SUPER KÄHLER STRUCTURES ON THE COMPLEX ABELIAN LIE SUPERGROUPS

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Abstract: We consider a real Abelian Lie supergroup G acting on its complexification M, equipped with a G–invariant super Kähler form. We extend the scheme of classical geometric quantization to this setting and construct a unitary G–representation. We show that the occurrences of its irreducible subrepresentations are governed by the image of the moment map of the super Kähler form. As an application, we construct a Gelfand model of G, namely a unitary G–representation in which every unitary irreducible representation occurs exactly once.

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1. Introduction

The theory of geometric quantization [14] associates the action of a Lie group G on a symplectic manifold M to a unitary G-representation $\mathcal{H}$, and one studies its irreducible subrepresentations. Super geometric quantization has been discussed through the prequantization stage [19], where $\mathcal{H}$ is an algebraic representation without a unitary structure. In view of recent developments in the notions of super Hilbert spaces and super unitary representations (see for example [9]), it becomes appropriate to study the unitary structure of $\mathcal{H}$.

The geometric quantization of the actions of connected Abelian Lie groups on their complexifications has been carried out successfully for [5, 6]. We now consider its super analogue. Let G be a connected Abelian Lie supergroup. Its even part $G_{\bar{0}}$ is a connected Lie group, so $G_{\bar{0}} \cong T_n \times \mathbb{R}^m$, where T_n is the n–dimensional torus. As for any other supergroup (cf. [10, 11]), we have a global splitting of G of the form

$$G = T_n \times \mathbb{R}^m \times \textstyle\bigwedge_k^{\mathbb{R}} \tag{1.1}$$

where $\bigwedge_k^{\mathbb{R}}$ is the supermanifold associated with the Grassmann $\mathbb{R}$–algebra in k odd indeterminates — roughly, it is a single point endowed with a purely odd, k–dimensional affine superstructure. Algebraically, this means that the defining superalgebra of global regular functions on G (real smooth, in the present case) factors into

$$\mathcal{O}_G(G) := C^\infty(G_{\bar{0}}) \otimes_{\mathbb{R}} \Lambda_{\mathbb{R}}(\xi_1, \dots, \xi_k) \ .$$

In particular, the local structure around a single point in $G_{\bar{0}}$ can be described by a local chart, denoted by (x, ξ) — cf. §2.3 later on.

To provide a fluent presentation, we first consider the super torus $G = T_n \times \bigwedge_k^{\mathbb{R}}$, namely $m = 0$ in (1.1). Let $\mathfrak{t}_n$ be the Lie algebra of T_n. Let $\lambda \in i\mathfrak{t}_n^*$, namely $\lambda : \mathfrak{t}_n \longrightarrow i\,\mathbb{R}$. We say that λ is an integral weight if it determines a character $\chi_\lambda : T_n \longrightarrow S^1$ such that the diagram commutes,

$$\begin{array}{ccc} \mathfrak{t}_n & \xrightarrow{\lambda} & i\,\mathbb{R} \\ \downarrow & & \downarrow \\ T_n & \xrightarrow{\chi_\lambda} & S^1 \end{array} \tag{1.2}$$

where the downward arrows are exponential maps.

Let $\widehat{T_n}$ denote the set of all irreducible unitary T_n–representations, up to equivalence. The members of $\widehat{T_n}$ are 1–dimensional. They are parametrized by the integral weights λ, where $V_\lambda \in \widehat{T_n}$ consists of vectors v which satisfy $t \cdot v = \chi_\lambda(t) v$ for all $t \in T_n$. We identify the integral weights with $\mathbb{Z}^n$ and write

$$i\,\mathfrak{t}^* \cong \mathbb{R}^n \quad , \quad \widehat{T_n} \cong \mathbb{Z}^n \quad .$$

A unitary representation of G is a super vector space with a super Hermitian form — see [9, §4] — and compatible actions by $G_{\bar{0}}$ and $\mathfrak{g} := \mathrm{Lie}(G)$, the tangent Lie superalgebra of G. Let $\widehat{G}$ denote the equivalence classes of irreducible unitary representations of G. In the following theorem, $\mathbb{Z}^n$ identifies with the set of all integral weights of T_n.

Theorem 1.1. *For the super torus $G = T_n \times \bigwedge_k^{\mathbb{R}}$, there exists a group isomorphism*

$$\widehat{G} \ \cong \ \widehat{G_{\bar{0}}} \times \mathbb{Z}_2 \ \cong \ \mathbb{Z}^n \times \mathbb{Z}_2$$

where the group structure of $\widehat{G}$ is given by the tensor product of representations.

The representation space parametrized by $(\lambda, \varepsilon) \in \mathbb{Z}^n \times \mathbb{Z}_2$ *has dimension* $1|0$ *(resp.* $0|1$*) if* $\varepsilon = \bar{0}$ *(resp.* $\varepsilon = \bar{1}$*), and its vectors* v *satisfy* $t \cdot v = \chi_\lambda(t)v$ *for all* $t \in T_n$ *and* $\xi \cdot v = 0$ *for all* $\xi \in \mathfrak{g}_{\bar{1}}$ *.*

We may express the group $\mathbb{Z}_2$ additively by $\{\bar{0}, \bar{1}\}$ or multiplicatively by $\{+, -\}$. By Theorem 1.1, we write

$$\widehat{G} \;=\; \left\{ V_\lambda^{\pm} \;\middle|\; \lambda \in \mathbb{Z}^n \,,\, \pm \in \mathbb{Z}_2 \right\} \;.$$

We construct $V_\lambda^{\pm}$ explicitly in Example 3.2.

Let M be the complexification of G. Thus M is a complex Lie supergroup that admits the following description: $M \cong M_{\bar{0}} \times M_{\bar{1}}$, where $M_{\bar{1}} := \bigwedge_k^{\mathbb{C}}$ is described through complex odd Grassmann variables $\zeta_1, \ldots, \zeta_k$, while $M_{\bar{0}} \cong \mathbb{C}^n \big/ i\,\mathbb{Z}^n$ is the underlying reduced classical complex Lie subgroup of M, with $\mathbb{C}^n \big/ i\,\mathbb{Z}^n$ denoting quotient on the imaginary part. In particular, $M_{\bar{0}}$ is the complexification of the real classical torus $T_n = G_{\bar{0}}$, that it contains as a maximal torus. From the splitting

$$M \;=\; M_{\bar{0}} \times M_{\bar{1}} \;=\; \mathbb{C}^n \big/ i\,\mathbb{Z}^n \times \textstyle\bigwedge_k^{\mathbb{C}} \tag{1.3}$$

we shall use local charts of the form $(\, z \,, \zeta \,) = (z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_k)$. Then, both for $\mathcal{O}_{M_{\bar{0}}}(M_{\bar{0}}) = C^\infty(M_{\bar{0}})$ and $\mathcal{O}_{M_{\bar{1}}}(M_{\bar{1}}) = \Lambda_{\mathbb{C}}(\zeta_1, \ldots, \zeta_k)$ we fix the real structure given by setting $z_r = x_r + i\, y_r$ and $\zeta_s = \xi_s + i\, \eta_s$, for all r and s; accordingly, as a real manifold M is described by local charts of the form

$$(\, x \,, y \,, \xi \,, \eta \,) \;:=\; (x_1, \ldots, x_n, y_1, \ldots, y_n, \xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k) \;. \tag{1.4}$$

Now G identifies with a real super subgroup of M, described by the (local chart) variables $(\, y \,, \xi \,)$; then we have the natural G–action on M, as left action of a super subgroup.

We shall define the super Kähler forms on M (Definition 4.2) and their moment maps $\Phi : M \longrightarrow \mathfrak{g}^*$ (Definition 4.4). We identify $\mathfrak{g}^* \cong \mathbb{R}^{n|k}$. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function. Its gradient map is

$$F' : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad , \quad F'(x) \;:=\; \left(\frac{\partial F}{\partial x_1}(x), \ldots, \frac{\partial F}{\partial x_n}(x) \right) \qquad \forall\;\, x \in \mathbb{R}^n \quad .$$

We say that F is strictly convex if its Hessian matrix $\left(\frac{\partial^2 F}{\partial x_p \, \partial x_q} \right)$ is positive definite everywhere. The next proposition uses local coordinates (x, y, ξ, η) as in (1.4).

Theorem 1.2. *Let* G *be the super torus. Every* G*–invariant exact super Kähler form on* M *can be expressed as*

$$\omega \;=\; \sum_{p,q=1}^{n} \frac{\partial^2 F}{\partial x_p \partial x_q} \, dx_p \wedge dy_q \;+\; \sum_{r=1}^{k} \Big((d\xi_r)^2 + (d\eta_r)^2 \Big) \;,$$

where $F \in C^\infty\big(\mathbb{R}^n\big)$ *is a strictly convex function. Its moment map is*

$$\Phi : M \longrightarrow \mathfrak{g}^* \quad , \quad (x, y, \xi, \eta) \,\mapsto\, \Phi(x, y, \xi, \eta) = \big(-F'(x) \,,\, 2\,\xi \,\big) \;. \tag{1.5}$$

Fix a super Kähler form ω as given above. We extend the standard machinery of geometric quantization [14] to the super setting and obtain a holomorphic Hermitian line bundle $\mathbb{L}$ on M. Let $\mathcal{H}(\mathbb{L})$ denote its holomorphic sections. We define the star operator $f \mapsto f^*$ on $C^\infty(\mathbb{L})$ — see (5.8) — then apply Berezin integration [20] to construct the super Hilbert space (see Definition 5.1)

$$\mathcal{H}^2(\mathbb{L}) \;:=\; \left\{ f \in \mathcal{H}(\mathbb{L}) \;\middle|\; \int_M f f^* \, d\mathcal{B} \quad \text{converges} \right\} \;. \tag{1.6}$$

The G–representation on $\mathcal{H}^2(\mathbb{L})$ is not unitary, nevertheless it has a unique largest subrepresentation in which G acts unitarily, and we study its irreducible subrepresentations. Let $\mathrm{Im}(\Phi)_{\bar{0}} \subset \mathbb{R}^n$ denote the even part of the image of Φ. Recall also that $\widehat{G} = \left\{ V_\lambda^\pm \,\middle|\, \lambda \in \mathbb{Z}^n \right\}$.

Theorem 1.3. *Let G be the super torus. Then $\mathcal{H}^2(\mathbb{L})$ is a super Hilbert space, and $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$ is its largest G–subrepresentation in which the G–action is unitary. Moreover, $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$ is multiplicity free, with V_λ^+ occurring if and only if $\lambda \in \mathrm{Im}(\Phi)_{\bar{0}}$. Also, V_λ^- does not occur in $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$, for any integral weight λ.*

Theorems 1.2 and 1.3 enable us to construct unitary G–representations of various sizes, depending on the images of F'. We shall illustrate this in Example 5.5, where $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$ can be 0, an irreducible G–representation, or a sum of all the even representations $\{V_\lambda^+\}_\lambda$.

The above discussions handle the super torus, and we now consider the general connected Abelian Lie supergroup (1.1). The Lie algebra of the additive group $\mathbb{R}^m$ is just $\mathbb{R}^m$ itself, and its exponential map $\mathbb{R}^m \longrightarrow \mathbb{R}^m$ is the identity map. In this way, (1.2) extends to

$$\begin{array}{ccc} \mathfrak{t}_n \times \mathbb{R}^m & \xrightarrow{\lambda} & i\,\mathbb{R} \\ \downarrow & & \downarrow \\ T_n \times \mathbb{R}^m & \xrightarrow{\chi_\lambda} & S^1 \end{array} \;. \tag{1.7}$$

We say that λ is integral if there exists χ_λ such that (1.7) is a commutative diagram. If we write $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \in i\,\mathfrak{t}^*$ and $\lambda_2 \in i\,\big(\mathbb{R}^m\big)^*$, then λ_2 does not impose any obstruction to the existence of χ_λ. So λ is integral if and only if λ_1 is integral. The integral weights are identified with $\widehat{G_{\bar{0}}}$, so

$$\widehat{G_{\bar{0}}} \cong \mathbb{Z}^n \times \mathbb{R}^m \;.$$

Theorem 1.1 generalizes to the following.

Theorem 1.4. *Let $G = T_n \times \mathbb{R}^m \times \bigwedge_k^{\mathbb{R}}$. There exists a group isomorphism*

$$\widehat{G} \;\cong\; \widehat{G_{\bar{0}}} \times \mathbb{Z}_2 \;\cong\; \mathbb{Z}^n \times \mathbb{R}^m \times \mathbb{Z}_2$$

where the group structure of $\widehat{G}$ is given by the tensor product of representations.

The representation space parametrized by $(\lambda, \varepsilon) \in (\mathbb{Z}^n \times \mathbb{R}^m) \times \mathbb{Z}_2$ has dimension $1|0$ (resp. $0|1$) if $\varepsilon = \bar{0}$ (resp. $\varepsilon = \bar{1}$), and its vectors v satisfy $t \cdot v = \chi_\lambda(t) v$ for all $t \in T_n \times \mathbb{R}^m$ and $\xi \cdot v = 0$ for all $\xi \in \mathfrak{g}_{\bar{1}}$.

Let M be the complexification of G. This is the Lie supergroup with Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}$, such that M and G have the same maximal compact subgroup. So (1.3) extends to

$$M = M_{\bar{0}} \times M_{\bar{1}} = \mathbb{C}^n / i\,\mathbb{Z}^n \times \mathbb{C}^m \times \textstyle\bigwedge_k^{\mathbb{C}} . \tag{1.8}$$

We again consider G–invariant Kähler forms on M, and prove the following theorem.

