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MULTIPARAMETER QUANTUM GROUPS AT ROOTS OF UNITY

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Abstract. We address the study of multiparameter quantum groups (=MpQG's) at roots of unity, namely quantum universal enveloping algebras $U_{\mathbf{q}}(\mathfrak{g})$ depending on a matrix of parameters $\mathbf{q} = \big(q_{ij}\big)_{i,j \in I}$. This is performed via the construction of quantum root vectors and suitable "integral forms" of $U_{\mathbf{q}}(\mathfrak{g})$, a *restricted* one — generated by quantum divided powers and quantum binomial coefficients — and an *unrestricted* one — where quantum root vectors are suitably renormalized. The specializations at roots of unity of either form are the "MpQG's at roots of unity" we look for. In particular, we study special subalgebras and quotients of our MpQG's at roots of unity — namely, the multiparameter version of small quantum groups — and suitable associated quantum Frobenius morphisms, that link the MpQG's at roots of 1 with MpQG's at 1, the latter being classical Hopf algebras bearing a well precise Poisson-geometrical content.

A key point in the discussion, often at the core of our strategy, is that every MpQG is actually a 2–cocycle deformation of the algebra structure of (a lift of) the "canonical" one-parameter quantum group by Jimbo-Lusztig, so that we can often rely on already established results available for the latter. On the other hand, depending on the chosen multiparameter $\mathbf{q}$ our quantum groups yield (through the choice of integral forms and their specializations) different semiclassical structures, namely different Lie coalgebra structures and Poisson structures on the Lie algebra and algebraic group underlying the canonical one-parameter quantum group.

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1. Introduction

In literature, by "quantum group" one usually means some deformation of an algebraic object that in turn encodes a geometrical object describing symmetries (such as a Lie or algebraic group or a Lie algebra): we are interested now in the case when the geometrical object is a Lie bialgebra $\mathfrak{g}$, and the algebraic one its universal enveloping algebra $U(\mathfrak{g})$, with its full structure of co-Poisson Hopf algebra.

In most cases, such a deformation depends on one single parameter, in a "formal" version, like with Drinfeld's $U_\hbar(\mathfrak{g})$, or in a "polynomial" one, for Jimbo-Lusztig's $U_q(\mathfrak{g})$. But since the dawn of the theory, more general deformations depending on many parameters have been considered too: one then talks of "multiparameter quantum groups" (in short,

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MpQG's) that again exist both in formal and polynomial versions; see for instance [BGH], [BW1, BW2], [CM], [CV1], [Hay], [HLT], [HP], [HPR], [Ko], [KT], [Man], [OY], [Re], [Su], [Ta] — and the list might be quite longer.

In the previously mentioned papers, multiparameter quantum enveloping algebras where often introduced via *ad hoc* constructions. A very general recipe, instead, was that devised by Reshetikhin (cf. [Re]), that consists in performing a so-called *deformation by twist* on a "standard" one-parameter quantum group.

Similarly, a dual method was developed, that starts again from a usual one-parameter quantum group and then performs on it a deformation by a 2–cocycle. In addition, as the usual uniparameter quantum group is a quotient of the Drinfeld's quantum double of two Borel quantum (sub)groups, one can start by deforming (e.g., by a 2–cocycle) the Borel quantum subgroups and then look at their quantum double and its quotient. This is the point of view adopted, for instance, in [AA], [AAR1, AAR2], [An1, An2, An3, An4], [AS1, AS2], [AY], [Gar], [He1, He2], [HK], [HY] and [Mas], where in addition the Borel quantum (sub)groups are always thought of as bosonizations of Nichols algebras.

In our forthcoming papers [GG1, GG2] we thoroughly compare deformations by twist or by 2–cocycles on the standard uniparameter quantum group; up to technicalities, it turns out that the two methods yield the same results. Taking this into account, we adopt the point of view of deformations by 2–cocycles, implemented on uniparameter quantum groups, that are realized as (quotients of) quantum doubles of Borel quantum (sub)groups. With this method, the multiparameter $\mathbf{q}$ encoding our MpQG is used from scratch as the core datum to construct the Borel quantum (sub)groups and eventually remains in the description of our MpQG by generators and relations. In this approach, a natural constraint arises for $\mathbf{q}$, namely that it be *of Cartan type*, to guarantee that our MpQG have finite Gelfand-Kirillov dimension.

In order to have meaningful specializations of a MpQG, one needs to choose a suitable integral form of that MpQG, and then specialize the latter: indeed, by "specialization of a MpQG" one means in short the specialization of such an integral form of it. The outcome of the specialization process then can strongly depend on the choice of the integral form. For the usual case of uniparameter "canonical" quantum groups, one usually considers two types of integral forms, namely *restricted* ones (after Lusztig's) and *unrestricted* ones (after De Concini and Procesi), whose specialization yield entirely different outcomes — dual to each other, in a sense. There also exist *mixed* integral forms (due to Habiro and Thang Le) that are very interesting for applications in algebraic topology.

For general MpQG's, we introduce integral forms of restricted, unrestricted and mixed type, by directly extending the construction of the canonical setup: this is quite a natural step, yet (to the best of the authors' knowledge) it had not been considered so far. Moreover, for restricted forms — for which the multiparameter has to be "integral", i.e. made of powers (with integral exponents) of just one single, "basic" parameter q — we consider two possible variants, which gives something new even in the canonical case. For these integral forms (of either type) we state and prove all those fundamental structure results (triangular decompositions, PBW Theorems, etc.) that one needs to work with them.

When taking specialization at $q = 1$ (where "q" is again sort of a "basic parameter" underlying the multiparameter $\mathbf{q}$), co-Poisson and Poisson Hopf structures pop up, yielding classical objects that bear some Poisson geometrical structure. In detail, when specializing the restricted form one gets the enveloping algebra of a Lie bialgebra, and when specializing the unrestricted one the function algebra of a Poisson group is found: this shows some duality phenomenon, which is not surprising because the two integral forms are in a sense related by Hopf duality. This feature already occurs in the uniparameter, canonical case: but in the present, multiparameter setup, the additional relevant fact is that the involved (co)Poisson structures directly depend on the multiparameter $\mathbf{q}$.

Now consider instead a non-trivial root of 1, say ε. Then the specialization of a MpQG at $q = \varepsilon$ is tightly related with its specialization at $q = 1$: this link is formalized in a so-called *quantum Frobenius morphism* — a Hopf algebra morphism with several remarkable properties between these two specialized MpQG's — moving to opposite directions in the restricted and the unrestricted case. We complete these morphisms to short exact sequences, whose middle objects are our MpQG's at $q = \varepsilon$; the new Hopf algebras we add to complete the sequences are named *small MpQG's*.

Remarkably enough, we prove that the above mentioned short exact sequences have the additional property of being *cleft*; thus, our specialized MpQG's at $q = \varepsilon$ are *cleft extensions* of the corresponding small MpQG's and the corresponding specialized MpQG's at $q = 1$ — which are *classical* geometrical objects, see above. Furthermore, implementing this construction in both cases — with restricted and with unrestricted forms — literally yields *two* small MpQG's: nevertheless, we eventually prove that they do coincide indeed.

To some extent, these results (at roots of 1) are a direct generalisation of what happens in the uniparameter case (i.e., for the canonical multiparameter). However, some of our results seem to be entirely new even for the uniparameter context.

Finally, here is the plan of the paper.

In section 2 we set some basic facts about Hopf algebras, the bosonization process, cocycle deformations, etc. — along with all the related notation.

Section 3 introduces our MpQG's: we define them by generators and relations, and we recall that we can get them as 2–cocycle deformations of the canonical one.

We collect in section 4 some fundamental results on MpQG's, such as the construction of quantum root vectors and PBW-like theorems (and related facts). In addition, we compare the multiplicative structure in the canonical MpQG with that in a general MpQG, the latter being thought of as 2–cocycle deformation of the former.

In section 5 we introduce integral forms of our MpQG's — of restricted type and of unrestricted type — providing all the basic results one needs when working with them. We also shortly discuss mixed integral forms.

Section 6 focuses on specializations at 1, and the semiclassical structures arising from MpQG's by means of this process.

At last, in section 7 we finally harvest our main results. Namely, we deal with specializations at non-trivial roots of 1, with quantum Frobenius morphisms and with small MpQG's, for both the restricted version and the unrestricted one.

2. Generalities on Hopf algebras and deformations

Throughout the paper, by $\Bbbk$ we denote a field of characteristic zero and by $\Bbbk^\times$ we denote the group of units of $\Bbbk$. By convention, $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$.

2.1. **Conventions for Hopf algebras.**

Our main references for the theory of Hopf algebras are [Mo], [Sw] and [Ra], for Lie algebras [Hu] and for quantum groups [Ja] and [BG]. We use standard notation for Hopf algebras; the comultiplication is denoted Δ and the antipode $\mathcal{S}$. For the first, we use the Heyneman-Sweedler notation, namely $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

Let H be a Hopf algebra. The *left adjoint representation* of H is the algebra morphism $\mathrm{ad}_\ell : H \longrightarrow \mathrm{End}(H)$ given by $\mathrm{ad}_\ell(x)(y) := x_{(1)}\, y\, \mathcal{S}\big(x_{(2)}\big)$ for $x, y \in H$; ,we drop the subscript ℓ unless needed; the *right* adjoint action $\mathrm{ad}_r : H \longrightarrow \mathrm{End}(H)$ is given by $\mathrm{ad}_r(x)(y) := \mathcal{S}\big(x_{(1)}\big)\, y\, x_{(2)}$ for $x, y \in H$. Any subalgebra K of H is said to be *normal* if $\mathrm{ad}_\ell(h)(k) \in K$, $\mathrm{ad}_r(h)(k) \in K$ for all $h \in H$, $k \in K$.

In any coalgebra C, the set of group-like elements of a coalgebra is denoted by $G(C)$; also, we denote by $C^+ := \mathrm{Ker}(\epsilon)$ the augmentation ideal of C, where $\epsilon : C \longrightarrow \Bbbk$ is the counit map of C. If $g, h \in G(H)$, the set of (g,h)–primitive elements is defined to be

$$P_{g,h}(H) \;:=\; \big\{\, x \in H \,|\, \Delta(x) = x \otimes g + h \otimes x \,\big\}$$

In particular, we call $P(H) := P_{1,1}(H)$ the set of primitive elements.

It is convenient to recall the notions of exact sequence and of *cleft extension*:

Definition 2.1.1. *(cf.* [AD]*)* A sequence of Hopf algebras maps over a field $\Bbbk$

$$1 \longrightarrow B \xrightarrow{\ \iota\ } A \xrightarrow{\ \pi\ } H \longrightarrow 1$$

where 1 denotes the Hopf algebra $\Bbbk$, is called *exact* if ι is injective, π is surjective, $\mathrm{Ker}(\pi) = AB^+$ and $B = {}^{\mathrm{co}\,\pi}A := \big\{ a \in A \,\big|\, (\pi \otimes \mathrm{id})\big(\Delta(a)\big) = 1 \otimes a \big\}$. We say that A is a *cleft extension* of B by H if there exists an H–colinear, convolution-invertible section γ of π . $\diamondsuit$

Finally, we recall the notions of *Hopf pairing* and *skew-Hopf pairing* of Hopf algebras:

Definition 2.1.2. *(cf.* [AY, §2.1]*)* Given two Hopf algebras H and K with bijective antipode over a ring R, an R–linear map $\eta : H \otimes_R K \longrightarrow R$ is called

— *Hopf pairing* (between H and K) if, for all $h \in H$, $k \in K$, one has

$$\eta\big(h\,,\,k_1\,k_2\big) \,=\, \eta\big(h_{(1)}\,,\,k_1\big)\,\eta\big(h_{(2)}\,,\,k_2\big) \;\;,\qquad \eta\big(h_1\,h_2\,,\,k\big) \,=\, \eta\big(h_1\,,\,k_{(1)}\big)\,\eta\big(h_2\,,\,k_{(2)}\big)$$
$$\eta\big(h\,,1\big) \,=\, \epsilon(h) \;\;,\quad \eta\big(1\,,k\big) \,=\, \epsilon(k) \;\;,\qquad \eta\big(S^{\pm 1}(h)\,,k\big) \,=\, \eta\big(h\,,S^{\pm 1}(k)\big)$$

— *skew-Hopf pairing* (between H and K) if, for all $h \in H$, $k \in K$, one has

$$\eta\big(h\,,\,k_1\,k_2\big) \,=\, \eta\big(h_{(1)}\,,\,k_1\big)\,\eta\big(h_{(2)}\,,\,k_2\big) \;\;,\qquad \eta\big(h_1\,h_2\,,\,k\big) \,=\, \eta\big(h_2\,,\,k_{(1)}\big)\,\eta\big(h_1\,,\,k_{(2)}\big)$$
$$\eta\big(h\,,1\big) \,=\, \epsilon(h) \;\;,\quad \eta\big(1\,,k\big) \,=\, \epsilon(k) \;\;,\qquad \eta\big(S^{\pm 1}(h)\,,k\big) \,=\, \eta\big(h\,,S^{\mp 1}(k)\big) \qquad \diamondsuit$$

Recall that, given two Hopf R–algebras H_+ and H_-, and a Hopf pairing among them, say $\pi : H_-^{\mathrm{cop}} \otimes_R H_+ \longrightarrow \Bbbk$, the *Drinfeld double* $D(H_-, H_+, \pi)$ is the quotient algebra $T(H_- \oplus H_+)\big/\mathcal{I}$ where $\mathcal{I}$ is the (two-sided) ideal generated by the relations

$$1_{H_-} \,=\, 1 \,=\, 1_{H_+} \;\;,\qquad a \otimes b \,=\, a\,b \qquad\qquad \forall\;\, a\,,b \in H_+ \;\;\text{ or }\;\; a\,,b \in H_- \;\;,$$
$$x_{(1)} \otimes y_{(1)}\; \pi\big(y_{(2)}, x_{(2)}\big) \,=\, \pi\big(y_{(1)}, x_{(1)}\big)\; y_{(2)} \otimes x_{(2)} \qquad \forall\;\, x \in H_+\,,\; y \in H_- \;\;;$$

such a quotient R–algebra is also endowed with a standard Hopf algebra structure, which is *consistent*, in that both H_+ and H_- are Hopf R–subalgebras of it.

2.1.3. Yetter-Drinfeld modules, bosonization and Hopf algebras with a projection. Let H be a Hopf algebra with bijective antipode. A Yetter-Drinfeld module over H is a left H–module and a left H–comodule V, with comodule structure denoted by $\delta : V \longrightarrow H \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$, such that

$$\delta(h \cdot v) \,=\, h_{(1)} v_{(-1)} \mathcal{S}\big(h_{(3)}\big) \otimes h_{(2)} \cdot v_{(0)} \qquad \text{for all } \; v \in V\,,\, h \in H\;.$$

Let ${}^H_H\mathcal{YD}$ be the category of Yetter-Drinfeld modules over H with H–linear and H–colinear maps as morphisms. The category ${}^H_H\mathcal{YD}$ is monoidal and braided. A Hopf algebra in the category ${}^H_H\mathcal{YD}$ is called a *braided Hopf algebra* for short.

Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$. The procedure to obtain a usual Hopf algebra from the (braided) Hopf algebras R and H is called *bosonization* or *Radford-Majid product*, and it is usually denoted by $R \,\#\, H$. As a vector space $R \,\#\, H := R \otimes H$, and the multiplication and comultiplication are given by the smash-product and smash-coproduct, respectively. That is, for all $r, s \in R$ and $g, h \in H$, we have

$$(r\#g)(s\#h) \;:=\; r\big(g_{(1)} \cdot s\big) \,\#\, g_{(2)} h$$
$$\Delta(r\#g) \;:=\; r^{(1)} \,\#\, \big(r^{(2)}\big)_{(-1)} g_{(1)} \otimes \big(r^{(2)}\big)_{(0)} \,\#\, g_{(2)}$$
$$\mathcal{S}(r\#g) \;:=\; \big(1 \,\#\, \mathcal{S}_H\big(r_{(-1)} g\big)\big)\big(\mathcal{S}_R\big(r_{(0)}\big) \,\#\, 1\big)$$

where $\Delta_R(r) \,=\, r^{(1)} \otimes r^{(2)}$ is the comultiplication in $R \in {}^H_H\mathcal{YD}$ and $\mathcal{S}_R$ the antipode. The map $\iota : H \longrightarrow R\#H \,\big(h \mapsto 1\#h\big)$, resp. $\pi : R\#H \longrightarrow H \,\big(r \,\#\, h \mapsto \epsilon_R(r)\, h\big)$, is a Hopf algebra monomorphism, resp. epimorhism, and $\pi \circ \iota = \mathrm{id}_H$. Moreover, we have $R = (R\#H)^{\mathrm{co}\,\pi} = \big\{\, x \in R \,\#\, H \,\big|\, (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1 \,\big\}$.

Conversely, let A be a Hopf algebra with bijective antipode and $\pi : A \longrightarrow H$ a Hopf algebra epimorphism. If there is a Hopf algebra map $\iota : H \longrightarrow A$, such that $\pi \circ \iota = \mathrm{id}_H$, then $R := A^{\mathrm{co}\,\pi}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$, called the *diagram* of A , and we have $A \cong R\#H$ as Hopf algebras. See [Ra, 11.6] for further details.

2.2. **Cocycle deformations.**

We recall now the standard procedure that, starting from a given Hopf algebra and a suitable 2–cocycle on it, gives us a new Hopf algebra structure on it, with the same coproduct and a new, "deformed" product. We shall then see the special form that this construction may take when the Hopf algebra is bigraded by some Abelian group and the 2–cocycle is induced by one of that group.

2.2.1. First construction. Let $\big(H, m, 1, \Delta, \epsilon\big)$ be a bialgebra over a ring R. A *normalized Hopf 2-cocycle* (see [Mo, Sec. 7.1]) is a map σ in $\mathrm{Hom}_{\Bbbk}(H \otimes H, R)$ which is convolution invertible and such that, for all $a, b, c \in H$, we have

$$\sigma\big(b_{(1)}, c_{(1)}\big)\, \sigma\big(a, b_{(2)}c_{(2)}\big) \,=\, \sigma\big(a_{(1)}, b_{(1)}\big)\, \sigma\big(a_{(2)}b_{(2)}, c\big)$$

and $\sigma(a, 1) = \varepsilon(a) = \sigma(1, a)$. We simply call it a 2–cocycle if no confusion arises.

Using a 2–cocycle σ it is possible to define a new algebra structure on H by deforming the multiplication: indeed, define $m_\sigma = \sigma * m * \sigma^{-1} : H \otimes H \longrightarrow H$ by

$$m_\sigma(a, b) \,=\, a \cdot_\sigma b \,=\, \sigma\big(a_{(1)}, b_{(1)}\big)\, a_{(2)}\, b_{(2)}\, \sigma^{-1}\big(a_{(3)}, b_{(3)}\big) \qquad \forall\; a, b \in H$$

If in addition H is a Hopf algebra with antipode $\mathcal{S}$, then define also $\mathcal{S}_\sigma : H \longrightarrow H$ as $\mathcal{S}_\sigma = \sigma * \mathcal{S} * \sigma^{-1} : H \longrightarrow H$ where

$$\mathcal{S}_\sigma(a) \,=\, \sigma\big(a_{(1)}, \mathcal{S}(a_{(2)})\big)\, \mathcal{S}(a_{(3)})\, \sigma^{-1}\big(\mathcal{S}(a_{(4)}), a_{(5)}\big) \qquad \forall\; a \in H$$

It is then known — see [DT] — that $\big(H, m_\sigma, 1, \Delta, \epsilon\big)$ is in turn a bialgebra, and also that $\big(H, m_\sigma, 1, \Delta, \epsilon, \mathcal{S}_\sigma\big)$ is a Hopf algebra: we shall call such a new structure on H a *cocycle deformation* of the old one, and we shall graphically denote it by H_σ.

When dealing with a Hopf algebra H and its deformed counterpart H_σ as above, we denote by ad_ℓ and ad_r the adjoint actions in H and by ad_ℓ^σ and ad_r^σ those in H_σ.

2.2.2. Second construction. There is a second type of cocycle twisting — of algebras, bialgebras and Hopf algebras — that we shall need (cf. [AST] and references therein). Let $\varGamma$ be an Abelian group, for which we adopt multiplicative notation, and H an algebra over a ring R that is $\varGamma$–bigraded (i.e., graded by $\varGamma \times \varGamma$): so $H = \bigoplus_{(\gamma,\eta) \in \varGamma \times \varGamma} H_{\gamma,\eta}$ with $R \subseteq H_{1,1}$ and $H_{\gamma,\eta} H_{\gamma',\eta'} \subseteq H_{\gamma\gamma',\eta\eta'}$. Given any group 2–cocycle $c : \varGamma \times \varGamma \longrightarrow R^\times$ where $R^\times$ is the group of units of R, define a new product on H, denoted by $\underset{c}{\star}$, by $h \underset{c}{\star} k := c\big(\eta', \kappa'\big)\, c(\eta, \kappa)^{-1}\, h \cdot k$ for all homogeneous $h, k \in H$ with degrees $\big(\eta, \eta'\big), \big(\kappa, \kappa'\big) \in \varGamma \times \varGamma$. Then $\big(H\,;\underset{c}{\star}\big)$ is (again) an associative algebra, with the same unit as H before.

