi}}_{\mathcal{R}} = \overleftarrow{\underline{\Psi}}_\rho \, , \;\; \overleftarrow{\underline{\Phi}}'_{\mathcal{R}} = \overleftarrow{\underline{\Psi}}'_\rho \, , \;\; \overleftarrow{\underline{\Phi}}^\vee_{\mathcal{R}} = \overleftarrow{\underline{\Psi}}^\vee_\rho \;\; \text{and} \;\; \overrightarrow{\underline{\Phi}}_{\mathcal{R}} = \overrightarrow{\underline{\Psi}}_\rho \, , \;\; \overrightarrow{\underline{\Phi}}'_{\mathcal{R}} = \overrightarrow{\underline{\Psi}}'_\rho \, , \;\; \overrightarrow{\underline{\Phi}}^\vee_{\mathcal{R}} = \overrightarrow{\underline{\Psi}}^\vee_\rho \qquad \square$$

**5.3.7. Comparing morphisms (2).** We shall now compare morphisms among quantum groups provided by a quasi–$R$–matrix as above.

Let us start with a QFSHA $F_\hbar[[G]]$, with dual QUEA denoted by $U_\hbar(\mathfrak{g})$, and a quasi–$R$–matrix $\mathcal{R}$ for $F_\hbar[[G]]$. Then Theorem 5.3.2 gives a couple of diagrams

$$\begin{tikzcd} F_\hbar[[G^*]] \arrow[r, "\overleftarrow{\underline{\Phi}}'_{\mathcal{R}}"] \arrow[d, squiggly, dash] & F_\hbar[[G]]^{\mathrm{cop}} \arrow[d, squiggly, dash] \\ U_\hbar(\mathfrak{g}^*) & U_\hbar(\mathfrak{g})^{\mathrm{op}} \arrow[l, "\overrightarrow{\underline{\Phi}}^\vee_{\mathcal{R}}"] \end{tikzcd} \qquad \begin{tikzcd} F_\hbar[[G^*]] \arrow[r, "\overrightarrow{\underline{\Phi}}'_{\mathcal{R}}"] \arrow[d, squiggly, dash] & F_\hbar[[G]]^{\mathrm{op}} \arrow[d, squiggly, dash] \\ U_\hbar(\mathfrak{g}^*) & U_\hbar(\mathfrak{g})^{\mathrm{cop}} \arrow[l, "\overleftarrow{\underline{\Phi}}^\vee_{\mathcal{R}}"] \end{tikzcd} \tag{5.12}$$

where the vertical, twisting lines denote a link of (Hopf) duality while the horizontal arrows are Hopf algebra morphisms. Next result, "quasi–analogue" of Theorem 5.2.9, tells us that the morphisms on top and bottom row are "as close as possible":

**Theorem 5.3.8.** *The two morphisms in left-hand side, resp. in right-hand side, of* (5.12) *are adjoint to each other, that is for all* $\eta \in U_\hbar(\mathfrak{g})$ *and* $f \in F_\hbar[[G^*]]$ *we have*

$$\Big\langle\, \overrightarrow{\underline{\Phi}}^\vee_{\mathcal{R}}(\eta)\,, f \Big\rangle \;=\; \Big\langle \eta\,, \overleftarrow{\underline{\Phi}}'_{\mathcal{R}}(f) \Big\rangle \qquad \textit{and} \qquad \Big\langle\, \overleftarrow{\underline{\Phi}}^\vee_{\mathcal{R}}(\eta)\,, f \Big\rangle \;=\; \Big\langle \eta\,, \overrightarrow{\underline{\Phi}}'_{\mathcal{R}}(f) \Big\rangle$$

*where by "$\langle\ ,\ \rangle$" we denote the pairing between any two Hopf algebras in duality.*

