QUANTUM GROUP DEFORMATIONS AND QUANTUM R–(CO)MATRICES VS. QUANTUM DUALITY PRINCIPLE

GASTÓN ANDRÉS GARCÍA[♭] , FABIO GAVARINI[♯]

INTRODUCTION

In Hopf algebra theory, there exists a well-established theory of "deformations" that are produced via specific tools, namely *twists* in one case and *2–cocycles* in the other case. Given a Hopf algebra H, a twist for it is a suitable element $\mathcal{F} \in H \otimes H$, while (dually) a 2–cocycle is a suitable 2–form $\sigma \in (H \otimes H)^*$. Deformation by $\mathcal{F}$ provides H with a new Hopf algebra structure, by modifying the coproduct (and the antipode) but not the product, while deformation by σ endows H with yet another Hopf structure by changing the product (and the antipode) but not the coproduct.

Quantum groups are Hopf algebras of special type, in two versions: QUEAs (= quantized universal enveloping algebras) and QFSHAs (= quantized formal series Hopf algebras). Roughly speaking, a QUEA is a (topological) Hopf algebra $U_\hbar$ over the $\Bbbk$–algebra of formal power series $\Bbbk[[\hbar]]$ such that $U_0 := U_\hbar \big/ \hbar\, U_\hbar$ is isomorphic to $U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$. Then $U(\mathfrak{g})$ inherits from $U_\hbar$ a Poisson cobracket, which makes it into a co-Poisson Hopf algebra, hence $\mathfrak{g}$ bears a Lie cobracket making it into a Lie bialgebra. One then says that $U_\hbar$ is a *quantization* of the co-Poisson Hopf algebra $U(\mathfrak{g})$, or just of the Lie bialgebra $\mathfrak{g}$. Dually, a QFSHA is a (topological) Hopf algebra $F_\hbar$ over $\Bbbk[[\hbar]]$ such that $F_0 := F_\hbar \big/ \hbar\, F_\hbar$ is isomorphic to $F[[G]]$ for some formal algebraic group G. Then $F[[G]]$ inherits from $F_\hbar$ a Poisson bracket, which makes it into a Poisson Hopf algebra, thus G bears a Poisson structure which makes it into a formal Poisson (algebraic) group. One says then that $F_\hbar$ is a *quantization* of the Poisson Hopf algebra $F[[G]]$, or just of the (formal) Poisson group G.

As a general philosophy, from any Hopf-theoretical notion — at the quantum level — one typically infers a Lie-theoretical counterpart — at the semiclassical level. When dealing with deformations, this leads to devising suitable notions of "twists" and "2–cocycles" for Lie bialgebras as well as "deformations" (of Lie bialgebras) by them. In particular, a deformation by twist yields a new Lie bialgebra structure where only the Lie cobracket is modified, whereas deformation by 2–cocycle defines yet another, similar structure where only the Lie bracket is changed.

Via this recipe, we expect the following: when we deform (as a Hopf algebra) a quantization $U_\hbar$ of $\mathfrak{g}$ by a twist which is trivial modulo $\hbar$, we get a quantization of $\mathfrak{g}'$, the latter being a deformation by twist (as a Lie bialgebra) of $\mathfrak{g}$: moreover, the (Lie) twist working on $\mathfrak{g}$ is "induced" by the (Hopf) twist that works upon $U_\hbar$, namely the former (Lie) twist is the "semiclassical limit" of the latter (Hopf) twist.

Dually, the following also should hold: when we deform (as a Hopf algebra) a quantization $F_\hbar$ of G by a 2–cocycle which is trivial modulo $\hbar$, we get a quantization of G', the latter being a (formal) Poisson group whose cotangent Lie bialgebra is a deformation by 2–cocycle of $\mathfrak{g}^* := \mathit{Lie}(G)^*$: moreover, the (Lie) 2–cocycle acting on $\mathfrak{g}^*$ is "induced" by the (Hopf) 2–cocycle that acts on $F_\hbar$, namely the former (Lie) 2–cocycle is the "semiclassical limit", in some sense, of the latter (Hopf) 2–cocycle.