As $\varGamma$ is free Abelian, each element of $H^2(\varGamma, R^\times)$ has a representative, say c, which is *bimultiplicative* and such that $c\big(\eta, \eta^{-1}\big) = 1$ for all $\eta \in \varGamma$ (see [AST, Proposition 1 and Lemma 4]); so we may assume that $c : \varGamma \times \varGamma \longrightarrow R^\times$ is such a cocycle. Thus

$$c\big(\gamma, \eta^{-1}\big) \,=\, c\big(\gamma^{-1}, \eta\big) \,=\, c(\gamma, \eta)^{-1} \;\;, \qquad c(\gamma, 1) \,=\, c(1, \gamma) \,=\, 1 \qquad \forall\; \gamma, \eta \in \varGamma$$

Now assume H is a bialgebra, with $\Delta\big(H_{\alpha,\beta}\big) \subseteq \sum_{\gamma \in \varGamma} H_{\alpha,\gamma} \otimes_R H_{\gamma,\beta}$ for all $(\alpha, \beta) \in \varGamma \times \varGamma$ and $\epsilon\big(H_{\alpha,\beta}\big) = 0$ if $\alpha \neq \beta$. Then H with the new product $\underset{c}{\star}$ and the old coproduct Δ is a bialgebra too. If in addition H is a Hopf algebra, whose antipode obeys $\mathcal{S}\big(H_{\alpha,\beta}\big) \subseteq \big(H_{\beta^{-1},\alpha^{-1}}\big)$ — for $(\alpha, \beta) \in \varGamma \times \varGamma$ — then the new bialgebra structure on H (with the new product and the old coproduct) makes it again into a Hopf algebra with antipode $\mathcal{S}^{(c)} := \mathcal{S}$ (the old one). In all cases, we will graphically denote by $H^{(c)}$ the new structure on H obtained by this (second) cocycle twisting.

In the sequel we shall compare computations in H with computations in $H^{(c)}$, in particular regarding the adjoint action(s); in such cases, we shall denote by ad_ℓ and ad_r the adjoint actions in H and by $\mathrm{ad}_\ell^{(c)}$ and $\mathrm{ad}_r^{(c)}$ those in $H^{(c)}$.

We shall make use of the following result (whose proof is straightforward):

Lemma 2.2.3. *(cf.* [CM, Lemma 3.2]*) Let a 2–cocycle* $c : \varGamma \times \varGamma \longrightarrow R^\times$ *as above be given, and assume in addition (with no loss of generality) that* c *is bimultiplicative. Let* $e, b \in H$ *be homogeneous with degrees* $(\gamma, 1)$ *and* $(\eta, 1)$ *respectively, and assume* e *is* $(1, h)$*–primitive with* $h \in H$ *homogeneous of degree* (γ, γ) *. Then*

$$\begin{aligned} \mathrm{ad}_\ell^{(c)}(e)(b) \; &= \; c(\gamma, \eta)^{-1} \, \mathrm{ad}_\ell(e)(b) \\ \mathrm{ad}_r^{(c)}(e)(b) \; &= \; c(\gamma, \gamma) \big(- h^{-1} e \, b \, + \, c(\gamma, \eta) \, c(\eta, \gamma)^{-1} h^{-1} b \, e \big) \end{aligned}$$

In particular, if $c(\gamma, \eta) \, c(\eta, \gamma)^{-1} \, = \, 1$ *, then* $\mathrm{ad}_r^{(c)}(e)(b) \, = \, c(\gamma, \gamma) \, \mathrm{ad}_r(e)(b)$ *.*

2.2.4. A relation between the two constructions Let H be a Hopf algebra with bijective antipode, R a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $A = R \# H$ its bosonization (see [Gar] for details). For any $a \in R$, set $\delta(a) = a_{(-1)} \otimes a_{(0)}$ for the left coaction of H.

Any Hopf 2–cocycle on H gives rise to a Hopf 2–cocycle on A which may deform the H-module structure of R and consequently its braided structure as well. Specifically, let $\sigma \in \mathcal{Z}^2(H, \Bbbk)$: then the map $\tilde{\sigma} : A \otimes A \longrightarrow \Bbbk$ given by

$$\tilde{\sigma}(r \# h, s \# k) \, = \, \sigma(h, k) \, \epsilon_R(r) \, \epsilon_R(s) \qquad \qquad \forall \; r, s \in R \, , \; h, k \in H$$

is a normalized Hopf 2–cocycle such that $\tilde{\sigma}\big|_{H \otimes H} = \sigma$. By [Mas, Prop. 5.2] we have $A_{\tilde{\sigma}} = R_\sigma \# H_\sigma$, where $R_\sigma = R$ as coalgebras, and the product is given by

$$a \cdot_\sigma b \, := \, \sigma\big(a_{(-1)}, b_{(-1)}\big) \, a_{(0)} \, b_{(0)} \qquad \qquad \text{for all} \quad a, b \in R \ .$$

Therefore, H_σ is a Hopf subalgebra of $A_{\tilde{\sigma}}$ and the map $\mathcal{Z}^2(H, \Bbbk) \longrightarrow \mathcal{Z}^2(A, \Bbbk)$ given by $\sigma \mapsto \tilde{\sigma}$ is a section of the map $\mathcal{Z}^2(A, \Bbbk) \longrightarrow \mathcal{Z}^2(H, \Bbbk)$ induced by the restriction; in particular, it is injective.

Now assume $H = \Bbbk\varGamma$, with $\varGamma$ a group. Then a normalized Hopf 2–cocycle on H is equivalent to a 2–cocycle $\varphi \in \mathcal{Z}^2(\varGamma, \Bbbk)$, i.e. a map $\varphi : \varGamma \times \varGamma \longrightarrow \Bbbk^\times$ such that

$$\varphi(g, h) \, \varphi(g \, h, t) \, = \, \varphi(h, t) \, \varphi(g, h \, t) \quad , \qquad \varphi(g, e) \, = \, 1 \, = \, \varphi(e, g) \qquad \forall \; g, h, t \in \varGamma \ .$$

Assume $A = R \,\#\, \Bbbk\varGamma$ is given by a bosonization over a free Abelian group $\varGamma$. Then the coaction of $\Bbbk\varGamma$ on the elements of R induces a $(\varGamma \times \varGamma)$–grading on A with $\deg(g) := (g, g)$ for all $g \in \varGamma$ and $\deg(a) := (g, 1)$ if $\delta(a) = g \otimes a$ with $a \in R$ a homogeneous element; in particular, a is $(1, g)$–primitive, since $\Delta(a) = a \otimes 1 + a_{(-1)} \otimes a_{(0)}$. If $\varphi \in \mathcal{Z}^2(\varGamma, \Bbbk)$, then $A^{(\varphi^{-1})} = A_{\tilde{\varphi}}$, where $\tilde{\varphi}$ is the Hopf 2–cocycle on A induced by φ . Indeed, this holds true because, for a, b homogeneous in R of degree $(g, 1)$ and $(h, 1)$ respectively, we have that

$$a \underset{\varphi^{-1}}{\star} b \, = \, \varphi(1, 1)^{-1} \, \varphi(g, h) \, a \, b \, = \, \varphi\big(a_{(-1)}, b_{(-1)}\big) \, a_{(0)} \, b_{(0)} \, = \, a \cdot_\sigma b.$$

2.3. **Basic constructions from multiparameters.**

The definition of multiparameter quantum groups requires a whole package of related material, involving root data, weight lattices, etc. This entails several different constructions, depending on "multiparameters", that we now go and present.

2.3.1. Root data. Hereafter we fix $\theta \in \mathbb{N}_+$ and $I := \{1, \dots, \theta\}$ as before. Let $A := \big(a_{ij}\big)_{i, j \in I}$ be a Cartan matrix of finite type; then there exists a unique diagonal matrix $D := \big(d_i \, \delta_{ij}\big)_{i, j \in I}$ with positive integral, pairwise coprime entries such that DA is symmetric. Let $\mathfrak{g}$ be the finite dimensional simple Lie algebra over $\mathbb{C}$ associated with A , let Φ be the (finite) root system of $\mathfrak{g}$, with $\Pi = \big\{ \alpha_i \,|\, i \in I \big\}$ as a set of simple roots, $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ the associated root lattice, Φ^+ the set of positive roots with respect to Π , $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ the positive root (semi)lattice. We denote by P the associated weight lattice, with basis $\big\{ \omega_i \big\}_{i \in I}$ dual to $\big\{ \alpha_j \big\}_{j \in I}$, namely $\omega_i(\alpha_j) = \delta_{ij}$ for all $i, j \in I$. Using an invariant non-degenerate bilinear form on the dual $\mathfrak{h}^*$ of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we identify Q with a sublattice of P ; in particular, we have $\alpha_i = \sum_{j \in I} a_{ji} \, \omega_j$ for all $i \in I$.

In this setup, we have two natural $\mathbb{Z}$–bilinear pairings $P \times Q \longrightarrow \mathbb{Z}$, that we denote by $\langle\ ,\ \rangle$ and $(\ ,\ )$, one given by the evaluation (of weights onto roots), and the other one by $(\omega_i, \alpha_j) := d_i \delta_{ij}$ for all $i, j \in I$. In particular, the restriction of $(\ ,\ )$ to $Q \times Q$ is a *symmetric* bilinear pairing on Q; moreover, both the given pairings uniquely extend to $\mathbb{Q}$–bilinear pairings, still denoted by $\langle\ ,\ \rangle$ and $(\ ,\ )$, onto $\big(\mathbb{Q} \otimes_{\mathbb{Z}} P\big) \times \big(\mathbb{Q} \otimes_{\mathbb{Z}} Q\big) = \big(\mathbb{Q} \otimes_{\mathbb{Z}} P\big) \times \big(\mathbb{Q} \otimes_{\mathbb{Z}} P\big)$. Then we define

$$Q^\circ := \big\{ \lambda \in \mathbb{Q}Q \,\big|\, (\lambda, \gamma) \in \mathbb{Z} \;\; \forall\, \gamma \in Q \big\} = \big\{ \rho \in \mathbb{Q}Q \,\big|\, (\gamma, \rho) \in \mathbb{Z} \;\; \forall\, \gamma \in Q \big\}$$

By construction $P \subseteq Q^\circ$, and equality holds true if and only if $\mathfrak{g}$ is simply-laced.

Note that, in terms of the above symmetric pairing on Q, one has $d_i = (\alpha_i, \alpha_i)\big/2$ for all $i \in I$. More in general, we shall use the notation $d_\alpha := (\alpha, \alpha)\big/2$ for every $\alpha \in \Phi^+$; in particular $d_{\alpha_i} = d_i$ $(i \in I)$. We denote by W the Weyl group associated with the root data (Φ, Π); it is generated by the simple reflections s_i given by $s_i(\beta) := \beta - \frac{2(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}\,\alpha_i$ $(i \in I)$; in particular $s_i(\alpha_j) = \alpha_j - a_{ij}\,\alpha_i$ for $i, j \in I$.

2.3.2. Multiparameters. Let $\Bbbk$ be our fixed ground field, and let $I := \{1, \dots, \theta\}$ be as in §2.3.1 above. We fix a matrix $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$, whose entries belong to $\Bbbk^\times$, that will play the role of "parameters" of our quantum groups. These can be used to construct diagonal braidings and braided spaces, see for example [ARS], [An4], [Gar], [He2].

We assume that $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ is of *finite Cartan type* A i.e. there is a Cartan matrix $A = \big(a_{ij}\big)_{i,j \in I}$ of finite type such that

$$q_{ij}\, q_{ji} = q_{ii}^{a_{ij}} \qquad\qquad \forall\;\, i, j \in I \tag{2.1}$$

To avoid some irrelevant technicalities, *we assume that A is indecomposable.*

For later use we fix in $\Bbbk$ some "square roots" of all the q_{ii}'s, as follows. From the relations in (2.1) one finds (since the Cartan matrix A is indecomposable) that there exists $j_0 \in I$ such that $q_{ii} = q_{j_0 j_0}^{e_i}$ for some $e_i \in \mathbb{N}$, for all $i \in I$. Now *we assume hereafter that $\Bbbk$ contains a square root of $q_{j_0 j_0}$, which we fix throughout and denote by $q_{j_0} := \sqrt{q_{j_0 j_0}}$.* Then we set $q_i := q_{j_0}^{e_i}$ (a square root of q_{ii}) for all $i \in I$.

As recorded in §2.3.1 above, the Cartan matrix A is diagonalizable, hence we fix positive, relatively prime integers $d_1, \dots, d_\theta$ such that the diagonal matrix $D = \mathit{diag}\,(d_1, \dots, d_\theta)$ symmetrizes A, i.e. $D\,A$ is symmetric; in fact, each of these d_i's coincides with the corresponding exponent e_i mentioned above.

We introduce now some special cases of Cartan type multiparameter matrices.

Integral type: We say that $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ is *of integral type* if it is of Cartan type and there exist $b_{ij} \in \mathbb{Z}$ such that $q_{ij} = q^{b_{ij}}$ for $i, j \in I$; then we may assume $b_{ii} = 2\,d_i$ and $b_{ij} + b_{ji} = 2\,d_i\,a_{ij}$ $(i, j \in I)$, with $q = q_{j_0}$ and the d_i's as above. To be precise, *we say also that $\mathbf{q}$ is "of integral type B"*, with $B := \big(b_{ij}\big)_{i,j \in I} \in M_\theta(\mathbb{Z})$.

Strongly integral type: We say that $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ is *of strongly integral type* if it is of integral type and in addition one has $b_{ij} \in d_i\,\mathbb{Z} \bigcap d_j\,\mathbb{Z}$ for all $i, j \in I$. In other words, $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ of Cartan type is *strongly integral* if and only if there exist integers $t^+_{ij}, t^-_{ij} \in \mathbb{Z}$ such that $q_{ij} = q^{d_i t^+_{ij}} = q^{d_j t^-_{ij}}$ for all $i, j \in I$; then we may assume $t^\pm_{ii} = 2 = a_{ii}$ and $t^+_{ij} + t^-_{ji} = 2\,a_{ij}$, for $i, j \in I$.

Canonical multiparameter: As a last (very) special case, given $q \in \Bbbk^\times$ consider

$$\check{q}_{ij} := q^{d_i a_{ij}} \qquad\qquad \forall\;\, i, j \in I \tag{2.2}$$

with d_i $(i \in I)$ given as above. These $q_{ij} = \check{q}_{ij}$'s obey condition (2.1), hence the matrix $\mathbf{q} = \check{\mathbf{q}}$ is of Cartan type A: we shall refer to it as to *the "canonical" case.*

Overall we have the following relations among different types of multiparameters:

$$\text{“canonical”} \Longrightarrow \text{“strongly integral”} \Longrightarrow \text{“integral”} \Longrightarrow \text{“Cartan”}$$

By the way, when the multiparameter matrix $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ is *symmetric*, i.e. $q_{ij} = q_{ji}$ (for all $i,j \in I$), then the conditions $q_{ij}\,q_{ji} = q_{ii}^{a_{ij}}$ read $q_{ij}^2 = q^{2\,d_i a_{ij}}$, hence $q_{ij} = \pm q^{d_i a_{ij}}$ (for all $i,j \in I$). This means that every symmetric multiparameter is "almost the canonical one", as indeed it is the canonical one "up to sign(s)".

Finally, we assume that for each $i,j \in I$ there exists in the ground field $\Bbbk$ a square root of q_{ij}, which we fix once and for all and denote hereafter by $q_{ij}^{1/2}$; in addition, we require that these square roots satisfy the "compatibility constraints" $q_{ii}^{1/2} = q_i \;\big(:= q^{d_i}\big)$ and $q_{ij}^{1/2}\,q_{ji}^{1/2} = \big(q_{ii}^{1/2}\big)^{a_{ij}}$ for all $i,j \in I$ — in short, we assume that "the signs of all square roots $q_{ij}^{1/2}$ are chosen in an overall consistent way".

Even more, *when $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ in particular is of integral type, say $q_{ij} = q^{b_{ij}}$, we fix a square root $q^{1/2}$ of q in $\Bbbk$ and we set $q_{ij}^{1/2} := \big(q^{1/2}\big)^{b_{ij}} \in \Bbbk$ for all $i\,,j \in I$.*

2.3.3. Multiparameter Lie bialgebras. Consider the complex Lie algebra $\mathfrak{g}$ associated with the Cartan matrix A as in §2.3.1, and let $\mathfrak{b}_+$ and $\mathfrak{b}_-$ be opposite Borel subalgebras in it, containing a Cartan subalgebra $\mathfrak{h}$ whose associated set of roots is identified with Φ. There is a canonical, non-degenerate pairing between $\mathfrak{b}_+$ and $\mathfrak{b}_-$, and using it one can construct a *Manin double* $\mathfrak{g}_{(D)} = \mathfrak{b}_+ \oplus \mathfrak{b}_-$, which is automatically endowed with a structure of Lie bialgebra. Roughly, $\mathfrak{g}_{(D)}$ is like $\mathfrak{g}$ *but* with *two copies of* $\mathfrak{h}$ inside it; see [Hal] for details (in particular Proposition 4.5 therein, with $\mathfrak{g}_{(D)}$ denoted $\mathfrak{c}$).

Now fix in $\mathfrak{b}_+$ and $\mathfrak{b}_-$ generators $\mathrm{e}_i, \mathrm{h}_i^+ \;(i \in I)$ and $\mathrm{f}_i, \mathrm{h}_i^- \;(i \in I)$ respectively as in the usual Serre's presentation of $\mathfrak{g}$. Then, thinking of these elements as living in $\mathfrak{g}_{(D)}$, the latter is just the Lie algebra over $\Bbbk$ with generators $\mathrm{e}_i, \mathrm{h}_i^+, \mathrm{h}_i^-, \mathrm{f}_i \;(i \in I)$ and relations

$$\begin{gathered}
\big[\mathrm{h}_i^+, \mathrm{e}_j\big] = +d_i\,a_{ij}\,\mathrm{e}_j \;, \quad \big[\mathrm{h}_i^+, \mathrm{f}_j\big] = -d_i\,a_{ij}\,\mathrm{f}_j \;, \quad \big[\mathrm{h}_i^-, \mathrm{e}_j\big] = +d_j\,a_{ji}\,\mathrm{e}_j \;, \quad \big[\mathrm{h}_i^-, \mathrm{f}_j\big] = -d_j\,a_{ji}\,\mathrm{f}_j \\
\big[\mathrm{h}_i^+, \mathrm{h}_j^+\big] = 0 \;, \quad \big[\mathrm{h}_i^-, \mathrm{h}_j^-\big] = 0 \;, \quad \big[\mathrm{h}_i^+, \mathrm{h}_j^-\big] = 0 \;, \quad \big[\mathrm{e}_i, \mathrm{f}_j\big] = \delta_{ij}\,2^{-1}\big(\mathrm{h}_i^+ + \mathrm{h}_i^-\big) \\
\mathrm{ad}(\mathrm{e}_i)^{1-a_{ij}}(\mathrm{e}_j) = 0 \quad , \qquad \mathrm{ad}(\mathrm{f}_i)^{1-a_{ij}}(\mathrm{f}_j) = 0 \qquad \big(i \neq j\big)
\end{gathered}$$

Moreover (cf. [Hal]), $\mathfrak{g}_{(D)}$ bears the unique Lie bialgebra structure given by the formulas

$$\begin{aligned}
\delta(\mathrm{e}_i) \;&=\; \big(d_i\mathrm{h}_i^+\big) \otimes \mathrm{e}_i - \mathrm{e}_i \otimes \big(d_i\mathrm{h}_i^+\big) \;, \qquad & \delta\big(\mathrm{h}_i^+\big) \;&=\; 0 \\
\delta\big(\mathrm{h}_i^-\big) \;&=\; 0 \;, & \delta(\mathrm{f}_i) \;&=\; \mathrm{f}_i \otimes \big(d_i\mathrm{h}_i^-\big) - \big(d_i\mathrm{h}_i^-\big) \otimes \mathrm{f}_i
\end{aligned}$$

Now, all this construction can be extended as follows. Instead of the symmetric matrix DA, consider any square matrix $B = \big(b_{ij}\big)_{i,j \in I} \in M_\theta(\mathbb{Z})$ such that $B + B^t = 2\,DA$. Then one can repeat the construction in [Hal] and then find a new Lie bialgebra $\mathfrak{g}_{{}_B}$ given as follows: it is the Lie algebra over $\mathbb{C}$ with generators $\mathrm{e}_i, \dot{\mathrm{k}}_i, \dot{\mathrm{l}}_i, \mathrm{f}_i \;(i \in I)$ and relations

$$\begin{gathered}
\big[\,\dot{\mathrm{k}}_i\,, \mathrm{e}_j\big] = +b_{ij}\,\mathrm{e}_j \;, \quad \big[\,\dot{\mathrm{k}}_i\,, \mathrm{f}_j\big] = -b_{ij}\,\mathrm{f}_j \;, \quad \big[\,\dot{\mathrm{l}}_i\,, \mathrm{e}_j\big] = +b_{ji}\,\mathrm{e}_j \;, \quad \big[\,\dot{\mathrm{l}}_i\,, \mathrm{f}_j\big] = -b_{ji}\,\mathrm{f}_j \\
\big[\,\dot{\mathrm{k}}_i\,, \dot{\mathrm{k}}_j\big] = 0 \;, \quad \big[\,\dot{\mathrm{l}}_i\,, \dot{\mathrm{l}}_j\big] = 0 \;, \quad \big[\,\dot{\mathrm{k}}_i\,, \dot{\mathrm{l}}_j\big] = 0 \;, \quad \big[\,\mathrm{e}_i\,, \mathrm{f}_j\big] = \delta_{ij}\,(2\,d_i)^{-1}\big(\,\dot{\mathrm{k}}_i + \dot{\mathrm{l}}_i\big) \\
\mathrm{ad}(\mathrm{e}_i)^{1-a_{ij}}(\mathrm{e}_j) = 0 \quad , \qquad \mathrm{ad}(\mathrm{f}_i)^{1-a_{ij}}(\mathrm{f}_j) = 0 \qquad \big(i \neq j\big)
\end{gathered} \tag{2.3}$$

and it bears the Lie bialgebra structure whose Lie cobracket is uniquely given by

$$\begin{aligned}
\delta(\mathrm{e}_i) \;&=\; \dot{\mathrm{k}}_i \otimes \mathrm{e}_i - \mathrm{e}_i \otimes \dot{\mathrm{k}}_i \;, \qquad & \delta\big(\dot{\mathrm{k}}_i\big) \;&=\; 0 \\
\delta\big(\dot{\mathrm{l}}_i\big) \;&=\; 0 \;, & \delta(\mathrm{f}_i) \;&=\; \mathrm{f}_i \otimes \dot{\mathrm{l}}_i - \dot{\mathrm{l}}_i \otimes \mathrm{f}_i
\end{aligned} \tag{2.4}$$

Note that the Lie bialgebra $\mathfrak{g}_{(D)}$ above is simply the special case of $\mathfrak{g}_{{}_B}$ for $B := DA$.