Theorem 1.5. *Let ω be a super Kähler form on M with a $G_{\bar{0}}$-invariant potential function. Then ω can be expressed as*

$$\omega = \sum_{p,q=1}^{n+m} \frac{\partial^2 F}{\partial x_p \partial x_q}\, dx_p \wedge dy_q + \sum_{r=1}^{k} \Big((d\xi_r)^2 + (d\eta_r)^2 \Big)$$

where $F \in C^\infty\big(\mathbb{R}^{n+m}\big)$ is a strictly convex function. Its moment map is

$$\Phi : M \longrightarrow \mathfrak{g}^* \ , \quad (x,y,\xi,\eta) \mapsto \Phi(x,y,\xi,\eta) = \big(-F'(x) \, , \, 2\,\xi \big) \, .$$

While Theorems 1.2 and 1.5 resemble each other, there is a subtle difference due to the topologies of T_n and $\mathbb{R}^m$. We explain this in Remark 6.1.

We similarly perform geometric quantization and obtain a super Hilbert space $\mathcal{H}^2(\mathbb{L})$. It contains $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$ as the largest G-subrepresentation in which the G–action is unitary, and we consider the irreducible unitary subrepresentations which occur in $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$, However, by Theorem 1.4, $\widehat{G}$ contains the factor $\mathbb{R}^m$, whose Plancherel measure provides zero measure on each member (unlike $\mathbb{Z}^n$, whose members have point mass). For this reason, the occurrence of a subrepresentation is understood as appearance in the direct integral decomposition of $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$, see Definition 6.2. With this in mind, Theorem 1.3 extends to the following theorem.

Theorem 1.6. *Let $G = T_n \times \mathbb{R}^m \times \bigwedge_k^{\mathbb{R}}$. Then $\mathcal{H}^2(\mathbb{L})$ is a super Hilbert space, and $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$ is its largest G–subrepresentation in which the G–action is unitary. Moreover, $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$ is multiplicity free, with V_λ^+ occurring if and only if $\lambda \in \mathrm{Im}(\Phi)_{\bar{0}}$. Also, V_λ^- does not occur in $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$, for any integral weight λ.*

According to Gelfand, a *model* of a Lie group is a unitary representation on a Hilbert space in which every irreducible representation occurs exactly once [12]. The model of $G_{\bar{0}}$ has been constructed in [6, Cor.3.3]. It is natural to extend this notion to the super setting, so we say that a model of G is a unitary representation on a super Hilbert space in which every member of $\widehat{G}$ occurs once. We now construct a model.

By Theorem 1.3, the odd representations V_λ^- do not occur in $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$. To remedy this defect, let us recall that for the category $(\text{sspaces})_{\mathbb{C}}$ of complex superspaces, there exists an involutive endofunctor $\Pi : (\text{sspaces})_{\mathbb{C}} \longrightarrow (\text{sspaces})_{\mathbb{C}}$ that is defined on objects by switching parity. Thus Π is the identity on each object as a vector space but reverses the built-in $\mathbb{Z}_2$–grading (and is the identity on morphisms). If $\mathfrak{g}$ is any Lie superalgebra and $(\mathfrak{g}\text{–smod})_{\mathbb{C}}$ is the category of complex $\mathfrak{g}$–supermodules, then Π actually restricts to an endofunctor of $(\mathfrak{g}\text{–smod})_{\mathbb{C}}$ too — the $\mathfrak{g}$–action on each $\mathfrak{g}$–module being kept untouched, namely

$$\Pi V_\lambda^+ \;=\; V_\lambda^- \;. \tag{1.9}$$

We apply Theorems 1.5 and 1.6 to construct a model of G as follows.

Corollary 1.7. *Let F be a strictly convex function such that F' is surjective. Then $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big) \oplus \Pi\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$ is a model of G.*

In view of Theorem 1.6, one might wonder if $\mathcal{H}\big(\mathbb{L}\big)$ contains any G–subrepresentation beyond $\mathcal{H}\big(\mathbb{L}_{\bar{0}}\big)$ which is irreducible or unitarizable (apart from using the L^2-structure (1.6)). In this respect, we find the following answer, in the negative:

Theorem 1.8. *Every irreducible or unitarizable G–subrepresentation of $\mathcal{H}(\mathbb{L})$ is contained in $\mathcal{H}(\mathbb{L}_{\bar{0}})$.*

We organize the sections of this article as follows. Section 2 recalls the notions and language of Lie superalgebras and Lie supergroups. Section 3 proves Theorem 1.1, which classifies the unitary irreducible representations of the real super torus G. Section 4 proves Theorem 1.2, which classifies the G–invariant super Kähler forms on the complex super torus, and studies their moment maps. Section 5 proves Theorem 1.3, and provides Example 5.5. Section 6 extends the above results to general connected Abelian Lie supergroups G and proves Theorems 1.4, 1.5 and 1.6. They lead to Corollary 1.7, which constructs a model of G in terms of $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$. Section 6 also proves Theorem 1.8, which restricts the irreducibility and unitarizability of subrepresentations of $\mathcal{H}\big(\mathbb{L}\big)$.

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2. Reminders of supergeometry

In this section, we recall the notions and language of Lie superalgebras and Lie supergroups. Everything indeed is standard matter, we just fix the terminology.

2.1. **Basic superobjects.** All throughout the paper, we work over a field $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$. By $\mathbb{K}$–supermodule, or $\mathbb{K}$–super vector space, we mean any $\mathbb{K}$-module V endowed with a $\mathbb{Z}_2$–grading $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is the group with two elements, which we also write as $\{+, -\}$ using then multiplicative notation. Then $V_{\bar{0}}$ and its elements are called even, $V_{\bar{1}}$ and its elements odd. By $|x| \in \mathbb{Z}_2$ we denote the parity of any non-zero homogeneous element, defined by the condition $x \in V_{|x|}$.

We call $\mathbb{K}$–superalgebra any associative, unital $\mathbb{K}$–algebra A which is $\mathbb{Z}_2$–graded: so A has a $\mathbb{Z}_2$–grading $A = A_{\bar{0}} \oplus A_{\bar{1}}$, and $A_{\bar{a}} A_{\bar{b}} \subseteq A_{\bar{a}+\bar{b}}$. Any such A is said to be commutative if $xy = (-1)^{|x||y|} yx$ for all homogeneous $x, y \in A$; so, in particular, $z^2 = 0$ for all $z \in A_{\bar{1}}$. All $\mathbb{K}$–superalgebras form a category, whose morphisms are those of unital $\mathbb{K}$–algebras preserving the $\mathbb{Z}_2$–grading; inside it, commutative $\mathbb{K}$–superalgebras form a subcategory, that we denote by $(\mathbf{salg})$. We denote by $(\mathbf{alg})$ the category of (associative, unital) commutative $\mathbb{K}$–algebras, and by $(\mathbf{mod})$ that of $\mathbb{K}$–modules. There exists an obvious functor $(\ )_{\bar{0}} \colon (\mathbf{salg}) \longrightarrow (\mathbf{alg})$ given on objects by $A \mapsto A_{\bar{0}}$.

2.2. **Lie superalgebras.** A Lie superalgebra over the field $\mathbb{K}$ is a $\mathbb{K}$–supermodule $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a Lie super bracket $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $(x, y) \mapsto [x, y]$, which is $\mathbb{K}$–bilinear, preserves the $\mathbb{Z}_2$–grading and obeys, for all homogenenous $x, y, z \in \mathfrak{g}$,

(a) $\quad [x, y] + (-1)^{|x||y|}[y, x] = 0$ (*anti-symmetry*),

(b) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ (*Jacobi identity*).

All Lie $\mathbb{K}$–superalgebras form a category, denoted by $(\mathbf{sLie})_{\mathbb{K}}$, whose morphisms are $\mathbb{K}$–linear, preserving the $\mathbb{Z}_2$–grading and the bracket. Note that if $\mathfrak{g}$ is a Lie $\mathbb{K}$–superalgebra, then its even part $\mathfrak{g}_{\bar{0}}$ is automatically a Lie $\mathbb{K}$–algebra.

2.3. **Supermanifolds and supergroups.** We now recall the notion of supermanifold and (Lie) supergroup, very quickly: see [2, 7, 20] for more details.

Superspaces. A superspace is a pair $S = (|S|, \mathcal{O}_S)$ of a topological space $|S|$ and a sheaf of commutative superalgebras $\mathcal{O}_S$ on it such that the stalk $\mathcal{O}_{S,x}$ of $\mathcal{O}_S$ at each $x \in |S|$ is a local superalgebra. A morphism $\phi \colon S \longrightarrow T$ between superspaces is a pair $(|\phi|, \phi^*)$ where $|\phi| \colon |S| \longrightarrow |T|$ is a continuous map and $\phi^* \colon \mathcal{O}_T \longrightarrow |\phi|_*(\mathcal{O}_S)$ is a morphism of sheaves on $|T|$ is such that $\phi^*_x(\mathfrak{m}_{|\phi|(x)}) \subseteq \mathfrak{m}_x$, where $\mathfrak{m}_{|\phi|(x)}$ and $\mathfrak{m}_x$ denote the unique maximal ideals in the stalks $\mathcal{O}_{T,|\phi|(x)}$ and $\mathcal{O}_{S,x}$, respectively.

As basic model, the holomorphic linear supervariety $\mathcal{H}_{\mathbb{C}}^{p|q}$ is, by definition, the topological space $\mathbb{C}^p$ endowed with the following sheaf of commutative superalgebras: $\mathcal{O}_{\mathcal{H}_{\mathbb{C}}^{p|q}}(U) := \mathcal{H}_{\mathbb{C}^p}(U) \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}(\xi_1, \dots, \xi_q) =: \mathcal{H}_{\mathbb{C}^{p|q}}(U)$ for any open set $U \subseteq \mathbb{C}^p$, where $\mathcal{H}_{\mathbb{C}^p}$ is the sheaf of holomorphic functions on $\mathbb{C}^p$, and $\Lambda_{\mathbb{C}}(\xi_1, \dots, \xi_q)$ is the complex Grassmann algebra on q variables $\xi_1, \dots, \xi_q$ of odd parity. A holomorphic supermanifold of super dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ such that $|M|$ is Hausdorff and second-countable, and M is locally isomorphic to $\mathcal{H}_{\mathbb{C}}^{p|q}$, i.e., for each $x \in |M|$ there is an open set $V_x \subseteq |M|$ with $x \in V_x$ and $U \subseteq \mathbb{C}^p$ such that $\mathcal{O}_M\Big|_{V_x} \cong \mathcal{O}_{\mathcal{H}_{\mathbb{C}}^{p|q}}\Big|_U$. A morphism between holomorphic supermanifolds is just a morphism between them as superspaces. We denote the category of holomorphic supermanifolds by (**hsmfd**) .

With a similar construction, one defines objects and morphisms in the category (**ssmfd**) of all real smooth supermanifolds. This is done by replacing the basic model $\mathcal{H}_{\mathbb{C}}^{p|q}$ with its real, smooth counterpart given by the topological space $\mathbb{R}^p$ endowed with the sheaf of commutative superalgebras $\mathcal{O}_{\mathcal{C}_{\mathbb{R}}^{p|q}}(U) := C^\infty_{\mathbb{R}^p}(U) \otimes_{\mathbb{R}} \Lambda_{\mathbb{R}}(\xi_1, \dots, \xi_q) =: C^\infty_{\mathbb{R}^{p|q}}(U)$ for any open set $U \subseteq \mathbb{R}^p$, with $C^\infty_{\mathbb{R}^p}$ being the sheaf of smooth functions on $\mathbb{R}^p$, and $\Lambda_{\mathbb{R}}(\xi_1, \dots, \xi_q)$ the real Grassmann algebra on q odd variables $\xi_1, \dots, \xi_q$.

Given a real smooth supermanifold M and an open subset U in $|M|$, let us choose a local chart around a point in U : then the even coordinates x_i in this chart along with the ξ_j's — which play the role of global odd coordinates — provide a smooth local chart for U (at the chosen point) as a superspace, which we will later denote by $(x, \xi) := (x_1, \dots, x_p, \xi_1, \dots, \xi_q)$. Similarly, for any given holomorphic supermanifold one defines the notion of holomorphic local chart around any point in $|M|$. See [3] for further details.

Let now M be a holomorphic supermanifold and U an open subset in $|M|$. Let $\mathcal{I}_M(U)$ be the ideal of $\mathcal{O}_M(U)$ generated by the odd part of the latter: then $\mathcal{O}_M \big/ \mathcal{I}_M$ defines a sheaf of purely even superalgebras over $|M|$, locally isomorphic to $\mathcal{H}_{\mathbb{C}^p}$. Then the reduced manifold $M_{\mathrm{rd}} := \big(|M| , \mathcal{O}_M / \mathcal{I}_M \big)$ is a classical holomorphic manifold, called the underlying holomorphic submanifold of M . The projection $s \mapsto \tilde{s} := s + \mathcal{I}_M(U)$, for $s \in \mathcal{O}_M(U)$, at the sheaf level yields an embedding $M_{\mathrm{rd}} \longrightarrow M$, so M_{rd} can be seen as an embedded sub-supermanifold of M . The whole construction is functorial in M .

The same construction applies to real smooth supermanifolds as well.

Finally, each classical manifold — either complex holomorphic or real smooth — can be seen as a supermanifold, just by regarding its structure sheaf as one of superalgebras that have trivial odd part. Conversely, any supermanifold enjoying the latter property is a classical manifold.

Lie supergroups. Any group object in the category (**hsmfd**) is called a holomorphic Lie supergroup. These objects, together with their obvious morphisms, form a subcategory among supermanifolds, denoted by $(\mathbf{Lsgrp})_{\mathbb{C}}$. Similarly, the group objects in the category (**ssmfd**) are called real smooth Lie supergroups: together with their obvious morphisms, they form a subcategory $(\mathbf{Lsgrp})_{\mathbb{R}}$ of (**ssmfd**) .

Much like in the classical setup, there exists a functor $\mathrm{Lie} : (\mathbf{Lsgrp})_{\mathbb{C}} \longrightarrow (\mathbf{sLie})_{\mathbb{C}}$ which links holomorphic Lie supergroups to complex Lie superalgebras, and a similar one $\mathrm{Lie} : (\mathbf{Lsgrp})_{\mathbb{R}} \longrightarrow (\mathbf{sLie})_{\mathbb{R}}$ linking real Lie supergroups to real Lie superalgebras. In both cases, the super version of the correspondence between Lie groups and Lie algebras hold true.

