— preprint arXiv:2312.00444 [math.DG] (2023) —

SUPER KÄHLER STRUCTURES ON THE COMPLEX ABELIAN LIE SUPERGROUPS

MENG-KIAT CHUAH[♭] , FABIO GAVARINI[♯]

1. Introduction

The theory of geometric quantization [14] associates the action of a Lie group G on a symplectic manifold M to a unitary G-representation $\mathcal{H}$, and one studies its irreducible subrepresentations. Super geometric quantization has been discussed through the prequantization stage [19], where $\mathcal{H}$ is an algebraic representation without a unitary structure. In view of recent developments in the notions of super Hilbert spaces and super unitary representations (see for example [9]), it becomes appropriate to study the unitary structure of $\mathcal{H}$.

The geometric quantization of the actions of connected Abelian Lie groups on their complexifications has been carried out successfully for [5, 6]. We now consider its super analogue. Let G be a connected Abelian Lie supergroup. Its even part $G_{\bar{0}}$ is a connected Lie group, so $G_{\bar{0}} \cong T_n \times \mathbb{R}^m$, where T_n is the n–dimensional torus. As for any other supergroup (cf. [10, 11]), we have a global splitting of G of the form

$$G \,=\, T_n \times \mathbb{R}^m \times {\textstyle\bigwedge}_k^{\mathbb{R}} \tag{1.1}$$

where ${\textstyle\bigwedge}_k^{\mathbb{R}}$ is the supermanifold associated with the Grassmann $\mathbb{R}$–algebra in k odd indeterminates — roughly, it is a single point endowed with a purely odd, k–dimensional affine superstructure. Algebraically, this means that the defining superalgebra of global regular functions on G (real smooth, in the present case) factors into

$$\mathcal{O}_G(G) \,:=\, C^\infty(G_{\bar{0}}) \otimes_{\mathbb{R}} \Lambda_{\mathbb{R}}(\xi_1, \dots, \xi_k) \; .$$

In particular, the local structure around a single point in $G_{\bar{0}}$ can be described by a local chart, denoted by (x, ξ) — cf. §2.3 later on.

To provide a fluent presentation, we first consider the super torus $G = T_n \times {\textstyle\bigwedge}_k^{\mathbb{R}}$, namely $m = 0$ in (1.1). Let $\mathfrak{t}_n$ be the Lie algebra of T_n. Let $\lambda \in i\mathfrak{t}_n^*$, namely $\lambda : \mathfrak{t}_n \longrightarrow i\,\mathbb{R}$. We say that λ is an integral weight if it determines a character $\chi_\lambda : T_n \longrightarrow S^1$ such that the diagram commutes,

$$\begin{array}{ccc} \mathfrak{t}_n & \xrightarrow{\lambda} & i\,\mathbb{R} \\ \downarrow & & \downarrow \\ T_n & \xrightarrow{\chi_\lambda} & S^1 \end{array} \tag{1.2}$$

where the downward arrows are exponential maps.

Let $\widehat{T_n}$ denote the set of all irreducible unitary T_n–representations, up to equivalence. The members of $\widehat{T_n}$ are 1–dimensional. They are parametrized by the integral

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2020 *MSC:* 22E45, 32A36, 53C55, 53D50, 58A50

Keywords: Lie supergroups, super Kähler forms, geometric quantization, unitary representations, moment map.

weights λ, where $V_\lambda \in \widehat{T_n}$ consists of vectors v which satisfy $t \cdot v = \chi_\lambda(t) v$ for all $t \in T_n$. We identify the integral weights with $\mathbb{Z}^n$ and write

$$i\,\mathfrak{t}^* \cong \mathbb{R}^n \quad , \quad \widehat{T_n} \cong \mathbb{Z}^n \quad .$$

A unitary representation of G is a super vector space with a super Hermitian form — see [9, §4] — and compatible actions by $G_{\bar{0}}$ and $\mathfrak{g} := \mathrm{Lie}(G)$, the tangent Lie superalgebra of G. Let $\widehat{G}$ denote the equivalence classes of irreducible unitary representations of G. In the following theorem, $\mathbb{Z}^n$ identifies with the set of all integral weights of T_n.