A more detailed, thorough construction of these Lie bialgebras is presented in [GG2].

Basing upon the e_i's and the f_i's we construct root vectors $\mathrm{e}_\alpha \in \mathfrak{b}_+$ and $\mathrm{f}_\alpha \in \mathfrak{b}_-$ (for all $\alpha \in \Phi^+$); this construction takes place inside the nilpotent part of $\mathfrak{b}_+$ and of $\mathfrak{b}_-$, hence these new elements are well-defined for each Lie bialgebra $\mathfrak{g}_{{}_B}$ as above. All these root vectors, together with the $\dot{\mathrm{k}}_i$'s and the $\dot{\mathrm{l}}_i$'s, form a *Chevalley-type basis* of $\mathfrak{g}_{{}_B}$, with $\mathrm{e}_{\alpha_i} = \mathrm{e}_i$ and $\mathrm{f}_{\alpha_i} = \mathrm{f}_i$ for all $i \in I$: indeed, up to signs this basis (hence the e_α's and the f_α's) is unique. We also recall (cf. §2.3.1) the notation $d_\alpha := (\alpha,\alpha)\big/2$ for all $\alpha \in \Phi^+$.

We introduce now some $\mathbb{Z}$–*integral forms* of $\mathfrak{g}_B$.

Definition 2.3.4. Keep notation as above, in particular $B + B^t = 2\,DA$. Then:

(a) We call $\dot{\mathfrak{g}}_B$ the Lie subalgebra over $\mathbb{Z}$ of $\mathfrak{g}_B$ generated by the elements e_i , f_i , $\dot{\mathrm{k}}_i$, $\dot{\mathrm{l}}_i$ and $\mathrm{h}_i^\circ := (2\,d_i)^{-1}\big(\dot{\mathrm{k}}_i + \dot{\mathrm{l}}_i\big)$ (for all $i \in I$); indeed, this is a *Lie bialgebra* over $\mathbb{Z}$, with

$$\delta(\mathrm{e}_i) \,=\, \dot{\mathrm{k}}_i \otimes \mathrm{e}_i - \mathrm{e}_i \otimes \dot{\mathrm{k}}_i \quad , \qquad \delta(\mathrm{f}_i) \,=\, \mathrm{f}_i \otimes \dot{\mathrm{l}}_i - \dot{\mathrm{l}}_i \otimes \mathrm{f}_i$$
$$\delta\big(\dot{\mathrm{k}}_i\big) = 0 \quad , \qquad \delta\big(\dot{\mathrm{l}}_i\big) = 0 \quad , \qquad \delta\big(\mathrm{h}_i^\circ\big) \,=\, 0$$

(b) We call $\tilde{\mathfrak{g}}_B$ the Lie subalgebra over $\mathbb{Z}$ of $\mathfrak{g}_B$ generated by the elements $\tilde{\mathrm{e}}_\alpha := 2\,d_\alpha\,\mathrm{e}_\alpha$, $\tilde{\mathrm{f}}_\alpha := 2\,d_\alpha\,\mathrm{f}_\alpha$ ($\alpha \in \Phi^+$), $\dot{\mathrm{k}}_i$ and $\dot{\mathrm{l}}_i$ ($i \in I$); indeed, this is a *Lie bialgebra* over $\mathbb{Z}$, with

$$\delta(\tilde{\mathrm{e}}_i) \,=\, \dot{\mathrm{k}}_i \otimes \tilde{\mathrm{e}}_i - \tilde{\mathrm{e}}_i \otimes \dot{\mathrm{k}}_i \quad , \qquad \delta\big(\tilde{\mathrm{f}}_i\big) \,=\, \tilde{\mathrm{f}}_i \otimes \dot{\mathrm{l}}_i - \dot{\mathrm{l}}_i \otimes \tilde{\mathrm{f}}_i$$
$$\delta\big(\dot{\mathrm{k}}_i\big) = 0 \quad , \qquad \delta\big(\dot{\mathrm{l}}_i\big) = 0$$

(c) Assume in addition that $b_{ij} = d_i t_{ij}^+ = d_j t_{ij}^-$ for some $t_{ij}^\pm \in \mathbb{Z}$ $(i,j \in I\,)$. Then we call $\hat{\mathfrak{g}}_B$ the Lie subalgebra over $\mathbb{Z}$ of $\mathfrak{g}_B$ generated by the elements e_i , f_i , $\mathrm{k}_i := d_i^{-1}\dot{\mathrm{k}}_i$, $\mathrm{l}_i := d_i^{-1}\dot{\mathrm{l}}_i$, $\mathrm{h}_i^\circ = 2^{-1}\big(\mathrm{k}_i + \mathrm{l}_i\big)$ (for all $i \in I$); indeed, this is a *Lie bialgebra* over $\mathbb{Z}$, with

$$\delta(\mathrm{e}_i) \,=\, d_i(\mathrm{k}_i \otimes \mathrm{e}_i - \mathrm{e}_i \otimes \mathrm{k}_i\,) \quad , \qquad \delta(\mathrm{f}_i) \,=\, d_i(\mathrm{f}_i \otimes \mathrm{l}_i - \mathrm{l}_i \otimes \mathrm{f}_i\,)$$
$$\delta(\mathrm{k}_i) \,=\, 0 \quad , \qquad \delta(\mathrm{l}_i) \,=\, 0 \quad , \qquad \delta\big(\mathrm{h}_i^\circ\big) \,=\, 0 \qquad \diamondsuit$$

Remarks 2.3.5. *(a)* It is clear by definition that $\dot{\mathfrak{g}}_B$, $\tilde{\mathfrak{g}}_B$ and $\hat{\mathfrak{g}}_B$ are all $\mathbb{Z}$–integral forms of the Lie algebra $\mathfrak{g}_B$ in §2.3.3, i.e. $\mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{a} \cong \mathfrak{g}_B$ *as Lie algebras* for $\mathfrak{a} \in \big\{\, \dot{\mathfrak{g}}_B \,,\, \tilde{\mathfrak{g}}_B \,,\, \hat{\mathfrak{g}}_B \big\}$.

We also remark that the elements $\tilde{\mathrm{e}}_i$, $\tilde{\mathrm{f}}_i$, $\dot{\mathrm{k}}_i$ and $\dot{\mathrm{l}}_i$ (with $i \in I$) are enough to generate the Lie algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{\mathfrak{g}}_B$ over $\mathbb{Q}$; therefore, the formulas given in Definition 2.3.4*(b)* are enough, though they do not display the values $\delta\big(\tilde{\mathrm{f}}_\alpha\big)$ nor $\delta(\tilde{\mathrm{e}}_\alpha)$, to determine a unique Lie cobracket on $\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{\mathfrak{g}}_B$, so by restriction on $\tilde{\mathfrak{g}}_B$ too.

(b) The fact that each of $\dot{\mathfrak{g}}_B$, $\tilde{\mathfrak{g}}_B$ and $\hat{\mathfrak{g}}_B$ be a Lie *sub-bialgebra* of $\mathfrak{g}_B$ (hence a $\mathbb{Z}$–integral form of it *as a Lie bialgebra*) is a direct check. It is also a consequence, though, of our results in §6.2 later on about specialization of suitable multiparameter quantum groups.

(c) Definitions imply that in each Lie bialgebra $\mathfrak{g}_B$ — as well as in its $\mathbb{Z}$–integral forms $\dot{\mathfrak{g}}_B$, $\tilde{\mathfrak{g}}_B$ and $\hat{\mathfrak{g}}_B$ — the Lie *algebra* structure *does* depend on B , whereas the Lie *coalgebra* structure *does not*. This follows from simple observations, namely that the root vectors e_α and f_α are independent of B, and that the formulas for the Lie cobracket of the $\dot{\mathrm{k}}_i$'s, the $\dot{\mathrm{l}}_i$'s, the e_α's and the f_α's are independent of B as well; this second fact requires a quick computation for non-simple α's, where the condition $B + B^t = 2\,DA$ makes the job.

This implies that if we consider two such Lie bialgebras $\mathfrak{g}_{B'}$ and $\mathfrak{g}_{B''}$, and their corresponding basis elements (over $\mathbb{Q}$) e'_α , e''_α , etc., *mapping* $\mathrm{e}'_\alpha \mapsto \mathrm{e}''_\alpha$, $\dot{\mathrm{k}}'_i \mapsto \dot{\mathrm{k}}''_i$, $\dot{\mathrm{l}}'_i \mapsto \dot{\mathrm{l}}''_i$ *and* $\mathrm{f}'_\alpha \mapsto \mathrm{f}''_\alpha$ *defines an isomorphism of Lie coalgebras* $\mathfrak{g}_{B'} \cong \mathfrak{g}_{B''}$, that on the other hand is *not* one of Lie algebras. The same occurs for the $\mathbb{Z}$–integral forms as well.

For later use we need yet another definition:

Definition 2.3.6. Given $B = \big(b_{ij}\big)_{i,j \in I} \in M_\theta(\mathbb{Z})$ such that $B + B^t = 2\,DA$, let $\mathfrak{g}_B$ be the complex Lie algebra mentioned in §2.3.3 above, and $U\big(\mathfrak{g}_B\big)$ its universal enveloping algebra. We define $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_B\big)$, resp. $U_{\mathbb{Z}}\big(\tilde{\mathfrak{g}}_B\big)$, the $\mathbb{Z}$–subalgebra of $U\big(\hat{\mathfrak{g}}_B\big)$ generated by

$$\left\{ \binom{\dot{\mathrm{k}}_i}{n} , \binom{\dot{\mathrm{l}}_i}{n} , \binom{\mathrm{h}_i^\circ}{n} , \mathrm{e}_i^{(n)} , \mathrm{f}_i^{(n)} \;\middle|\; i \in I,\, n \in \mathbb{N} \right\} ,$$

$$\text{resp.} \qquad \left\{ \binom{\mathrm{k}_i}{n} , \binom{\mathrm{l}_i}{n} , \binom{\mathrm{h}_i^\circ}{n} , \mathrm{e}_i^{(n)} , \mathrm{f}_i^{(n)} \;\middle|\; i \in I,\, n \in \mathbb{N} \right\} ,$$

where $\binom{\mathrm{t}}{n}$ and $\mathrm{a}^{(n)}$ denote standard binomial coefficients and divided powers, and in the second case we are assuming that $b_{ij} = d_i t_{ij}^+ = d_j t_{ij}^-$ for some $t_{ij}^\pm \in \mathbb{Z}$ $(\,i,j \in I\,)$. $\diamondsuit$

Remarks 2.3.7. *(a)* By Remarks 2.3.5 above, it is easily seen that $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{{}_B}\big)$ and $U_{\mathbb{Z}}\big(\hat{\mathfrak{g}}_{{}_B}\big)$ are $\mathbb{Z}$–integral forms of $U\big(\mathfrak{g}_{{}_B}\big)$; one can also find a *presentation* of each of them by generators (the given ones) and relations. Indeed, for both $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{{}_B}\big)$ and $U_{\mathbb{Z}}\big(\hat{\mathfrak{g}}_{{}_B}\big)$ this is a simple variation of the well-known presentation of the Kostant $\mathbb{Z}$–integral form of $U(\mathfrak{g})$, generated are binomial coefficients and divided powers of the Chevalley generators.

Moreover, as $\mathfrak{g}_{{}_{(D)}}$ is a Lie bialgebra, $U\big(\mathfrak{g}_{{}_{(D)}}\big)$ is in fact a *co-Poisson* Hopf algebra; then $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{{}_B}\big)$ and $U_{\mathbb{Z}}\big(\hat{\mathfrak{g}}_{{}_B}\big)$ are in fact $\mathbb{Z}$–integral forms of $U\big(\mathfrak{g}_{{}_{(D)}}\big)$ *as co-Poisson Hopf algebras*.

(b) By a standard fact in the "arithmetic of binomial coefficients", $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{{}_B}\big)$ contains also all "translated" binomial coefficients, of the form $\begin{pmatrix} \dot{\mathrm{k}}_i + z \\ n \end{pmatrix}$, $\begin{pmatrix} \dot{\mathrm{l}}_i + z \\ n \end{pmatrix}$ and $\begin{pmatrix} \mathrm{h}_i + z \\ n \end{pmatrix}$ for $i \in I$, $n \in \mathbb{N}$, $z \in \mathbb{Z}$; then one has also a presentation of $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{{}_B}\big)$ including these extra generators, and corresponding extra relations too. And similarly for $U_{\mathbb{Z}}\big(\hat{\mathfrak{g}}_{{}_B}\big)$ as well.

(c) The definition of the $\mathbb{Z}$–integral forms $\dot{\mathfrak{g}}_{{}_B}$, $\hat{\mathfrak{g}}_{{}_B}$ and $\tilde{\mathfrak{g}}_{{}_B}$ — of $\mathfrak{g}_{{}_B}$ — and of the forms $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{{}_B}\big)$ and $U_{\mathbb{Z}}\big(\hat{\mathfrak{g}}_{{}_B}\big)$ — of $U\big(\mathfrak{g}_{{}_B}\big)$ — may seem to come out of the blue, somehow. Nevertheless, we will show in §6.2 that they occur as direct output of a "specialization process" of multiparameter quantum groups *once suitable integral forms of them are chosen*.

2.3.8. Some q–numbers. Throughout the paper we shall need to consider several kinds of "q–numbers". Let $\mathbb{Z}\big[q,q^{-1}\big]$ be the ring of Laurent polynomials with integral coefficients in the indeterminate q. For every $n \in \mathbb{N}$ we define

$$(0)_q := 1 \;,\quad (n)_q := \frac{q^n - 1}{q-1} = 1 + q + \cdots + q^{n-1} = \sum_{s=0}^{n-1} q^s \qquad \big(\in \mathbb{Z}[q]\big)$$

$$(n)_q! := (0)_q(1)_q \cdots (n)_q := \prod_{s=0}^{n} (s)_q \;,\quad \binom{n}{k}_q := \frac{(n)_q!}{(k)_q!(n-k)_q!} \qquad \big(\in \mathbb{Z}[q]\big)$$

$$[0]_q := 1 \;,\; [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-(n-1)} + \cdots + q^{n-1} = \sum_{s=0}^{n-1} q^{2s-n+1} \quad \big(\in \mathbb{Z}\big[q,q^{-1}\big]\big)$$

$$[n]_q! := [0]_q[1]_q \cdots [n]_q = \prod_{s=0}^{n} [s]_q \;,\quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \qquad \big(\in \mathbb{Z}\big[q,q^{-1}\big]\big)$$

Moreover, we have $(n)_{q^2} = q^{n-1}[n]_q$, $(n)_{q^2}! = q^{\frac{n(n-1)}{2}}[n]_q$, $\binom{n}{k}_{q^2} = q^{k(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Furthermore, thinking of Laurent polynomials as functions on $\Bbbk^\times$, for any $q \in \Bbbk^\times$ we shall read every symbol above as representing the corresponding element in $\Bbbk$.

3. Multiparameter quantum groups

In this section we present the notion of *multiparameter quantum group*, or *MpQG* for short. We introduce it by a direct definition by generators and relations as it suits better for our purposes. There exists also a realization in terms of Nichols algebras of diagonal type, see for example [ARS], [An4]. [Gar], [He2]. Finally, we connect them with cocycle deformations of their simplest example, the "canonical" one.

3.1. **Defining multiparameter quantum groups (=MpQG's).**

In this subsection we introduce the multiparameter quantum group $U_{\mathbf{q}}(\mathfrak{g})$, or *MpQG* for short, associated with a matrix of parameters $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ of Cartan type (cf. §2.3.2). We fix also scalars q_i ($i \in I$) as in §2.3.2, *with the additional assumption that* $q_{ii}^k = q_i^{2k} \neq 1$ for all $k = 1, \dots, 1 - a_{ij}$, with $i, j \in I$ and $i \neq j$.

Definition 3.1.1. (cf. [HPR]) We denote by $U_{\mathbf{q}}(\mathfrak{g})$ the unital associative $\Bbbk$–algebra generated by elements E_i, F_i, $K_i^{\pm 1}$, $L_i^{\pm 1}$ with $i \in I$ obeying the following relations:

$$
\begin{array}{lll}
(a) & K_i^{\pm 1} L_j^{\pm 1} = L_j^{\pm 1} K_i^{\pm 1} \ , & K_i^{\pm 1} K_i^{\mp 1} = 1 = L_i^{\pm 1} L_i^{\mp 1} \\
(b) & K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1} \ , & L_i^{\pm 1} L_j^{\pm 1} = L_j^{\pm 1} L_i^{\pm 1} \\
(c) & K_i \, E_j \, K_i^{-1} = q_{ij} \, E_j \ , & L_i \, E_j \, L_i^{-1} = q_{ji}^{-1} \, E_j \\
(d) & K_i \, F_j \, K_i^{-1} = q_{ij}^{-1} \, F_j \ , & L_i \, F_j \, L_i^{-1} = q_{ji} \, F_j \\
(e) & [E_i, F_j] = \delta_{i,j} \, q_{ii} \, \dfrac{K_i - L_i}{q_{ii} - 1} & \\
(f) & \displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\binom{k}{2}} \, q_{ij}^k \, E_i^{1-a_{ij}-k} E_j \, E_i^k = 0 & (\, i \neq j \,) \\
(g) & \displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\binom{k}{2}} \, q_{ij}^k \, F_i^{1-a_{ij}-k} F_j \, F_i^k = 0 & (\, i \neq j \,)
\end{array}
$$

Moreover, $U_{\mathbf{q}}(\mathfrak{g})$ is a Hopf algebra with coproduct, counit and antipode determined for all $i, j \in I$ by

$$
\begin{array}{rlll}
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \ , & \epsilon(E_i) = 0 \ , & \mathcal{S}(E_i) = -K_i^{-1} E_i \\
\Delta(F_i) = F_i \otimes L_i + 1 \otimes F_i \ , & \epsilon(F_i) = 0 \ , & \mathcal{S}(F_i) = -F_i L_i^{-1} \\
\Delta\big(K_i^{\pm 1}\big) = K_i^{\pm 1} \otimes K_i^{\pm 1} \ , & \epsilon\big(K_i^{\pm 1}\big) = 1 \ , & \mathcal{S}\big(K_i^{\pm 1}\big) = K_i^{\mp 1} \\
\Delta\big(L_i^{\pm 1}\big) = L_i^{\pm 1} \otimes L_i^{\pm 1} \ , & \epsilon\big(L_i^{\pm 1}\big) = 1 \ , & \mathcal{S}\big(L_i^{\pm 1}\big) = L_i^{\mp 1}
\end{array}
$$

Finally, for later use we introduce also, for every $\lambda = \sum_{i \in I} \lambda_i \, \alpha_i \in Q$, the notation $K_\lambda := \prod_{i \in I} K_i^{\lambda_i}$ and $L_\lambda := \prod_{i \in I} L_i^{\lambda_i}$. $\diamondsuit$

Remark 3.1.2. Assume that $q \in \Bbbk^\times$ is not a root of unity and fix the "canonical" multiparameter $\check{\mathbf{q}} := \big(\check{q}_{ij} = q^{d_i a_{ij}}\big)_{i,j \in I}$ like in (2.2). Then we can define the corresponding MpQG, denoted $U_{\check{\mathbf{q}}}(\mathfrak{g})$: the celebrated one-parameter quantum group $U_q(\mathfrak{g})$ by Jimbo and Lusztig is (up to a minimal, irrelevant change of generators) just the quotient of $U_{\check{\mathbf{q}}}(\mathfrak{g})$ by the (Hopf) ideal generated by $\big\{ L_i - K_i^{-1} \,\big|\, i = 1, \dots, \theta \big\}$.