*Proof.* The proof follows from Theorem 5.2.9 along with Theorem 5.1.5. □

Similarly, let $U_\hbar(\mathfrak{g})$ be a QUEA, with dual QFSHA denoted by $F_\hbar[[G]]$, and let $\rho$ be a quasi–$\varrho$–comatrix $U_\hbar(\mathfrak{g})$. Then Theorem 5.3.4 yields a couple of diagrams

$$\begin{tikzcd} F_\hbar[[G^*]] \arrow[r, "\overleftarrow{\underline{\Psi}}'_{\rho}"] \arrow[d, squiggly, dash] & F_\hbar[[G]]^{\mathrm{cop}} \arrow[d, squiggly, dash] \\ U_\hbar(\mathfrak{g}^*) & U_\hbar(\mathfrak{g})^{\mathrm{op}} \arrow[l, "\overrightarrow{\underline{\Psi}}^\vee_{\rho}"] \end{tikzcd} \qquad \begin{tikzcd} F_\hbar[[G^*]] \arrow[r, "\overrightarrow{\underline{\Psi}}'_{\rho}"] \arrow[d, squiggly, dash] & F_\hbar[[G]]^{\mathrm{op}} \arrow[d, squiggly, dash] \\ U_\hbar(\mathfrak{g}^*) & U_\hbar(\mathfrak{g})^{\mathrm{cop}} \arrow[l, "\overleftarrow{\underline{\Psi}}^\vee_{\rho}"] \end{tikzcd} \tag{5.13}$$

akin to (5.12). We get now the "quasi–analogue" of Theorem 5.2.10, which claims that the morphisms on top and bottom row of (5.13) are "as close as possible":

**Theorem 5.3.9.** *The two morphisms in left-hand side, resp. in right-hand side, of* (5.13) *are adjoint to each other, that is for all* $\eta \in U_\hbar(\mathfrak{g})$ *and* $f \in F_\hbar[[G^*]]$ *we have*

$$\Big\langle\, \overrightarrow{\underline{\Psi}}^\vee_{\rho}(\eta)\,, f \Big\rangle \;=\; \Big\langle \eta\,, \overleftarrow{\underline{\Psi}}'_{\rho}(f) \Big\rangle \qquad \textit{and} \qquad \Big\langle\, \overleftarrow{\underline{\Psi}}^\vee_{\rho}(\eta)\,, f \Big\rangle \;=\; \Big\langle \eta\,, \overrightarrow{\underline{\Psi}}'_{\rho}(f) \Big\rangle$$

*where "$\langle\ ,\ \rangle$" denotes the pairing between any two Hopf algebras in duality.*

*Proof.* Here again, the proof follows from Theorem 5.2.9 and Theorem 5.1.5. □

## 5.4. **Semiclassical morphisms induced by specialization.**

We will now go and study the semiclassical limit of the various morphisms among quantum groups, considered in §§5.2 and 5.3 above.

First we consider the case of an $R$–matrix $\mathcal{R}$ for a given QUEA $U_\hbar(\mathfrak{g})$, whose dual QFSHA is $F_\hbar[[G]]$. With this assumptions, we recall the existence of the Hopf algebra morphisms in (5.7), which by Theorem 5.2.9 are pairwise mutually adjoint.

Specialising $\hbar$ to 0, the left-hand side of (5.7) provides two mutually adjoint morphisms $F[[G]] \xrightarrow{\;\overleftarrow{\Phi}'_{\mathcal{R}}\big|_{\hbar=0}\;} F[[G^*]]^{\mathrm{cop}}$ and $U(\mathfrak{g}^*)^{\mathrm{op}} \xrightarrow{\;\overrightarrow{\Phi}^\vee_{\mathcal{R}}\big|_{\hbar=0}\;} U(\mathfrak{g})$, the first being a morphism of *Poisson* Hopf algebras, the second one of *co-Poisson* Hopf algebras. As they are mutually adjoint, each one of them defines one and the same morphism of formal Poisson groups $\phi_{\mathcal{R}}^{+} : G_{\mathrm{op}}^{*} \longrightarrow G$ where $G_{\mathrm{op}}^{*}$ is the *opposite* (i.e., with opposite product) formal Poisson group to $G^*$. Note that $\phi_{\mathcal{R}}^{+} : G_{\mathrm{op}}^{*} \longrightarrow G$ is directly defined by $\overleftarrow{\Phi}'_{\mathcal{R}}\big|_{\hbar=0}$, while the morphism of Lie bialgebras $d\phi_{\mathcal{R}}^{+} : \mathfrak{g}_{\mathrm{op}}^{*} \longrightarrow \mathfrak{g}$ can be deduced directly from $\overrightarrow{\Phi}_{\mathcal{R}}^{\vee}\big|_{\hbar=0}$, by restriction to $\mathfrak{g}_{\mathrm{op}}^{*}$ and corestriction to $\mathfrak{g}$.