Theorem 1.1. *For the super torus $G = T_n \times \bigwedge_k^{\mathbb{R}}$, there exists a group isomorphism*

$$\widehat{G} \;\cong\; \widehat{G_{\bar{0}}} \times \mathbb{Z}_2 \;\cong\; \mathbb{Z}^n \times \mathbb{Z}_2$$

where the group structure of $\widehat{G}$ is given by the tensor product of representations.

The representation space parametrized by $(\lambda, \epsilon) \in \mathbb{Z}^n \times \mathbb{Z}_2$ has dimension $1|0$ (resp. $0|1$) if $\epsilon = \bar{0}$ (resp. $\epsilon = \bar{1}$), and its vectors v satisfy $t \cdot v = \chi_\lambda(t) v$ for all $t \in T_n$ and $\xi \cdot v = 0$ for all $\xi \in \mathfrak{g}_{\bar{1}}$.

We may express the group $\mathbb{Z}_2$ additively by $\{\bar{0}, \bar{1}\}$ or multiplicatively by $\{+, -\}$. By Theorem 1.1, we write

$$\widehat{G} \;=\; \left\{ V_\lambda^{\pm} \;\middle|\; \lambda \in \mathbb{Z}^n \, , \, \pm \in \mathbb{Z}_2 \right\} \quad .$$

We construct $V_\lambda^{\pm}$ explicitly in Example 3.2.

Let M be the complexification of G. Thus M is a complex Lie supergroup that admits the following description: $M \cong M_{\bar{0}} \times M_{\bar{1}}$, where $M_{\bar{1}} := \bigwedge_k^{\mathbb{C}}$ is described through complex odd Grassmann variables $\zeta_1, \dots, \zeta_k$, while $M_{\bar{0}} \cong \mathbb{C}^n \big/ i\,\mathbb{Z}^n$ is the underlying reduced classical complex Lie subgroup of M, with $\mathbb{C}^n \big/ i\,\mathbb{Z}^n$ denoting quotient on the imaginary part. In particular, $M_{\bar{0}}$ is the complexification of the real classical torus $T_n = G_{\bar{0}}$, that it contains as a maximal torus. From the splitting

$$M \;=\; M_{\bar{0}} \times M_{\bar{1}} \;=\; \mathbb{C}^n \big/ i\,\mathbb{Z}^n \times \textstyle\bigwedge_k^{\mathbb{C}} \tag{1.3}$$

we shall use local charts of the form $(z, \zeta) = (z_1, \dots, z_n, \zeta_1, \dots, \zeta_k)$. Then, both for $\mathcal{O}_{M_{\bar{0}}}(M_{\bar{0}}) = C^\infty(M_{\bar{0}})$ and $\mathcal{O}_{M_{\bar{1}}}(M_{\bar{1}}) = \Lambda_{\mathbb{C}}(\zeta_1, \dots, \zeta_k)$ we fix the real structure given by setting $z_r = x_r + i\,y_r$ and $\zeta_s = \xi_s + i\,\eta_s$, for all r and s; accordingly, as a real manifold M is described by local charts of the form

$$(x, y, \xi, \eta) \;:=\; (x_1, \dots, x_n, y_1, \dots, y_n, \xi_1, \dots, \xi_k, \eta_1, \dots, \eta_k) \,. \tag{1.4}$$

Now G identifies with a real super subgroup of M, described by the (local chart) variables (y, ξ); then we have the natural G–action on M, as left action of a super subgroup.

We shall define the super Kähler forms on M (Definition 4.2) and their moment maps $\Phi : M \longrightarrow \mathfrak{g}^*$ (Definition 4.4). We identify $\mathfrak{g}^* \cong \mathbb{R}^{n|k}$. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function. Its gradient map is

$$F' : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad , \quad F'(x) \;:=\; \left(\frac{\partial F}{\partial x_1}(x), \dots, \frac{\partial F}{\partial x_n}(x) \right) \qquad \forall \;\, x \in \mathbb{R}^n \quad .$$

We say that F is strictly convex if its Hessian matrix $\left(\frac{\partial^2 F}{\partial x_p \, \partial x_q}\right)$ is positive definite everywhere. The next proposition uses local coordinates (x, y, ξ, η) as in (1.4).