As a matter of fact, that we shall deeply exploit in the present work, *most constructions usually carried on for $U_q(\mathfrak{g})$ — like construction of (quantum) root vectors, of integral forms, etc. — actually makes sense and apply the same to $U_{\check{\mathbf{q}}}(\mathfrak{g})$ as well.*

We introduce now a family of subalgebras of any MpQG, say $U_{\mathbf{q}}(\mathfrak{g})$, as follows:

Definition 3.1.3. Given $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ and $U_{\mathbf{q}}(\mathfrak{g})$ as in §3.1, we define $U_{\mathbf{q}}^0 := U_{\mathbf{q}}(\mathfrak{h} \oplus \mathfrak{h})$, $U_{\mathbf{q}}^{+,0}$, $U_{\mathbf{q}}^{-,0}$, $U_{\mathbf{q}}^- := U_{\mathbf{q}}(\mathfrak{n}_-)$, $U_{\mathbf{q}}^+ := U_{\mathbf{q}}(\mathfrak{n}_+)$, $U_{\mathbf{q}}^{\leq} := U_{\mathbf{q}}(\mathfrak{b}_-)$ and $U_{\mathbf{q}}^{\geq} := U_{\mathbf{q}}(\mathfrak{b}_+)$ to be the $\Bbbk$–subalgebra of $U_{\mathbf{q}}(\mathfrak{g})$ respectively generated as

$$
U_{\mathbf{q}}^0 := \Big\langle K_i^{\pm 1}, L_i^{\pm 1} \Big\rangle_{i \in I} \ , \qquad U_{\mathbf{q}}^{+,0} := \Big\langle K_i^{\pm 1} \Big\rangle_{i \in I} \ , \qquad U_{\mathbf{q}}^{-,0} := \Big\langle L_i^{\pm 1} \Big\rangle_{i \in I}
$$

$$
U_{\mathbf{q}}^- := \big\langle F_i \big\rangle_{i \in I} \ , \quad U_{\mathbf{q}}^{\leq} := \Big\langle F_i \, , L_i^{\pm 1} \Big\rangle_{i \in I} \ , \quad U_{\mathbf{q}}^{\geq} := \Big\langle K_i^{\pm 1} \, , E_i \Big\rangle_{i \in I} \ , \quad U_{\mathbf{q}}^+ := \big\langle E_i \big\rangle_{i \in I}
$$

We shall refer to $U_{\mathbf{q}}^{\leq}$ and $U_{\mathbf{q}}^{\geq}$ as to the *positive* and *negative* multiparameter quantum Borel (sub)algebras, and $U_{\mathbf{q}}^0$, $U_{\mathbf{q}}^{+,0}$ and $U_{\mathbf{q}}^{-,0}$ as to the *global*, *positive* and *negative* multiparameter Cartan (sub)algebras. $\diamondsuit$

Recall the notion of "skew-Hopf pairing" (cf. Definition 2.1.2). From [He2, Proposition 4.3] — see also [HPR, Theorem 20] and [AY, Propostion 2.4] — we have:

Proposition 3.1.4. *With the assumptions above, assume in addition that* $q_{ii} \neq 1$ *for all indices* $i \in I$ *. Then there exists a unique skew-Hopf pairing* $\eta : U_{\mathbf{q}}^{\geq} \underset{\Bbbk}{\otimes} \big(U_{\mathbf{q}}^{\leq}\big)^{\mathrm{cop}} \longrightarrow \Bbbk$ *that is non-degenerate and such that, for all* $1 \leq i,j \leq \theta$ *, one has*

$$\eta(K_i, L_j) \,=\, q_{ij} \ , \qquad \eta(E_i, F_j) \,=\, \delta_{i,j} \, \frac{-\,q_{ii}}{q_{ii}-1} \ , \qquad \eta(E_i, L_j) \,=\, 0 \,=\, \eta(K_i, F_j)$$

Moreover, for every $E \in U_{\mathbf{q}}^{+}$ *,* $F \in U_{\mathbf{q}}^{-}$ *, and every Laurent monomials* K *in the* K_i*'s and* L *in the* L_j*'s, we have*

$$\eta\big(E\,K, F\,L\big) \,=\, \eta(E\,,F)\,\eta(K,L)$$

The following result states that there exist special "tensor product factorizations" of MpQG's (the last ones are usually referred to as "triangular decompositions"):

Proposition 3.1.5. (*cf.* [HPR, Corollary 22], [BGH, Corollary 2.6])
The multiplication in $U_{\mathbf{q}}(\mathfrak{g})$ *provides* $\Bbbk$*–linear isomorphisms*

$$U_{\mathbf{q}}^{-} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{0} \,\cong\, U_{\mathbf{q}}^{\leq} \,\cong\, U_{\mathbf{q}}^{0} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{-} \quad , \qquad U_{\mathbf{q}}^{+} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{0} \,\cong\, U_{\mathbf{q}}^{\geq} \,\cong\, U_{\mathbf{q}}^{0} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{+}$$
$$U_{\mathbf{q}}^{+,0} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{-,0} \,\cong\, U_{\mathbf{q}}^{0} \,\cong\, U_{\mathbf{q}}^{-,0} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{+,0} \quad , \qquad U_{\mathbf{q}}^{\leq} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{\geq} \,\cong\, U_{\mathbf{q}} \,\cong\, U_{\mathbf{q}}^{\geq} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{\leq}$$
$$U_{\mathbf{q}}^{+} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{0} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{-} \,\cong\, U_{\mathbf{q}} \,\cong\, U_{\mathbf{q}}^{-} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{0} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{+}$$

Remark 3.1.6. It is clear from definitions that $U_{\mathbf{q}}^{0} = U_{\mathbf{q}}(\mathfrak{h} \oplus \mathfrak{h})$ has the set of monomials in the $K_i^{\pm 1}$'s and the $L_i^{\pm 1}$'s as $\Bbbk$–basis. It follows then that each triangular decompositions of $U_{\mathbf{q}}(\mathfrak{g})$ as above induces also a splitting $U_{\mathbf{q}}(\mathfrak{g}) = U_{\mathbf{q}}(\mathfrak{h} \oplus \mathfrak{h}) \oplus U_{\mathbf{q}}(\mathfrak{g})^{\oplus}$ where
$$U_{\mathbf{q}}(\mathfrak{g})^{\oplus} \,:=\, \big(U_{\mathbf{q}}(\mathfrak{n}_-)^{+} \cdot U_{\mathbf{q}}(\mathfrak{h}_D) \cdot U_{\mathbf{q}}(\mathfrak{n}_+) \,+\, U_{\mathbf{q}}(\mathfrak{n}_-) \cdot U_{\mathbf{q}}(\mathfrak{h}_D) \cdot U_{\mathbf{q}}(\mathfrak{n}_+)^{+} \,.$$

3.2. **MpQG's as cocycle deformations.**

Now we want to perform on the Hopf algebras $U_{\mathbf{q}}(\mathfrak{g})$ a cocycle deformation process, via special types of 2–cocycles, like in §2.2, following [AST], [DT] and [Mo].

Let us consider $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ and $U_{\mathbf{q}}(\mathfrak{g})$ as in §3.1. As explained in §2.3.2, we fix a special element $q_{j_0} \in \Bbbk^{\times}$, also denoted by $q := q_{j_0}$; for this choice of q, we consider the canonical "one parameter" quantum group $U_{\check{\mathbf{q}}}(\mathfrak{g})$ as in Remark 3.1.2.

Recall from Definition 3.1.1 the notation $K_\lambda := \prod_{i \in I} K_i^{\lambda_i}$ and $L_\lambda := \prod_{i \in I} L_i^{\lambda_i}$ for every $\lambda = \sum_{i \in I} \lambda_i \alpha_i \in Q$. Similarly, we shall also write

$$q_{\mu\,\nu} \,:=\, \prod_{i,j \in I} q_{ij}^{\mu_i \nu_j} \ , \quad q_{\mu\,\nu}^{1/2} \,:=\, \prod_{i,j \in I} \big(q_{ij}^{1/2}\big)^{\mu_i \nu_j} \qquad \forall \ \mu = \sum_{i \in I} \mu_i \, \alpha_i \, , \ \nu = \sum_{j \in I} \nu_j \, \alpha_j \,\in\, Q$$

Likewise, we define also $q_\beta := q_i$ for every positive root $\beta \in \Phi^{+}$ which belongs to the same orbit as the simple root α_i for the action of the Weyl group of $\mathfrak{g}$ onto Q (which is well-defined, by standard theory of root systems).

Definition 3.2.1. With the above conventions, let $U_{\check{\mathbf{q}}}(\mathfrak{g})$ be the MpQG of Remark 3.1.2, and let $\sigma : U_{\check{\mathbf{q}}}(\mathfrak{g}) \otimes U_{\check{\mathbf{q}}}(\mathfrak{g}) \longrightarrow \Bbbk$ be the unique $\Bbbk$–linear map given by

$$\sigma(x,y) \,:=\, q_{\mu\,\nu}^{1/2} \qquad \text{if} \quad x = K_\mu \ \text{ or } \ x = L_\mu \ , \quad y = K_\nu \ \text{ or } \ y = L_\nu$$
$$\sigma\big(U_{\check{\mathbf{q}}}(\mathfrak{g})\,, U_{\check{\mathbf{q}}}(\mathfrak{g})^{\oplus}\big) \,:=\, 0 \,=:\, \sigma\big(U_{\check{\mathbf{q}}}(\mathfrak{g})^{\oplus}\,, U_{\check{\mathbf{q}}}(\mathfrak{g})\big)$$

(by Remark 3.1.6 above, this is enough to determine a unique σ as requested). $\diamondsuit$

The key result that we shall rely upon in the sequel is the following:

Theorem 3.2.2. *(cf.* [HPR, Theorem 28.]*) Let* $\mathbf{q} := \big(q_{ij}\big)_{i,j\in I}$ *and* q *be as above. Then the map* σ *in Definition 3.2.1 is a normalized* 2*–cocycle of the Hopf algebra* $U_{\check{\mathbf{q}}}(\mathfrak{g})$ *and there exists a Hopf algebra isomorphism (with notation of §2.2.1)*

$$U_{\mathbf{q}}(\mathfrak{g}) \;\cong\; \big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)_\sigma$$

Remark 3.2.3. A similar result is given in [HLR, Theorem 4.5], but using another σ.

As a last result in this section, we can show that the 2–cocycle deformation considered in Theorem 3.2.2 can be also realized as a cocycle deformation in the sense of §2.2.2 as well. Indeed, let $\varGamma := \mathbb{Z}^{2\theta}$ be the free Abelian group generated by the K_i's and L_i's ($i \in I$), and V_E, resp. V_F, be the $\Bbbk$–vector space generated by the E_i's, resp. the F_i's ($i \in I$). Then, by [Gar], we know that $U_{\check{\mathbf{q}}}(\mathfrak{g})$ is a quotient of $T\big(V_E \oplus V_F\big)\,\#\,\Bbbk\varGamma$ by the two-sided ideal generated by the relations (e), (f) and (g) in Definition 3.1.1. We have a $(Q\times Q)$–grading on $T\big(V_E \oplus V_F\big)\,\#\,\Bbbk\varGamma$ given by

$$\deg(K_i) \,=\, (\alpha_i,\alpha_i) \,=\, \deg\big(L_i\big) \;\;, \qquad \deg(E_i) \,=\, (1,\alpha_i) \;\;, \qquad \deg(F_i) \,=\, (\alpha_i,1)$$

for all $i \in I$; it coincides with the grading induced by the coaction on the Yetter-Drinfeld modules V_E and V_F such that $\deg(K_i) = \deg(L_i)$. As the defining relations are homogeneous with respect to this grading, we get a $(Q\times Q)$–grading on $U_{\check{\mathbf{q}}}(\mathfrak{g})$.

Consider now the group 2–cocycle $\varphi \in \mathcal{Z}^2(\varGamma,\Bbbk)$ given by $\varphi := \sigma\big|_{\varGamma\times\varGamma}$, that is

$$\varphi(h,k) \,:=\, q_{\mu\nu}^{1/2} \qquad \text{if} \quad h = K_\mu \;\text{ or }\; h = L_\mu \;, \quad k = K_\nu \;\text{ or }\; k = L_\nu$$

and let $\tilde{\varphi}$ be the 2–cocycle defined on $T(V\oplus W)\,\#\,\Bbbk\varGamma$ as in §2.2.4. Since $\varGamma$ is Abelian and $E_i \underset{\tilde{\varphi}}{\cdot} F_j \,=\, E_i\,F_j$ for all $i,j \in I$, we have that $E_i \cdot_{\tilde{\varphi}} F_j \,-\, F_j \cdot_{\tilde{\varphi}} E_i \,=\, [\,E_i\,,F_j\,]$, hence $\tilde{\varphi}$ defines a Hopf 2–cocycle $\widehat{\varphi}$ on $U_{\check{\mathbf{q}}}(\mathfrak{g})$. Finally, a direct comparison shows that $\widehat{\varphi} = \sigma$. Thus, using §2.2.4, we conclude that the following holds:

Proposition 3.2.4. *There exists a Hopf algebra identification*

$$\big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)_\sigma \;=\; \big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)^{(\tilde{\varphi})}$$

hence, by Theorem 3.2.2, a Hopf algebra isomorphism $U_{\mathbf{q}}(\mathfrak{g}) \cong \big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)^{(\tilde{\varphi})}$.

3.3. **Multiparameter quantum groups with larger torus.**

The MpQG's $U_{\mathbf{q}}(\mathfrak{g})$ that we considered so far have a toral part (i.e., the subalgebra $U_{\mathbf{q}}^0$ generated by the $K_i^{\pm1}$'s and the $L_j^{\pm1}$'s) that is nothing but the group algebra of a double copy of the root lattice Q of $\mathfrak{g}$, much like in the one-parameter case — but for the duplication of Q, say. Now, in that (uniparameter) case, one also considers MpQG's with a larger toral part, namely the group algebra of any intermediate lattice between Q and P; similarly, we can introduce MpQG's whose toral part is the group algebra of any lattice $\varGamma_\ell \times \varGamma_r$ with $Q \subseteq \varGamma_\ell$ and $Q \subseteq \varGamma_r$.

3.3.1. Larger tori for MpQG's. Recall that the definition of the "toral parts" of $U_{\mathbf{q}}(\mathfrak{g})$ — cf. Definition 3.1.3 — is independent of $\mathbf{q}$: indeed, $U_{\mathbf{q}}^{+,0}$ is the group algebra over $\Bbbk$ for the group Q — identifying $\pm\alpha_i \simeq K_i^{\pm1}$ and $\alpha \simeq K_\alpha$ ($i \in I$, $\alpha \in Q$); similarly, $U_{\mathbf{q}}^{-,0}$ is the group algebra (over $\Bbbk$) of Q again with $\alpha \simeq L_\alpha$, and $U_{\mathbf{q}}(\mathfrak{h}\oplus\mathfrak{h}) := U_{\mathbf{q}}^0$ is the group algebra (over $\Bbbk$) of $Q\times Q$ — with $\big(\alpha',\alpha''\big) \simeq K_{\alpha'}\,L_{\alpha''}$.

Let us denote by $\mathbb{Q}Q$ and $\mathbb{Q}P$ the scalar extension from $\mathbb{Z}$ to $\mathbb{Q}$ of the lattices $\mathbb{Z}Q$ and $\mathbb{Z}P$, respectively; note that $\mathbb{Q}Q = \mathbb{Q}P$. For any other sublattice $\varGamma$ in $\mathbb{Q}Q$ $\big(= \mathbb{Q}P\big)$ of rank θ — the same as Q and P — we can define toral quantum groups $U_{\mathbf{q},\varGamma}^{\pm,0}$ akin to $U_{\mathbf{q}}^{\pm,0}$ but now associated with the lattice $\varGamma$, again as group algebras; similarly, we have an analogue $U_{\mathbf{q},\varLambda}^0$ of $U_{\mathbf{q}}^0$ associated with any sublattice $\varLambda$ in $\mathbb{Q}Q\times\mathbb{Q}Q$ of rank 2θ. Moreover, all these bear a natural Hopf algebra structure. Any sublattice inclusion $\varGamma' \leq \varGamma''$ yields a unique Hopf embedding $U^{\pm,0}_{\mathbf{q},\Gamma'} \subseteq U^{\pm,0}_{\mathbf{q},\Gamma''}$, and similar embeddings exist for the $U^0_{\mathbf{q},\Lambda}$'s. We aim to use these "larger toral MpQG's" as toral parts of larger MpQG's; this requires some compatibility constraints on $\mathbf{q}$, and some preliminary facts that we now settle.

Let Γ be a sublattice of $\mathbb{Q}Q$ of rank θ with $Q \leq \Gamma$. For any basis $\{\gamma_1, \dots \gamma_\theta\}$ of Γ, let $C := \big(c_{ij}\big)_{i,j\in I}$ be the matrix, with entries in $\mathbb{Z}$, that describes the change of basis (for $\mathbb{Q}Q$ as a $\mathbb{Q}$–vector space) from $\{\gamma_i\}_{i\in I}$ to $\{\alpha_i\}_{i\in I}$, so $\alpha_i = \sum_{j=1}^{\theta} c_{ij}\,\gamma_j$ for each $i \in I = \{1, \dots, \theta\}$. Let also $c := \big|det(C)\big| \in \mathbb{N}_+$ be the absolute value of the determinant of C; this is equal to the index (as a subgroup) of Q in Γ, hence it is independent of any choice of basis. If $C^{-1} = \big(c'_{ij}\big)_{i,j\in I}$ is the inverse matrix to C, then $\gamma_i = \sum_{j=1}^{\theta} c'_{ij}\,\alpha_j$ and $c''_{ij} := c \cdot c'_{ij} \in \mathbb{Z}$ for all $i, j \in I = \{1, \dots, \theta\}$.

Let now $U^{+,0}_{\mathbf{q},\Gamma}$ be given, as group algebra of Γ over $\Bbbk$ with generators $K^{\pm 1}_{\gamma_i}$ corresponding to the basis elements γ_i (and their opposite) in Γ (for $i \in I$). Define $K_{\alpha_i} := \prod_{j\in I} K^{c_{ij}}_{\gamma_j}$ for all $i \in I$: then the $\Bbbk$–subalgebra of $U^{+,0}_{\mathbf{q},\Gamma}$ generated by the $K^{\pm 1}_{\alpha_i}$'s is an isomorphic copy of $U^{+,0}_{\mathbf{q},Q}$, which provides a realization of the Hopf algebra embedding $U^{+,0}_{\mathbf{q},Q} \subseteq U^{+,0}_{\mathbf{q},\Gamma}$ corresponding to the group embedding $Q \leq \Gamma$.

In the obvious symmetric way we define also the "negative counterpart" $U^{-,0}_{\mathbf{q},\Gamma}$ of $U^{+,0}_{\mathbf{q},\Gamma}$, generated by elements $L^{\pm 1}_{\gamma_i}$ corresponding to the $\pm\gamma_i$'s in Γ, along with a suitable embedding $U^{-,0}_{\mathbf{q},Q} \subseteq U^{-,0}_{\mathbf{q},\Gamma}$ corresponding to the group embedding $Q \leq \Gamma$.

Finally, given any two sublattices $\Gamma_\pm$ of rank θ in $\mathbb{Q}Q$ containing Q, letting $\Gamma_\bullet := \Gamma_+ \times \Gamma_-$ we define $U^0_{\mathbf{q},\Gamma_\bullet} := U^{+,0}_{\mathbf{q},\Gamma_+} \underset{\Bbbk}{\otimes} U^{-,0}_{\mathbf{q},\Gamma_-}$; in this case, the basis elements for $\Gamma_\pm$ will be denoted by $\gamma^\pm_i$ ($i \in I$). The previous embeddings of $U^{\pm,0}_{\mathbf{q},Q}$ into $U^{\pm,0}_{\mathbf{q},\Gamma_\pm}$ then induces a similar embedding of $U^0_{\mathbf{q}} = U^{+,0}_{\mathbf{q},Q} \underset{\Bbbk}{\otimes} U^{-,0}_{\mathbf{q},Q}$ into $U^0_{\mathbf{q},\Gamma_\bullet}$ as well.

3.3.2. Special root pairings (in the integral case). *Let us now assume that the multiparameter* $\mathbf{q} := \big(q_{ij} = q^{b_{ij}}\big)_{i,j\in I}$ *is of integral type*; we therefore use notation $B := \big(b_{ij}\big)_{i,j\in I} \in M_\theta(\mathbb{Z})$. Then a $\mathbb{Z}$–bilinear pairing $(\ ,\ )_B : Q \times Q \longrightarrow \mathbb{Z}$ is defined via the matrix B by $(\alpha_i\,, \alpha_j)_B := b_{ij}$ for all $i, j \in I$. Moreover, by Proposition 3.1.4, we know that the pairing $U^{\geq}_{\mathbf{q}} \underset{\Bbbk}{\otimes} U^{\leq}_{\mathbf{q}} \longrightarrow \Bbbk$ is non-degenerate; but then (by the special properties of this pairing) its restriction to $U^0_{\mathbf{q}} \underset{\Bbbk}{\otimes} U^0_{\mathbf{q}}$ is non-degenerate too. Finally, from $\big\langle K_\alpha\,, L_\beta \big\rangle = q^{(\alpha,\,\beta)_B}$ (for all $\alpha, \beta \in Q$) we get that $(\ ,\ )_B : Q\times Q \longrightarrow \mathbb{Z}$ is non-degenerate as well, which forces B to be invertible.

By scalar extension, $(\ ,\ )_B$ yields also a $\mathbb{Q}$–bilinear pairing on $\mathbb{Q}Q$, which again is non-degenerate; we denote it also by $(\ ,\ )_B$. It is then meaningful to consider, for any sublattice Γ in $\mathbb{Q}Q$, its *left-dual* $\dot{\Gamma}^{(\ell)}$ and its *right-dual* $\dot{\Gamma}^{(r)}$, defined by

$$\begin{aligned} \dot{\Gamma}^{(\ell)} &:= \big\{ \lambda \in \mathbb{Q}Q \,\big|\, (\lambda\,, \gamma)_B \in \mathbb{Z}\,,\ \forall\, \gamma \in \Gamma \big\} \\ \dot{\Gamma}^{(r)} &:= \big\{ \rho \in \mathbb{Q}Q \,\big|\, (\gamma, \rho)_B \in \mathbb{Z}\,,\ \forall\, \gamma \in \Gamma \big\} \end{aligned} \tag{3.1}$$

that are sublattices in $\mathbb{Q}Q$ and coincide iff B is symmetric; then restricting the $\mathbb{Q}$–bilinear pairing $(\ ,\ )_B : \mathbb{Q}Q \times \mathbb{Q}Q \longrightarrow \mathbb{Q}$ to $\dot{\Gamma}^{(\ell)} \times \Gamma$ and $\Gamma \times \dot{\Gamma}^{(r)}$ one gets $\mathbb{Z}$-valued pairings $\dot{\Gamma}^{(\ell)} \times \Gamma \longrightarrow \mathbb{Z}$ and $\Gamma \times \dot{\Gamma}^{(r)} \longrightarrow \mathbb{Z}$, still denoted by $(\ ,\ )_B$.

