Annales de l'Institut Fourier (to appear)
— preprint `arXiv:2203.11023 [math.QA]` (2022) —

FORMAL MULTIPARAMETER QUANTUM GROUPS, DEFORMATIONS AND SPECIALIZATIONS

GASTÓN ANDRÉS GARCÍA$^\flat$, FABIO GAVARINI$^\sharp$

1. Introduction

Quantum groups can be thought of, roughly speaking, as Hopf algebras depending on one "parameter" such that, for a "special value" of this parameter, they turn isomorphic either to the universal enveloping algebra of some Lie algebra $\mathfrak{g}$ or to the function algebra of some algebraic group G. In the first case the quantum group is called "quantized universal enveloping algebra" (or QUEA in short) and in the second "quantized function algebra" (or QFA in short).

Quite soon, people also began to introduce new quantum groups depending on two or more parameters, whence the terminology "multiparameter quantum groups" came in use: see, e.g., [BGH], [BW1], [BW2], [CM], [CV1], [CV2], [DPW], [GG1], [HLT], [HPR], [Jn], [Kh], [KT], [Ma], [OY], [Re], [Su], [Tk] — and the list might be longer. Nevertheless, one can typically describe a multiparameter quantum group so that one single parameter stands "distinguished", as the *continuous* one that can be specialized. The other parameters instead (seen as *discrete*) parametrize different structures on a common "socle" underlying the semiclassical limit of the quantum group, that is achieved when the continuous parameter is specialized. Indeed, this already occurs with one-parameter quantum groups: for example, the celebrated Drinfeld's QUEA $U_\hbar(\mathfrak{g})$ associated with a complex, finite-dimensional, semisimple Lie algebra $\mathfrak{g}$ has a description where the *continuous* parameter $\hbar$ bears the quantization nature of $U_\hbar(\mathfrak{g})$, while other *discrete* parameters, namely the entries of the Cartan matrix of $\mathfrak{g}$, describe the Lie algebra structure on $\mathfrak{g}$ itself.

In this paper we focus onto the study of multiparameter QUEAs; then it will be possible to realize a parallel study and to achieve the corresponding results for multiparameter QFA's by suitably applying duality. Recall that QUEAs (and QFA's alike) are usually considered in two versions: the so-called "formal" one — dealing with topological Hopf algebras over $\Bbbk[[\hbar]]$ — and the "polynomial" one — dealing with Hopf algebras over a field $\mathbb{K}$ with some $q \in \mathbb{K}$ entering the game as parameter.

One of the first general examples of multiparameter QUEA, hereafter mentioned as MpQUEA, was provided by Reshetikhin in [Re]. This extends Drinfeld's definition of $U_\hbar(\mathfrak{g})$ to a new object $U_\hbar^\Psi(\mathfrak{g})$ that shares the same algebra structure of $U_\hbar(\mathfrak{g})$ but bears a new *coalgebra* structure, depending on a matrix Ψ that collects the new, *discrete* parameters of $U_\hbar^\Psi(\mathfrak{g})$. At the semiclassical limit, these new parameters (hence Ψ) describe the new Lie coalgebra structure inherited by $\mathfrak{g}$ from $U_\hbar^\Psi(\mathfrak{g})$ itself. Note that $U_\hbar^\Psi(\mathfrak{g})$ is defined from scratch as being the outcome of a *deformation by twist* of Drinfeld's $U_\hbar(\mathfrak{g})$, using a twist of a specific type (that we shall call "toral")

2020 *MSC:* 17B37, 17B62 — *Keywords:* Quantum Groups, Quantum Enveloping Algebras.

defined via Ψ. It follows that the class of all Reshetikhin's MpQUEAs is stable under deformation by toral twists, i.e. any such deformation of an object of this kind is again an object of the same kind. Even more, this class is "homogeneous", in that each $U_\hbar^\Psi(\mathfrak{g})$ is nothing but a twist deformation of Drinfeld's $U_\hbar(\mathfrak{g})$.

With a parallel approach, a *polynomial* version of Reshetikhin's MpQUEAs was introduced and studied by Costantini-Varagnolo: see [CV1], [CV2], and also [Ga1]; on the other hand, these works do not consider deformations. Alternatively, using the duality with quantum coordinate algebras, two-parameters quantum envelopling algebras of polynomial type are considered in Dobrev-Parashar [DoP] and in Dobrev-Tahri [DoT]. The effect of the twist can be seen in the description of the coproduct after a change the presentation à la Drinfeld-Jimbo type.