Theorem 1.2. *Let G be the super torus. Every G–invariant exact super Kähler form on M can be expressed as*

$$\omega \;=\; \sum_{p,q=1}^{n} \frac{\partial^2 F}{\partial x_p \partial x_q}\, dx_p \wedge dy_q \;+\; \sum_{r=1}^{k} \Big((d\xi_r)^2 + (d\eta_r)^2 \Big) \;,$$

where $F \in C^\infty\big(\mathbb{R}^n\big)$ is a strictly convex function. Its moment map is

$$\Phi : M \longrightarrow \mathfrak{g}^* \;\;, \quad (x, y, \xi, \eta) \mapsto \Phi(x, y, \xi, \eta) = \big(-F'(x) \,,\, 2\,\xi \,\big) \;. \tag{1.5}$$

Fix a super Kähler form ω as given above. We extend the standard machinery of geometric quantization [14] to the super setting and obtain a holomorphic Hermitian line bundle $\mathbb{L}$ on M. Let $\mathcal{H}(\mathbb{L})$ denote its holomorphic sections. We define the star operator $f \mapsto f^*$ on $C^\infty(\mathbb{L})$ — see (5.8) — then apply Berezin integration [20] to construct the super Hilbert space (see Definition 5.1)

$$\mathcal{H}^2(\mathbb{L}) \;:=\; \left\{ f \in \mathcal{H}(\mathbb{L}) \;\middle|\; \int_M f f^* \, d\mathcal{B} \quad \text{converges} \right\} \;. \tag{1.6}$$

The G–representation on $\mathcal{H}^2(\mathbb{L})$ is not unitary, nevertheless it has a unique largest subrepresentation in which G acts unitarily, and we study its irreducible subrepresentations. Let $\mathrm{Im}(\Phi)_{\bar 0} \subset \mathbb{R}^n$ denote the even part of the image of Φ. Recall also that $\widehat{G} = \big\{ V_\lambda^\pm \,\big|\, \lambda \in \mathbb{Z}^n \big\}$.

Theorem 1.3. *Let G be the super torus. Then $\mathcal{H}^2(\mathbb{L})$ is a super Hilbert space, and $\mathcal{H}^2(\mathbb{L}_{\bar 0})$ is its largest G–subrepresentation in which the G–action is unitary. Moreover, $\mathcal{H}^2\big(\mathbb{L}_{\bar 0}\big)$ is multiplicity free, with V_λ^+ occurring if and only if $\lambda \in \mathrm{Im}(\Phi)_{\bar 0}$. Also, V_λ^- does not occur in $\mathcal{H}^2\big(\mathbb{L}_{\bar 0}\big)$, for any integral weight λ.*

Theorems 1.2 and 1.3 enable us to construct unitary G–representations of various sizes, depending on the images of F'. We shall illustrate this in Example 5.5, where $\mathcal{H}^2(\mathbb{L}_{\bar 0})$ can be 0, an irreducible G–representation, or a sum of all the even representations $\{V_\lambda^+\}_\lambda$.

The above discussions handle the super torus, and we now consider the general connected Abelian Lie supergroup (1.1). The Lie algebra of the additive group $\mathbb{R}^m$ is just $\mathbb{R}^m$ itself, and its exponential map $\mathbb{R}^m \longrightarrow \mathbb{R}^m$ is the identity map. In this way, (1.2) extends to

$$\begin{array}{ccc} \mathfrak{t}_n \times \mathbb{R}^m & \stackrel{\lambda}{\longrightarrow} & i\,\mathbb{R} \\ \downarrow & & \downarrow \\ T_n \times \mathbb{R}^m & \stackrel{\chi_\lambda}{\longrightarrow} & S^1 \end{array} \;. \tag{1.7}$$

We say that λ is integral if there exists χ_λ such that (1.7) is a commutative diagram. If we write $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \in i\,\mathfrak{t}^*$ and $\lambda_2 \in i\,\big(\mathbb{R}^m\big)^*$, then λ_2 does not impose any obstruction to the existence of χ_λ. So λ is integral if and only if λ_1 is integral.