Using the matrix $B^{-1} = \big(b'_{ij}\big)_{i,j\in I}$, we define in $\mathbb{Q}Q$ the elements

$$\dot{\varpi}^{(\ell)}_i := \sum\nolimits_{k\in I} b'_{ik}\,\alpha_k \qquad\qquad \forall\ i \in I \tag{3.2}$$

which are characterized by the property that $\big(\dot{\varpi}^{(\ell)}_i, \alpha_j\big)_B = \delta_{ij}$; in short, $\{\dot{\varpi}^{(\ell)}_i\}_{i\in I}$ is the $\mathbb{Q}$–basis of $\mathbb{Q}Q$ which is *left*-dual to the basis $\{\alpha_j\}_{j\in I}$ w.r.t. $(\ ,\ )_B$; in particular,

$\big\{\dot{\varpi}_i^{(\ell)}\big\}_{i\in I}$ is a $\mathbb{Z}$–basis of $\dot{Q}^{(\ell)}$, the left-dual to Q w.r.t. $(\ ,\ )_B$. Definitions give also $Q \subseteq \dot{Q}^{(\ell)}$ with $\alpha_i = \sum_{k\in I} b_{ik}\, \dot{\varpi}_k^{(\ell)}$ for all $i \in I$.

The left-right symmetrical counterpart is given once we define the elements

$$\dot{\varpi}_i^{(r)} := \textstyle\sum_{k\in I} b'_{ki}\, \alpha_k \qquad\qquad \forall\ i \in I \tag{3.3}$$

characterized by the property that $\big(\alpha_j\,, \dot{\varpi}_i^{(r)}\big)_B = \delta_{ji}$; thus $\big\{\dot{\varpi}_i^{(r)}\big\}_{i\in I}$ is the $\mathbb{Q}$–basis of $\mathbb{Q}Q$ which is *right*-dual to the basis $\big\{\alpha_j\big\}_{j\in I}$ with respect to $(\ ,\ )_B$; in particular, $\big\{\dot{\varpi}_i^{(r)}\big\}_{i\in I}$ is a $\mathbb{Z}$–basis of $\dot{Q}^{(r)}$, the right-dual to Q w.r.t. $(\ ,\ )_B$. Furthermore, definitions give also $Q \subseteq \dot{Q}^{(r)}$ with $\alpha_i = \sum_{k\in I} b_{ki}\, \dot{\varpi}_k^{(r)}$ for all $i \in I$.

The strongly integral case: The previous construction has a sort of "refinement" when the integral type multiparameter $\mathbf{q} := \big(q_{ij} = q^{b_{ij}}\big)_{i,j\in I}$ is actually *strongly integral*, with $b_{ij} = d_i\, t_{ij}^+ = d_j\, t_{ij}^-$ for all $i, j \in I$ (cf. §2.3.2). In this case, consider the two $\mathbb{Z}$–bilinear pairings $\langle\ ,\ \rangle_{T^\pm} : Q \times Q \longrightarrow \mathbb{Z}$ defined by the matrices T^+ and T^- — thus given by $\langle \alpha_i\,, \alpha_j \rangle_{T^\pm} := t_{ij}^\pm$ for all $i, j \in I$ — that are obviously non-degenerate (as $(\ ,\ )_B$ is, and $D\, T^+ = B = T^-\, D$), and extend them to same-name $\mathbb{Q}$–bilinear pairings on $\mathbb{Q}Q \times \mathbb{Q}Q$ by scalar extension. Then define, for any sublattice $\varGamma$ in $\mathbb{Q}Q$, its *left-dual* and *right-dual* (w.r.t. T^- and T^+ respectively) as

$$\begin{aligned} \varGamma^{(\ell)} &:= \big\{\, \lambda \in \mathbb{Q}Q \;\big|\; \langle\lambda\,,\gamma\rangle_{T^-} \in \mathbb{Z}\,,\ \forall\ \gamma \in \varGamma \,\big\} \\ \varGamma^{(r)} &:= \big\{\, \rho \in \mathbb{Q}Q \;\big|\; \langle\gamma,\rho\rangle_{T^+} \in \mathbb{Z}\,,\ \forall\ \gamma \in \varGamma \,\big\} \end{aligned} \tag{3.4}$$

that both are sublattices in $\mathbb{Q}Q$; the pairings $\langle\ ,\ \rangle_{T^\pm}$ then restrict to $\mathbb{Z}$-valued pairings $\langle\ ,\ \rangle_{T^-} : \varGamma^{(\ell)} \times \varGamma \longrightarrow \mathbb{Z}$ and $\langle\ ,\ \rangle_{T^+} \varGamma \times \varGamma^{(r)} \longrightarrow \mathbb{Z}$. Now consider the elements

$$\varpi_i^{(\ell)} := d_i\, \dot{\varpi}_i^{(\ell)} = \textstyle\sum_{k\in I} t_{ik}^{-,\prime}\, \alpha_k \ \ ,\quad \varpi_i^{(r)} := d_i\, \dot{\varpi}_i^{(r)} = \sum_{k\in I} t_{ki}^{+,\prime}\, \alpha_k \qquad \forall\ i \in I \tag{3.5}$$

where $\big(t_{ij}^{\pm,\prime}\big)_{i\in I}^{j\in I} = \big(T^\pm\big)^{-1}$, which are characterized by the properties $\big\langle \varpi_i^{(\ell)}, \alpha_j \big\rangle_{T^-} = \delta_{ij}$ and $\big\langle \alpha_i\,, \varpi_j^{(r)} \big\rangle_{T^+} = \delta_{ij}$; in a nutshell, $\big\{\varpi_i^{(\ell)}\big\}_{i\in I}$ is the $\mathbb{Q}$–basis of $\mathbb{Q}Q$ which is *left*-dual to the basis $\big\{\alpha_j\big\}_{j\in I}$ w.r.t. $\langle\ ,\ \rangle_{T^-}$, while $\big\{\varpi_i^{(r)}\big\}_{i\in I}$ is the *right*-dual to $\big\{\alpha_j\big\}_{j\in I}$ w.r.t. $\langle\ ,\ \rangle_{T^+}$. In particular, $\big\{\varpi_i^{(\ell)}\big\}_{i\in I}$ is a $\mathbb{Z}$–basis of $Q^{(\ell)}$, and $\big\{\varpi_i^{(r)}\big\}_{i\in I}$ is a $\mathbb{Z}$–basis of $Q^{(r)}$ with notation of (3.4). Note also that definitions give $Q \subseteq Q^{(\ell)} \bigcap Q^{(r)}$ with $\alpha_i = \sum_{k\in I} t_{ik}^-\, \varpi_k^{(\ell)}$ and $\alpha_i = \sum_{k\in I} t_{ki}^+\, \varpi_k^{(r)}$ for all $i \in I$.

3.3.3. MpQG's with larger tori. Let $\varGamma_+$ and $\varGamma_-$ be any two lattices in $\mathbb{Q}Q$ such that $Q \leq \varGamma_\pm$; then set $\varGamma_\bullet := \varGamma_+ \times \varGamma_-$. From §3.3.1, with notation fixed therein, we can consider the corresponding "multiparameter quantum torus" $U_{\mathbf{q},\varGamma_\bullet}^0$, that contains $U_{\mathbf{q}}^0 = U_{\mathbf{q},Q}^0$. For either lattice $\varGamma_\pm$, we have a matrix $C_\pm = \big(c_{ij}^\pm\big)_{i,j\in I}$ and $C_\pm^{-1} = \big(c_{ij}^{\pm,\prime}\big)_{i,j\in I}$, with $c_\pm := \big|det(C_\pm)\big| \in \mathbb{Z}_+$ and $c_{ij}^{\pm,\prime\prime} := c_\pm \cdot c_{ij}^{\pm,\prime} \in \mathbb{Z}$ $(\, i, j \in I\,)$.

For the rest of this subsection, we make now the following assumption concerning the ground field $\Bbbk$ *and the multiparameter (of Cartan type)* $\mathbf{q} := \big(q_{ij}\big)_{i,j\in I}$*: for every* $i, j \in I$*, the field* $\Bbbk$ *contains a* $c_\pm$*–th root of* q_{ij}*, hereafter denoted by* $q_{ij}^{1/c_\pm}$*; moreover, we assume that* $\mathbf{q}^{1/c_\pm} := \Big(q_{ij}^{1/c_\pm}\Big)_{i,j\in I}$ *is of Cartan type too.*

The natural (adjoint) action of $U_{\mathbf{q}}^0$ onto $U_{\mathbf{q}}$ extends, in a unique manner, to a $U_{\mathbf{q},\varGamma_\bullet}^0$–action $\cdot : U_{\mathbf{q},\varGamma_\bullet}^0 \times U_{\mathbf{q}} \longrightarrow U_{\mathbf{q}}$, given by

$$K_{\gamma_i^+} \cdot E_j = q_{ij}^{\varGamma_+} E_j \ ,\quad L_{\gamma_i^-} \cdot E_j = \big(q_{ji}^{\varGamma_-}\big)^{-1} E_j \ ,\quad K_{\gamma_i^+} \cdot K_{\alpha_j} = K_{\alpha_j} \ ,\quad L_{\gamma_i^-} \cdot K_{\alpha_j} = K_{\alpha_j}$$

$$K_{\gamma_i^+} \cdot L_{\alpha_j} = L_{\alpha_j} \ ,\quad L_{\gamma_i^-} \cdot L_{\alpha_j} = L_{\alpha_j} \ ,\quad K_{\gamma_i^+} \cdot F_j = \big(q_{ij}^{\varGamma_+}\big)^{-1} F_j \ ,\quad L_{\gamma_i^-} \cdot F_j = q_{ji}^{\varGamma_-} F_j$$

where $q_{rs}^{\Gamma_+} := \prod_{k\in I} \big(q_{ks}^{1/c_+}\big)^{c_{rk}^{+,\prime\prime}}$ and $q_{ae}^{\Gamma_-} := \prod_{k\in I} \big(q_{ak}^{1/c_-}\big)^{c_{ek}^{-,\prime\prime}}$; this makes $U_{\mathbf{q}}$ into a $U^0_{\mathbf{q},\Gamma_\bullet}$–module Hopf algebra. This allows us to consider the Hopf algebra $U^0_{\mathbf{q},\Gamma_\bullet} \ltimes U_{\mathbf{q}}$ given by the *smash product* of $U^0_{\mathbf{q},\Gamma_\bullet}$ and $U_{\mathbf{q}}$: the underlying vector space is just $U^0_{\mathbf{q},\Gamma_\bullet} \otimes U_{\mathbf{q}}$, the coalgebra structure is the one given by the tensor product of the corresponding coalgebras, and the product is given by the formula

$$(h \ltimes x)\,(k \ltimes y) \;=\; h\,k_{(1)} \ltimes \big(\mathcal{S}(k_{(1)}) \cdot x\big)\, y \qquad \text{for all } \; h,k \in U^0_{\mathbf{q},\Gamma_\bullet} \, , \; x,y \in U_{\mathbf{q}} \;\; .$$

Since $U^0_{\mathbf{q},\Gamma_\bullet}$ contains $U^0_{\mathbf{q}} \; \big(=: U^0_{\mathbf{q},Q\times Q}\big)$ as a Hopf subalgebra, it follows that $U^0_{\mathbf{q},\Gamma_\bullet}$ itself is a right $U^0_{\mathbf{q}}$–module Hopf algebra with respect to the adjoint action. It is not difficult to see that, under these hypotheses, the smash product $U^0_{\mathbf{q},\Gamma_\bullet} \ltimes U_{\mathbf{q}}$ maps onto a Hopf algebra structure on the vector space $U^0_{\mathbf{q},\Gamma_\bullet} \underset{U^0_{\mathbf{q}}}{\otimes} U_{\mathbf{q}}$, which hereafter we denote by $U^0_{\mathbf{q},\Gamma_\bullet} \underset{U^0_{\mathbf{q}}}{\ltimes} U_{\mathbf{q}}$, see [Le, Theorem 2.8]. We define then

$$U_{\mathbf{q},\Gamma_\bullet}(\mathfrak{g}) \;\equiv\; U_{\mathbf{q},\Gamma_\bullet} \;:=\; U^0_{\mathbf{q},\Gamma_\bullet} \underset{U^0_{\mathbf{q}}}{\ltimes} U_{\mathbf{q}} \;=\; U^0_{\mathbf{q},\Gamma_\bullet} \underset{U^0_{\mathbf{q}}}{\ltimes} U_{\mathbf{q}}(\mathfrak{g}) \tag{3.6}$$

It is easy to check that $U_{\mathbf{q},\Gamma_\bullet}(\mathfrak{g})$ and its Hopf algebra structure can be described with a presentation by generators and relations like that for $U_{\mathbf{q}}(\mathfrak{g})$ in Definition 3.1.1. Indeed, since the coalgebra structure is the one given by the tensor product, one has to describe only the algebra structure. For this, first one has to replace the generators $K_i^{\pm 1} = K_{\pm\alpha_i}$ and $L_i^{\pm 1} = L_{\pm\alpha_i}$ with the generators $\mathcal{K}_i^{\pm 1} = K_{\pm\gamma_i^+}$ and $\mathcal{L}_i^{\pm 1} = L_{\pm\gamma_i^-}$. Second, replace relations *(c)* and *(d)* of Definition 3.1.1 with the following, generalized relations:

$$(c') \qquad K_{\gamma_i^+}\, E_j\, K_{\gamma_i^+}^{-1} \;=\; q_{ij}^{\Gamma_+}\, E_j \;\;, \qquad L_{\gamma_i^-}\, E_j\, L_{\gamma_i^-}^{-1} \;=\; \big(q_{ji}^{\Gamma_-}\big)^{-1} E_j$$

$$(d') \qquad K_{\gamma_i^+}\, F_j\, K_{\gamma_i^+}^{-1} \;=\; \big(q_{ij}^{\Gamma_+}\big)^{-1} F_j \;\;, \qquad L_{\gamma_i^-}\, F_j\, L_{\gamma_i^-}^{-1} \;=\; q_{ji}^{\Gamma_-}\, F_j$$

Then, in relation *(e)* write each $K_i^{\pm 1} = K_{\alpha_i}$, resp. $L_i^{\pm 1} = L_{\pm\alpha_i}$, in terms of the $\mathcal{K}_j^{\pm 1} := K_{\pm\gamma_j^+}$'s, resp. $\mathcal{L}_j^{\pm 1} := L_{\pm\gamma_j^-}$'s; finally, leave relations *(f)* and *(g)* unchanged.

With much the same approach, one defines also the "(multiparameter) quantum subgroups" of $U_{\mathbf{q},\Gamma_\bullet}(\mathfrak{g})$ akin to those of $U_{\mathbf{q}}(\mathfrak{g})$ (cf. Definition 3.1.3), that we denote by adding a subscript $\Gamma_\bullet$, namely $U^+_{\mathbf{q},\Gamma_\bullet}$, $U^-_{\mathbf{q},\Gamma_\bullet}$, $U^{\geq}_{\mathbf{q},\Gamma_\bullet}$, $U^{\leq}_{\mathbf{q},\Gamma_\bullet}$, $U^{+,0}_{\mathbf{q},\Gamma_\bullet}$ and $U^{-,0}_{\mathbf{q},\Gamma_\bullet}$.

The integral case: When $\mathbf{q}$ is of integral type, the above construction may have a simpler description. Indeed, *assume also that the lattices Γ_+ and Γ_- (both containing Q) are such that $\Gamma_+ \leq \dot{Q}^{(\ell)}$ and $\Gamma_- \leq \dot{Q}^{(r)}$* , that is $\big(\Gamma_+ , Q\big)_B \subseteq \mathbb{Z}$ and $\big(Q , \Gamma_-\big)_B \subseteq \mathbb{Z}$ — notation of §3.3.2. Then in the presentation of the MpQG $U_{\mathbf{q},\Gamma_\bullet}$ of (3.6) the modified relations *(c')* and *(d')* mentioned above take the form

$$(c') \qquad K_{\gamma_i^+}\, E_j\, K_{\gamma_i^+}^{-1} \;=\; q^{+(\gamma_i^+,\,\alpha_j)_B}\, E_j \;\;, \qquad L_{\gamma_i^-}\, E_j\, L_{\gamma_i^-}^{-1} \;=\; q^{-(\alpha_j,\,\gamma_i^-)_B}\, E_j$$

$$(d') \qquad K_{\gamma_i^+}\, F_j\, K_{\gamma_i^+}^{-1} \;=\; q^{-(\gamma_i^+,\,\alpha_j)_B}\, F_j \;\;, \qquad L_{\gamma_i^-}\, F_j\, L_{\gamma_i^-}^{-1} \;=\; q^{+(\alpha_j,\,\gamma_i^-)_B}\, F_j$$

In particular, this means that $U_{\mathbf{q},\Gamma_\bullet}$ *is actually well-defined over the (possibly smaller) ground field generated in $\Bbbk$ by q — and similarly for $U^+_{\mathbf{q},\Gamma_\bullet}$, $U^{\geq}_{\mathbf{q},\Gamma_\bullet}$, etc.* Therefore, the assumption that $\Bbbk$ contain c_+–th and c_-–th roots of q_{ij} , that is required in the non-integral case, is not necessary in the integral one.

3.3.4. Duality among MpQG's with larger tori. Let again $\Gamma_\pm$ be two lattices of rank θ in $\mathbb{Q}Q$ containing Q , and set $\Gamma_\bullet := \Gamma_+ \times \Gamma_-$; then we have "toral MpQG's" $U^{\pm,0}_{\mathbf{q},\Gamma_\pm}$ and $U^0_{\mathbf{q},\Gamma_\bullet}$ as in §3.3.1. Moreover, we have bases $\big\{\, \gamma_s^\pm \,\big\}_{s\in I}$ of $\Gamma_\pm$ and corresponding matrices $C_\pm = \big(c_{ij}^\pm\big)_{i,j\in I}$ and $C_\pm^{-1} = \big(c_{ij}^{\pm,\prime}\big)_{i,j\in I}$, and the integers $c_\pm := \big|det(C_\pm)\big|$ and $c_{ij}^{\pm,\prime\prime} := c_\pm \cdot c_{ij}^{\pm,\prime}$ $\;(\, i,j \in I \,)$ as in §3.3.3. In addition, we assume that $\Bbbk$ contain a $(c_+\, c_-)$–th root of q_{ij}, say $q_{ij}^{1/(c_+c_-)}$, and that overall the multiparameter $\mathbf{q}^{1/(c_+c_-)} := \Big(q_{ij}^{1/(c_+c_-)} \Big)_{i,j \in I}$ be of Cartan type.

It is straightforward to check that the skew-Hopf pairing $\eta : U_{\mathbf{q}}^{\geq} \underset{\Bbbk}{\otimes} U_{\mathbf{q}}^{\leq} \longrightarrow \Bbbk$ in Proposition 3.1.4 actually extends to a similar pairing $U_{\mathbf{q},\varGamma_+}^{\geq} \underset{\Bbbk}{\otimes} U_{\mathbf{q},\varGamma_-}^{\leq} \longrightarrow \Bbbk$ given by

$E_i \otimes L_{\gamma^-} \mapsto 0\,,\ K_{\gamma^+} \otimes F_j \mapsto 0\,,\ E_i \otimes F_j \mapsto -\delta_{ij} \frac{q_{ii}}{q_{ii}-1}\,,\ K_{\gamma_i^+} \otimes L_{\gamma_j^-} \mapsto \prod_{h,k \in I} \Big(q_{hk}^{1/(c_+c_-)} \Big)^{c_{ih}^{+,\prime\prime} c_{jk}^{-,\prime\prime}}$

for all $i, j \in I$, and still denoted η. In particular, this $\eta : U_{\mathbf{q},\varGamma_+}^{\geq} \underset{\Bbbk}{\otimes} U_{\mathbf{q},\varGamma_-}^{\leq} \longrightarrow \Bbbk$ is still non-degenerate, like its restrictions $U_{\mathbf{q},\varGamma_+}^{+} \underset{\Bbbk}{\otimes} U_{\mathbf{q},\varGamma_-}^{-} \longrightarrow \Bbbk$ and $U_{\mathbf{q},\varGamma_+}^{0} \underset{\Bbbk}{\otimes} U_{\mathbf{q},\varGamma_-}^{0} \longrightarrow \Bbbk$.

When $\mathbf{q} := \big(q_{ij} = q^{b_{ij}} \big)_{i \in I}^{j \in I}$ *is of integral type*, and $(\ ,\ )_B : Q \times Q \longrightarrow \mathbb{Z}$ is the associated pairing — cf. §3.3.2 — *the previous construction may have a simpler description, under the additional assumption that* $\big(\varGamma_+ , \varGamma_- \big)_B \subseteq \mathbb{Z}$ — that is equivalent to either of $\varGamma_+ \subseteq \dot{\varGamma}_-^{(\ell)}$ and $\varGamma_- \subseteq \dot{\varGamma}_+^{(r)}$ — so that $(\ ,\ )_B$ induces a pairing $(\ ,\ )_B : \varGamma_+ \times \varGamma_- \longrightarrow \mathbb{Z}$. In the following, we shall briefly refer to such a situation by saying that $\big(\varGamma_+ , \varGamma_- \big)$ *is a pair in duality* (w.r.t. B), or that *the lattices* $\varGamma_+$ *and* $\varGamma_-$ *are in duality* (w.r.t. B). Indeed, under these assumptions we have $\eta\big(K_{\gamma_i^+}, L_{\gamma_j^-} \big) = \prod_{h,k \in I} \Big(q_{hk}^{1/(c_+c_-)} \Big)^{c_{ih}^{+,\prime\prime} c_{jk}^{-,\prime\prime}} = q^{(\gamma_i^+, \gamma_j^-)_B}$; in particular, requiring a (c_+c_-)–th root in $\Bbbk$ of every q_{hk} is no longer necessary.