In another direction, a different version of *polynomial* MpQUEA (still working over $\mathfrak{g}$ as above), call it $U_{\mathbf{q}}(\mathfrak{g})$, has been developed in the works of Andruskiewitsch-Schneider, Rosso, and many others — see for instance [AS1], [AS2], [HPR], [Ro]. In this case, the "multiparameter" is cast into a matrix $\mathbf{q} = \big(q_{ij}\big)_{i,j \in I}$ whose entries take part in the description of the *algebra* structure of $U_{\mathbf{q}}(\mathfrak{g})$. Under mild, additional conditions, this yields a very general family of MpQUEAs which is very well-behaved: in particular, it is stable under deformations by 2–cocycles of "toral" type. Even better, this family is "homogeneous", in that each $U_{\mathbf{q}}(\mathfrak{g})$ is a 2–cocycle deformation of Jimbo-Lusztig's polynomial version $U_q(\mathfrak{g})$ of Drinfeld's $U_\hbar(\mathfrak{g})$.

Note that, in Hopf theory, *twist* and *2–cocycle* are notions dual to each other. Thus the constructions of MpQUEAs by Reshetikhin and by Andruskiewitsch-Schneider (besides the difference in being "formal" or "polynomial") are somehow *dual* to each other — and, as such, seem definitely different from each other.

The purpose of this paper is to introduce a new notion of MpQUEA that encompass both Reshtikhin's one and Andruskiewitsch-Schneider's one. Indeed, we achieve this goal introducing a new family of MpQUEAs which incorporates Andruskiewitsch-Schneider's one, hence in particular it includes Drinfeld's standard example (see Definition 4.2.2, Theorem 4.3.2 and §4.5). We show that this new family is stable by toral 2–cocycle deformations (Theorem 5.2.12), just as Andruskiewitsch-Schneider's, *and* it is also stable by toral twist deformations (Theorem 5.1.4), hence it incorporates Reshetikhin's family as well. In particular, we show that every MpQUEA of the Reshetikhin's family is actually isomorphic to one of the Andruskiewitsch-Schneider's family, and viceversa: the isomorphism is especially meaningful in itself, in that it amounts to a suitable change of presentation via a well-focused change of generators (see Theorem 5.1.4). In this sense, we really end up with a single, *homogeneous* family — not just a collage of two distinct families; this can be seen as a byproduct of the intrinsic "self-duality" of Drinfeld's standard $U_\hbar(\mathfrak{g})$.

For each one of these MpQUEAs, then, one can decide to focus the dependence on the *discrete* multiparameters either on the coalgebra structure (which amounts to adopt Reshetikhin's point of view) or on the algebra structure (thus following Andruskiewitsch-Schneider's approach). In our definition we choose to adopt the latter point of view, as it is definitely closer to the classical Serre's presentation of $U(\mathfrak{g})$ — or even to the presentation of Drinfeld's standard $U_\hbar(\mathfrak{g})$ — where the *discrete* multiparameters given by the Cartan matrix entries rule the algebra structure.

Technically speaking, we adopt the setting and language of *formal* quantum groups, thus our newly minted objects are "formal MpQUEAs", in short "FoMpQUEAs". This is indeed a necessary option: in fact, the setup of polynomial MpQUEAs is well-suited when one deals with (toral) 2–cocycle deformations, but behaves quite poorly under deformations by (toral) twists. Roughly speaking, the toral part in a polynomial MpQUEA (in the sense of Andruskiewitsch-Schneider, say) happens to be too rigid, in general, under twist deformations; this is shown in our previous paper [GG2], where we pursued the same goal by means of "polynomial MpQUEAs", which eventually prove to be a somewhat less suitable tool.

Thus, one needs to allow a more flexible notion of "toral part" in our would-be MpQUEA in order to get a notion that is stable under deformation by (toral) twists. We obtain this by choosing to define our *formal* MpQUEA as having a toral part with two distinguished sets of "coroots" and "roots", whose mutual interaction is encrypted in a "multiparameter matrix" P whose role generalizes that of the Cartan matrix. We formalize all this via the notion of *realization* of the matrix P, which is a natural extension of Kac' notion of realization of a generalized Cartan matrix (cf. Definition 5.1.2); our FoMpQUEA then is defined much like Drinfeld's standard one, with the entries of P playing the role of *discrete* multiparameters.