The integral weights are identified with $\widehat{G_{\bar{0}}}$, so

$$\widehat{G_{\bar{0}}} \cong \mathbb{Z}^n \times \mathbb{R}^m .$$

Theorem 1.1 generalizes to the following.

Theorem 1.4. *Let $G = T_n \times \mathbb{R}^m \times \bigwedge_k^{\mathbb{R}}$. There exists a group isomorphism*

$$\widehat{G} \;\cong\; \widehat{G_{\bar{0}}} \times \mathbb{Z}_2 \;\cong\; \mathbb{Z}^n \times \mathbb{R}^m \times \mathbb{Z}_2$$

where the group structure of $\widehat{G}$ is given by the tensor product of representations.

The representation space parametrized by $(\lambda, \epsilon) \in (\mathbb{Z}^n \times \mathbb{R}^m) \times \mathbb{Z}_2$ has dimension $1|0$ (resp. $0|1$) if $\epsilon = \bar{0}$ (resp. $\epsilon = \bar{1}$), and its vectors v satisfy $t \cdot v = \chi_\lambda(t) v$ for all $t \in T_n \times \mathbb{R}^m$ and $\xi \cdot v = 0$ for all $\xi \in \mathfrak{g}_{\bar{1}}$.

Let M be the complexification of G . This is the Lie supergroup with Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}$, such that M and G have the same maximal compact subgroup. So (1.3) extends to

$$M \;=\; M_{\bar{0}} \times M_{\bar{1}} \;=\; \mathbb{C}^n \big/ i\, \mathbb{Z}^n \times \mathbb{C}^m \times \textstyle\bigwedge_k^{\mathbb{C}} . \tag{1.8}$$

We again consider G–invariant Kähler forms on M, and prove the following theorem.

Theorem 1.5. *Let ω be a super Kähler form on M with a $G_{\bar{0}}$-invariant potential function. Then ω can be expressed as*

$$\omega \;=\; \sum_{p,q=1}^{n+m} \frac{\partial^2 F}{\partial x_p \partial x_q}\, dx_p \wedge dy_q \;+\; \sum_{r=1}^{k} \Big((d\xi_r)^2 + (d\eta_r)^2 \Big)$$

where $F \in C^\infty\big(\mathbb{R}^{n+m}\big)$ is a strictly convex function. Its moment map is

$$\Phi : M \longrightarrow \mathfrak{g}^* \;\;,\quad (x, y, \xi, \eta) \;\mapsto\; \Phi(x, y, \xi, \eta) = \big(-F'(x) \,,\, 2\,\xi \,\big) .$$

While Theorems 1.2 and 1.5 resemble each other, there is a subtle difference due to the topologies of T_n and $\mathbb{R}^m$. We explain this in Remark 6.1.

We similarly perform geometric quantization and obtain a super Hilbert space $\mathcal{H}^2(\mathbb{L})$. It contains $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$ as the largest G-subrepresentation in which the G–action is unitary, and we consider the irreducible unitary subrepresentations which occur in $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$. However, by Theorem 1.4, $\widehat{G}$ contains the factor $\mathbb{R}^m$, whose Plancherel measure provides zero measure on each member (unlike $\mathbb{Z}^n$, whose members have point mass). For this reason, the occurrence of a subrepresentation is understood as appearance in the direct integral decomposition of $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$, see Definition 6.2. With this in mind, Theorem 1.3 extends to the following theorem.