Remark 3.3.5. It is easy to see that, using the skew-Hopf pairing η between (suitably chosen) quantum Borel subgroups $U_{\mathbf{q},\varGamma_+}^{\geq}$ and $U_{\mathbf{q},\varGamma_-}^{\leq}$ mentioned in §3.3.4 above, *every MpQG with larger torus, say* $U_{\mathbf{q},\varGamma_\bullet}(\mathfrak{g})$, *can be realized as a Drinfeld double (of those quantum Borel subgroups)*, so extending what happens for MpQG's with "standard" torus.

4. Quantum root vectors and PBW theorems for MpQG's

The first purpose of this section is to introduce root vectors for MpQG's. Second, we show that PBW theorems hold true for an MpQG and all its relevant subalgebras.

4.1. **Quantum root vectors in MpQG's.**

For the one-parameter quantum group $U_q(\mathfrak{g})$ of Lusztig several authors introduced quantum analogues of root vectors — or "quantum root vectors" — in different ways, the most common ones being via iterated q–brackets or iterated adjoint action. Lusztig gave (cf. [Lu]) a general procedure, using an action on $U_q(\mathfrak{g})$ of the braid group associated with $\mathfrak{g}$; later, it was extended to the multiparameter case in [He2].

To begin with, let W the Weyl group of $\mathfrak{g}$, generated by reflections $s_i = s_{\alpha_i}$ associated with the simple roots α_i of $\mathfrak{g}$ ($i \in I$), and let $w_0 \in W$ be the longest element in W. Then the number $N := \big|\Phi^+\big|$ of positive roots (cf. 2.3.1) of $\mathfrak{g}$ is also the length of any reduced expression of w_0. Let us fix now one such reduced expression, say $w_0 = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_{N-1}} s_{i_N}$, so that all the following constructions will actually depend on this specific choice.

Set $\beta^k := s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \big(\alpha_{i_k} \big)$ for all $k = 1, \dots, N$: then one has $\big\{ \beta^k \big\}_{k=1,2,\dots,N} = \Phi^+$; in particular, all positive roots are recovered starting from the fixed reduced expression of w_0, and in addition this also endows Φ^+ with a total order, namely $\beta^k \preceq \beta^h \iff k \leq h$. The same method of course can be applied to negative roots.

A similar procedure allows to construct a root vector in $\mathfrak{g}$ for each positive root. First consider the braid group $\mathbb{B}$ associated with W, generated by elements $\overline{T}_i$ which lift the simple reflections $s_i = s_{\alpha_i}$ ($i \in I$). There is a standard way (cf. for instance [Hu]) to define a group action of $\mathbb{B}$ onto $\mathfrak{g}$ that on root space yields $\overline{T}_i(\mathfrak{g}_\beta) = \mathfrak{g}_{s_i(\beta)}$; using this action one can define root vectors via

$$x_{\beta^k} := \overline{T}_{i_1} \overline{T}_{i_2} \cdots \overline{T}_{i_{k-1}} \big(x_{i_k} \big) \in \mathfrak{g}_{\beta^k} \qquad \forall\ k = 1, 2, \dots, N$$

where each x_i is a Chevalley generator in $\mathfrak{g}_{\alpha_i}$. It is worth remarking that if β^k is a simple root, say $\beta^k = \alpha_j$, then the root vector x_{β^k} defined above actually coincides with the generator x_j given from scratch, so the entire construction is overall consistent. The same argument can be used to construct negative root vectors.

This type of procedure was "lifted" to the one-parameter quantum case by Lusztig (cf. [Lu]), who did it introducing a suitable braid group action on $U_q(\mathfrak{g})$; his construction was later extended by Heckenberger to the multiparameter case, that is to $U_{\mathbf{q}}(\mathfrak{g})$, as we shall now shortly recall. One defines — see [HY], formulas (4.3–4) — isomorphisms $T_1, \dots, T_\theta$ which yield a $\mathbb{B}$–action that lifts that on $U(\mathfrak{g})$; using this action one defines "quantum root vectors" E_{β^k} as given by

$$E_{\beta^k} := T_{i_1} T_{i_2} \cdots T_{i_{k-1}}\big(E_{i_k}\big) \in U_{\mathbf{q}}^+ \qquad \forall\ k = 1, 2, \dots, N \tag{4.1}$$

where one finds that $E_{\beta^k} = E_j$ whenever $\beta^k = \alpha_j$; similarly one also constructs "(quantum) negative root vectors" $F_{\beta^k} \in U_{\mathbf{q}}^-$. In the following, we shall refer to the E_{β^k}'s or the F_{β^k}'s by loosely calling them "(quantum) root vectors".

It is also remarkable that these quantum root vectors can be realized as iterated braided brackets (e.g., like in [HY, Section 4]*).* This will be of key importance, by the following:

Proposition 4.1.1. *Every quantum root vector in $U_{\mathbf{q}}(\mathfrak{g})$ is proportional to the corresponding quantum root vector in $U_{\check{\mathbf{q}}}(\mathfrak{g})$ by a coefficient that is a monomial in the $q_{ij}^{\pm 1/2}$'s.*

Proof. By Theorem 3.2.2 and Proposition 3.2.4 together we know that

$$U_{\mathbf{q}}(\mathfrak{g}) \cong U_{\check{\mathbf{q}}}(\mathfrak{g})_\sigma = U_{\check{\mathbf{q}}}(\mathfrak{g})^{(\tilde{\varphi})}$$

for the 2–cocycle σ of $U_{\check{\mathbf{q}}}(\mathfrak{g})$ and a suitable group bicharacter φ of Q. Now denote by E_α a quantum root vector in $U_{\mathbf{q}}(\mathfrak{g})$ and by $\check{E}_\alpha$ the corresponding (i.e., built in the same way, for the same root) quantum root vector in $U_{\check{\mathbf{q}}}(\mathfrak{g})$. Since $\check{q}_{ij} = q^{d_i a_{ij}} = q^{d_j a_{ji}} = \check{q}_{ji}$ for all $i, j \in I$, in $U_{\check{\mathbf{q}}}(\mathfrak{g})_\sigma$ we have that

$$\begin{aligned}
\mathrm{ad}(E_j)(E_i) &= \mathrm{ad}_\sigma(\check{E}_j)(\check{E}_i) = (\check{E}_j)_{(1)} \cdot_\sigma \check{E}_i \cdot_\sigma \mathcal{S}_\sigma\big((\check{E}_j)_{(2)}\big) = \\
&= \check{E}_j \cdot_\sigma \check{E}_i + K_j \cdot_\sigma \check{E}_i \cdot_\sigma \big(K_j^{-1} \cdot_\sigma \check{E}_j\big) = \\
&= \sigma(K_j, K_i)\, \check{E}_j \check{E}_i + \big(\sigma(K_j, K_i)\, K_j \check{E}_i\big) \cdot_\sigma \big(\sigma(K_j^{-1}, K_j)\, K_j^{-1} \check{E}_j\big) = \\
&= \sigma(K_j, K_i)\big(\check{E}_j \check{E}_i + \sigma\big(K_j^{-1}, K_j\big)\, \sigma(K_j K_i, 1)\, K_j \check{E}_i K_j^{-1} \check{E}_j\, \sigma^{-1}\big(K_j, K_j^{-1}\big)\big) = \\
&= q_{ji}^{1/2}\big(\check{E}_j \check{E}_i + \check{q}_{ij} \check{E}_i \check{E}_j\big) = q_{ji}^{1/2}\big((\check{E}_j)_{(1)} \cdot \check{E}_i \cdot \mathcal{S}\big((\check{E}_j)_{(2)}\big)\big) = q_{ji}^{1/2}\, \mathrm{ad}(\check{E}_j)(\check{E}_i)
\end{aligned}$$

Therefore, although the adjoint action is not preserved under the 2–cocycle deformation, both elements differ only by a coefficient which is a monomial in the $q_{ij}^{\pm 1/2}$'s. Since both quantum root vectors are defined by an iteration of adjoint actions (because of the very definition of the T_i's) by Lemma 2.2.3 we infer, taking into account the explicit form of σ (whose values are monomials in the $q_{ij}^{\pm 1/2}$'s), that the quantum root vectors E_α and $\check{E}_\alpha$ associated with any root α in $U_{\mathbf{q}}(\mathfrak{g})$ and in $U_{\check{\mathbf{q}}}(\mathfrak{g})_\sigma$, respectively, are linked by an identity $E_\alpha = m_\alpha^+\big(\mathbf{q}^{\pm 1/2}\big)\, \check{E}_\alpha$ for some monomial $m_\alpha^+\big(\mathbf{q}^{\pm 1/2}\big)$ in the $q_{ij}^{\pm 1/2}$'s, as claimed.

The above accounts for all (quantum) *positive* root vectors. A similar argument proves the claim for *negative* root vectors as well. □

4.2. Poincaré-Birkhoff-Witt (=PBW) theorems for MpQG's.

Once we have quantum root vectors, some Poincaré-Birkhoff-Witt (=PBW) theorems hold too, stating that suitable ordered products of quantum root vectors and/or toral generators do form a $\Bbbk$–basis of $U_{\mathbf{q}}(\mathfrak{g})$ itself. Here is the exact claim:

Theorem 4.2.1. *("PBW Theorem" for $U_{\mathbf{q}}(\mathfrak{g})$ — cf.* [An4, Theorem 3.6], [HY, Theorem 4.5]*, and references therein) Assume quantum root vectors in $U_{\mathbf{q}}(\mathfrak{g})$ have been defined as above. Then the set of ordered monomials*

$$\left\{ \prod_{k=N}^{1} F_{\beta^k}^{f_k} \prod_{j\in I} L_j^{a_j} \prod_{i\in I} K_i^{b_i} \prod_{h=1}^{N} E_{\beta^h}^{e_h} \;\middle|\; f_k, a_j, b_i, e_h \in \mathbb{N} \right\}$$

is a $\Bbbk$–basis of $U_{\mathbf{q}}(\mathfrak{g})$, and similarly if we take the opposite order in Φ^+.

Similar results hold for the subalgebras $U_{\mathbf{q}}^{\geq}$, $U_{\mathbf{q}}^{\leq}$, $U_{\mathbf{q}}^{+}$, $U_{\mathbf{q}}^{-}$, $U_{\mathbf{q}}^{+,0}$, $U_{\mathbf{q}}^{-,0}$ and $U_{\mathbf{q}}^{0}$.

Proof. This is proved in [An4, Theorem 3.6] (also for $\mathbf{q}$ not of Cartan type). □

Remark 4.2.2. It is easy to see that a suitable "PBW Theorem" holds as well for any generalized MpQG with larger torus $U_{\mathbf{q},\Gamma_\bullet}(\mathfrak{g})$ — cf. §3.3.

4.3. **Hopf duality among quantum Borel subgroups.**

Proposition 3.1.4 provides a skew-Hopf pairing between the two MpQG's of Borel type $U_{\mathbf{q}}^{\geq}$ and $U_{\mathbf{q}}^{\leq}$, that we denote by η. Again, from [AY, Proposition 4.6], we have a complete description of this pairing, in terms of PBW bases (of both sides), namely the following:

Proposition 4.3.1. *Keep notation as above. Then*

$$\eta\left(\prod_{k=1}^{M} E_{\beta_k}^{e_k} K \,,\, \prod_{k=1}^{M} F_{\beta_k}^{f_k} L \right) \;=\; \prod_{k=1}^{M} \delta_{e_k,f_k} \left(\frac{(-1)^{h(\beta_k)}\, q_{\beta^k\beta^k}}{q_{\beta^k\beta^k}-1} \right)^{e_k} (e_k)_{q_{\beta^k\beta^k}}! \cdot \eta(K,L)$$

for all $e_k, f_k \in \mathbb{N}$ and all $K \in U_{\mathbf{q}}^{+,0}$, $L \in U_{\mathbf{q}}^{-,0}$, where $h(\beta_k)$ is the height of the root β_k and $q_{\beta^k\beta^k}$ is defined as in §3.2.

Remark 4.3.2. It is straightforward to see that the result above actually extends to the case when — under suitable assumptions — one considers the pairing η between two multiparameter quantum Borel subgroups $U_{\mathbf{q},\Gamma_\bullet}^{\geq}$ and $U_{\mathbf{q},\Gamma_\bullet}^{\leq}$ like in §3.3.4.

4.4. **Special products in $U_{\mathbf{q}}(\mathfrak{g}) = \big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)_\sigma$.**

When performing calculations in our MpQG's, a convenient strategy is to reduce ourselves to similar calculations in the simpler framework of uniparameter quantum groups. The basic point to start from is the existence of a Hopf algebra isomorphism

$$U_{\mathbf{q}}(\mathfrak{g}) \;\cong\; \big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)_\sigma$$

(cf. Theorem 3.2.2) where σ is the 2–cocycle given in Definition 3.2.1. Therefore, we can describe $U_{\mathbf{q}}(\mathfrak{g})$ as being the coalgebra $\big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)_\sigma$ *endowed with the new, deformed product* $* := \cdot_\sigma$ (defined as in §2.2.1) and the corresponding, deformed antipode $\mathcal{S}_\sigma$. The "old" product in $U_{\check{\mathbf{q}}}(\mathfrak{g})$ instead will be denoted by $\check{\cdot}$. So hereafter by Y^{*z} or $Y^{\check{\cdot} z}$ we shall denote the z–th power of any $Y \in U_{\check{\mathbf{q}}}(\mathfrak{g})$ with respect to either the deformed product $*$ or the old product $\check{\cdot}$, respectively, for any exponent $z \in \mathbb{N}$, or even $z \in \mathbb{Z}$ when Y is invertible.

For later use, we need to introduce some more notation:

Definition 4.4.1.

(a) Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$, and let $p \in \mathbb{F}$ be *not* a root of unity. For every $H \in \mathcal{A}$, $n \in \mathbb{N}$ and $c \in \mathbb{Z}$, define the elements

$$\binom{H\,;c}{n}_p := \prod_{s=1}^{n} \frac{p^{c+1-s}H-1}{p^s-1} \quad , \quad \binom{H}{n}_p := \binom{H\,;0}{n}_p \tag{4.2}$$

that are called *p–binomial coefficients (or just "p–binomials") in H.*

(b) For every $i \in I$, $\alpha \in \Phi^+$, $X_i \in \{E_i, F_i\}$, $Y_\alpha \in \{E_\alpha, F_\alpha\}$ — notation as in §4.1 — and all $n \in \mathbb{N}$, the elements in $U_{\mathbf{q}}(\mathfrak{g})$

$$X_i^{(n)} := \frac{X_i^n}{(n)_{q_{ii}}!} \qquad , \qquad Y_\alpha^{(n)} := \frac{Y_\alpha^n}{(n)_{q_{\alpha\alpha}}!} \tag{4.3}$$

are called *quantum divided powers*, or *q–divided powers*. $\diamondsuit$

Note that if in $U_{\check{\mathbf{q}}}(\mathfrak{g})$ we consider the two products $\cdot$ and $*$ we have *two* corresponding types of q–binomial coefficients, hereafter denoted by $\binom{X;0}{n}_p^{\check{\cdot}}$ and $\binom{X;0}{n}_p^{*}$. Similarly, we shall consider two types of q–divided powers, for which we use notation $Y^{\check{\cdot}(n)}$ and $Y^{*(n)}$; indeed, the first type denotes a q–divided power in $\big(U_{\check{\mathbf{q}}}(\mathfrak{g}), \check{\cdot}\,\big)$, and the second one a q–divided power in $U_{\mathbf{q}}(\mathfrak{g}) = \big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)_\sigma = \big(U_{\check{\mathbf{q}}}(\mathfrak{g}), *\big)$.

4.4.2. Comparison formulas. Some elementary calculations lead to explicit formulas linking same type objects in $\big(U_{\check{\mathbf{q}}}(\mathfrak{g}), \check{\cdot}\,\big)$ and in $U_{\mathbf{q}}(\mathfrak{g}) = \big(U_{\check{\mathbf{q}}}(\mathfrak{g}), *\big)$; we shall use them later on when studying integral forms of $U_{\mathbf{q}}(\mathfrak{g})$.

Concretely, for all $i \in \{1, \dots, \theta\}$, $n, m \in \mathbb{N}$, $z, z', z'' \in \mathbb{Z}$ $p_s \in \{q, q_s\}$, $X, Y \in \{K, L\}$ and $G_i^{\pm 1} := K_i^{\pm 1} L_i^{\mp 1}$ we have — cf. (4.2) for notation —

$$E_i^{*(n)} = q_i^{+\binom{n}{2}} E_i^{\check{\cdot}(n)} \qquad , \qquad F_i^{*(n)} = q_i^{-\binom{n}{2}} F_i^{\check{\cdot}(n)}$$

$$K_i^{*z} = K_i^{\check{\cdot}\,z} \quad , \quad L_i^{*z} = L_i^{\check{\cdot}\,z} \quad , \quad G_i^{*z} = G_i^{\check{\cdot}\,z}$$

$$\binom{X_i}{n}_{p_i}^{*} = \binom{X_i}{n}_{p_i}^{\check{\cdot}} \quad , \quad \binom{G_i^{\pm 1}}{n}_{q_{ii}}^{*} = \binom{G_i^{\pm 1}}{n}_{q_{ii}}^{\check{\cdot}}$$

$$E_i^{*(n)} * E_j^{*(m)} = q_i^{+\binom{n}{2}} q_{n\alpha_i,\, m\alpha_j}^{+1/2} q_j^{+\binom{m}{2}} E_i^{\check{\cdot}(n)} \,\check{\cdot}\, E_j^{\check{\cdot}(m)} = q_i^{+\binom{n}{2}} \big(q_{ij}^{+1/2}\big)^{nm} q_j^{+\binom{m}{2}} E_i^{\check{\cdot}(n)} \,\check{\cdot}\, E_j^{\check{\cdot}(m)}$$

$$E_{i_1}^{*(n_1)} * E_{i_2}^{*(n_2)} * \cdots * E_{i_s}^{*(n_s)} = \bigg(\prod_{j=1}^{s} q_{i_j}^{+\binom{n_j}{2}}\bigg) \bigg(\prod_{j<k} q_{n_{i_j}\alpha_{i_j},\, n_k \alpha_{i_k}}^{+1/2}\bigg) E_{i_1}^{\check{\cdot}(n_1)} \,\check{\cdot}\, E_{i_2}^{\check{\cdot}(n_2)} \,\check{\cdot} \cdots \check{\cdot}\, E_{i_s}^{\check{\cdot}(n_s)}$$

$$F_i^{*(n)} * F_j^{*(m)} = q_i^{-\binom{n}{2}} q_{n\alpha_i,\, m\alpha_j}^{-1/2} q_j^{-\binom{m}{2}} F_i^{\check{\cdot}(n)} \,\check{\cdot}\, F_j^{\check{\cdot}(m)} = q_i^{-\binom{n}{2}} \big(q_{ij}^{-1/2}\big)^{nm} q_i^{-\binom{m}{2}} F_i^{\check{\cdot}(n)} \,\check{\cdot}\, F_j^{\check{\cdot}(m)}$$

$$F_{i_1}^{*(n_1)} * F_{i_2}^{*(n_2)} * \cdots * F_{i_s}^{*(n_s)} = \bigg(\prod_{j=1}^{s} q_{i_j}^{-\binom{n_j}{2}}\bigg) \bigg(\prod_{j<k} q_{n_{i_j}\alpha_{i_j},\, n_{i_k} \alpha_{i_k}}^{-1/2}\bigg) F_{i_1}^{\check{\cdot}(n_1)} \,\check{\cdot}\, F_{i_2}^{\check{\cdot}(n_2)} \,\check{\cdot} \cdots \check{\cdot}\, F_{i_s}^{\check{\cdot}(n_s)}$$

$$X_i^{*z'} * Y_j^{*z''} = X_i^{\check{\cdot}\,z'} \,\check{\cdot}\, Y_j^{\check{\cdot}\,z''} \; , \quad X_i^{*z'} * G_j^{*z''} = X_i^{\check{\cdot}\,z'} \,\check{\cdot}\, Y_j^{\check{\cdot}\,z''} \; , \quad G_i^{*z'} * Y_j^{*z''} = G_i^{\check{\cdot}\,z'} \,\check{\cdot}\, Y_j^{\check{\cdot}\,z''}$$

$$\binom{X_i}{n}_{p_i}^{*} * \binom{Y_j}{m}_{p_j}^{*} = \binom{X_i}{n}_{p_i}^{\check{\cdot}} \,\check{\cdot}\, \binom{Y_j}{m}_{p_j}^{\check{\cdot}}$$

$$\binom{G_i^{\pm 1}}{n}_{q_{ii}}^{*} * \binom{G_j^{\pm 1}}{m}_{q_{jj}}^{*} = \binom{G_i^{\pm 1}}{n}_{q_{ii}}^{\check{\cdot}} \,\check{\cdot}\, \binom{G_j^{\pm 1}}{m}_{q_{jj}}^{\check{\cdot}}$$

$$\binom{X_i}{n}_{p_i}^{*} * \binom{G_j^{\pm 1}}{m}_{q_{jj}}^{*} = \binom{X_i}{n}_{p_i}^{\check{\cdot}} \,\check{\cdot}\, \binom{G_j^{\pm 1}}{m}_{q_{jj}}^{\check{\cdot}}$$

$$E_i^{*(n)} * F_j^{*(m)} = q_i^{+\binom{n}{2}} q_j^{-\binom{m}{2}} E_i^{\check{\cdot}(n)} \,\check{\cdot}\, F_j^{\check{\cdot}(m)} \; , \quad F_j^{*(m)} * E_i^{*(n)} = q_j^{-\binom{m}{2}} q_i^{+\binom{n}{2}} F_j^{\check{\cdot}(m)} \,\check{\cdot}\, E_i^{\check{\cdot}(n)}$$