By looking at semiclassical limits, we find that our new class of FoMpQUEAs gives rise to a new family of multiparameter Lie bialgebras (in short MpLbA's) that come equipped with a presentation "à la Serre" in which the parameters — i.e., the entries of P, again — rule the Lie algebra structure (cf. §3.2.3). Again, we prove that this family is stable by deformations — in Lie bialgebra theoretical sense — both via "toral" 2–cocycles and via "toral" twists (see Theorem 3.4.3 and Theorem 3.3.3). In particular, every such MpLbA admits an alternative presentation in which the Lie algebra structure stands fixed (always being ruled by a fixed generalized Cartan matrix) while the Lie *coalgebra* structure does vary according to the multiparameter matrix P. Like in the quantum setup, the isomorphism between the two presentations is quite meaningful, as it boils down to a well-chosen change of generators (cf. Theorem 3.3.3). The very definition of these MpLbA's, as well as the just mentioned results about them, can be deduced as byproducts of those for FoMpQUEAs (via the process of specialization); otherwise, they can be introduced and proved directly; in short, we do both (cf. §3 and Theorem 6.1.4). These MpLbA's were possibly known in literature, at least in part: yet our construction yields a new, systematic presentation of their whole family in its full extent, also proving its stability under deformations by both (toral) 2–cocycles and (toral) twists.

As a final, overall comment, we recall that a close relation between multiparameters and deformations is ubiquitous in several applications, e.g. in the classification of complex finite-dimensional pointed Hopf algebras over abelian groups [AS2], [AGI] — where deformations by 2–cocycle play a central role. Moreover, MpQUEAs may also serve as interpolating objects in the study of the representation theory of quantum groups associated with Langlands dual semi-simple Hopf algebras [FH] — where deformations by twist instead are a key tool.

A last word about the organization of the paper.

In section 2, we introduce the "combinatorial data" underlying our constructions of MpLbA's and FoMpQUEAs alike: the notion of *realization* of a multiparameter matrix, and the process of deforming realizations either by twists or by 2–cocycles.

In section 3 we introduce our MpLbA's and study their deformations by (toral) twists and by (toral) 2–cocycles.

Section 4 is dedicated to introduce our newly minted FoMpQUEAs, in particular using different, independent approaches, and to prove their basic properties.

With section 5 we discuss deformations of FoMpQUEAs by (toral) twists and by (toral) 2–cocycles: we prove that these deformations turn FoMpQUEAs into new FoMpQUEAs again, the case by twist being possibly the more surprising.

Finally, in section 6 we perform specializations of FoMpQUEAs and look at their resulting semiclassical limit: we find that this limit is always a MpLbA (in short, by the very definition of MpLbA's), with the same multiparameter matrix P as the FoMpQUEA it comes from. Conversely, any possible MpLbA does arise as such a limit — in other words, any MpLbA has a FoMpQUEA which is quantization of it. Then — more important — we compare deformations (by toral twists or 2–cocycle) before and after specialization: the outcome is, in a nutshell, that *"specialization and deformation (of either type) commute with each other"* (cf. Theorem 6.2.2 and Theorem 6.2.4). In fact, this last result can be deduced also as a special instance of a more general one, which in turn is an outcome of a larger study about deformations (of either type) of formal quantum groups — i.e., Drinfeld's-like QUEAs and their dual, the so-called QFSHA's — and of their semiclassical limits. This is a more general chapter in quantum group theory, with its own reasons of interest, thus we shall treat it in a separate publication — cf. [GG3].

ACKNOWLEDGEMENTS

The authors thank Marco Farinati for several fruitful conversations. They also especially thank the Referee, whose shrewd remarks and comments helped a lot to improve the paper.

This work was supported by CONICET, ANPCyT, Secyt (Argentina) and by INdAM/GNSAGA, ICTP and University of Padova (Italy), as well as by the MIUR *Excellence Department Project MatMod@TOV (CUP E83C23000330006)* awarded to the Department of Mathematics, University of Rome "Tor Vergata".

References

[AD] N. Andruskiewitsch, J. Devoto, *Extensions of Hopf algebras*, St. Petersburg Math. J. **7** (1996), no. 1, 17–52.

[AGI] I. E. Angiono, A. García Iglesias, *Liftings of Nichols algebras of diagonal type II: all liftings are cocycle deformations*, Selecta Math. (N.S.) **25** (2019), no. 1, Paper No. 5, 95 pp.

[ApS] A. Appel, F. Sala, *Quantization of continuum Kac-Moody algebras*, Pure Appl. Math. Q. **16** (2020), no. 3, 439–493.