Similarly, specialising $\hbar$ to 0 the right-hand side of (5.7) yields two mutually adjoint morphisms $F[[G]] \xrightarrow{\overrightarrow{\Phi}'_{\mathcal{R}}|_{\hbar=0}} F[[G^*]]^{\mathrm{op}}$ and $U(\mathfrak{g}^*)^{\mathrm{cop}} \xrightarrow{\overleftarrow{\Phi}_{\mathcal{R}}^{\vee}|_{\hbar=0}} U(\mathfrak{g})$ which in turn defines one single morphism of formal Poisson groups $\phi_{\mathcal{R}}^{-} : G_{\mathrm{cop}}^{*} \longrightarrow G$ where now $G_{\mathrm{cop}}^{*}$ denotes the *co-opposite* formal Poisson group to $G^*$ — i.e., with same product but opposite Poisson structure. This goes along with its associated morphism of Lie bialgebras $d\phi_{\mathcal{R}}^{-} : \mathfrak{g}_{\mathrm{cop}}^{*} \longrightarrow \mathfrak{g}$. In short, we have pairs of morphisms

$$G_{\mathrm{op}}^{*} \xrightarrow{\phi_{\mathcal{R}}^{+}} G \; , \quad G_{\mathrm{cop}}^{*} \xrightarrow{\phi_{\mathcal{R}}^{-}} G \qquad \text{and} \qquad \mathfrak{g}_{\mathrm{op}}^{*} \xrightarrow{d\phi_{\mathcal{R}}^{+}} \mathfrak{g} \; , \quad \mathfrak{g}_{\mathrm{cop}}^{*} \xrightarrow{d\phi_{\mathcal{R}}^{-}} \mathfrak{g} \tag{5.14}$$

of formal Poisson groups and of Lie bialgebras, respectively.

Second, we consider the case of a $\varrho$–comatrix $\rho$ for a given QFSHA $F_\hbar[[G]]$, with dual QUEA $U_\hbar(\mathfrak{g})$. In this case, there exist Hopf algebra morphisms as in (5.11), which are pairwise mutually adjoint due to Theorem 5.2.10.

Acting as before, specialising $\hbar$ to 0 we find that the semiclassical limits of these (quantum) morphisms eventually define two pairs of morphisms

$$G_{\mathrm{op}}^{*} \xrightarrow{\psi_{\rho}^{+}} G \; , \quad G_{\mathrm{cop}}^{*} \xrightarrow{\psi_{\rho}^{-}} G \qquad \text{and} \qquad \mathfrak{g}_{\mathrm{op}}^{*} \xrightarrow{d\psi_{\rho}^{+}} \mathfrak{g} \; , \quad \mathfrak{g}_{\mathrm{cop}}^{*} \xrightarrow{d\psi_{\rho}^{-}} \mathfrak{g} \tag{5.15}$$

of formal Poisson groups and of Lie bialgebras, respectively.

Third, to compare the two constructions, assume that, given mutually dual quantum groups $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$, we pick a single element $\mathcal{R} = \rho$, thought of simultaneously as an $R$–matrix for $U_\hbar(\mathfrak{g})$ and as a $\varrho$–comatrix for $F_\hbar[[G]]$, much in the spirit of Proposition 2.2.7 and Theorem 2.2.12. Then morphisms as in (5.14) and (5.15) are defined: but in addition, directly by Theorem 5.2.7 we get at once that

$$\phi_{\mathcal{R}}^{+} = \psi_{\rho}^{+} \; , \quad \phi_{\mathcal{R}}^{-} = \psi_{\rho}^{-} \qquad \text{and} \qquad d\phi_{\mathcal{R}}^{+} = d\psi_{\rho}^{+} \; , \quad d\phi_{\mathcal{R}}^{-} = d\psi_{\rho}^{-}$$

If one works instead with quasi–$R$–matrices and quasi– $\varrho$–comatrices, the roles of $G$ and $G^*$ are reversed, but for the rest the analysis is entirely similar (so we may be more sketchy). Therefore, assume we have dual quantum groups $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$.