$$X_i^{*z} * E_j^{*(n)} = q_{ij}^{+z\,n/2} X_i^{\check{\cdot}\,z} \,\check{\cdot}\, E_j^{\check{\cdot}(n)} \quad , \quad E_j^{*(n)} * X_i^{*z} = q_{ji}^{+z\,n/2} E_j^{\check{\cdot}(n)} \,\check{\cdot}\, X_i^{\check{\cdot}\,z}$$

$$G_i^{*z} * E_j^{*(n)} = G_i^{\check{\cdot}\,z} \,\check{\cdot}\, E_j^{\check{\cdot}(n)} \quad , \quad E_j^{*(n)} * G_i^{*z} = E_j^{\check{\cdot}(n)} \,\check{\cdot}\, G_i^{\check{\cdot}\,z}$$

$$X_i^{*z} * F_j^{*(m)} = q_{ij}^{-z\,m/2} X_i^{\check{\cdot}\,z} \,\check{\cdot}\, F_j^{\check{\cdot}(m)} \quad , \quad F_j^{*(m)} * X_i^{*z} = q_{ji}^{-z\,m/2} F_j^{\check{\cdot}(m)} \,\check{\cdot}\, X_i^{\check{\cdot}\,z}$$

$$G_i^{*z} * F_j^{*(m)} = G_i^{\check{\cdot}\,z} \,\check{\cdot}\, F_j^{\check{\cdot}(m)} \quad , \quad F_j^{*(m)} * G_i^{*z} = F_j^{\check{\cdot}(m)} \,\check{\cdot}\, G_i^{\check{\cdot}\,z}$$

$$E_i^{*(n)} * \binom{X_j}{m}_{p_j}^{*} = q_i^{+\binom{n}{2}} \sum_{c=0}^{m} p_j^{-c(m-c)} \prod_{s=1}^{c} \frac{p_j^{1-s} \big(q_{ij}^{+1/2}\big)^{n} - 1}{p_j^{s} - 1} E_i^{\check{\cdot}(n)} \binom{X_j}{m-c}_{p_j}^{\check{\cdot}} X_j^{\check{\cdot}\,c}$$

$$
\begin{aligned}
\binom{X_j}{m}_{p_j}^{\!*} * E_i^{*(n)} \;&=\; q_i^{+\binom{n}{2}} \sum_{c=0}^{m} p_j^{-c(m-c)} \prod_{s=1}^{c} \frac{p_j^{1-s}\,\big(q_{ji}^{+1/2}\big)^n - 1}{p_j^s - 1}\; E_i^{\check{*}(n)} \binom{X_j}{m-c}_{p_j}^{\!\check{*}} X_j^{\check{*}c} \\
E_i^{*(n)} * \binom{G_j}{m}_{q_{jj}}^{\!*} \;&=\; q_i^{+\binom{n}{2}}\, E_i^{\check{*}(n)} \,\check{*}\, \binom{G_j}{m}_{q_{jj}}^{\!\check{*}} \quad , \quad \binom{G_j}{m}_{q_{jj}}^{\!*} * E_i^{*(n)} = \; q_i^{+\binom{n}{2}} \binom{G_j}{m}_{q_{jj}}^{\!\check{*}} \,\check{*}\, E_i^{\check{*}(n)} \\
F_i^{*(n)} * \binom{X_j}{m}_{p_j}^{\!*} \;&=\; q_i^{-\binom{n}{2}} \sum_{c=0}^{m} p_j^{-c(m-c)} \prod_{s=1}^{c} \frac{p_j^{1-s}\,\big(q_{ij}^{-1/2}\big)^n - 1}{p_j^s - 1}\; F_i^{\check{*}(n)} \binom{X_j}{m-c}_{p_j}^{\!\check{*}} X_j^{\check{*}c} \\
\binom{X_j}{m}_{p_j}^{\!*} * F_i^{*(n)} \;&=\; q_i^{-\binom{n}{2}} \sum_{c=0}^{m} p_j^{-c(m-c)} \prod_{s=1}^{c} \frac{p_j^{1-s}\,\big(q_{ji}^{-1/2}\big)^n - 1}{p_j^s - 1} \binom{X_j}{m-c}_{p_j}^{\!\check{*}} X_j^{\check{*}c}\, F_i^{\check{*}(n)} \\
F_i^{*(n)} * \binom{G_j}{m}_{q_{jj}}^{\!*} \;&=\; q_i^{-\binom{n}{2}}\, F_i^{\check{*}(n)} \,\check{*}\, \binom{G_j}{m}_{q_{jj}}^{\!\check{*}} \quad , \quad \binom{G_j}{m}_{q_{jj}}^{\!*} * F_i^{*(n)} = \; q_i^{-\binom{n}{2}} \binom{G_j}{m}_{q_{jj}}^{\!\check{*}} \,\check{*}\, F_i^{\check{*}(n)}
\end{aligned}
$$

In addition, more in general for root vectors we have the following. From Proposition 4.1.1 and its proof, recall that — keeping notation from there — if we denote by E_α a quantum root vector in $U_{\mathbf{q}}(\mathfrak{g})$ and by $\check{E}_\alpha$ the corresponding (i.e., for the same root) quantum root vector in $U_{\check{\mathbf{q}}}(\mathfrak{g})$ we have

$$
E_\alpha \,=\, m_\alpha^+\, \check{E}_\alpha \qquad\quad \text{and} \qquad\quad F_\alpha \,=\, m_\alpha^-\, \check{F}_\alpha
$$

for some Laurent monomials $m_\alpha^+ \,=\, m_\alpha^+\big(\mathbf{q}^{\pm 1/2}\big)$ and $m_\alpha^- \,=\, m_\alpha^-\big(\mathbf{q}^{\pm 1/2}\big)$ in the $q_{ij}^{\pm 1/2}$'s. Then an analysis like above (just a bit finer), for $\alpha, \beta, \in \Phi^+$, $j \in \{1, \dots, \theta\}$, yields

$$
\begin{aligned}
E_\alpha^{*(n)} * E_\beta^{*(m)} \;&=\; q_\alpha^{+\binom{n}{2}} \big(q_{\alpha\beta}^{+1/2}\big)^{nm} q_\beta^{+\binom{m}{2}} \big(m_\alpha^+\big)^n \big(m_\beta^+\big)^m\, \check{E}_\alpha^{\check{*}(n)} \,\check{*}\, \check{E}_\beta^{\check{*}(m)} \\
E_{\alpha_1}^{*(n_1)} * \cdots * E_{\alpha_s}^{*(n_s)} \;&=\; \Bigg(\prod_{j=1}^{s} q_{\alpha_j}^{+\binom{n_j}{2}}\Bigg)\Bigg(\prod_{j<k} q_{n_j\alpha_j,\, n_k\alpha_k}^{+1/2}\Bigg)\Bigg(\prod_{j=1}^{s} \big(m_{\alpha_j}^+\big)^{n_j}\Bigg) \check{E}_{\alpha_1}^{\check{*}(n_1)} \,\check{*}\, \cdots \,\check{*}\, \check{E}_{\alpha_s}^{\check{*}(n_s)} \\
F_\alpha^{*(n)} * F_\beta^{*(m)} \;&=\; q_\alpha^{-\binom{n}{2}} \big(q_{\alpha\beta}^{-1/2}\big)^{nm} q_\beta^{-\binom{m}{2}} \big(m_\alpha^-\big)^n \big(m_\beta^-\big)^m\, \check{F}_\alpha^{\check{*}(n)} \,\check{*}\, \check{F}_\beta^{\check{*}(m)} \\
F_{\alpha_1}^{*(n_1)} * \cdots * F_{\alpha_s}^{*(n_s)} \;&=\; \Bigg(\prod_{j=1}^{s} q_{\alpha_j}^{-\binom{n_j}{2}}\Bigg)\Bigg(\prod_{j<k} q_{n_j\alpha_j,\, n_k\alpha_k}^{-1/2}\Bigg)\Bigg(\prod_{j=1}^{s} \big(m_{\alpha_j}^-\big)^{n_j}\Bigg) \check{F}_{\alpha_1}^{\check{*}(n_1)} \,\check{*}\, \cdots \,\check{*}\, \check{F}_{\alpha_s}^{\check{*}(n_s)} \\
E_\alpha^{*(n)} * F_\beta^{*(m)} \;&=\; q_\alpha^{+\binom{n}{2}} q_\beta^{-\binom{m}{2}} \big(m_\alpha^+\big)^n \big(m_\beta^-\big)^m\, \check{E}_\alpha^{\check{*}(n)} \,\check{*}\, \check{F}_\beta^{\check{*}(m)} \\
F_\beta^{*(m)} * E_\alpha^{*(n)} \;&=\; q_\beta^{-\binom{m}{2}} q_\alpha^{+\binom{n}{2}} \big(m_\beta^-\big)^m \big(m_\beta^+\big)^n\, \check{F}_\beta^{\check{*}(m)} \,\check{*}\, \check{E}_\alpha^{\check{*}(n)} \\
X_i^{*z} * E_\beta^{*(n)} \;&=\; q_{\alpha_i\,\beta}^{+z\,n/2} \big(m_\beta^+\big)^n X_i^{\check{*}z} \,\check{*}\, \check{E}_\beta^{\check{*}(n)} \;\;,\;\; E_\beta^{*(n)} * X_i^{*z} \;=\; q_{\beta\,\alpha_i}^{+z\,n/2} \big(m_\beta^+\big)^n\, \check{E}_\beta^{\check{*}(n)} \,\check{*}\, X_i^{\check{*}z} \\
G_i^{*z} * E_\beta^{*(n)} \;&=\; \big(m_\beta^+\big)^n G_i^{\check{*}z} \,\check{*}\, \check{E}_\beta^{\check{*}(n)} \quad , \qquad E_\beta^{*(n)} * G_i^{*z} \;=\; \big(m_\beta^+\big)^n\, \check{E}_\beta^{\check{*}(n)} \,\check{*}\, G_i^{\check{*}z} \\
X_i^{*z} * F_\beta^{*(m)} \;&=\; q_{\alpha_i\,\beta}^{-z\,m/2} \big(m_\beta^-\big)^m X_i^{\check{*}z} \,\check{*}\, \check{F}_\beta^{\check{*}(m)} \;,\;\; F_\beta^{*(m)} * X_i^{*z} \;=\; q_{\beta\,\alpha_i}^{+z\,m/2} \big(m_\beta^-\big)^m \check{F}_\beta^{\check{*}(m)} \,\check{*}\, X_i^{\check{*}z} \\
G_i^{*z} * F_\beta^{*(m)} \;&=\; \big(m_\beta^-\big)^m G_i^{\check{*}z} \,\check{*}\, \check{F}_\beta^{\check{*}(m)} \quad , \qquad F_\beta^{*(m)} * G_i^{*z} \;=\; \big(m_\beta^-\big)^m\, \check{F}_\beta^{\check{*}(m)} \,\check{*}\, G_i^{\check{*}z} \\
E_\alpha^{*(n)} * \binom{X_j}{m}_{p_j}^{\!*} &= q_\alpha^{+\binom{n}{2}} \big(m_\alpha^+\big)^n \sum_{c=0}^{m} p_j^{-c(m-c)} \prod_{s=1}^{c} \frac{p_j^{1-s}\big(q_{\alpha\,\alpha_j}^{+1/2}\big)^n - 1}{p_j^s - 1}\; \check{E}_\alpha^{\check{*}(n)} \binom{X_j}{m-c}_{p_j}^{\!\check{*}} X_j^{\check{*}c} \\
\binom{X_j}{m}_{p_j}^{\!*} * E_\alpha^{*(n)} &= q_\alpha^{+\binom{n}{2}} \big(m_\alpha^+\big)^n \sum_{c=0}^{m} p_j^{-c(m-c)} \prod_{s=1}^{c} \frac{p_j^{1-s}\,\big(q_{\alpha_j\,\alpha}^{+1/2}\big)^n - 1}{p_j^s - 1} \binom{X_j}{m-c}_{p_j}^{\!\check{*}} X_j^{\check{*}c} \check{E}_\alpha^{\check{*}(n)} \\
E_\alpha^{*(n)} * \binom{G_j}{m}_{q_{jj}}^{\!*} \;&=\; q_\alpha^{+\binom{n}{2}} \big(m_\alpha^+\big)^n\, E_\alpha^{\check{*}(n)} \,\check{*}\, \binom{G_j}{m}_{q_{jj}}^{\!\check{*}} \\
\binom{G_j}{m}_{q_{jj}}^{\!*} * E_\alpha^{*(n)} \;&=\; q_\alpha^{+\binom{n}{2}} \big(m_\alpha^+\big)^n \binom{G_j}{m}_{q_{jj}}^{\!\check{*}} \,\check{*}\, E_\alpha^{\check{*}(n)}
\end{aligned}
$$

$$
\begin{aligned}
F_\alpha^{\,*(n)} {}_* \binom{X_j}{m}_{p_j}^{\!*} &= q_\alpha^{-\binom{n}{2}} \big(m_\alpha^-\big)^n \sum_{c=0}^{m} p_j^{-c(m-c)} \prod_{s=1}^{c} \frac{p_j^{1-s} \big(q_{\alpha\,\alpha_j}^{-1/2}\big)^n - 1}{p_j^s - 1} \check{F}_\alpha^{\,\check{:}\,(n)} \binom{X_j}{m-c}_{p_j}^{\!\check{:}} X_j^{\,\check{:}\,c} \\
\binom{X_j}{m}_{p_j}^{\!*} {}_* F_\alpha^{\,*(n)} &= q_\alpha^{-\binom{n}{2}} \big(m_\alpha^-\big)^n \sum_{c=0}^{m} p_j^{-c(m-c)} \prod_{s=1}^{c} \frac{p_j^{1-s} \big(q_{\alpha_j\,\alpha}^{-1/2}\big)^n - 1}{p_j^s - 1} \binom{X_j}{m-c}_{p_j}^{\!\check{:}} X_j^{\,\check{:}\,c} \, \check{F}_\alpha^{\,\check{:}\,(n)}
\end{aligned}
$$

$$
\begin{aligned}
F_\alpha^{\,*(n)} \, {}_* \binom{G_j}{m}_{q_{jj}}^{\!*} \;&=\; q_\alpha^{-\binom{n}{2}} \big(m_\alpha^-\big)^n \, F_\alpha^{\,\check{:}\,(n)} \,\check{:}\, \binom{G_j}{m}_{q_{jj}}^{\!\check{:}} \\
\binom{G_j}{m}_{q_{jj}}^{\!*} {}_* \, F_\alpha^{\,*(n)} \;&=\; q_\alpha^{-\binom{n}{2}} \big(m_\alpha^-\big)^n \binom{G_j}{m}_{q_{jj}}^{\!\check{:}} \,\check{:}\, F_\alpha^{\,\check{:}\,(n)}
\end{aligned}
$$

5. Integral forms of MpQG's

The main purpose of the present section is to introduce integral forms of our MpQG's; in particular, we shall also provide suitable PBW-like theorems for them.

5.1. **Preliminaries on integral forms.**

In this subsection we fix the ground for our discussion of integral forms of MpQG's.

5.1.1. Integral forms. Let S be any ring, and M any S–module. If R is any subring of S, we call *R–integral form* (or "integral form over R") of M any R–submodule M_R of M whose scalar extension from R to S is M, i.e. $M_R \otimes_R S = M$. When M has some richer structure (than the S–module one) by "R–integral form" we mean an R–integral form that in addition respects the additional structure; in other words, the definition is like above but one has to replace the words "module" and "submodule" with the words referring to the additional, richer structure. For instance, if H is a *Hopf algebra* over S by "R–integral form" of it we mean any *Hopf subalgebra* H_R over R such that $S \otimes_R H_R = H$.

5.1.2. The ground ring. The integral forms of our MpQG's will be defined over a suitable ground ring. To define it, we begin fixing a multiparameter matrix $\mathbf{q}$ of Cartan type with entries in the field $\Bbbk$, assuming again that the Cartan matrix is indecomposable. Starting from $\mathbf{q}$, we fix in $\Bbbk$ an element $q_{j_0} \in \Bbbk^\times$, now denoted by $q := q_{j_0}$, like in §2.3.2, and square roots $q_{ij}^{1/2}$ of all the q_{ij}'s, like in §2.3.2.

We denote by $\mathcal{F}_\mathbf{q}$ the subfield of $\Bbbk$ generated by all the $q_{ij}^{\pm 1}$'s ($i, j \in I$) along with $q^{\pm 1}$; moreover, we denote by $\mathcal{F}_\mathbf{q}^{\sqrt{\ }}$ the subfield of $\Bbbk$ generated by all the $q_{ij}^{\pm 1/2}$'s ($i, j \in I$) and $q^{\pm 1/2}$: then $\mathcal{F}_\mathbf{q}^{\sqrt{\ }}$ is a field extension of $\mathcal{F}_\mathbf{q}$, that contains also all the square roots $q_i^{\pm 1/2}$'s and $q_\beta^{\pm 1/2}$'s ($\beta \in \Phi^+$), for all the q_i's and q_β's defined at the beginning of §3.2. As ground ring for our integral forms, we fix the subring $\mathcal{R}_\mathbf{q}$ of $\Bbbk$ generated by all the $q_{ij}^{\pm 1}$'s (for all $i, j \in I$) and $q^{\pm 1}$; moreover, we denote by $\mathcal{R}_\mathbf{q}^{\sqrt{\ }}$ the subring of $\Bbbk$ generated by all the $q_{ij}^{\pm 1/2}$'s ($i, j \in I$) and $q^{\pm 1/2}$: this is a ring extension of $\mathcal{R}_\mathbf{q}$, that contains all the square roots $q_i^{\pm 1/2}$'s and $q_\beta^{\pm 1/2}$'s ($\beta \in \Phi^+$). The field of fractions of $\mathcal{R}_\mathbf{q}$ is just $\mathcal{F}_\mathbf{q}$, and similarly that of $\mathcal{R}_\mathbf{q}^{\sqrt{\ }}$ is just $\mathcal{F}_\mathbf{q}^{\sqrt{\ }}$.

When $\mathbf{q}$ is of integral type we have that $\mathcal{R}_\mathbf{q}$ and $\mathcal{F}_\mathbf{q}$ are generated (as a ring and as a field, respectively) by $q^{\pm 1}$ alone, while $\mathcal{R}_\mathbf{q}^{\sqrt{\ }}$ and $\mathcal{F}_\mathbf{q}^{\sqrt{\ }}$ are generated by $q^{\pm 1/2}$.

Finally, if we consider MpQG's with larger tori, then we take a ground field $\mathcal{F}_{\mathbf{q}_c}$ and a ground ring $\mathcal{R}_{\mathbf{q}_c}$ defined like $\mathcal{F}_\mathbf{q}$ and $\mathcal{R}_\mathbf{q}$ but replacing the q_{ij}^{-1}'s with the $q_{ij}^{\pm 1/c}$'s and $q^{\pm 1}$ by $q^{\pm 1/c}$, with $c_\pm := \big|det(C_\pm)\big|$ and $c := c_+ \, c_-$ (cf. §3.3.3, §3.3.4).

5.2. **Integral forms of "restricted" type.**

Following §3, we consider the multiparameter quantum group $U_{\mathbf{q}}(\mathfrak{g})$ associated with $\mathbf{q}$, defined over $\Bbbk$; also, for the special value of $q \in \Bbbk$ fixed above (depending on $\mathbf{q}$), we pick the MpQG of "canonical type" $U_{\check{\mathbf{q}}}(\mathfrak{g})$ as in Remark 3.1.2. Moreover, for each $\beta \in \Phi^+$ we consider quantum root vectors E_β and F_β — within $U_{\mathbf{q}}(\mathfrak{g})$ and within $U_{\check{\mathbf{q}}}(\mathfrak{g})$ — as in §4.1.

Lusztig's quantum groups of "restricted type" were introduced (cf. [Lu]) as special integral forms of his uniparameter quantum group — which is "almost" $U_{\check{\mathbf{q}}}(\mathfrak{g})$ — defined in terms of the so-called "q–binomial coefficients" and "q–divided powers". We shall now perform a similar construction in the multiparameter case.

5.2.1. ***q*–binomial coefficients and their arithmetic.** Let p be any formal indeterminate, $m \in \mathbb{N}$ and $J := \{1, 2, \dots, m\}$. We consider the two algebras

$$\boldsymbol{E}_m := \mathbb{Q}(p)\big[\{X_i^{\pm 1}\}_{i \in J}\big] \quad , \qquad \mathcal{E}_m := \mathbb{Q}(p)\big[\{\chi_i^{\pm 1}\}_{i \in J}\big]$$

of Laurent polynomials in the set of indeterminates $\{X_i^{\pm 1}\}_{i \in J}$ and $\{\chi_i^{\pm 1}\}_{i \in J}$ respectively on the field $\mathbb{Q}(p)$ of rational functions in p with coefficients in $\mathbb{Q}$. Both these bear unique Hopf algebra structures — over $\mathbb{Q}(p)$ — for which the $X_i^{\pm 1}$'s and the $\chi_i^{\pm 1}$'s are group-like, i.e. $\Delta\big(X_i^{\pm 1}\big) = X_i^{\pm 1} \otimes X_i^{\pm 1}$, $\epsilon\big(X_i^{\pm 1}\big) = 1$, $S\big(X_i^{\pm 1}\big) = X_i^{\mp 1} = 1$ for $\boldsymbol{E}_m$ and $\Delta\big(\chi_i^{\pm 1}\big) = \chi_i^{\pm 1} \otimes \chi_i^{\pm 1}$, $\epsilon\big(\chi_i^{\pm 1}\big) = 1$, $S\big(\chi_i^{\pm 1}\big) = \chi_i^{\mp 1} = 1$ for $\mathcal{E}_m$.