A relevant aspect of the theory of holomorphic or smooth supermanifolds is that they can be entirely studied in terms of the algebra of global sections of their structure sheaf, i.e. $\mathcal{O}_M(M)$ for any supermanifold M. In addition, a key feature of supergroups is that they admit a global splitting, i.e. for any supergroup G we have superalgebra isomorphisms $\mathcal{O}_G(G) \cong \mathcal{O}_{G_{\bar{0}}}\big(G_{\bar{0}}\big) \underset{\mathbb{K}}{\otimes} \Lambda_{\mathbb{K}}(\xi_1, \dots, \xi_q)$ — with q such that the super dimension of G is $p|q$. Geometrically, this means that there exist supermanifold splittings $G \cong G_{\bar{0}} \times \bigwedge_q^{\mathbb{K}}$, where $\bigwedge_q^{\mathbb{K}}$ is the supermanifold given by $\big|\bigwedge_q^{\mathbb{K}}\big| := \{*\}$ and $\mathcal{O}_{\bigwedge_q^{\mathbb{K}}}\big(\{*\}\big) := \Lambda_{\mathbb{K}}(\xi_1, \dots, \xi_q)$ — i.e., a single point endowed with a purely odd, q–dimensional affine superstructure; see, e.g., [3] and [10, 11].

2.4. **Alternative approaches.** In the present work we adopt the approach to (differential or holomorphic) supergeometry that is centered on the viewpoint of superringed spaces. We base our construction of supergeometry on the category of commutative superalgebras, as its classical counterpart is based upon the category of commutative algebras. However, everything works equally well — including whatever we do in the present paper — if one adopts instead the similar construction based upon the category of Weil superalgebras (and Weil algebras in the non-super setup), which are mild generalizations of Grassmann algebras: see [2] and related works for more details on this point of view.

Furthermore, essentially any other approach — such as that of the functor of points, or that of manifolds with super-calculus, like in the viewpoint of DeWitt [8], Rogers [16] or Tuynman [18] — will work fine as well: that is because the supergroups we will be dealing with, namely supertori, are so "nice" that simply there is no room left for relevant differences among different paths. Technically speaking, the key steps boil down to a local analysis around single points: this makes use of (smooth or holomorphic) local charts, which can be done equally well with either one of the above mentioned approaches.

2.5. **Unitarity issues.** Let us now introduce the question of unitarity. We recall unitary Lie superalgebras in a generalized sense, as considered in [9], §4. Let V be a complex super vector space endowed with a generalized real form B (in the sense of [9]) and with non-degenerate, consistent Hermitian form with respect to B. Accordingly, a generalized real structure is defined in the linear Lie superalgebra $\mathfrak{gl}(V)$ by taking for any $M \in \mathfrak{gl}(V)$ the adjoint operator $M^\star \in \mathfrak{gl}(V)$ with respect to the Hermitian form. The real form of $\mathfrak{gl}(V)$ associated with such a real structure — i.e. the Lie sub-superalgebra of fixed points of the real structure in $\mathfrak{gl}(V)$ — is called the unitary Lie superalgebra of (V, B), denoted by $\mathfrak{u}_B(V)$. Given a complex Lie superalgebra $\mathfrak{g}$, any representation V of $\mathfrak{g}$ is said to be unitary if there exists a suitable non-degenerate, consistent Hermitian form B on V such that the action of $\mathfrak{g}$ on V factors through $\mathfrak{u}_B(V)$, i.e. $\mathfrak{g}$ acts on V via unitary operators with respect to B.

All these notions are given for generalized real structures and real forms that might be of two types, known as *standard* or *graded* type. They are also formulated for complex Lie superalgebras, their functorial versions, and for complex Lie supergroups. The standard type is the simpler one, where one has

$$\mathfrak{u}_B(V) \;=\; \Big\{ u \in \mathfrak{gl}(V) \,\Big|\, B\big(u(v), w\big) + (-1)^{|u||v|} B\big(v, u(w)\big) = 0 \,,\; v, w \in V_{\bar{0}} \cup V_{\bar{1}} \Big\} \,. \tag{2.1}$$

In other words, we may also express the same by saying that $\mathfrak{u}_B(V)$ is the subset of all elements of $\mathfrak{gl}(V)$ which fix the form B.

In this paper, we will always assume to be in the standard case. Namely, unitary will be always meant with respect to some non-degenerate, consistent Hermitian form B on a superspace V endowed with a standard real structure.

The following are basic results about unitary representations, which are proved exactly as in the non-super framework:

Lemma 2.1. *Let V be a unitary representation of a complex Lie superalgebra or a complex Lie supergroup with respect to some form B . If W is a subrepresentation of V, then its orthogonal space*

$$W^{\perp} \;:=\; \big\{\, v \in V \,\big|\, B(v, w) = 0 = B(w, v) \;\; \forall\, w \in W \,\big\}$$

is a submodule as well. □

A representation is said to be completely reducible if it is the direct sum of irreducible subrepresentations. The previous lemma, together with an induction argument, yields the following:

Proposition 2.2. *Every finite dimensional unitary representation of a complex Lie superalgebra or a complex Lie supergroup is completely reducible.*

2.6. **Abelian connected Lie supergroups.** We shall work now with a connected Abelian real Lie supergroup $G := T_n \times \mathbb{R}^m \times \bigwedge_k^{\mathbb{R}}$, with tangent Lie superalgebra $\mathfrak{g} := \mathrm{Lie}(G)$.

Inside G , we consider the normal subsupergroups

$$G_+ \;:=\; T_n \times \mathbb{R}^m \times \big\{ 1_{\bigwedge_k^{\mathbb{R}}} \big\} \quad , \qquad G_- \;:=\; \big\{ 1_{T_n \times \mathbb{R}^m} \big\} \times \textstyle\bigwedge_k^{\mathbb{R}}$$

for which hereafter we will adopt standard identifications $G_+ \cong T_n \times \mathbb{R}^m$ and $G_- \cong \bigwedge_k^{\mathbb{R}}$. Note that G_+ coincides with $G_{\bar{0}}$ in previous notation. On the other hand, in a general Lie supergroup G' there exists no canonical analogue of the subsupergroup G_- that is a natural counterpart of $G'_{\bar{0}}$: indeed G_- is a specific peculiarity of the case under study.

By definition $G_- = \bigwedge_k^{\mathbb{R}}$ is the spectrum of $\Lambda_{\mathbb{R}}^k\big(\underline{\xi}\big) := \Lambda_{\mathbb{R}}\big(\xi_1, \dots, \xi_k\big)$, the real Grassmann algebra in k generators $\xi_1 , \dots , \xi_k$, which are assumed to be homogeneous with odd parity. In particular, when one thinks at G_- as a group–valued functor, for every commutative $\mathbb{R}$–superalgebra A the A–points of G_- are given by

$$G_-(A) \;=\; \big\{\, (\alpha_1, \dots, \alpha_k) \,\big|\, \alpha_i \in A_{\bar{1}} \;\forall\, i = 1, \dots, k \,\big\} \;=:\; A_{\bar{1}}^{k}$$

thought of as an Abelian group for the additive structure of the $\mathbb{R}$–module $A_{\bar{1}}^{k}$. In other words, G_- is nothing but the real affine, entirely odd superspace of dimension $0|k$, usually denoted by $\mathbb{A}_{\mathbb{R}}^{0|k}$, that we now regard as a Lie supergroup.

Both G_+ and G_- are connected Abelian real Lie supergroups on their own, with G_+ being entirely even and G_- entirely odd — in that this is what occurs with their tangential Lie superalgebras. Indeed, for the latter we have $\mathfrak{g}_+ := \mathrm{Lie}(G_+) = \mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_- := \mathrm{Lie}(G_-) = \mathfrak{g}_{\bar{1}}$. Finally, G is isomorphic to the direct product

$$G \;=\; G_+ \times G_- \qquad \textit{(direct product of supergroups)} \tag{2.2}$$

at the tangent level, the splitting $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is indeed decomposition of $\mathfrak{g}$ into direct sum of the Lie superalgebras $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$.

In the sequel, we denote by M the complexification of G: hence M is a complex Lie supergroup, whose tangent Lie superalgebra is $\mathrm{Lie}(M) = \mathfrak{g}_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$, i.e. the complexification of $\mathfrak{g}$.

3. Irreducible unitary representations

Let $G = T_{n|k} := T_n \times \bigwedge_k^{\mathbb{R}}$ be the real super torus of super dimension $n|k$, and let $\mathfrak{g} := \mathrm{Lie}(G) = \mathfrak{t}_{n|k}$ be its tangent Lie superalgebra. In algebraic terms, $T_{n|k}$ is the real Lie supergroup associated with $\mathcal{O}\big(T_{n|k}\big) = C^\infty\big(T_n\big) \otimes \Lambda_{\mathbb{R}}(\xi_1, \dots, \xi_k)$, where $\Lambda_{\mathbb{R}}(\xi_1, \dots, \xi_k)$ is the Grassmann algebra in the k odd variables $\xi_1 , \dots , \xi_k$, with the unique Hopf structure for which every ξ_i is primitive. Here $G_{\bar{0}} = T_n$ is the classical Lie group associated with G, namely the n–dimensional real torus.

A unitary representation of G is a super vector space with a G–invariant super Hermitian form — cf. [9, §4.1]; see also (4.3). Let $\widehat{G}$ be the set of unitary irreducible representations of G , up to equivalence. In this section, we prove Theorem 1.1, which shows that $\widehat{G} \cong \widehat{G_{\bar{0}}} \times \mathbb{Z}_2 \cong \mathbb{Z}^n \times \mathbb{Z}_2$. It identifies $\widehat{G}$ with the set of pairs (λ , ϵ) , where $\lambda \in \mathbb{Z}^n$ is an integral weight — see (1.2) — and $\varepsilon \in \mathbb{Z}_2$ denotes the parity of the representation space under scrutiny. We also provide a realization of the representation space of (λ , ε) in terms of holomorphic functions on the complexification of T_n , see Example 3.2.

Let $V \in \widehat{G}$; then V is a $\mathfrak{g}$–module, irreducible and unitary. By $\rho : G \longrightarrow GL(V)$ and $d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ we denote the associated representation maps. In the sequel, $V = V_{\bar{0}} \oplus V_{\bar{1}}$ will be the super $\mathbb{Z}_2$–grading of V. We need the following result:

Lemma 3.1. *Let $\mathcal{L}$ be an Abelian Lie sub-superalgebra of $\mathfrak{gl}(r|s)$. Then $\mathcal{L}$ can be put in triangular form. In other words, for some suitable change of basis in $V := \mathbb{R}^{r|s}$ one has that $\mathcal{L}$ turns into a subalgebra of upper triangular matrices.*

The result above is [22, Lem.2.1] in a slightly simplified (weaker) form. In turn, that lemma follows from [21, Lem.6.3], which is a suitable formulation of Engel's

Theorem adapted to Lie superalgebras, claiming that for any Lie super-subalgebra $\mathcal{L}$ of $\mathfrak{gl}(W)$ acting on some superspace W by nilpotent transformations, there exists a common eigenvector (of eigenvalue 0) — namely there exists $w \in W \setminus \{0\}$ such that $\mathcal{L}.w = 0$.

Proof of Theorem 1.1:

Let V be an irreducible G–representation. By definition, G is Abelian, hence $\mathfrak{g} = \mathrm{Lie}(G)$ is Abelian too. Then we apply Lemma 3.1 to $\mathcal{L} := d\rho(\mathfrak{g})$, a subalgebra of $\mathfrak{gl}(V)$. Setting $r|s := \mathrm{sdim}(V)$, if $r + s > 1$ then Lemma 3.1 implies that V is reducible, contrary to our assumption. Therefore it is $r + s = 1$, hence

$$\mathrm{sdim} V = 1|0 \quad \text{or} \quad \mathrm{sdim} V = 0|1 \; . \tag{3.1}$$

We write $V = V^+$ and $V = V^-$ accordingly, for the even and odd cases respectively.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be the $\mathbb{Z}_2$–grading, in particular, $\mathfrak{g}_{\bar{0}} = Lie\big(G_{\bar{0}}\big)$. For any G–module $W = W_{\bar{0}} \oplus W_{\bar{1}}$ one has

$$G_{\bar{0}} \, . \, W_{\bar{s}} \, = \, W_{\bar{s}} \;\; , \quad \mathfrak{g}_{\bar{0}} \, . \, W_{\bar{s}} \, = \, W_{\bar{s}} \;\; , \quad \mathfrak{g}_{\bar{1}} \, . \, W_{\bar{s}} \, = \, W_{\bar{s}+\bar{1}} \qquad \forall \;\, \bar{s} \in \big\{ \bar{0} \, , \bar{1} \big\} \; .$$

Applying this to the irreducible G–modules $W = V^{\pm}$, by (3.1), we get that $\mathfrak{g}_{\bar{1}}$ acts trivially, namely

$$\xi \cdot v = 0 \qquad \forall \;\, \xi \in \mathfrak{g}_{\bar{1}} \; , \; v \in V^{\pm} \; . \tag{3.2}$$

Hence $V^{\pm}$ are no more than sheer $G_{\bar{0}}$–modules equipped with trivial $\mathfrak{g}_{\bar{1}}$–action.

If we regard $V = V^{\pm}$ as an ordinary vector space with $G_{\bar{0}}$–action, then by compactness of $G_{\bar{0}}$, V has a $G_{\bar{0}}$–invariant inner product H. If $V = V^+$, then by (3.2) it is a unitary G–representation with respect to H. If $V = V^-$, then $i\,H$ is a super Hermitian form on V^-, cf. [9, §4]. By (3.2), V^- is a unitary G–representation with respect to $i\,H$; thus we have shown that $V^{\pm}$ is unitarizable.