Theorem 1.6. *Let $G = T_n \times \mathbb{R}^m \times \bigwedge_k^{\mathbb{R}}$. Then $\mathcal{H}^2(\mathbb{L})$ is a super Hilbert space, and $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$ is its largest G–subrepresentation in which the G–action is unitary. Moreover, $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$ is multiplicity free, with V_λ^+ occurring if and only if $\lambda \in \mathrm{Im}(\Phi)_{\bar{0}}$. Also, V_λ^- does not occur in $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$, for any integral weight λ .*

According to Gelfand, a *model* of a Lie group is a unitary representation on a Hilbert space in which every irreducible representation occurs exactly once [12]. The model of $G_{\bar{0}}$ has been constructed in [6, Cor.3.3]. It is natural to extend this notion to the super setting, so we say that a model of G is a unitary representation on a super Hilbert space in which every member of $\widehat{G}$ occurs once. We now construct a model.

By Theorem 1.3, the odd representations V_λ^- do not occur in $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$. To remedy this defect, let us recall that for the category $(\text{sspaces})_{\mathbb{C}}$ of complex superspaces, there exists an involutive endofunctor $\Pi : (\text{sspaces})_{\mathbb{C}} \longrightarrow (\text{sspaces})_{\mathbb{C}}$ that is defined on objects by switching parity. Thus Π is the identity on each object as a vector space but reverses the built-in $\mathbb{Z}_2$–grading (and is the identity on morphisms). If $\mathfrak{g}$ is any Lie superalgebra and $(\mathfrak{g}\text{–smod})_{\mathbb{C}}$ is the category of complex $\mathfrak{g}$–supermodules, then Π actually restricts to an endofunctor of $(\mathfrak{g}\text{–smod})_{\mathbb{C}}$ too — the $\mathfrak{g}$–action on each $\mathfrak{g}$–module being kept untouched, namely

$$\Pi V_\lambda^+ \;=\; V_\lambda^- \;. \tag{1.9}$$

We apply Theorems 1.5 and 1.6 to construct a model of G as follows.

Corollary 1.7. *Let F be a strictly convex function such that F' is surjective. Then $\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big) \oplus \Pi\mathcal{H}^2\big(\mathbb{L}_{\bar{0}}\big)$ is a model of G.*

In view of Theorem 1.6, one might wonder if $\mathcal{H}\big(\mathbb{L}\big)$ contains any G–subrepresentation beyond $\mathcal{H}\big(\mathbb{L}_{\bar{0}}\big)$ which is irreducible or unitarizable (apart from using the L^2-structure (1.6)). In this respect, we find the following answer, in the negative:

Theorem 1.8. *Every irreducible or unitarizable G–subrepresentation of $\mathcal{H}(\mathbb{L})$ is contained in $\mathcal{H}(\mathbb{L}_{\bar{0}})$.*

We organize the sections of this article as follows. Section 2 recalls the notions and language of Lie superalgebras and Lie supergroups. Section 3 proves Theorem 1.1, which classifies the unitary irreducible representations of the real super torus G. Section 4 proves Theorem 1.2, which classifies the G–invariant super Kähler forms on the complex super torus, and studies their moment maps. Section 5 proves Theorem 1.3, and provides Example 5.5. Section 6 extends the above results to general connected Abelian Lie supergroups G and proves Theorems 1.4, 1.5 and 1.6. They lead to Corollary 1.7, which constructs a model of G in terms of $\mathcal{H}^2(\mathbb{L}_{\bar{0}})$. Section 6 also proves Theorem 1.8, which restricts the irreducibility and unitarizability of subrepresentations of $\mathcal{H}\big(\mathbb{L}\big)$.

ACKNOWLEDGEMENTS

The first author is partially supported by the National Science and Technology Council of Taiwan. The second author is partially supported by the MIUR *Excellence Department Project MatMod@TOV (CUP E83C23000330006) awarded to the Department of Mathematics, University of Rome Tor Vergata.*

Also, this article is based upon work from *COST Action 21109 CaLISTA*, supported by COST (European Cooperation in Science and Technology), see www.cost.eu .

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Department of Mathematics
National Tsing Hua University
Hsinchu 300, TAIWAN
chuah@math.nthu.edu.tw

Dipartimento di Matematica,
Università degli Studi di Roma "Tor Vergata"
Via della ricerca scientifica 1 — I-00133 Roma, ITALY
gavarini@mat.uniroma2.it