Given a quasi–$R$–matrix $\mathcal{R}$ for $F_\hbar[[G]]$, the Hopf algebra morphisms in Theorem 5.3.2 give rise (through their semiclassical limit) to two pairs of morphisms

$$G_{\mathrm{op}} \xrightarrow{\underline{\phi}_{\mathcal{R}}^{+}} G^* \; , \quad G_{\mathrm{cop}} \xrightarrow{\underline{\phi}_{\mathcal{R}}^{-}} G^* \qquad \text{and} \qquad \mathfrak{g}_{\mathrm{op}} \xrightarrow{d\underline{\phi}_{\mathcal{R}}^{+}} \mathfrak{g}^* \; , \quad \mathfrak{g}_{\mathrm{cop}} \xrightarrow{d\underline{\phi}_{\mathcal{R}}^{-}} \mathfrak{g}^* \tag{5.16}$$

of formal Poisson groups and of Lie bialgebras, respectively.

Similarly, if $\rho$ is a quasi–$\varrho$–comatrix for $U_\hbar(\mathfrak{g})$, the Hopf algebra morphisms in Theorem 5.3.4 define (via their semiclassical limit) two pairs of morphisms

$$G_{\mathrm{op}} \xrightarrow{\underline{\psi}_{\rho}^{+}} G^* \; , \quad G_{\mathrm{cop}} \xrightarrow{\underline{\psi}_{\rho}^{-}} G^* \qquad \text{and} \qquad \mathfrak{g}_{\mathrm{op}} \xrightarrow{d\underline{\psi}_{\rho}^{+}} \mathfrak{g}^* \; , \quad \mathfrak{g}_{\mathrm{cop}} \xrightarrow{d\underline{\psi}_{\rho}^{-}} \mathfrak{g}^* \tag{5.17}$$

of formal Poisson groups and of Lie bialgebras, respectively.

Finally, if $\mathcal{R} = \rho$ — in the spirit of Proposition 2.2.7, more precisely like in Theorem 5.3.6 — then Theorem 5.3.6 gives at once

$$\underline{\phi}_{\mathcal{R}}^{+} = \underline{\psi}_{\rho}^{+} \ , \quad \underline{\phi}_{\mathcal{R}}^{-} = \underline{\psi}_{\rho}^{-} \qquad \text{and} \qquad d\underline{\phi}_{\mathcal{R}}^{+} = d\underline{\psi}_{\rho}^{+} \ , \quad d\underline{\phi}_{\mathcal{R}}^{-} = d\underline{\psi}_{\rho}^{-}$$

Studying in depth all the morphisms introduced above seems to be quite an interesting problem; we cannot, however, cope with in the present paper — we just finish with a comparison with previous results.

Assume we have an $R$–matrix $\mathcal{R}$ for a given QUEA $U_\hbar(\mathfrak{g})$, whose dual QFSHA is $F_\hbar[[G]] := U_\hbar(\mathfrak{g})^*$. It is well-known that the "semiclassical limit" of $\mathcal{R}$, that is $r := \dfrac{\mathcal{R} - 1^{\otimes 2}}{\hbar} \pmod{\hbar}$, is in turn a "classical $r$–matrix" for the Lie bialgebra $\mathfrak{g}$. Then Lie bialgebra morphisms $\mathfrak{g}^*_{\mathrm{op}} \xrightarrow{\varphi_r^+} \mathfrak{g}$ and $\mathfrak{g}^*_{\mathrm{cop}} \xrightarrow{\varphi_r^-} \mathfrak{g}$ are defined directly through $r$ itself — with no need of $\mathcal{R}$, nor of $U_\hbar(\mathfrak{g})$, nor $F_\hbar[[G]]$, cf. [CP], §2.1, or [Mj], §8.1. Tracking the various constructions involved — in particular, the functor $F_\hbar[[G]] \mapsto F_\hbar[[G]]^\vee =: U_\hbar(\mathfrak{g}^*)$ — by direct comparison one immediately sees that

$$\varphi_r^+ = d\underline{\phi}_{\mathcal{R}}^{+} \qquad \text{and} \qquad \varphi_r^- = d\underline{\phi}_{\mathcal{R}}^{-}$$