Conversely, every irreducible $G_{\bar{0}}$–representation is 1-dimensional, and so is the restriction of some irreducible G–module V^+ to $G_{\bar{0}}$. Hence

$$\widehat{G} = \widehat{G_{\bar{0}}} \times \mathbb{Z}_2 \; . \tag{3.3}$$

In (3.3) the $\mathbb{Z}_2$ component controls the parity of the representation space. The $\widehat{G_{\bar{0}}}$ component amounts to the integral weights λ of T_n, or equivalently their characters χ_λ, see (1.2). We write $V_\lambda^\varepsilon \in \widehat{G}$ accordingly, where $\varepsilon \in \mathbb{Z}_2 = \{+, -\}$. Its elements v satisfy $g \cdot v = \chi_\lambda(g) v$ for all $g \in G_{\bar{0}}$; for integral weights λ and μ, their characters satisfy $\chi_\lambda \chi_\mu = \chi_{\lambda+\mu}$. So the tensor product of representations leads to $V_\lambda^\varepsilon \otimes V_\mu^\delta = V_{\lambda+\mu}^{\varepsilon\delta}$. In other words, identifying the integral weights of T_n with $\mathbb{Z}^n$, the tensor product of representations yields a group isomorphism $\widehat{G} \cong \mathbb{Z}^n \times \mathbb{Z}_2$. All this eventually proves Theorem 1.1. $\square$

The following example provides a realization of the representation space associated with $(\lambda \, , \varepsilon) \in \widehat{G}$.

Example 3.2. We consider

$$T_n := \mathbb{R}^n/\mathbb{Z}^n \quad , \qquad X := \mathbb{C}^n/i\,\mathbb{Z}^n$$

where the quotient on $\mathbb{C}^n$ is made on the imaginary part. We denote their elements by $[r] \in T_n$ and $\big(x + i\,[y]\big) \in X$, where $r, x, y \in \mathbb{R}^n$. The Lie algebra of T_n then is $\mathfrak{t}_n \cong \mathbb{R}^n$. Since we have written T_n as an additive group, its exponential map $e : \mathfrak{t}_n \longrightarrow T_n$ reads simply $e^r = [r]$.

Let $\lambda \in \mathbb{Z}^n$. We regard it as $\lambda \in i\,\mathfrak{t}^*$ by

$$\lambda * r = i\,(\lambda_1 r_1 + ... + \lambda_n r_n) \quad \forall\, r \in \mathbb{R}^n \cong \mathfrak{t}\,.$$

In this example only, we use the notation $\lambda * r \in i\,\mathbb{R}$ to distinguish it from $\lambda\, r = \lambda_1 r_1 + ... + \lambda_n r_n \in \mathbb{R}$. Its character (see (1.2)) is

$$\chi_\lambda : T_n \longrightarrow S^1 \quad , \quad \chi_\lambda\big([r]\big) := e^{\lambda * r} \tag{3.4}$$

where we fix the normalization $2\pi \sim 1$. Note that (3.4) is well-defined, because $\lambda \in \mathbb{Z}^n$, and it implies $\chi_\lambda(e^r) = e^{\lambda * r}$ as required by (1.2).

Let $\mathcal{H}(X)$ denote the set of all the holomorphic functions on X. Let T_n act on the imaginary part of X, namely

$$[r] * \big(x + i\,[y]\big) := x + i\,\big([r + y]\big) \qquad \forall\ [r] \in T_n\,,\ \ x + i\,[y] \in X\,.$$

We use the holomorphic coordinates $z = x + i\,[y]$ on X. The T_n–action on X induces a T_n–representation on $\mathcal{H}(X)$ by

$$\big([r] * f\big)(z) := f\big(-[r] * z\big) \qquad \forall\ [r] \in T_n\,,\ \ f \in \mathcal{H}(X)\,.$$

Consider the holomorphic function

$$f : X \longrightarrow \mathbb{C} \quad , \quad f(z) := e^{-\lambda\, z}\,.$$

We have

$$\big([r] * f\big)(z) = f\big(x + i\,[-r + y]\big) = e^{-\lambda(x - i\,r + i\,y)} = \chi_\lambda\big([r]\big) f(z)\,. \tag{3.5}$$

Hence the span of f is the irreducible $G_{\bar 0}$–representation V_λ: as such, we can then identify it with V_λ^+ or V_λ^- depending on whether we assign to it even or odd parity.

4. Super Kähler structures

Let $G = T_n \times \bigwedge_k^{\mathbb{R}}$ be the real super torus, and let M be its complexification. In this section, we prove Theorem 1.2, which characterizes G-invariant super Kähler forms on M.

Let $T_{\mathbb{C}} = \mathbb{R}^n \times T_n$ be the complexification of T_n. We use the holomorphic coordinates $z = x + i\,y$ on $T_{\mathbb{C}}$, as given in (1.4), where T_n acts on the imaginary part. Since G has super dimension $n|k$, the summations below are made for n even indices and k odd indices.

Proposition 4.1. *Every T_n–invariant exact Kähler form on $T_{\mathbb{C}}$ can be expressed as*

$$\omega_{\bar{0}} \;=\; \sum_{p,q} \frac{\partial^2 F}{\partial x_p \partial x_q}\, dx_p \wedge dy_q$$

where $F \in C^\infty(\mathbb{R}^n)$ is a strictly convex function.

Proof. Let $\omega_{\bar{0}}$ be a T_n–invariant Kähler form on $T_{\mathbb{C}}$. Since it is a 2–form of type $(1,1)$, we write $\omega_{\bar{0}} = \sum_{p,q} f_{pq}\, dz_p \wedge d\bar{z}_q$ for some functions f_{pq}. Let $h_{p,q} = -i\,(f_{p,q} + f_{q,p})$. Then

$$\begin{aligned} \omega_{\bar{0}} &= \textstyle\sum_{p,q} f_{pq}\big(dx_p + i\,dy_p\big) \wedge \big(dx_q - i\,dy_q\big) \;= \\ &= \textstyle\sum_{p,q} h_{p,q} dx_p \wedge dy_q + \sum_{p<q} \big(f_{pq} - f_{qp}\big)\big(dx_p \wedge dx_q + dy_p \wedge dy_q\big) \;= \\ &= \textstyle\sum_{p,q} h_{p,q} dx_p \wedge dy_q + R \end{aligned} \tag{4.1}$$

where R denotes the terms containing $\big(f_{pq} - f_{qp}\big)$. Here y are the coordinates on T_n induced from the linear coordinates of $\mathbb{R}^n$, so the 2–forms $dy_p \wedge dy_q$ are not exact. Hence if $R \neq 0$, then $\omega_{\bar{0}}$ is not exact.

For $R = 0$, we follow the arguments of [5, §2]. They show that $\sum_{p,q} h_{p,q}\, dx_p \wedge dy_q$ is exact and has the desired expression of this proposition. $\square$

<u>N.B.</u>: The above proposition is a correction of the arguments in [5, §2], which overlooks the summand R of (4.1).

Let $\Omega^p(T_{\mathbb{C}}, \mathbb{C})$ denote the set of all differential p–forms of $T_{\mathbb{C}}$ with complex coefficients. Then $\Omega^\bullet(T_{\mathbb{C}}, \mathbb{C})$ is a chain complex with the deRham operator d.

Let $\bigwedge^p_{\xi,\eta}$ denote the summand of degree p in the super Grassmann algebra generated by the $d\xi_r$'s and $d\eta_s$'s. Then $\bigwedge^\bullet_{\xi,\eta}$ is also a chain complex with exterior derivative d — see [4, p.234]. The differential forms of M are the tensor product of chain complexes, namely

$$\Omega^\bullet(M, \mathbb{C}) \;=\; \Omega^\bullet(T_{\mathbb{C}}, \mathbb{C}) \otimes \textstyle\bigwedge^\bullet_{\xi,\eta}\,.$$

We still let d denote its chain map. We say that ω is closed if $d\omega = 0$, and it is exact if $\omega = d\beta$ for some β. We have $d^2 = 0$, so exact forms are closed. Since M is complex, there exists a decomposition

$$\Omega^p(M, \mathbb{C}) \;=\; \sum_{r+s=p} \Omega^{r,s}(M, \mathbb{C})\,. \tag{4.2}$$

We have $d = \partial + \bar{\partial}$, where the Dolbeault operator ∂ (resp. $\bar{\partial}$) raises the degree of r (resp. s). We say that $\omega \in \Omega^2(M, \mathbb{C})$ is a $(1,1)$–form if $r = 1 = s$ in (4.2).

We intend to define a super Kähler form ω as the imaginary part of a super Hermitian metric, with $d\omega = 0$. To do this, we recall from [9, §4.1] that a super Hermitian metric on a complex super vector space V is a map $H : V \times V \longrightarrow \mathbb{C}$ which is $\mathbb{C}$–linear (resp. $\mathbb{C}$–antilinear) in the first (resp. the second) entry, such that for all non-zero homogeneous vectors $u, v \in V$, one has

$$\begin{array}{lll} \textit{(a)} & H(u,v) = 0 \ \text{ if } \ |u| \neq |v| & \textit{(consistent)} \\ \textit{(b)} & H(u,v) = (-1)^{|u|\cdot|v|}\overline{H(v,u)} & \textit{(super Hermitian symmetric)} \\ \textit{(c)} & H(v,v) \in i^{|v|}\mathbb{R}_+ \ \ \forall\, v \neq 0 & \textit{(super positive)}\,. \end{array} \tag{4.3}$$

In (4.3)*(c)*, we make the convention that

$$i^{|v|} \in \left\{ i^0, i^1 \right\} = \{1, i\,\} \tag{4.4}$$

to avoid ambiguity arising from $i^{[2]}$ and so on.

We treat V as a real vector space, and replace multiplication by i with $J : V \longrightarrow V$, where $J^2 = -I$. Let ω be the imaginary part of H : the condition $H(iu, iv) = H(u, v)$ implies that $\omega(Ju, Jv) = \omega(u, v)$, so ω is a $(1,1)$–form. Condition (4.3)*(b)* says that ω is super skew-symmetric, namely

$$\omega(u,v) \;=\; -(-1)^{|u|\cdot|v|}\,\omega(v,u).$$

Condition (4.3)*(c)* says that $H(iv, v) \in i\,\mathbb{R}^+$ if v is even, and $H(v, v) \in i\,\mathbb{R}^+$ if v is odd, namely

$$\begin{array}{lcl} |v| = [0] & \Longrightarrow & \omega(Jv,v) \in \mathbb{R}^+ \\ |v| = [1] & \Longrightarrow & \omega(v,v) \in \mathbb{R}^+ \;. \end{array} \tag{4.5}$$

We say that ω is positive if it satisfies (4.5). The above conditions motivate the following definition.

Definition 4.2. A super Kähler form ω on M is a real closed consistent super skew-symmetric positive form $\omega \in \Omega^{1,1}(M)$.

Concerning super Kähler forms, we shall need the following result:

Proposition 4.3. *Every G–invariant exact super Kähler form on M can be written as*

$$\omega \;=\; \sum_{p,q} \frac{\partial^2 F}{\partial x_p \partial x_q}\, dx_p \wedge dy_q \,+ \sum_r \left((d\xi_r)^2 + (d\eta_r)^2 \right)$$

where $F \in C^\infty\big(\mathbb{R}^n\big)$ is a strictly convex function.

Proof. Let ω be a G–invariant exact super Kähler form on M. We write $\omega = \omega_{\bar 0} + \omega_{\bar 1}$. The description of $\omega_{\bar 0}$ is handled by Proposition 4.1; we now focus on $\omega_{\bar 1}$.

By [19, Prop.4.3], there exist real odd variables θ so that

$$\omega_{\bar 1} \;=\; \sum_{r=1}^{\ell} (d\theta_r)^2 - \sum_{s=1}^{t} (d\theta_s)^2$$

with $\ell + t = 2\,k$, the real odd dimension of M. By (4.5), the negative part $-\sum_{s=1}^{t} (d\theta_s)^2$ vanishes, i.e. $t = 0$ and $\ell = 2\,k$. We express these θ_r in terms of the coordinates ξ_r, η_r of (1.4) and obtain

$$\omega_{\bar 1} \;=\; \sum_r \left(a_r\,(d\xi_r)^2 \,+\, b_r\,(d\eta_r)^2 \right) \;, \qquad a_r\,, b_r > 0 \;. \tag{4.6}$$

Note that $d\xi_r$ commutes with $d\eta_r$, so

$$d\zeta_r \wedge d\bar{\zeta_r} \;=\; \big(d\xi_r + i\,d\eta_r\big) \wedge \big(d\xi_r - i\,d\eta_r\big) \;=\; \big(d\xi_r\big)^2 + \big(d\eta_r\big)^2.$$

Since $\omega_{\bar 1}$ is a positive $(1,1)$–form, it is a positive linear combination of $d\zeta_r \wedge d\bar{\zeta_r}$, so $a_r = b_r$ in (4.6). Due to the last step, we can also normalize each ζ_r so that $a_r = 1 = b_r$. This proves the proposition. □

Fix ω as given above. We briefly review the moment map [13, §26] of the ordinary setting, where a Lie group T acts on a manifold $M_{\bar 0}$ and preserves a symplectic form $\omega_{\bar 0}$ on $M_{\bar 0}$. Let $v \in \mathfrak{t}$, and let $v^\sharp$ be its infinitesimal vector field on $M_{\bar 0}$ defined by

$$(v^\sharp f)(m) \;:=\; \left(\frac{d}{dt} f\big(e^{tv}.m\big)\right)\bigg|_{t=0} \quad , \quad f \in C^\infty\big(M_{\bar 0}\big) \ , \ \ m \in M_{\bar 0} \ .$$

Then $\imath(v^\sharp)\omega_{\bar 0}$ is a closed 1–form on $M_{\bar 0}$. Suppose that it is exact, and there exists a T-equivariant map Φ — with respect to the coadjoint action on $\mathfrak{t}^*$ — such that

$$\Phi : M_{\bar 0} \longrightarrow \mathfrak{t}^* \quad , \quad d(\Phi, v) := \imath(v^\sharp)\omega_{\bar 0} \tag{4.7}$$

where we regard (Φ, v) as a member of $C^\infty\big(M_{\bar 0}\big)$ by $m \mapsto \big(\Phi(m)\big)(v) \in \mathbb{R}$, so that $d(\Phi, v) \in \Omega^1\big(M_{\bar 0}\big)$. Then Φ is called the *moment map* of $\omega_{\bar 0}$. It may not exist, for instance one needs the closed 1–form $\imath(v^\sharp)\omega_{\bar 0}$ to be exact, so an obstruction is the cohomology $H^1\big(M_{\bar 0}\big)$. For example, the ordinary 2–torus acting on itself preserving the invariant volume form has no moment map.

