

FORMAL MULTIPARAMETER QUANTUM GROUPS, DEFORMATIONS AND SPECIALIZATIONS

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ABSTRACT. We introduce the notion of *formal multiparameter QUEA* — in short *FoMpQUEA* — as a straightforward generalization of Drinfeld’s quantum group $U_{\hbar}(\mathfrak{g})$. Then we show that the class of FoMpQUEAs is closed under deformations by (“toral”) twists and deformations by (“toral”) 2-cocycles: as a consequence, all “multiparameter formal QUEAs” considered so far are recovered, as falling within this class. In particular, we prove that any FoMpQUEA is isomorphic to a suitable deformation, by twist or by 2-cocycle, of Drinfeld’s standard QUEA.

We introduce also multiparameter Lie bialgebras (in short, MpLbA’s), and we consider their deformations, by twist and by 2-cocycle. The semiclassical limit of every FoMpQUEA is a suitable MpLbA, and conversely each MpLbA can be quantized to a suitable FoMpQUEA. In the end, we prove that, roughly speaking, the two processes of “specialization” — of a FoMpQUEA to a MpLbA — and of “deformation (by toral twist or toral 2-cocycle)” — at the level of FoMpQUEAs or of MpLbA’s — do commute with each other.

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