2020 *MSC:* 17B37, 17B62 — *Keywords:* Quantum Groups, Quantum Enveloping Algebras.

Nevertheless, neither of the two results mentioned above seems to be published anywhere in literature (to the best of the authors' knowledge, say). Therefore, as a first contribution in this paper we provide a full, complete statement and proof for the above sketched results, turning them into well-established theorems.

As a second step — our main contribution in this paper — we extend the notions of (Hopf) twist and 2–cocycle, as well as the construction of (Hopf) deformations by them, to a wider setup. Namely, we introduce the notions of *quasi–twist* for a QFSHA and of *quasi–2–cocycle* for a QUEA: roughly speaking, a quasi–twist for $F_\hbar$ has the formal Hopf properties of a twist but has the form $\exp\big(\hbar^{-1}\varphi\big)$, while any twist (trivial modulo $\hbar$) looks like $\exp\big(\hbar^{+1}\phi\big)$ — and similarly for the link between quasi–2–cocycles and 2–cocycles. Thus even the very definition of these "quasi–objects", at least in this form, seems to be problematic — as multiplying by $\hbar^{-1}$ is meaningless. In spite of this, we show that the recipe defining deformations still makes sense if we replace "twists" with "quasi–twists", resp. "2–cocycles" with "quasi–2–cocycles". Moreover, we can describe the semiclassical limit of these deformations (by "quasi–objects"), again in terms of deformations of Lie bialgebras by some (Lie) twist, resp. 2–cocycle, that can be explicitly read out as the semiclassical limit of the quantum (Hopf) quasi–twist, resp. quasi–2–cocycle, that we started with. In a nutshell, we find the perfect "quasi–versions" of the results mentioned above for standard quantum group deformations, i.e. those by twist or by 2–cocycle.

The fact that "deformations by *quasi*–objects" do make sense can be explained in light of the *Quantum Duality Principle (=QDP)*. In fact, the latter provides functorial recipes (via *Drinfeld's functors*) which turn any QUEA into a QFSHA and any QFSHA into a QUEA. Then, through the QDP lens, every "quasi-twist" for a QFSHA, resp. every "quasi–2–cocycle" for a QUEA, is just a sheer standard twist, resp. 2–cocycle, for the QUEA, resp. the QFSHA, obtained when applying the appropriate Drinfeld functor. In this way, our deformations "by quasi–objects" turn out to be tightly related with standard ones, but applied to different quantum groups. Nevertheless, one still has to prove that the (standard) deformation applied to the new quantum group can actually be adapted to the original quantum group.

Finally, we consider some constructions of morphisms that, in general Hopf algebra theory, are provided by R–matrices or ϱ–comatrices. We apply these constructions to quantum groups, showing that their outcome is much finer than expected from the general theory, and bringing to light their geometrical meaning at the semiclassical level. In addition, we improve those results as follows: we introduce the notions of *quasi–R–matrices* and *quasi–ϱ–comatrices* (much in the same spirit as with quasi–twists and quasi–2–cocycles), and then we extend the construction of the above morphisms to quasi–R–matrices and quasi–ϱ–comatrices, again involving the QDP.

The paper is organized as follows.

In §2 we quickly recall the material we work with. In §3 we present the bulk of the paper First we study deformations by twist and by 2–cocycles, then we introduce *quasi–2–cocycles* and *quasi–twists* and the procedures of deformations by these. All this material is discussed again in §4, in light of the *Quantum Duality Principle*. Finally, in §5 we study the morphisms associated with R–matrices or ϱ–comatrices in the case of quantum groups, also explaining their meaning at the semiclassical limit. Moreover, we extend those constructions and results to the newly minted notions of quasi–R–matrices and quasi–ϱ–comatrices.