We now turn to the super setting, where there are peculiar phenomena. We consider $\omega_{\bar 1} = \sum_r \big((d\xi_r)^2 + (d\eta_r)^2\big)$ of Proposition 4.3. If the supermanifold is merely symplectic but not Kähler, this formula becomes $\sum_{r=1}^k \pm (d\theta_r)^2$ where k can be odd; hence the superdimension of a symplectic supermanifold can be odd, unlike its ordinary counterpart. The expression of $\omega_{\bar 1}$ also prevents the existence of a G–equivariant moment map, as we shall discuss in Remark 4.5 below. For this reason, we omit the G–equivariance property in the following definition.

Definition 4.4. Given a G–action on a symplectic supermanifold (M, ω) , we call *moment map* for it a map $\Phi : M \longrightarrow \mathfrak{g}^*$ which satisfies condition $d(\Phi, v) = \imath(v^\sharp)\omega$ for all $v \in \mathfrak{g}$.

Proof of Theorem 1.2:

The first part of the theorem follows from Proposition 4.3. Let ω be as given, and we go and compute its moment map. We use the coordinates (x, y, ξ, η) as in (1.4).

Let $u + v \in \mathbb{R}^{n|k} \cong \mathfrak{g}$. Its associated infinitesimal vector field on M is

$$u^\sharp + v^\sharp \;=\; \sum\nolimits_q u_q \,\frac{\partial}{\partial y_q} \;+\; \sum\nolimits_s v_s \,\frac{\partial}{\partial \xi_s} \ . \tag{4.8}$$

By Proposition 4.3, we have

$$\imath(u^\sharp + v^\sharp)\omega \;=\; \sum\nolimits_{p,q} u_q \,\frac{\partial^2 F}{\partial x_p \partial x_q}\, dx_p \;+\; 2 \sum\nolimits_s v_s \, d\xi_s \tag{4.9}$$

where hereafter we identify $\mathfrak{g}^* \cong \mathbb{R}^{n|k}$. Recall that $F'(x) = (\frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n})$. By Definition 4.4 and (4.9), a canonical moment map is

$$\Phi : M \longrightarrow \mathbb{R}^{n|k} \quad , \quad \Phi(x, y, \xi, \eta) = \big(-F'(x)\,, 2\,\xi\big)$$

as it obeys $d(\Phi, u+v) = \imath(u^\sharp + v^\sharp)\omega$. This completes the proof of Theorem 1.2. □

Remark 4.5. We end this section by discussing equivariant properties of the moment map Φ. While we require Φ to be equivariant in the ordinary setting (4.7), we remove this condition in the super setting in Definition 4.4. Using the coordinates (x, y, ξ, η) of (1.4), the super torus G acts on the even variable y and odd variable ξ. In Theorem 1.2, the moment map is $\big(-F'(x)\,, 2\,\xi\big)$, so it is $G_{\bar{0}}$–invariant (because y does not appear) but it is not invariant for the action of the group "outside $G_{\bar{0}}$" (because ξ appears) — in terms of the action of the associated super Harish-Chandra pair $\big(G_{\bar{0}}\,, \mathfrak{g}\big)$, this means that it is not invariant for the action of vectors in $\mathfrak{g}_{\bar{1}}$. Note also that here G–invariance and G–equivariance are the same because G, being Abelian, has trivial coadjoint action on $\mathfrak{g}^*$; thus Φ is not G–equivariant. There is no G–equivariant moment map because $\omega_{\bar{1}} = \sum_r \big((d\xi_r)^2 + (d\eta_r)^2\big)$ behaves differently from even symplectic forms. The functions f which satisfy $df = \imath\,(\frac{\partial}{\partial \xi_r})(d\xi_r)^2 = 2\,d\xi_r$ are $f = 2\xi_r + c$ for constants c, and they are not G–invariant. For this reason, in Definition 4.4 we omit G–equivariance condition.

5. Geometric quantization

In this section, we prove Theorem 1.3. Let ω be the super Kähler form on M with moment map Φ as given by Theorem 1.2. We first recall geometric quantization in the ordinary setting (cf. [14]). There exists a line bundle $\mathbb{L}_{\bar{0}}$ on $M_{\bar{0}}$ whose Chern class is the cohomology class $\big[\omega_{\bar{0}}\big]$. Since $\omega_{\bar{0}}$ is exact, we have $\big[\omega_{\bar{0}}\big] = 0$, so the bundle $\mathbb{L}_{\bar{0}}$ is topologically trivial; yet it has interesting geometry given by a connection ∇ with curvature $\omega_{\bar{0}}$. Let $C^\infty\big(\mathbb{L}_{\bar{0}}\big)$ be the set of all smooth sections of $\mathbb{L}_{\bar{0}}$: we define the set of all holomorphic sections by

$$\mathcal{H}(\mathbb{L}_{\bar{0}}) \;:=\; \Big\{\, s \in C^\infty\big(\mathbb{L}_{\bar{0}}\big) \;\Big|\; \nabla_v s = 0 \;\text{ for all anti-holomorphic vector fields } v \,\Big\}\,.$$

The T_n–action lifts to a T_n–representation on $\mathcal{H}\big(\mathbb{L}_{\bar{0}}\big)$. The line bundle has an invariant Hermitian form, namely $(s,t) \in C^\infty\big(M_{\bar{0}}\big)$ for all sections $s, t \in C^\infty\big(\mathbb{L}_{\bar{0}}\big)$, and $v(s,t) = (\nabla_v s\,, t) + (s\,, \nabla_v t)$ for all vector fields v.

We now extend the above construction to the super setting. Recall from (1.4) that the holomorphic even and odd variables are respectively

$$z_p \;=\; x_p + i\, y_p \;\;,\;\; \zeta_q \;=\; \xi_q + i\,\eta_q \;\;, \qquad \forall\;\, p = 1, \dots, n\;,\;\; q = 1, \dots, k\;.$$

Hereafter we use standard multi-index notation, for example

$$\zeta_{P,Q} \;:=\; \zeta_{p_1} \cdots \zeta_{p_r}\, \bar{\zeta}_{q_1} \cdots \bar{\zeta}_{q_s} \qquad \forall\;\, P = (p_1, ..., p_r)\,,\; Q = (q_1, ..., q_s)\,.$$

The odd variables anti-commute, so we use ascending indices $p_1 < \cdots < p_r$ and $q_1 < \cdots < q_s$ so that the set of all $\zeta_{P,Q}$'s is linearly independent.

We let the subscript "top" denote the presence of all odd variables, so that

$$\zeta_{\mathrm{top}} \;=\; \zeta_1 \cdots \zeta_k \, \bar{\zeta}_1 \cdots \bar{\zeta}_k \;.$$

We define the star operator $\zeta_{P,Q} \mapsto \zeta^*_{P,Q}$, where $\zeta^*_{P,Q}$ consists of the odd variables missing from $\zeta_{P,Q}$, oriented and normalized so that

$$\zeta_{P,Q}\, \zeta^*_{P,Q} \;=\; i^{|P|+|Q|}\, \zeta_{\mathrm{top}} \;. \tag{5.1}$$

Here $i^{|P|+|Q|} \in \{1, i\}$, following (4.4). For $Q = \emptyset$, we also write $\zeta_P = \zeta_{p_1} \cdots \zeta_{p_r}$ and $\zeta_P\, \zeta^*_P \;=\; i^{|P|}\, \zeta_{\mathrm{top}}$.

Consider now $\sum_{P,Q} f_{P,Q}\, \zeta_{P,Q} \in C^\infty(M)$. The Berezin integration (see [20, §4.6]) keeps only the top monomial, namely

$$\int_M \sum\nolimits_{P,Q} f_{P,Q}\, \zeta_{P,Q}\, d\mathcal{B} \;:=\; \int_{M_{\bar{0}}} f_{\mathrm{top}}\, dx\, dy \tag{5.2}$$

where $dx\, dy$ is the Haar measure of $M_{\bar{0}}$. The line bundle $\mathbb{L}_{\bar{0}}$ extends to a super line bundle $\mathbb{L}$ over M , whose smooth sections form the set

$$C^\infty(\mathbb{L}) \;=\; C^\infty\big(\mathbb{L}_{\bar{0}}\big) \otimes {\textstyle\bigwedge_{\mathbb{R}}}\big(\, \zeta_1 \,, \ldots , \zeta_k \,, \bar{\zeta}_1 \,, \ldots , \bar{\zeta}_k\big)$$

consisting of linear combinations of $s_{\bar{0}}\, \zeta_{P,Q}$ where $s_{\bar{0}} \in C^\infty\big(\mathbb{L}_{\bar{0}}\big)$. We construct an L^2–structure on $C^\infty(\mathbb{L})$ by

$$\big\langle s_{\bar{0}}\, \zeta_{P,Q} \,, t_{\bar{0}}\, \zeta_{R,S} \big\rangle \;=\; \int_M \big(s_{\bar{0}} \,, t_{\bar{0}}\big)\, \zeta_{P,Q}\, \zeta^*_{R,S}\, d\mathcal{B} \;\;, \qquad \forall\;\; s_{\bar{0}}\, \zeta_{P,Q} \,, \, t_{\bar{0}}\, \zeta_{R,S} \;\in\; C^\infty(\mathbb{L}). \tag{5.3}$$

The space of holomorphic sections of $\mathbb{L}$ is given by

$$\mathcal{H}(\mathbb{L}) \;:=\; \left\{ \sum\nolimits_P s_P\, \zeta_P \;\middle|\; s_P \text{ is a holomorphic section of } \mathbb{L}_{\bar{0}} \right\} . \tag{5.4}$$

Note that in (5.4), we write ζ_P instead of $\zeta_{P,Q}$ because the anti-holomorphic odd variables do not appear. The set of square integrable holomorphic sections is defined by

$$\mathcal{H}^2(\mathbb{L}) \;:=\; \big\{\, s \in \mathcal{H}(\mathbb{L}) \;\big|\; \langle s \,, s \rangle \text{ converges} \,\big\} \;.$$

Recall that a super vector space V is said to have a super Hermitian metric H if it satisfies (4.3). It implies that H (resp. $-iH$) is positive definite on $V_{\bar{0}}$ (resp. on $V_{\bar{1}}$), therefore $H\big|_{V_{\bar{0}}} \oplus (-iH)\big|_{V_{\bar{1}}}$ is an ordinary inner product on $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

Definition 5.1. A super Hilbert space is a super vector space V equipped with a super Hermitian metric H such that the ordinary inner product $H\big|_{V_{\bar{0}}} \oplus (-iH)\big|_{V_{\bar{1}}}$ makes $V = V_{\bar{0}} \oplus V_{\bar{1}}$ into a complete metric space.

Proposition 5.2. *$\mathcal{H}^2(\mathbb{L})$ is a super Hilbert space.*

Proof. Let $s = s_{\bar{0}}\, \zeta_{P,Q}$ and $t = t_{\bar{0}}\, \zeta_{R,S}$. By (5.1), (5.2) and (5.3), we have

$$\big\langle s\,, t \big\rangle \;=\; i^{|s|}\, \delta_{P,R}\, \delta_{Q,\,S} \int_{M_{\bar{0}}} \big(s_{\bar{0}}\,, t_{\bar{0}}\big)\, dx\, dy \tag{5.5}$$

where $\delta_{P,R}$ and $\delta_{Q,\,S}$ are Kronecker deltas. We want to show that this is a super Hermitian metric on the elements that converge, namely it satisfies (4.3).

If s and t have different parities, then $\delta_{P,R}\, \delta_{Q,\,S} = 0$, and so $\big\langle s\,, t \big\rangle = 0$; this implies the consistency property (4.3)*(a)*.

Next we check the super Hermitian symmetric property (4.3)*(b)*. Here $\delta_{P,R}\, \delta_{Q,\,S} = 1$ only if s and t have the same parity, and in that case $(-1)^{|s|\cdot|t|} = (-1)^{|s|}$. Hence

$$(-1)^{|s|\cdot|t|}\, \overline{i^{|s|}}\, \delta_{P,R}\, \delta_{Q,\,S} \;=\; i^{|s|}\, \delta_{P,R}\, \delta_{Q,\,S}\;. \tag{5.6}$$

By (5.5) and (5.6), we have

$$\begin{aligned} (-1)^{|s|\cdot|t|}\, \overline{\big\langle t\,, s \big\rangle} \;&=\; (-1)^{|s|\cdot|t|}\, \overline{i^{|s|}}\, \delta_{P,R}\, \delta_{Q,\,S}\, \overline{\textstyle\int_{M_{\bar{0}}} \big(t_{\bar{0}}\,, s_{\bar{0}}\big)\, dx\, dy} \;=\; \\ &=\; i^{|s|}\, \delta_{P,R}\, \delta_{Q,\,S} \textstyle\int_{M_{\bar{0}}} \big(s_{\bar{0}}\,, t_{\bar{0}}\big)\, dx\, dy \;=\; \big\langle s\,, t \big\rangle\;. \end{aligned}$$

This proves (4.3)*(b)*.