[AS1] N. Andruskiewitsch, H.-J. Schneider *A characterization of quantum groups*, J. Reine Angew. Math. **577** (2004), 81–104.

[AS2] ———, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. Math. **171** (2010), no. 1, 375–417.

[AT] A. Appel, V. Toledano-Laredo, *A 2–categorical extension of Etingof-Kazhdan quantisation*, Sel. Math. New Series **24** (2018), 3529–3617.

[AY] I. E. Angiono, H. Yamane, *The R–matrix of quantum doubles of Nichols algebras of diagonal type*, J. Math. Phys. **56** (2015), no. 2, 021702, 19 pp.

[BGH] N. Bergeron, Y. Gao, N. Hu, *Drinfel'd doubles and Lusztig's symmetries of two-parameter quantum groups*, J. Algebra **301** (2006), no. 1, 378–405.

[Bor] R. Borcherds, *Generalized Kac-Moody algebras* J. Algebra **115** (1988), no. 2, 501–512.

[BW1] G. Benkart, S. Witherspoon, *Restricted two-parameter quantum groups*, Representations of finite-dimensional algebras and related topics in Lie theory and geometry, 293–318, Fields Inst. Commun. **40**, Amer. Math. Soc., Providence, RI, 2004.

[BW2] ———, *Two-parameter quantum groups and Drinfel'd doubles*, Algebr. Represent. Theory **7** (2004), no. 3, 261–286.

[CG] N. Ciccoli, L. Guerra, *The Variety of Lie Bialgebras*, J. Lie Theory **13** (2003), no. 2, 577–588.

[CM] W. Chin, I. Musson, *Multiparameter quantum enveloping algebras*, Contact Franco-Belge en Algèbre (Diepenbeek, 1993), J. Pure Appl. Algebra **107** (1996), no. 2-3, 171–191.

[CP] V. Chari, A. Pressley, *A guide to quantum group*, Cambridge University Press, Cambridge, 1995.

[CV1] M. Costantini, M. Varagnolo, *Quantum double and multiparameter quantum groups*, Comm. Algebra **22** (1994), no. 15, 6305–6321.

[CV2] ———, *Multiparameter quantum function algebra at roots of 1*, Math. Ann. **306** (1996), no. 4, 759–780.

[Doi] Y. Doi, *Braided bialgebras and quadratic bialgebras*, Comm. Algebra **21** (1993), no. 5, 1731–1749.

[DoP] V. K. Dobrev, P. Parashar, *Duality for multiparametric quantum $GL(n)$*, J. Phys. **A26** (1993), 6991–7002 & Addendum, **32** (1999), 443–444.

[DoT] V. K. Dobrev, E. H. Tahri, *Duality for Multiparametric Quantum Deformation of the Supergroup $GL(m/n)$*, Int. J. Mod. Phys. **A13** (1998), 4339–4366.

[DPW] J. Du, B. Parshall, J.P. Wang, *Two-parameter quantum linear groups and the hyperbolic invariance of q–Schur algebras*, J. London Math. Soc. (2) **44** (1991), no. 3, 420–436.

[ES] P. Etingof, O. Schiffmann, *Lectures on quantum groups*, Second edition. Lectures in Mathematical Physics, International Press, Somerville, MA, 2002, xii+242 pp.

[ESS] P. Etingof, T. Schedler, O. Schiffmann, *Explicit quantization of dynamical r-matrices for finite dimensional semisimple Lie algebras*, J. Amer. Math. Soc. **13** (2000), no. 3, 595–609.

[FH] E. Frenkel, D. Hernandez, *Langlands duality for representations of quantum groups*, Math. Ann. **349** (2011), no. 3, 705–746.

[Ga1] F. Gavarini, *Quantization of Poisson groups*, Pacific J. Math. **186** (1998), no. 2, 217–266.

[Ga2] ———, *The quantum duality principle*, Ann. Inst. Fourier **52** (2002), no. 3, 809–834.

[Gar] G. A. García, *Multiparameter quantum groups, bosonizations and cocycle deformations*, Rev. Un. Mat. Argentina **57** (2016), no. 2, 1–23.

[GG1] G. A. García, F. Gavarini, *Multiparameter quantum groups at roots of unity*, J. Noncommut. Geom. **16** (2022), 839–926.

[GG2] ______, *Twisted deformations vs. cocycle deformations for quantum groups*, Comm. Cont. Math. **23** (2021), no. 8, Paper no. 2050084, 56 pp.

