INTRODUCTION

Roughly speaking, quantum groups — in the form of quantized universal enveloping algebras — are Hopf algebra deformations of the universal enveloping algebra $U(\mathfrak{g})$ of some Lie algebra $\mathfrak{g}$. From this deformation, $\mathfrak{g}$ itself inherits (as “semiclassical limit” of the deformed coproduct) a Lie cobracket that makes it into a Lie bialgebra — the infinitesimal counterpart of a Poisson group whose tangent Lie algebra is $\mathfrak{g}$.

When $\mathfrak{g}$ is a complex simple Lie algebra, a quantum group in this sense, depending on a single parameter, was introduced by Drinfeld [Dr] as a formal series deformation $U_\hbar(\mathfrak{g})$ defined over a ring of formal power series (in the formal parameter $\hbar$) and by Jimbo and Lusztig (see [Ji], [Lu]) as a deformation $U_q(\mathfrak{g})$ defined over a ring of rational series (in the formal parameter $q$). Indeed, Jimbo’s $U_q(\mathfrak{g})$ is actually a “polynomial version” of Drinfeld’s $U_\hbar(\mathfrak{g})$.

Later on, several authors (cf. [BGH], [BW1,BW2], [CM], [CV1], [Hay], [HLT], [HPR], [Ko], [KT], [Ma], [OY], [Re], [Su], [Ta], to name a few) introduced many types of deformations of $U(\mathfrak{g})$ depending on several parameters, usually referred to as “multiparameter quantum groups”. In turn, these richer deformations induce as semiclassical limits corresponding “multiparameter” bialgebra structures on $\mathfrak{g}$. The construction of these multi-parameter deformations applies a general procedure, always available for Hopf algebras, following two patterns that we recall hereafter.

Let $H$ be any Hopf algebra (in a broad sense, in particular in some suitable category). Among all possible deformations of the Hopf structure of $H$, we look at those in which only one of either the product or the coproduct is actually modified, while the other one is kept fixed. The general deformation will then be, somehow, an intermediate case between two such extremes.

On the one hand, a twist deformation of $H$ is a (new) Hopf algebra structure on $H$ where the multiplicative structure is unchanged, whereas a new coproduct is defined by $\Delta^F(x) := F \Delta(x) F^{-1}$ for $x \in H$: here $F$ is an invertible element in $H^{\otimes 2}$ satisfying suitable axioms, called a “twist” for $H$. On the other hand, a 2-cocycle deformation of $H$ is one where the coproduct is unchanged, while a new product is defined via a formula which only depends on the old product and on a 2-cocycle $\sigma$ of $H$ (as an algebra).

Inasmuch as a meaningful notion of “duality” applies to the Hopf algebras one is dealing with, these two constructions of deformations (by twist and by 2-cocycle) are dual to each other, directly by definition — in particular, by the very conditions on $F$ and on $\sigma$. In
detail, if $H^*$ is the Hopf algebra dual to $H$ (in a proper sense), then the dual of the deformation by twist, resp. by 2-cocycle, of $H$ is a deformation by 2-cocycle, resp. by twist, of $H^*$; in addition, the 2-cocycle, resp. the twist, on $H^*$ is uniquely determined by the twist, resp. the 2-cocycle, on $H$.

It so happens that the large majority of multiparameter quantizations of $U(g)$ considered in literature actually occur as either twist deformations or 2-cocycle deformations of a one-parameter quantization of Drinfeld’s type or Jimbo-Lusztig’s type. Indeed, in both cases the twists and the 2-cocycles taken into account are of special type, namely “toral” ones, in that (roughly speaking) they are defined only in terms of the (quantum) toral part of the one-parameter deformation of $U(g)$.

Technically speaking, Drinfeld’s $U_h(g)$ is better suited for twisted deformations, while Jimbo-Lusztig’s $U_q(g)$ is typically used for 2-cocycle deformations (see [Re], [Ma], [Su], [HPR], [HLT], [CV1], [Ta]). As we aim to compare both kinds of deformations, we adapt the notion of “twist deformation” to polynomial one-parameter quantum groups $U_q(g)$. Then we consider both twist deformations and 2-cocycle deformations (of “toral type”, in both cases) of $U_q(g)$ — hereafter called “twisted quantum groups (=TwQG’s)” and “multiparameter quantum groups (=MpQG’s)”, respectively — and compare them. Moreover, because of a natural assumption we restrict our analysis to those twists and cocycles that are defined by a rational datum, namely a matrix with rational entries.

As a first result, we find a neat description of the link twist $\leftrightarrow$ 2-cocycle under duality. Namely, quantum Borel (sub)groups $U_q(b_{\pm})$ of opposite signs are in Hopf duality (in a proper sense): then we prove that the deformation on one side — by twist or by cocycle — and the dual one on the other side — by cocycle or by twist, respectively — are described by the same rational datum.

As a second, more striking result (the core of our paper, indeed), we find that, in short, twisted quantum groups and multiparameter quantum groups coincide: namely, any TwQG can be realized as a MpQG, and viceversa. Even more precisely, the twist and the 2-cocycle involved in either realization are described by the same (rational) datum. This result is, in a sense, a side effect of the “autoduality” of quantum groups (in particular Borel ones).

The proof of this equivalence is constructive, and quite explicit: indeed, switching from the realization as TwQG to that as MpQG and viceversa is a sheer change of presentation; hereafter we sketch the underlying motivation (at least in one direction). Any “standard” (=undeformed) quantum group is pointed (as a Hopf algebra); then any TwQG of “toral type” is pointed as well, and it is generated by the quantum torus and $(1, g)$—skew primitive elements: these new “homogeneous” generators yield a new presentation, which realizes the TwQG as a MpQG.

The direct consequence of this result is that (roughly speaking, and within the borders of our restrictions) there exists only one type of multiparameter quantization of $U(g)$, and consequently — taking semiclassical limit, as in [GG] — only one type of corresponding multiparameter Lie bialgebra structure on $g$.

It is worth remarking that all key elements that lead us to the above mentioned results for TwQG’s and MpQG’s are also available for Hopf algebras that are bosonizations of Nichols algebras of diagonal type (indeed, Borel quantum subgroups are such bosoniza-
Therefore, we can replicate our work in that context as well: we deal with this task in a forthcoming paper.

We finish with a few words on the structure of the paper.

In section 2 we collect the material on Hopf algebras and their deformations that will be later applied to quantum groups. Section 3 is devoted to introduce quantum groups (both in Drinfeld’s version and in Jimbo-Lusztig’s one) and their twist deformation (of rational, toral type): strictly speaking, the part on Drinfeld’s quantum groups here could be dropped, yet we present it to explain (half of) the deep-rooting motivations of our work, that otherwise would remain obscure. In section 4, instead, we present the 2-cocycle deformations (of rational, toral type) of Jimbo-Lusztig’s quantum groups, later referred to as MpQG’s.

Finally, in section 5 we compare TwQG’s and MpQG’s (in Jimbo-Lusztig’s formulation), proving that — in a proper sense — they actually coincide.

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REFERENCES