Finally, we check for super positivity (4.3)*(c)*. Let $s \neq 0$. By (5.5), if s is even (resp. odd), then $\big\langle s\,, s \big\rangle \in \mathbb{R}^+$ (resp. $\big\langle s\,, s \big\rangle \in i\,\mathbb{R}^+$). This proves (4.3)*(c)*. Thus we have shown that the L^2–structure (5.5) is indeed a super Hermitian metric on the elements that converge. As ordinary vector spaces, both $C^\infty(\mathbb{L})_{\bar{0}}$ and $C^\infty(\mathbb{L})_{\bar{1}}$ have ordinary inner products $\langle\, \cdot\,, \cdot \,\rangle$ and $-i\, \langle\, \cdot\,, \cdot \,\rangle$ respectively. This induces a metric space structure $C^\infty(\mathbb{L})$, so its completion $L^2(\mathbb{L})$ is a Hilbert space. The Bergman space $\mathcal{H}^2(\mathbb{L}) = L^2(\mathbb{L}) \cap \mathcal{H}(\mathbb{L})$ is a closed subspace of $L^2(\mathbb{L})$, so $\mathcal{H}^2(\mathbb{L})$ is complete as well, and hence it is a super Hilbert space. $\square$

We shall study $\mathcal{H}(\mathbb{L})$ and $\mathcal{H}^2(\mathbb{L})$ under the G–action. However, it is more convenient to work with functions than sections; to this end, the next proposition provides a global trivialization. As before, F denotes the Kähler potential of ω.

Proposition 5.3. *There exists a nowhere vanishing G–invariant section $u \in \mathcal{H}(\mathbb{L})$ such that $(u\,, u) = e^{-2F}$.*

Proof. In the ordinary setting, there exists a nowhere vanishing T–invariant section $u \in \mathcal{H}\big(\mathbb{L}_{\bar{0}}\big)$ such that $(u\,, u) = e^{-2F}$ — cf. [5, (3.13)] (we see e^{-F} in [5] because its $\omega_{\bar{0}}$ is half of ours). It extends naturally to a holomorphic section of $\mathbb{L}$, where u is independent of odd variables. So for all $\theta \in \mathfrak{g}_{\bar{1}}$, we have $\partial_\theta u = 0$, and this together with the T–invariance guarantees that u is G–invariant. $\square$

We consider the set of all holomorphic functions on M, namely

$$\mathcal{H}(M) \;:=\; \left\{\, \sum_P f_P\, \zeta_P \;\middle|\; f_P \in \mathcal{H}\big(M_{\bar{0}}\big)\;\; \forall\, P \,\right\}\;.$$

The section u of Proposition 5.3 is holomorphic and nowhere vanishing, so it leads to a G–equivariant global trivialization

$$\mathcal{H}(\mathbb{L}) \cong \mathcal{H}(M) \ , \quad f u \mapsto f \tag{5.7}$$

that allows us to study $\mathcal{H}^2(\mathbb{L})$ in terms of holomorphic functions.

Extend the star operator of (5.1) to $C^\infty(M)$ by

$$* : C^\infty(M) \longrightarrow C^\infty(M) \ , \quad \big(f_{\bar{0}}\, \zeta_{P,Q}\big)\big(f_{\bar{0}}\, \zeta_{P,Q}\big)^* \ = \ i^{|P|+|Q|}\, f_{\bar{0}}\, \bar{f}_{\bar{0}}\, \zeta_{\text{top}} \ . \tag{5.8}$$

Then define an L^2–structure on $C^\infty(M)$ by

$$\big\langle f\,, h \big\rangle \ := \ \int_M f\, h^*\, e^{-2F}\, d\mathcal{B} \qquad \forall\, f, h \in C^\infty(M) \ . \tag{5.9}$$

Using arguments similar to those applied for Proposition 5.2, one sees that this is a super Hermitian metric on the elements which converge, hence we let $L^2\big(M, e^{-2F}\big)$ denote its completion. Then we define the Bergman space

$$\mathcal{H}^2\big(M, e^{-2F}\big) \ := \ L^2\big(M, e^{-2F}\big) \cap \mathcal{H}(M) \ .$$

Note that we have inserted the weight e^{-2F} in (5.9) because, by Proposition 5.3, the L^2–structures of (5.3) and (5.9) are related by

$$\big\langle f u\,, h u \big\rangle \ = \ \int_M f\, h^*(u\,, u)\, d\mathcal{B} \ = \ \int_M f\, h^*\, e^{-2F}\, d\mathcal{B} \ = \ \big\langle f, h \big\rangle \tag{5.10}$$

for all $f, h \in C^\infty(M)$. So we have an isomorphism of G–modules and super Hilbert spaces, namely

$$\mathcal{H}^2(\mathbb{L}) \ \cong \ \mathcal{H}^2\big(M, e^{-2F}\big) \ , \quad f\, u \mapsto f \ . \tag{5.11}$$

We shall now see that $\mathcal{H}^2(\mathbb{L})$ and $\mathcal{H}^2\big(M, e^{-2F}\big)$ are not unitary G–representations, as the G–actions do not preserve their L^2–structures. The next proposition proves this claim, and moreover it characterizes the largest subrepresentation of $\mathcal{H}^2\big(M, e^{-2F}\big)$ which is indeed unitary.

Proposition 5.4. *$\mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\big)$ is the largest G–subrepresentation of $\mathcal{H}^2\big(M, e^{-2F}\big)$ in which the G–action is unitary.*

Proof. Let us suppose that V is a G–subrepresentation of $\mathcal{H}^2\big(M, e^{-2F}\big)$, and also that $V \not\subseteq \mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\big)$. Then there exists $f \in V$ which is dependent of the odd variables, namely $0 \neq \mathcal{D}_\xi f \in V$ for some ξ, where hereafter $\mathcal{D}_\xi$ denotes the left-invariant vector field on M (here realized as a derivation of functions on M) associated with the vector $\xi \in \mathfrak{g}_1$. Then let us consider $h := \mathcal{D}_\xi f$; we have

$$\begin{aligned} &(a) \qquad \big\langle \mathcal{D}_\xi f, h \big\rangle \ = \ \langle h\,, h \rangle \ \neq \ 0 \\ &(b) \qquad \big\langle\, f, \mathcal{D}_\xi h \big\rangle \ = \ \langle\, f, 0 \rangle \ = \ 0 \ . \end{aligned} \tag{5.12}$$

Indeed, super positivity implies (5.12)*(a)*. On the other hand, recall that mapping $\xi \mapsto \mathcal{D}_\xi$ defines a Lie superalgebra morphism from $\mathfrak{g}$ to $Der_{\mathcal{H},M}$, the superalgebra of superderivations of $\mathcal{H}^2\big(M, e^{-2F}\big)$; this canonically extends to a morphism of (unital, associative) superalgebras $\mathcal{D} : U(\mathfrak{g}) \longrightarrow Der_{\mathcal{H},M}$. Now, in $U(\mathfrak{g})$ we have

$$\xi \cdot \xi \; = \; 2^{-1} \, [\xi \, , \xi] \; = \; 0 \qquad\quad \forall \;\, \xi \in \mathfrak{g}_{\bar{1}} \tag{5.13}$$

because $\mathfrak{g}$ is Abelian; therefore, applying $\mathcal{D}$ to (5.13) we get

$$\mathcal{D}_\xi \circ \mathcal{D}_\xi \; = \; \mathcal{D}(\xi) \circ \mathcal{D}(\xi) \; = \; \mathcal{D}(\xi \cdot \xi) \; = \; 0 \qquad\quad \forall \;\, \xi \in \mathfrak{g}_{\bar{1}} \; . \tag{5.14}$$

But then 5.12*(b)* follows at once, because

$$\big\langle\, f, \mathcal{D}_\xi h \,\big\rangle \; = \; \big\langle\, f, \mathcal{D}_\xi\big(\mathcal{D}_\xi f\big) \,\big\rangle \; = \; \big\langle\, f, \big(\mathcal{D}_\xi \circ \mathcal{D}_\xi\big) f \,\big\rangle \; = \; \big\langle\, f, 0 \,\big\rangle \; = \; 0 \; .$$

Now, (5.12) implies $\big\langle \mathcal{D}_\xi f, h \,\big\rangle + (-1)^{|\xi||f|} \big\langle\, f, \mathcal{D}_\xi h \,\big\rangle \neq 0$, thus by (2.1) we conclude that $\mathfrak{g}$ does not act unitarily on V — hence G neither — a contradiction.

On the other hand, consider now $f, h \in \mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\big)$; for any $\xi \in \mathfrak{g}$, we need to describe $\mathcal{D}_\xi f$ and $\mathcal{D}_\xi h$. The group product in M induces a generalized coproduct

$$\Delta : C^\infty(M) \longrightarrow C^\infty(M \times M) \;\;\; , \quad f \mapsto \Delta(f) \;\; \big((m_1 \, , m_2) \mapsto f(m_1 \cdot m_2) \big) \; .$$

It makes sense to consider this a coproduct map in generalized coalgebra theory, because there exists a canonical identification

$$C^\infty(M \times M) \;\; \cong \;\; C^\infty(M) \,\widehat{\otimes}\, C^\infty(M)$$

where the right-hand side is a suitable completion of the algebraic tensor product of $C^\infty(M) \otimes C^\infty(M)$ with respect to some topology. Even more, there exists a dense subalgebra C' of $C^\infty(M)$ such that $\Delta\big(C'\big) \subseteq C' \otimes C'$. Hence Δ is uniquely determined by its restriction to C' ; moreover, the advantage is that for any $f \in C'$ its coproduct can be written as

$$\Delta(f) \; = \; \sum_{i=1}^k f'_i \otimes f''_i \qquad \text{for suitable } \; f'_i, f''_i \in C' \tag{5.15}$$

(cf. [1, p.161], and references therein). In this setup, every left invariant vector field $\mathcal{D}_\xi$ on $C^\infty(M)$ can be described as follows. Let $m_{{}_{C^\infty(M)}}$ be the multiplication in $C^\infty(M)$, and we have $\mathcal{D}_\xi := m_{{}_{C^\infty(M)}} \circ \big(\mathrm{id}_{{}_{C^\infty(M)}} \otimes \xi \big) \circ \Delta$. By (5.15), this means

$$\mathcal{D}_\xi f \; = \; \sum_{i=1}^k f'_i \cdot \big(\xi . f''_i\big) \qquad\qquad \forall \;\, f \in C' \tag{5.16}$$

where $\xi . f''_i$ denotes the scalar obtained by applying $\xi \in \mathfrak{g} \cong T_e(G)$ to the germ of function of f''_i at e . Now, assume that $f \in \mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\big)$ as above; then $f \in C^\infty\big(M_{\bar{0}}\big) \subseteq C^\infty(M)$, hence we can assume that $f \in C'$. In addition, since $f \in C^\infty\big(M_{\bar{0}}\big)$ we have also $f'_i, f''_i \in C^\infty\big(M_{\bar{0}}\big)$ in (5.15), so these f'_i and f''_i are independent of any odd variable. The very last claim implies that $\xi . f''_i = 0$ for all index i . But then, by (5.16) we have

$$\mathcal{D}_\xi f \;\; = \;\; \sum_{i=1}^k f'_i \cdot \big(\xi . f''_i\big) \;\; = \;\; 0 \qquad\qquad \forall \;\, f \in \mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\big) \; .$$

Eventually, for all $f, h \in \mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\,\big)$, the above gives

$$\big\langle\, \mathcal{D}_\xi f, h\,\big\rangle \;+\; (-1)^{|f|}\big\langle\, f, \mathcal{D}_\xi h\,\big\rangle \;\;=\;\; 0\ .$$

All in all, we have shown that $\mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\,\big)$ is the largest G–subrepresentation of $\mathcal{H}^2\big(M, e^{-2F}\,\big)$ in which the G–action is unitary. □

Proof of Theorem 1.3:

By Proposition 5.2 and (5.11), $\mathcal{H}^2(\mathbb{L})$ and $\mathcal{H}^2\big(M, e^{-2F}\,\big)$ are super Hilbert spaces. By Proposition 5.4, $\mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\,\big)$ is the largest G–subrepresentation of $\mathcal{H}^2\big(M, e^{-2F}\,\big)$ in which the G–action is unitary. By (5.11), we have an isomorphism of unitary G–modules

$$\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big) \;\cong\; \mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\,\big)\ , \tag{5.17}$$

so $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$ is the largest subrepresentation of $\mathcal{H}^2(\mathbb{L})$ in which the G–action is unitary.