[GG3] ______, *Quantum Group Deformations and Quantum R–(co)matrices vs. Quantum Duality Principle*, work in progress.

[GM] L. Grunenfelder, M. Mastnak, *Pointed Hopf algebras as cocycle deformations*, preprint `arXiv:1010.4976v1` (2010).

[Hal] G. Halbout, *Construction par dualité des algèbres de Kac-Moody syméetrisables*, J. Algebra **222** (1999), no. 1, 65–81.

[HK] J. Hong, S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Graduate Studies in Mathematics **42** (2002), American Mathematical Society, Providence, RI.

[HLT] T. J. Hodges, T. Levasseur, M. Toro, *Algebraic structure of multiparameter quantum groups*, Adv. Math. **126** (1997), no. 1, 52–92.

[HPR] N. Hu, Y. Pei, M. Rosso, *Multi-parameter quantum groups and quantum shuffles, I.* Quantum affine algebras, extended affine Lie algebras, and their applications, 145–171, Contemp. Math. **506**, Amer. Math. Soc., Providence, RI, 2010.

[Ja] J. C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Mathematics **6** (1996), American Mathematical Society, Providence, RI.

[Jn] N. H. Jing, *Quantum groups with two parameters*, Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), 129–138, Contemp. Math. **134**, Amer. Math. Soc., Providence, RI, 1992.

[Ka] V. G. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge, 1990.

[Kh] V. K. Kharchenko, *Quantizations of Kac-Moody algebras*, J. Pure Appl. Algebra **218** (2014), no. 4, 666–683.

[Ks] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics **155**, Springer-Verlag, New York, 1995, xii+531 pp.

[KS] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997, xx+552 pp

[KT] C. Kassel, V. Turaev, *Biquantization of Lie bialgebras*, Pacific J. Math. **195** (2000), no. 2, 297–369.

[Lu] G. Lusztig, *Introduction to Quantum Groups*, Progr. Math. **110**, Birkhäuser Boston Inc., MA, 1993.

[Ma] Y. I. Manin, *Multiparametric quantum deformation of the general linear supergroup*, Comm. Math. Phys. **123** (1989), no. 1, 163–175.

[Mj] S. Majid, *Foundations of quantum groups*, Cambridge University Press, Cambridge, 1995.

[Mo] S. Montgomery, *Hopf Algebras and their Actions on Rings*, CBMS Reg. Conf. Ser. Math. **82**, American Mathematical Society, Providence, RI, 1993.

[MW] S. Merkulov, T. Willwacher, *Deformation theory of Lie bialgebra properads*, Geometry and physics, Vol. I, 219–247, Oxford Univ. Press, Oxford, 2018.

[OY] M. Okado, H. Yamane, *R–matrices with gauge parameters and multi-parameter quantized enveloping algebras*, Special functions (Okayama, 1990), 289–293, ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991.

[PV] B. Pogorelsky, C. Vay, *Verma and simple modules for quantum groups at non-abelian groups*, Adv. Math. 301 (2016) 423–457.

[Ra] D. E. Radford, *Hopf algebras*, Series on Knots and Everything **49**, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[Re] N. Reshetikhin, *Multiparameter quantum groups and twisted quasitriangular Hopf algebras*, Lett. Math. Phys. **20** (1990), no. 4, 331–335.

[Ro] M. Rosso, *Quantum groups and quantum shuffles*, Invent. Math. **133** (1998), no. 2, 399–416.

[Su] H. J. Sudbery, *Consistent multiparameter quantisation of* $\mathrm{GL}(n)$, J. Phys. A **23** (1990), no. 15, L697–L704.

[Sw] M. E. Sweedler, *Cohomology of algebras over Hopf algebras*, Trans. Amer. Math. Soc. **133** (1968), 205–239.

[Tk] M. Takeuchi, *A two-parameter quantization of* $\mathrm{GL}(n)$ (summary), Proc. Japan Acad. Ser. A Math. Sci. **66** (1990), no. 5, 112–114.

Departamento de Matemática, Facultad de Ciencias Exactas
Universidad Nacional de La Plata — CMaLP-CIC-CONICET
1900 La Plata, ARGENTINA
ggarcia@mate.unlp.edu.ar

Dipartimento di Matematica,
Università degli Studi di Roma "Tor Vergata"
Via della ricerca scientifica 1 — I-00133 Roma, ITALY
gavarini@mat.uniroma2.it