N.B.: a longer version of this work, including full-detailed computations, is available on-line as electronic preprint [GaGa3].

ACKNOWLEDGEMENTS

This article was partially supported by: *(1)* the European research network *COST Action CaLISTA CA21109*; *(2)* CONICET, ANPCyT, Secyt (Argentina), *(3)* the MIUR *Excellence Department Project MatMod@TOV (CUP E83C23000330006)* awarded to the Department of Mathematics, University of Rome "Tor Vergata" (Italy), *(4)* INdAM research branch GNSAGA (Italy).

The authors thank Benjamin Enriquez for sparking their interest in the topics of §5.

References

[Ch] H.-X. Chen, *Quasitriangular Structures of Bicrossed Coproducts*, J. Algebra **204** (1998), 504–531.

[CG] N. Ciccoli, L. Guerra, *The Variety of Lie Bialgebras*, J. Lie Theory **13** (2003), no. 2, 577–588.

[CP] V. Chari, A. Pressley, *A guide to quantum group*, Cambridge University Press, Cambridge, 1995.

[Doi] Y. Doi, *Braided bialgebras and quadratic bialgebras*, Comm. Algebra **21** (1993), no. 5, 1731–1749.

[Dr] V. G. Drinfeld, *Quantum groups*, Proc. Int. Congr. Math., Berkeley 1986, vol. **1** (1987), 798–820.

[EH] B. Enriquez, G. Halbout, *An $\hbar$–adic valuation property of universal R–matrices*, J. Algebra **261** (2003), no. 2, 434–447.

[EK] P. Etingof, D. Kazhdan, *Quantization of Lie bialgebras I*, Selecta Math. (N.S.) **2** (1996), no. 1, 1–41.

[ES] P. Etingof, O. Schiffmann, *Lectures on quantum groups*, Second edition. Lectures in Mathematical Physics, International Press, Somerville, MA, 2002, xii+242 pp.

[Ga1] F. Gavarini, *The quantum duality principle*, Ann. Inst. Fourier **52** (2002), no. 3, 809–834.

[Ga2] ———, *Quantum duality principle for quantum continuous Kac-Moody algebras*, J. Lie theory **32** (2022), no. 3, 839–862.

[GaGa1] G. A. García, F. Gavarini, *Twisted deformations vs. cocycle deformations for quantum groups*, Commun. Contemp. Math. **23** (2021), no. 8, Paper No. 2050084, 56 pp.

[GaGa2] G. A. García, F. Gavarini, *Formal multiparameter quantum groups, deformations and specializations*, Annales de l'Institut Fourier, 117 pages (to appear) — preprint `arXiv:2203.11023 [math.QA]` (2022).

[GaGa3] G. A. García, F. Gavarini, *Quantum group deformations and quantum R–(co)matrices vs. Quantum Duality Principle*, 72 pages — preprint `arXiv:2403.15096 [math.QA]` (2024).

[Ks] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics **155**, Springer-Verlag, New York, 1995, xii+531 pp.

[KS] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997, xx+552 pp

[Mj] S. Majid, *Foundations of quantum groups*, Cambridge University Press, Cambridge, 1995.

[MW] S. Merkulov, T. Willwacher, *Deformation theory of Lie bialgebra properads*, Geometry and physics, Vol. I, 219–247, Oxford Univ. Press, Oxford, 2018.

[Ra] D. E. Radford, *Hopf algebras*, Series on Knots and Everything **49**, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

Departamento de Matemática, Facultad de Ciencias Exactas
Universidad Nacional de La Plata — CMaLP-CIC-CONICET
1900 La Plata, Pcia. de Buenos Aires, ARGENTINA / `ggarcia@mate.unlp.edu.ar`

Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata"
Via della ricerca scientifica 1, I-00133 Roma, ITALY / `gavarini@mat.uniroma2.it`
Istituto Nazionale di Alta Matematica "Francesco Severi" — GNSAGA