As we noticed in the proof of Proposition 5.4, the Hilbert space $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$ has trivial $\mathfrak{g}_{\bar{1}}$–action because its elements are independent of odd variables. It decomposes into

$$\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big) \;=\; \sum_\lambda \mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)_\lambda$$

under the T_n–action, where $\lambda \in i\,\mathfrak{t}_n^*$ are the integral weights. Now, for each integral weight λ, let $\chi_\lambda : T_n \longrightarrow S^1$ be its character, given by (1.2). Let $s \in \mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)_\lambda$, and let $z = x + i\,[y]$ be the coordinates in (1.4). By (3.5) and (5.17), we have

$$s \;=\; f\,u\ \ , \quad f(z) \;:=\; e^{-\lambda z} \tag{5.18}$$

up to a scalar multiple. Then

$$\begin{aligned}
\langle s\,, s\rangle \;&=\; \textstyle\int_M f\, f^*\, e^{-2F}\, d\mathcal{B} && \text{by } (5.10)\\
&=\; \textstyle\int_M f\,\bar{f}\,\zeta_{\mathrm{top}}\, e^{-2F}\, d\mathcal{B} && \text{by } (5.8)\\
&=\; \textstyle\int_{M_{\bar{0}}} f\,\bar{f}\, e^{-2F}\, dx\, dy && \text{by } (5.2)\\
&=\; \textstyle\int_{M_{\bar{0}}} e^{-2\lambda x - 2F}\, dx\, dy && \text{by } (5.18).
\end{aligned}$$

Note that the last integral converges if and only if $-2\lambda \in \mathrm{Im}\big(2F'\big)$, by [5, Prop.3.3]. By Theorem 1.2, this is equivalent to $\lambda \in \mathrm{Im}(\Phi)_{\bar{0}}$. We conclude that $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)_\lambda \neq 0$ if and only if $\lambda \in \mathrm{Im}(\Phi)_{\bar{0}}$, and in that case we have an isomorphism of irreducible unitary G–modules

$$\mathcal{H}^2(\mathbb{L}_{\bar{0}})_\lambda \;\cong\; \mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\,\big)_\lambda \;\cong\; V_\lambda^+$$

— see Example 3.2. The elements of $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$ are independent of odd variables, so V_λ^- does not occur in $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$. This proves Theorem 1.3. □

Example 5.5. By Theorems 1.2 and 1.3, a key ingredient is the strictly convex function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$. We consider two examples, namely

$$\begin{aligned}
&(a)\quad F_1(x) \;:=\; x_1^2 + \ldots + x_n^2\\
&(b)\quad F_2(x) \;:=\; \textstyle\sum_{j=1}^n \Big(-\mu_j x_j + \varepsilon\sqrt{x_j^2+1}\,\Big) \quad \text{for some fixed } \mu \in \mathbb{Z}^n\,,\ \varepsilon > 0\ .
\end{aligned} \tag{5.19}$$

We shall show that these lead to super Hilbert spaces in two extremes: (5.19)*(a)* provides a sum of all even members of $\widehat{G}$, while (5.19)*(b)* provides finitely many even members of $\widehat{G}$, including V_μ^+ alone when ε is small enough.

One easily checks that

$$\frac{\partial F_1}{\partial x_j} = 2x_j \quad , \qquad \frac{\partial F_2}{\partial x_j} = -\mu_j + \varepsilon\, x_j \left(x_j^2+1\right)^{-\frac{1}{2}} , \tag{5.20}$$

where $\lim\limits_{x_j \to \pm\infty} \varepsilon\, x_j \big(x_j^2+1\big)^{-\frac{1}{2}} = \pm\varepsilon$, so the gradient maps $F_i' : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ have images

$$\mathrm{Im}\big(F_1'\big) = \mathbb{R}^n \quad , \qquad \mathrm{Im}\big(F_2'\big) = \big\{ -\mu + x \;\big|\; |x_j| < \varepsilon \big\} \ . \tag{5.21}$$

By (5.20), the Hessian matrices of F_i are diagonal matrices with diagonal entries

$$\frac{\partial^2 F_1}{\partial x_j^2} = 2 \ , \qquad \frac{\partial^2 F_2}{\partial x_j^2} = \varepsilon \left(x_j^2+1\right)^{-\frac{3}{2}} > 0 \ .$$

So for $i = 1$ and $i = 2$, our F_i is strictly convex. They lead to super Kähler forms on M as constructed in Theorem 1.2; then by Theorem 1.3, they provide G–representations on $\mathcal{H}_i(\mathbb{L})$. Their unitary subrepresentations $\mathcal{H}_i^2\big(\mathbb{L}_{\bar{0}}\big)$ are multiplicity free, and they contain V_λ^+ if and only if $\lambda \in \mathrm{Im}(\Phi)_{\bar{0}}$. Since $\mathrm{Im}(\Phi)_{\bar{0}} = \mathrm{Im}\big(-F'\big)$, by (5.21), we have

$$\mathcal{H}_1^2\big(\mathbb{L}_{\bar{0}}\big) = \sum_{\lambda \in \mathbb{Z}^n} V_\lambda^+ \quad , \qquad \mathcal{H}_2^2\big(\mathbb{L}_{\bar{0}}\big) = \sum_{\lambda \in \mathbb{Z}^n ,\, |\lambda_j - \mu_j| < \varepsilon} V_\lambda^+ \ .$$

Hence $\mathcal{H}_1^2\big(\mathbb{L}_{\bar{0}}\big)$ is the sum of all the even elements of $\widehat{G}$. On the contrary, $\mathcal{H}_2^2\big(\mathbb{L}_{\bar{0}}\big)$ is the finite sum of those V_λ^+ parametrized by integral weights in the cube $\big|\lambda_j - \mu_j\big| < \varepsilon$; in particular, for $0 < \varepsilon < 1$ we obtain an irreducible representation $\mathcal{H}_2^2\big(\mathbb{L}_{\bar{0}}\big) = V_\mu^+$. Note that if we replace (5.19)*(b)* by $\mu \in \mathbb{R}^n \backslash \mathbb{Z}^n$ and choose $\varepsilon > 0$ sufficiently small, then $\mathrm{Im}\big(F_2'\big) \cap \mathbb{Z}^n = \emptyset$ in (5.21), and so $\mathcal{H}_2^2\big(\mathbb{L}_{\bar{0}}\big) = 0$.

6. Abelian Lie supergroups

In this section, we extend the previous results to the more general setting, where $G = T_n \times \mathbb{R}^m \times \bigwedge_k^{\mathbb{R}}$ is a connected Abelian Lie supergroup. We prove Theorems 1.4, 1.5 and 1.6; as an application, we construct a Gelfand model of G and prove Corollary 1.7. We also prove Theorem 1.8, which restricts the irreducibility and unitarizability of subrepresentations of $\mathcal{H}\big(\mathbb{L}\big)$.

Proof of Theorem 1.4:

The same arguments used to prove Theorem 1.1 apply again.

First, by Lemma 3.1 we find that every irreducible G–representation V has either $\mathrm{sdim} V = 1|0$ or $\mathrm{sdim} = 0|1$; we will then write $V = V^+$ or $V = V^-$, accordingly.

Second, letting $\mathfrak{g} := \mathrm{Lie}(G) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, for any G–module $W = W_{\bar{0}} \oplus W_{\bar{1}}$ we have

$$G_{\bar{0}} \,.\, W_{\bar{s}} = W_{\bar{s}} \ , \quad \mathfrak{g}_{\bar{0}} \,.\, W_{\bar{s}} = W_{\bar{s}} \ , \quad \mathfrak{g}_{\bar{1}} \,.\, W_{\bar{s}} = W_{\bar{s}+\bar{1}} \qquad \forall\ \bar{s} \in \big\{\bar{0}\,,\bar{1}\big\} \ .$$

So when $W = V^{\pm}$ we get that $\mathfrak{g}_{\bar{1}}$ acts trivially on $V^{\pm}$, hence the latter is nothing but a $G_{\bar{0}}$–module, formally endowed with a trivial $\mathfrak{g}_{\bar{1}}$–action.

Recall now that $G_{\bar{0}} \cong T_n \times \mathbb{R}^m$. If we regard $V = V^{\pm}$ as a T_n–module (restricting the G–action) then, by compactness of T_n, this V has a T_n–invariant inner product; therefore, it is a unitary irreducible representation for T_n, hence the T_n–action is described by some character $\lambda_1 \in \widehat{T_n} \cong \mathbb{Z}^n$. Similarly, the restricted action by the subgroup $\mathbb{R}^m$ makes V into an irreducible unitary representation of $\mathbb{R}^m$, which then — by classical theory — is described by some character in $\lambda_2 \in \widehat{\mathbb{R}^m} \cong \mathbb{R}^m$. Overall, the action of $G_{\bar{0}} \cong T_n \times \mathbb{R}^m$ onto $V^{\pm}$ is described by

$$\lambda := \big(\lambda_1, \lambda_2\big) \in \widehat{G_{\bar{0}}} \;\cong\; \widehat{T_n} \times \widehat{\mathbb{R}^m} \;\cong\; \mathbb{Z}^n \times \mathbb{R}^m \; .$$

To keep track of the parity of $V^{\pm}$, we complete the description by saying that the above G–representation $V = V^{\pm}$ is fully described — including for its superspace structure — by the pair $(\lambda\,,\epsilon)$, where $\lambda \in \widehat{G_{\bar{0}}} \cong \mathbb{Z}^n \times \mathbb{R}^m$ is found as above and $\epsilon = +$ (resp. $\epsilon = -$) if $V = V^{+}$ (resp. $V = V^{-}$). Accordingly, we denote such a G–representation by $V_\lambda^\varepsilon \in \widehat{G}$.

The above provides a set-theoretical bijection $\mathbb{Z}^n \times \mathbb{R}^m \times \mathbb{Z}_2 \cong \widehat{G}$ which is given by $(\lambda\,,\epsilon) \mapsto V_\lambda^\varepsilon$. As to the group structure, much like in the case of Theorem 1.1 we notice that tensor product of representations yields $V_\lambda^\varepsilon \otimes V_\mu^\delta = V_{\lambda+\mu}^{\varepsilon\delta}$; thus, the previous map is indeed a group isomorphism too, which ends the proof. $\square$

Proof of Theorem 1.5:

Let ω be a super Kähler form on M. It is consistent (see Definition 4.2), so we write $\omega = \omega_{\bar{0}} + \omega_{\bar{1}}$. Let F be a $G_{\bar{0}}$–invariant potential function of $\omega_{\bar{0}}$, namely $\omega_{\bar{0}} = i\,\partial\bar{\partial}F$. By direct computation,

$$\omega_{\bar{0}} \;=\; i \sum\nolimits_{j,k} \frac{\partial^2 F}{\partial z_j \partial \bar{z}_k}\, dz_j \wedge d\bar{z}_k \; . \tag{6.1}$$

Since F is $G_{\bar{0}}$–invariant, it depends only on x, so together with $\frac{\partial}{\partial z_j} = \frac{1}{2}\big(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\big)$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\big(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\big)$, we have $\frac{\partial^2 F}{\partial z_j \partial \bar{z}_k} = \frac{1}{4}\frac{\partial^2 F}{\partial x_j \partial x_k}$. Hence (6.1) becomes

$$\omega_{\bar{0}} \;=\; \frac{i}{4} \sum\nolimits_{j,k} \frac{\partial^2 F}{\partial x_j \partial x_k} \big(dx_j + i\,dy_j\big) \wedge \big(dx_k - i\,dy_k\big) \; . \tag{6.2}$$

The wedge product is skew-symmetric, so the summands $\frac{\partial^2 F}{\partial x_j \partial x_k}\big(dx_j \wedge dx_k + dy_j \wedge dy_k\big)$ cancel off pairwise when we switch j and k. So (6.2) becomes

$$\omega_{\bar{0}} \;=\; \frac{1}{2} \sum\nolimits_{j,k} \frac{\partial^2 F}{\partial x_j \partial x_k}\, dx_j \wedge dy_k \; .$$

We replace F by $2F$ and get the desired expression of the proposition. It is positive definite if and only if F is strictly convex.

Similar to the arguments of Theorem 1.2, we have $\omega_{\bar{1}} = \sum_{r=1}^{k} \big((d\xi_r)^2 + (d\eta_r)^2\big)$. Let $H = -i \sum_r \zeta_r \bar{\zeta}_r$. Then

$$i\,\partial\,\bar{\partial}\,H \;=\; \sum\nolimits_r d\zeta_r \wedge d\bar{\zeta}_r \;=\; \sum\nolimits_r \big((d\xi_r)^2 \wedge (d\eta_r)^2\big) \;=\; \omega_1 \; .$$

We have $i\,\partial\bar{\partial}(F+H)=\omega$, so $F+H$ is a potential function of ω . The formula of the moment map is derived in the same way as Theorem 1.2. $\square$

Remark 6.1. While Theorems 1.2 and 1.5 look alike, they have slight differences due to the topologies of T_n and $\mathbb{R}^m$. In Theorem 1.2, x_j are the variables of T_n , so if ω is exact, then the non-exact terms $dx_j\wedge dx_k$ do not appear. On the other hand in Theorem 1.5, if T_n is trivial, then x_j are variables of $\mathbb{R}^m$, and $dx_j\wedge dx_k$ are exact. Furthermore, since $\mathbb{R}^m$ is not compact, the $G_{\bar{0}}$–invariance of potential function does not follow automatically from G–invariance of ω by the averaging process.

Let dr be the product of point mass of $\mathbb{Z}^n$ and Lebesgue measure of $\mathbb{R}^m$. Each element of $\mathbb{R}^m$ has zero measure, so in the decomposition of unitary G-representation into $\widehat{G}$-components, we replace the direct sum by direct integral [15]. In the following definition, $\int$ denotes the combination of summation on $\mathbb{Z}^n$ and integration on $\mathbb{R}^m$.

Definition 6.2. Let $\mathcal{H}$ be a unitary super representation of G . We say that $(\lambda,\varepsilon)\in(\mathbb{Z}^n\times\mathbb{R}^m)\times\mathbb{Z}_2\cong\widehat{G}$ occurs in $\mathcal{H}$ if there exists $f\in\mathcal{H}$ with parity ε of the form

$$f(z)\ =\ \int_{\mathbb{Z}^n\times\mathbb{R}^m} h(r)\,e^{irz}\,dr\ , \tag{6.3}$$

where $h(r)\neq 0$ for all r sufficiently near λ . If $h(r)$ is almost unique for all r near λ , we say that (λ,ε) occurs with multiplicity one.

The phrase "almost unique" in Definition 6.2 means that if we replace h by h_1 in (6.3), then there is a neighborhood U of λ such that $h(r)=h_1(r)$ for almost all $r\in U$ — i.e., for all those r outside some subset of measure zero.

Example 6.3. The Fourier transform expresses every $f\in L^2(\mathbb{R})$ almost uniquely as $f(z)=\int_{\mathbb{R}} h(r)\,e^{irz}\,dr$ — see for instance [17, §7]; hence every member of $\widehat{\mathbb{R}}$ occurs with multiplicity one in $L^2(\mathbb{R})$.

We next perform geometric quantization and prove Theorem 1.6. Many arguments are similar to Section 5, and in such cases we merely sketch the ideas.