In particular, we get that *the morphisms* $d\underline{\phi}_{\mathcal{R}}^{\pm}$ *depend on* $r$ *alone, rather than on* $\mathcal{R}$, *hence the same is true for the morphisms* $\underline{\phi}_{\mathcal{R}}^{\pm}$; indeed, both facts can also be easily proved by direct inspection. Similarly, one can prove, via direct analysis again, or by a duality argument from the previous result, that *the morphisms* $\underline{\psi}_{\rho}^{\pm}$ *and* $d\underline{\psi}_{\rho}^{\pm}$ *depend only on the "classical $\varrho$–comatrix"* $\rho_0 := \dfrac{\rho - \epsilon^{\otimes 2}}{\hbar} \pmod{\hbar}$ *alone*, rather than on $\rho$.

## References


[Ch] H.-X. Chen, *Quasitriangular Structures of Bicrossed Coproducts*, J. Algebra **204** (1998), 504–531.

[CG] N. Ciccoli, L. Guerra, *The Variety of Lie Bialgebras*, J. Lie Theory **13** (2003), no. 2, 577–588.

[CP] V. Chari, A. Pressley, *A guide to quantum group*, Cambridge University Press, Cambridge, 1995.

[Doi] Y. Doi, *Braided bialgebras and quadratic bialgebras*, Comm. Algebra **21** (1993), no. 5, 1731–1749.

[Dr] V. G. Drinfeld, *Quantum groups*, Proc. Int. Congr. Math., Berkeley 1986, vol. **1** (1987), 798–820.

[EH] B. Enriquez, G. Halbout, *An $\hbar$–adic valuation property of universal $R$–matrices*, J. Algebra **261** (2003), no. 2, 434–447.

[EK] P. Etingof, D. Kazhdan, *Quantization of Lie bialgebras I*, Selecta Math. (N.S.) **2** (1996), no. 1, 1–41.

[ES] P. Etingof, O. Schiffmann, *Lectures on quantum groups*, Second edition. Lectures in Mathematical Physics, International Press, Somerville, MA, 2002, xii+242 pp.

[Ga1] F. Gavarini, *The quantum duality principle*, Ann. Inst. Fourier **52** (2002), no. 3, 809–834.

[Ga2] ———, *Quantum duality principle for quantum continuous Kac-Moody algebras*, J. Lie theory **32** (2022), no. 3, 839–862.

[GaGa1] G. A. García, F. Gavarini, *Twisted deformations vs. cocycle deformations for quantum groups*, Commun. Contemp. Math. **23** (2021), no. 8, Paper No. 2050084, 56 pp.

[GaGa2] G. A. García, F. Gavarini, *Formal multiparameter quantum groups, deformations and specializations*, Annales de l'Institut Fourier, 117 pages (to appear) — preprint `arXiv:2203.11023 [math.QA]` (2022).

[GaGa3] G. A. García, F. Gavarini, *Quantum group deformations and quantum R–(co)matrices vs. Quantum Duality Principle*, 72 pages — preprint `arXiv:2403.15096 [math.QA]` (2024).

[Ks] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics **155**, Springer-Verlag, New York, 1995, xii+531 pp.

[KS] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997, xx+552 pp

[Mj] S. Majid, *Foundations of quantum groups*, Cambridge University Press, Cambridge, 1995.

[MW] S. Merkulov, T. Willwacher, *Deformation theory of Lie bialgebra properads*, Geometry and physics, Vol. I, 219–247, Oxford Univ. Press, Oxford, 2018.

[Ra] D. E. Radford, *Hopf algebras*, Series on Knots and Everything **49**, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

Departamento de Matemática, Facultad de Ciencias Exactas
Universidad Nacional de La Plata — CMaLP-CIC-CONICET
1900 La Plata, Pcia. de Buenos Aires, ARGENTINA / `ggarcia@mate.unlp.edu.ar`

Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata"
Via della ricerca scientifica 1, I-00133 Roma, ITALY / `gavarini@mat.uniroma2.it`
Istituto Nazionale di Alta Matematica "Francesco Severi" — GNSAGA