Proof of Theorem 1.6:

Let ω be a super Käher form on M as characterized by Theorem 1.5. There exists a line bundle $\mathbb{L}_{\bar{0}}$ on $M_{\bar{0}}$ which corresponds to $\omega_{\bar{0}}$, cf. [14]; it extends to a super line bundle $\mathbb{L}$ on M, whose holomorphic sections $\mathcal{H}(\mathbb{L})$ consists of $\sum_P s_P\,\zeta_P$, where s_P are holomorphic sections of $\mathbb{L}_{\bar{0}}$. Much like with (5.1) and (5.2), we consider the star operator and Berezin integration, and let

$$\mathcal{H}^2(\mathbb{L})\ =\ \left\{\ \sum\nolimits_P s_P\,\zeta_P\in\mathcal{H}(\mathbb{L})\ \middle|\ \int_M\sum\nolimits_{P,Q}(s_P,s_Q)\,\zeta_P\,\zeta_Q^*\,d\mathcal{B}\ \text{ converges}\ \right\}.$$

This is a super Hilbert space, but its super Hermitian metric is not G–invariant.

By Propositions 5.3 and 5.4, the largest G–subrepresentation of $\mathcal{H}^2(\mathbb{L})$ with unitary G-action is

$$\mathcal{H}^2(\mathbb{L}_{\bar{0}}) \;\cong\; \mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\,\big)\;,$$

where F is the potential function of $\omega_{\bar{0}}$.

The elements of $\mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\,\big)$ are independent of odd variables, so $\mathfrak{g}_{\bar{1}}$ acts trivially on it. We consider the irreducible $G_{\bar{0}}$–representations which occur in its direct integral decomposition, in the sense of Definition 6.2. By [6, Thm.1.2], $V_\lambda^+ \in \widehat{G_{\bar{0}}}$ occurs in $\mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\,\big)$ if and only if $\lambda \in \mathrm{Im}\big(-F'\big)$ ([6, (1.4)] differs from us by a factor of -2). By Theorem 1.5, this is equivalent to $\lambda \in \mathrm{Im}(\Phi)_{\bar{0}}$. Finally, as the elements of $\mathcal{H}^2\big(M_{\bar{0}}, e^{-2F}\,\big)$ are independent of odd variables, it does not contain any V_λ^- . This concludes the proof of Theorem 1.6. □

For $G_{\bar{0}} = T_n$, Example 5.5 shows that $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$ can be 0 or 1-dimensional, because $\mathrm{Im}(\Phi)_{\bar{0}} \cap \mathbb{Z}^n$ can be $\emptyset$ or $\{\mu\}$. On the contrary, for $G_{\bar{0}} = \mathbb{R}^m$, $\mathcal{H}^2(\mathbb{L}_{\bar{0}}) = \int_U V_\lambda^+$ cannot be 0 or irreducible because U is an open subset of $\mathbb{R}^m$. This is because for a strictly convex function F, the image of F' (and hence $\mathrm{Im}(\Phi)_{\bar{0}}$) is an open set.

We extend Gelfand's notion of model of Lie group [12], by saying that a model of the connected Abelian Lie supergroup G is a unitary G–representation on a super Hilbert space in which every member of $\widehat{G}$ occurs with multiplicity one. To construct such a model, we need a strictly convex function whose gradient mapping is surjective: an example is given by (5.19)*(a)*.

Proof of Corollary 1.7:

Let $G = T_n \times \mathbb{R}^m \times \bigwedge_k^{\mathbb{R}}$. Let $F \in C^\infty\big(\mathbb{R}^{n+m}\big)$ be a strictly convex function whose gradient mapping F' is surjective, for instance $F(x) = \sum_1^{n+m} x_i^2$. By Theorems 1.5 and 1.6, we have $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big) \;\cong\; \int_{\mathbb{Z}^n\times\mathbb{R}^m} V_\lambda^+$.

Recall that we have the involutive endofunctor Π — see (1.9) — which switches parity. Then every member of $\widehat{G}$ occurs exactly once in

$$\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big) \oplus \Pi\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big) \;\cong\; \int_{\mathbb{Z}^n\times\mathbb{R}^m} V_\lambda^+ \oplus V_\lambda^-$$

and therefore this is a model of G. □

6.4. Beyond irreducibility and unitarity. By Theorem 1.6, $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$ is the largest G–subrepresentation of $\mathcal{H}^2(\mathbb{L})$ in which the G–action is unitary, and it decomposes into irreducible subrepresentations indexed by the image of the moment map. We now address the problem of whether $\mathcal{H}(\mathbb{L})$ contains any subrepresentation beyond $\mathcal{H}(\mathbb{L}_{\bar{0}})$ which is irreducible, or is unitarizable with respect to any super Hilbert space structure. In view of the trivialization $\mathcal{H}(\mathbb{L}) \;\cong\; \mathcal{H}(M)$ provided by the invariant section of $\mathbb{L}$, we may conduct our discussion on $\mathcal{H}(M)$.

To simplify notations, we let

$$\Lambda_{\mathbb{C}}^k = \Lambda_{\mathbb{C}}\big(\xi_1\,, \ldots, \xi_k\big) = \mathbb{C} \otimes_{\mathbb{R}} \Lambda_{\mathbb{R}}\big(\xi_1\,, \ldots, \xi_k\big)$$

denote the complex Grassmannian generated by $\xi_1, \dots, \xi_k$. Let us consider the following factorization

$$\mathcal{H}(M) = \mathcal{H}(M_{\bar{0}}) \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}^k . \tag{6.4}$$

Since G is connected, the G–action on $\mathcal{H}(M)$ is uniquely determined by the $\mathfrak{g}$–action, which is by super derivations. We consider the splitting of $\mathfrak{g}$ into direct sum of Lie superalgebras $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, discussed in §2.6. The elements in $\mathcal{H}(M_{\bar{0}})$ depend only on even variables, hence are annihilated by the super derivations of elements of $\mathfrak{g}_-$. In other words, $\mathfrak{g}_-$ acts trivially on $\mathcal{H}(M_{\bar{0}})$, and the same holds for G_-. Similarly, the elements in $\Lambda_{\mathbb{C}}^k$ depend only on odd variables, so are annihilated by the elements of $\mathfrak{g}_+$. Thus $\mathfrak{g}_+$ acts trivially on $\Lambda_{\mathbb{C}}^k$, and the same holds for G_+ too. This means that, through the splittings (2.2) and (6.4), the G–module $\mathcal{H}(M)$ arises from tensoring the G_+–module $\mathcal{H}(M_{\bar{0}})$ and the G_-–module $\Lambda_{\mathbb{C}}^k$.

Let us analyze the G_-–module structure of $\Lambda_{\mathbb{C}}^k$. To simplify this task, we work instead with $\mathfrak{g}_-$. Acting like in the proof of Proposition 5.4, the action of any $\eta \in \mathfrak{g}_-$ on $\Lambda_{\mathbb{C}}^k$ is given by the left invariant vector field $\mathcal{D}_\eta$, so that

$$\mathcal{D}_\eta f := \textstyle\sum_{i=1}^s f_i' \cdot \left(\xi . f_i''\right)$$

where $\sum_{i=1}^s f_i' \otimes f_i'' = \Delta(f)$ is the coproduct of f as in (5.15). Since $\mathcal{D}_\eta$ is a superderivation, we will know its actions on any f when we know it on the generators ξ_j for all $j = 1, \dots, k$. Moreover, it is enough to consider the case of η ranging in an $\mathbb{R}$–basis of $\mathfrak{g}_-$, which we can choose to be $\left\{ \partial_{\xi_i} := \partial / \partial \xi_i \,\middle|\, i = 1, \dots, k \right\}$. Therefore, taking into account that $\Delta(\xi_j) = \xi_j \otimes 1 + 1 \otimes \xi_j$, we end up with

$$\mathcal{D}_{\partial_{\xi_i}} \xi_j = \xi_j \cdot \partial_{\xi_i} 1 + 1 \cdot \partial_{\xi_i} \xi_j = \delta_{ij} \qquad \forall\ i, j = 1, \dots, k . \tag{6.5}$$

Recall that $\Lambda_{\mathbb{C}}^k$ is $\mathbb{N}$–graded algebra, with $\left|\xi_j\right| := 1$ for all j. Consider the associated filtration

$$\Lambda_{\mathbb{C}}^k = \Lambda_{\mathbb{C}}^{\leq k} \supseteq \Lambda_{\mathbb{C}}^{\leq k-1} \supseteq \cdots \supseteq \Lambda_{\mathbb{C}}^{\leq 1} \supseteq \Lambda_{\mathbb{C}}^{\leq 0} = \mathbb{C} \cdot 1$$

where $\Lambda_{\mathbb{C}}^{\leq s} := \left\{ f \in \Lambda_{\mathbb{C}}^k \,\middle|\, |f| \leq s \right\}$ for all $s = 0, 1, \dots, k$. Then (6.5) tells us that $\mathcal{D}_\eta\big(\Lambda_{\mathbb{C}}^{\leq s}\big) \subseteq \Lambda_{\mathbb{C}}^{\leq s-1}$ for all s and for all $\eta \in \mathfrak{g}_-$, hence in short

$$\mathfrak{g}_- . \Lambda_{\mathbb{C}}^{\leq s} \subseteq \Lambda_{\mathbb{C}}^{\leq s-1} \qquad \forall\ s = 0, 1, \dots, k . \tag{6.6}$$

Recall that as a category, the Lie supergroups G are equivalent to the super Harish-Chandra pairs $(G_{\bar{0}}, \mathfrak{g})$; see for instance [10], [11] and references therein. In particular, any superspace is a G–module if and only if it is a $(G_{\bar{0}}, \mathfrak{g})$–module, the action of G being uniquely determined by that of $(G_{\bar{0}}, \mathfrak{g})$, cf. [3], §8.3, for details.

In the present case, the super Harish-Chandra pair corresponding to the Lie supergroup G_- is $\big(\{1\}, \mathfrak{g}_-\big)$. Moreover, the action of $\mathfrak{g}_-$ on $\Lambda_{\mathbb{C}}^k$ has been described above. Thus we do know $\Lambda_{\mathbb{C}}^k$ as a G–representation space.

We say that a representation of G_- is completely reducible if it is the direct sum of irreducible subrepresentations.

Proposition 6.5. *The representation* $\Lambda_{\mathbb{C}}^k$ *of* G_- *is not completely reducible. Moreover, the only irreducible* G_-*–subrepresentation of* $\Lambda_{\mathbb{C}}^k$ *is* $\mathbb{C}\,1_{\Lambda_{\mathbb{C}}^k}$ *.*

Proof. Assume there is an isomorphism $\Lambda_{\mathbb{C}}^k \cong \oplus_{i\in I} V_i$ for some family $\big\{ V_i \big\}_{i\in I}$ of irreducible modules. By Theorem 1.4 and its proof, cf. §6, each V_i is 1–dimensional. But the action of $\mathfrak{g}_-$ switches the parity in V_i, hence such an action is necessarily trivial. Likewise, G_- acts trivially on each V_i. But this contradicts (6.6). Hence $\Lambda_{\mathbb{C}}^k$ is not completely reducible.

Finally, (6.5) implies that any non-zero subrepresentation of $\Lambda_{\mathbb{C}}^k$ necessarily contains $\mathbb{C}\,1_{\Lambda_{\mathbb{C}}^k}$; then the latter is the unique irreducible subrepresentation of $\Lambda_{\mathbb{C}}^k$. □

Proposition 6.6. *The only unitarizable* G_-*–subrepresentation of* $\Lambda_{\mathbb{C}}^k$ *is* $\mathbb{C}\,1_{\Lambda_{\mathbb{C}}^k}$ *.*

Proof. Let W_- be a non-trivial G_-–subrepresentation of $\Lambda_{\mathbb{C}}^k$. We apply the notion of $\mathfrak{u}_B\big(W_-\big)$, introduced in (2.1). By Proposition 2.2 and Proposition 6.5, there exists no form B on W_- such that the $\mathfrak{g}_-$–action on W_- factors through $\mathfrak{u}_B\big(W_-\big)$, i.e. no such B is fixed by the action of G_-. □

We are now ready to prove the last result of this article:

Proof of Theorem 1.8:
By (5.11) we have $\mathcal{H}(\mathbb{L}) \cong \mathcal{H}(M)$, as well as $\mathcal{H}\big(\mathbb{L}_{\bar{0}}\big) \cong \mathcal{H}\big(M_{\bar{0}}\big)$: therefore, it is enough to prove the claim with M replacing $\mathbb{L}$.

Let W be any irreducible $\mathfrak{g}$–subrepresentation of $\mathcal{H}\big(M\big) = \mathcal{H}\big(M_{\bar{0}}\big) \otimes \Lambda_{\mathbb{C}}^k$. Since $\mathfrak{g}$ splits into direct sum of Lie superalgebras as $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ (cf. §2.6), W is necessarily of the form $W = W_+ \otimes W_-$, where $W_\pm$ are some irreducible $\mathfrak{g}_\pm$–subrepresentations with $W_+ \subset \mathcal{H}\big(M_{\bar{0}}\big)$ and $W_- \subset \Lambda_{\mathbb{C}}^k$. Now Proposition 6.5 yields $W_- = \mathbb{C}\,1_{\Lambda_{\mathbb{C}}^k}$, so $W = W_+ \otimes W_- \subset \mathcal{H}\big(M_{\bar{0}}\big) \otimes \mathbb{C}\,1_{\Lambda_{\mathbb{C}}^k} = \mathcal{H}\big(M_{\bar{0}}\big)$.

Similarly, if $W = W_+ \otimes W_-$ is a unitarizable $\mathfrak{g}$–subrepresentation of $\mathcal{H}\big(M\big)$, then by Proposition 6.6, we have $W_- = \mathbb{C}\,1_{\Lambda_{\mathbb{C}}^k}$ and hence $W \subset \mathcal{H}\big(M_{\bar{0}}\big) \otimes \mathbb{C}\,1_{\Lambda_{\mathbb{C}}^k} = \mathcal{H}\big(M_{\bar{0}}\big)$. This completes the proof of Theorem 1.8. □

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