The following result lists some properties of p–binomial coefficients (cf. Definition 4.4.1*(a)*), taken from [DL, §3] (anyway, everything comes easily by induction):

Lemma 5.2.2. *Let $\mathcal{A}$ be any algebra over a field $\mathbb{F}$, and let $p \in \mathbb{F}$ be not a root of unity. Let $X, Y, M^{\pm 1} \in \mathcal{A}$ with $XY = YX$. Then for $t, s \in \mathbb{N}$, $c \in \mathbb{Z}$ we have*

$$\binom{XY\,;\,c}{t}_{\!p} = \sum_{s=0}^{t} p^{(s-c_y)(s-t)} \binom{X\,;\,c_X}{t-s}_{\!p} Y^{t-s} \binom{Y\,;\,c_Y}{s}_{\!p} \qquad \forall\; c_X + c_Y = c$$

$$M\,M^{-1} = 1 = M^{-1}\,M \quad , \qquad \binom{M\,;c}{0}_{\!p} = 1 \quad , \qquad (p-1)\binom{M\,;0}{1}_{\!p} = M - 1$$

$$M^{\pm 1}\binom{M\,;c}{t}_{\!p} = \binom{M\,;c}{t}_{\!p} M^{\pm 1} \,, \qquad \binom{M\,;c}{t}_{\!p}\binom{M\,;c-t}{s}_{\!p} = \binom{t+s}{t}_{\!p}\binom{M\,;c}{t+s}_{\!p}$$

$$\binom{M\,;c+1}{t}_{\!p} - p^{t}\binom{M\,;c}{t}_{\!p} = \binom{M\,;c}{t-1}_{\!p} \qquad \forall\; t \geq 1$$

$$\binom{M\,;c+1}{t}_{\!p} - \binom{M\,;c}{t}_{\!p} = p^{c-t+1}\,M\binom{M\,;c}{t-1}_{\!p} \qquad \forall\, t \geq 1$$

$$\binom{M\,;c}{t}_{\!p} = \sum_{s \geq 0}^{s \leq c,t} p^{(c-s)(t-s)} \binom{c}{s}_{\!p}\binom{M\,;0}{t-s}_{\!p} \qquad \forall\; c \geq 0$$

$$\binom{M\,;-c}{t}_{\!p} = \sum_{s=0}^{t} (-1)^s p^{-t(c+s)+\binom{s+1}{2}} \binom{s+c-1}{s}_{\!p}\binom{M\,;0}{t-s}_{\!p} \qquad \forall\; c \geq 1$$

If in addition $\mathcal{A}$ is a Hopf algebra and $M^{\pm 1}$ is group-like, then

$$\Delta\left(\binom{M\,;\,c}{t}_{\!p}\right) = \sum_{r+s=t} p^{-r(s-c_2)} \binom{M\,;\,c_1}{r}_{\!p} \otimes M^r \binom{M\,;\,c_2}{s}_{\!p} \qquad \forall\; c_1 + c_2 = c$$

$$\epsilon\left(\binom{M\,;\,c}{t}_{\!p}\right) = \binom{c}{t}_{\!p} \,, \quad \mathcal{S}\left(\binom{M\,;\,c}{t}_{\!p}\right) = (-1)^t p^{ct-\binom{t}{2}} M^{-t} \binom{M\,;\,t-c-1}{t}_{\!p}$$

$$\Delta\big(M^{\pm 1}\big) = M^{\pm 1} \otimes M^{\pm 1} \quad , \qquad \epsilon\big(M^{\pm 1}\big) = 1 \quad , \qquad \mathcal{S}\big(M^{\pm 1}\big) = M^{\mp 1}$$

Inside the $\mathbb{Q}(p)$–vector spaces $\boldsymbol{E}_m$ and $\mathcal{E}_m$ we consider the $\mathbb{Z}\big[p,p^{-1}\big]$–integral form of Laurent polynomials with coefficients in $\mathbb{Z}\big[p,p^{-1}\big]$, namely

$$\boldsymbol{E}_{m,\mathbb{Z}} \;:=\; \mathbb{Z}\big[p,p^{-1}\big]\big[\big\{X_i^{\pm 1}\big\}_{i\in J}\big] \quad , \qquad \mathcal{E}_{m,\mathbb{Z}} \;:=\; \mathbb{Z}\big[p,p^{-1}\big]\Big[\big\{\chi_i^{\pm 1}\big\}_{i\in J}\Big]$$

which in fact are both Hopf subalgebras (of $\boldsymbol{E}_m$ and $\mathcal{E}_m$) over $\mathbb{Z}\big[p,p^{-1}\big]$.

Fix some $d_i \in \mathbb{Z}\setminus\{0\}$ and powers $p_i := p^{d_i}$ for each $i \in J$. Then a unique $\mathbb{Q}(p)$–bilinear pairing $\big\langle\ ,\ \big\rangle : \boldsymbol{E}_m \times \mathcal{E}_m \longrightarrow \mathbb{Q}(p)$ exists, given by $\big\langle X_i^{z_i}, \chi_j^{\zeta_j}\big\rangle := p_i^{\delta_{ij} z_i \zeta_j}$ (for all $z_i, \zeta_j \in \mathbb{Z}$ and $i,j \in J$). By restriction, this clearly yields a similar $\mathbb{Z}\big[p,p^{-1}\big]$–valued pairing between the $\mathbb{Z}\big[p,p^{-1}\big]$–integral forms $\boldsymbol{E}_{m,\mathbb{Z}}$ and $\mathcal{E}_{m,\mathbb{Z}}$; indeed, this is even a Hopf pairing (cf. Definition 2.1.2). Finally, define

$$\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ \;:=\; \Big\{\, f \in \boldsymbol{E}_m \,\Big|\, \big\langle f\,,\mathcal{E}_{m,\mathbb{Z}}\big\rangle \subseteq \mathbb{Z}\big[p,p^{-1}\big] \,\Big\}.$$

It follows from definitions and $\displaystyle\binom{n}{k}_{p_i} \in \mathbb{Z}\big[p,p^{-1}\big]$ $(n,k \in \mathbb{N})$, cf. 2.3.8, that

$$X_i^{\pm 1}\,,\ \binom{X_i^{\pm 1}\,;\,c}{n}_{p_i} \in\ \big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ \qquad\qquad \forall\ i \in J\,,\ c \in \mathbb{Z}\,,\ n \in \mathbb{N} \tag{5.1}$$

Now set $\binom{X_J\,;\,0}{\underline{n}}_{\underline{p}} X_J^{-\lfloor \underline{n}/2\rfloor} := \prod_{i\in J} \binom{X_i\,;\,0}{n}_{p_i} X_i^{-\lfloor n_i/2\rfloor}$ for every $\underline{n} := \big(n_i\big)_{i\in J} \in \mathbb{N}^J$, where $\big\lfloor n_i/2\big\rfloor$ is the greatest natural number less or equal than $n/2$. Then we have

Proposition 5.2.3. [DL, Theorem 3.1]

(a) $\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$ *is a free* $\mathbb{Z}\big[p,p^{-1}\big]$*–module, with basis*

$$\mathbb{B}_m \;:=\; \left\{ \binom{X_J}{\underline{n}}_{\underline{p}} X_J^{-\lfloor \underline{n}/2\rfloor} \;\middle|\; \underline{n} \in \mathbb{N}^J \right\}$$

(b) $\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$ *is the* $\mathbb{Z}\big[p,p^{-1}\big]$*–subalgebra of* $\boldsymbol{E}_m$ *generated by all the* $\binom{X_i\,;\,c}{n}_{p_i}$*'s and the* X_i^{-1}*'s, or by all the* $\binom{X_i^{-1}\,;\,c}{n}_{p_i}$*'s and the* X_i*'s. In fact, it can be presented as the Hopf* $\mathbb{Z}\big[p,p^{-1}\big]$*–algebra with generators* $\binom{X_i\,;\,c}{n}_{p_i}$, $X_i^{\pm 1}$ *— for all* $i \in I$, $n \in \mathbb{N}$, $c \in \mathbb{Z}$ *— and relations stating that all generators commute with each other plus all relations like in Lemma 5.2.2 but with* $\binom{X_i\,;\,c}{n}_{p_i}$, $X_i^{\pm 1}$ *and* p_i *replacing* $\binom{M\,;\,c}{n}_{p}$, $M^{\pm 1}$ *and* p *respectively, for all* $i \in I$ *; the Hopf structure then is given again by the same formulas as in Lemma 5.2.2 now applied to the given generators.*

Proof. Due to (5.1), the $\mathbb{Z}\big[p,p^{-1}\big]$–subalgebra of $\boldsymbol{E}_m$ generated by all the $\binom{X_i\,;\,c}{n}_{p_i}$'s and the X_i^{-1}'s is contained in $\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$ — and similarly if we replace each $X_i^{\pm 1}$ with its inverse $X_i^{\mp 1}$. Thus to prove the whole claim it is enough to show that $\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$ admits $\mathbb{B}_m$ as $\mathbb{Z}\big[p,p^{-1}\big]$–basis: indeed, we already have that the $\mathbb{Z}\big[p,p^{-1}\big]$–span of $\mathbb{B}_m$ is contained in $\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$, so to prove *(a)* it is enough to show that any $f \in \big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$ can be written uniquely as a $\mathbb{Z}\big[p,p^{-1}\big]$–linear combination of elements in $\mathbb{B}_m$.

To begin with, $\boldsymbol{E}_m$ over $\mathbb{Q}(p)$ has basis the set $\Big\{ X_J^{\underline{z}} := \prod\limits_{i\in J} X_i^{z_i} \,\Big|\, \underline{z} := \big(z_i\big)_{i\in J} \in \mathbb{Z}^J \Big\}$ and from this one easily sees that the set $\mathbb{B}_m := \Big\{ \binom{X_J}{\underline{n}}_{\underline{p}} X_J^{-\lfloor \underline{n}/2\rfloor} \,\Big|\, \underline{n} \in \mathbb{N}^J \Big\}$ is a $\mathbb{Q}(p)$–basis too, which is contained in $\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$ by (5.1). Now consider the monomials $\chi_{\underline{\nu}} := \prod_{j\in J} \chi_j^{\nu_j}$

(with $\underline{\nu} := \big(\nu_j\big)_{j \in J} \in \mathbb{N}^J$) in the χ_i's. By construction one has

$$\left\langle \binom{X_J}{\underline{n}}_{\underline{p}} X_J^{-\lfloor \underline{n}/2 \rfloor} \, , \, \chi_{\underline{\nu}} \right\rangle \;=\; \prod_{i \in J} \binom{\nu_i}{n_i}_{p_i} p_i^{-\nu_i \lfloor n_i/2 \rfloor} \qquad \forall\; \underline{n} \, , \, \underline{\nu} \, \in \, \mathbb{N}^J \tag{5.2}$$

Let $\preceq$ be the order relation in $\mathbb{N}^J$ given by the product of the standard order in $\mathbb{N}$, so $\underline{n} \preceq \underline{\nu} \iff n_i \leq \nu_i \;\; \forall \, i \in \mathfrak{J}$. As $\binom{\nu_i}{n_i}_{p_i} \neq 0 \iff n_i \leq \nu_i$, by (5.2) one has

$$\left\langle \binom{X_J}{\underline{n}}_{\underline{p}} X_J^{-\lfloor \underline{n}/2 \rfloor} \, , \, \chi_{\underline{\nu}} \right\rangle \;\neq\; 0 \qquad \Longleftrightarrow \qquad \underline{n} \leq \underline{\nu} \qquad \big(\underline{n} \, , \, \underline{\nu} \, \in \, \mathbb{N}^J \big) \tag{5.3}$$

Now pick an $f \in \big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ \setminus \{0\}$, and expand it (uniquely) as a $\mathbb{Q}(p)$–linear combination of elements in $\mathbb{B}_m$, say $f = \sum_{s=1}^N c_s \binom{X_J}{\underline{n}^{(s)}}_{\underline{p}} X^{-\lfloor \underline{n}^{(s)}/2 \rfloor}$ for some $c_s \in \mathbb{Q}(p) \setminus \{0\}$. Choose any index $\sigma \in \{1, \dots, N\}$ such that $\underline{n}_\sigma$ is *minimal* in $\big\{ \underline{n}^{(1)}, \dots, \underline{n}^{(N)} \big\}$: then by (5.3) above and by $\binom{n}{n}_{p_i} = 1$ we get

$$\Big\langle f \, , \underline{\chi}_{\underline{n}^{(\sigma)}} \Big\rangle \;=\; \sum_{s=1}^N c_s \left\langle \binom{X_J}{\underline{n}^{(s)}}_{\underline{p}} X_J^{-\lfloor \underline{n}^{(s)}/2 \rfloor} \, , \, \underline{\chi}_{\underline{n}^{(\sigma)}} \right\rangle \;=\; c_\sigma \prod_{i \in J} p_i^{-\underline{n}_i^{(\sigma)} \lfloor \underline{n}_i^{(\sigma)}/2 \rfloor}$$

so that $\Big\langle f \, , \underline{\chi}_{\underline{n}^{(\sigma)}} \Big\rangle \in \mathbb{Z}\big[p, p^{-1}\big]$ — because $f \in \big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ \setminus \{0\}$ — implies at once $c_\sigma \in \mathbb{Z}\big[p, p^{-1}\big]$. By induction on N, one then concludes that *all* coefficients $c_s \, (s = 1, \dots, N)$ in the expansion of f actually lie in $\mathbb{Z}\big[p, p^{-1}\big]$, q.e.d.

Finally, the presentation mentioned in claim *(b)* follows from [DL, §3.4]. □

As a direct consequence of the above Lemma, we have the following:

Proposition 5.2.4. $\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$ *is a Hopf* $\mathbb{Z}\big[p, p^{-1}\big]$*–subalgebra of* $\boldsymbol{E}_m$ *. Therefore, the former is a* $\mathbb{Z}\big[p, p^{-1}\big]$*–integral form of the latter.*

For later use, we finish the present discussion with another result that gives the dual, somehow, of what we found for $\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$: it concerns the "bidual" space

$$\Big(\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ\Big)^\circ \;:=\; \Big\{\, t \in \mathcal{E}_m \,\Big|\, \big\langle \big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ, t \,\big\rangle \subseteq \mathbb{Z}\big[p, p^{-1}\big] \,\Big\}$$

Proposition 5.2.5. *The "bidual space"* $\Big(\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ\Big)^\circ$ *coincides with* $\mathcal{E}_{m,\mathbb{Z}}$ *.*

Proof. By definition $\Big(\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ\Big)^\circ \supseteq \mathcal{E}_{m,\mathbb{Z}}$, we have to prove the converse inclusion.

Let $t \in \Big(\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ\Big)^\circ$ and expand it with respect to the $\mathbb{Q}(p)$–basis of $\mathcal{E}_m$ made of the Laurent monomials $\chi_{\underline{\zeta}} := \prod_{j \in J} \chi_j^{\zeta_j}$ (with $\underline{\zeta} := \big(\zeta_j\big)_{j \in J} \in \mathbb{Z}^J$) in the χ_j's. This means writing t as $t = \sum_{\underline{\zeta} \in \mathbb{Z}^J} c_{\underline{\zeta}} \, \chi_{\underline{\zeta}}$ for suitable $c_{\underline{\zeta}} \in \mathbb{Q}(p)$, almost all being zero: we denote by $n(t) \in \mathbb{N}$ the number of all such non-zero coefficients. We must show that $t \in \mathcal{E}_{m,\mathbb{Z}}$, i.e. all the $c_{\underline{\zeta}}$'s belong to $\mathbb{Z}\big[p, p^{-1}\big]$; we do it by induction on $n(t)$.

As a first step, we assume that for all $\underline{\zeta} := \big(\zeta\big)_{j \in J}$ such that $c_{\underline{\zeta}} \neq 0$ we have $\zeta_j \geq 0$ for all $j \in J$. Then choose a $\underline{\zeta}^\uparrow \in \mathbb{Z}^J$ such that $c_{\underline{\zeta}^\uparrow} \neq 0$ and $\underline{\zeta}^\uparrow$ is *maximal* for that property with respect to the standard product order on $\mathbb{Z}^J$; in other words, there exists no $\underline{\zeta} \neq \underline{\zeta}^\uparrow$ such that $c_{\underline{\zeta}} \neq 0$ and $\underline{\zeta}_j \geq \underline{\zeta}_j^\uparrow$ for all $j \in J$. Then we have

$$\mathbb{Z}\big[p, p^{-1}\big] \;\ni\; \left\langle \binom{X_J}{\underline{\zeta}^\uparrow}_{\underline{p}} , \, t \right\rangle \;=\; \sum_{\underline{\zeta} \in \mathbb{Z}^J} c_{\underline{\zeta}} \left\langle \binom{X_J}{\underline{\zeta}^\uparrow}_{\underline{p}} , \, \chi_{\underline{\zeta}} \right\rangle \;=\; \sum_{\underline{\zeta} \in \mathbb{Z}^J} c_{\underline{\zeta}} \prod_{j \in J} \binom{\zeta_j}{\zeta_j^\uparrow}_{p}$$

by the maximality of $\underline{\zeta}^\uparrow$ — and the properties of q–binomial coefficients — we have $\binom{\zeta_j}{\zeta_j^\uparrow}_p = \delta_{\underline{\zeta}, \underline{\zeta}^\uparrow}$, thus the above eventually gives $c_{\underline{\zeta}^\uparrow} \in \mathbb{Z}\big[p, p^{-1}\big]$, q.e.d. Now look at $t' := t - c_{\underline{\zeta}^\uparrow} \chi_{\underline{\zeta}^\uparrow}$: by construction we have $n\big(t'\big) = n(t) - 1 \lneqq n(t)$, hence we may assume by induction that $t' \in \mathcal{E}_{m,\mathbb{Z}}$. Then $t = t' + c_{\underline{\zeta}^\uparrow} \chi_{\underline{\zeta}^\uparrow} \in \mathcal{E}_{m,\mathbb{Z}}$ too, q.e.d.

At last, notice that $\Big(\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ\Big)^\circ$ is a $\mathbb{Z}\big[p, p^{-1}\big]$–subalgebra: in fact, this follows at once from the fact that $\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ$ is a $\mathbb{Z}\big[p, p^{-1}\big]$–coalgebra — and we have the perfect Hopf pairing $\langle\ ,\ \rangle$ between $\mathcal{E}_m$ and $\boldsymbol{E}_m$. As clearly all the $\chi_{\underline{\zeta}}$'s do belong to $\Big(\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ\Big)^\circ$ and are invertible in it, it follows that for any $t \in \mathcal{E}_m$ and for any $\underline{\zeta}' \in \mathbb{Z}^J$ one has $t \in \Big(\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ\Big)^\circ$ if and only if $t\,\chi_{\underline{\zeta}'} \in \Big(\big(\mathcal{E}_{m,\mathbb{Z}}\big)^\circ\Big)^\circ$. Now, choosing a proper $\underline{\zeta}' \in \mathbb{Z}^J$ we can get $t\,\chi_{\underline{\zeta}'}$ such that in its $\mathbb{Q}(p)$–linear expansion in the $\chi_{\underline{\zeta}}$'s, say $t\,\chi_{\underline{\zeta}'} = \sum_{\underline{\zeta} \in \mathbb{Z}^J} c_{\underline{\zeta}}\, \chi_{\underline{\zeta}}$, for all the $\underline{\zeta} = \big(\zeta_j\big)_{j \in J}$'s such that $c_{\underline{\zeta}} \neq 0$ we have $\zeta_j \geq 0$ for all $j \in J$. But then $t' := t\,\chi_{\underline{\zeta}'}$ is of the type we considered above, for which we did prove that $t' := t\,\chi_{\underline{\zeta}'} \in \mathcal{E}_{m,\mathbb{Z}}$; so the previous analysis gives $t \in \mathcal{E}_{m,\mathbb{Z}}$ too. □

5.2.6. The toral part of restricted MpQG's. The restricted integral form of a uniparameter quantum group $U_q(\mathfrak{g})$ was introduced by Lusztig as q–analogue of Chevalley's $\mathbb{Z}$–form of $U(\mathfrak{g})$: we consider here its modified version as in [DL], where specific changes were done in the choice of the toral part. The construction in [DL] immediately extends to $U_{\check{\mathbf{q}}}(\mathfrak{g})$, hence now we want to further extend it to the general case of any multiparameter $U_{\mathbf{q}}(\mathfrak{g})$; nevertheless, a (mild) restriction on $\mathbf{q}$ is necessary, in the following terms:

♦ — *From now on, all along the present section* <u>*we assume that* $\mathbf{q}$ *is of integral type*</u> *(as well as Cartan, as usual)*, say $\mathbf{q} = \big(q_{ij} = q^{b_{ij}}\big)_{i,j \in I}$ as in §2.3.2. *Therefore (cf. §5.1.2)* $\mathcal{R}_{\mathbf{q}}$ *, resp.* $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ *, is just the subring of* $\Bbbk$ *generated by* $q^{\pm 1}$ *, resp.* $q^{\pm 1/2}$ *, and* $\mathcal{F}_{\mathbf{q}}$ *, resp.* $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$ *, is the subfield of* $\Bbbk$ *generated by* $q^{\pm 1}$ *, resp.* $q^{\pm 1/2}$ *.*

In the following, whatever object we shall introduce that bear a structure of module over $\mathcal{R}_{\mathbf{q}}$, resp. over $\mathcal{F}_{\mathbf{q}}$, will also have its natural counterpart defined over $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$, resp. over $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$, that is also a scalar extension of the previous one.

In the following, $U_{\mathbf{q}}(\mathfrak{g})$ will be the MpQG associated with $\mathbf{q}$ as in Definition 3.1.1. Inside it — more precisely, inside its toral part — we want to apply the construction presented in §5.2.1, for suitable choices of the X_i's, the χ_i's and the p_i's.

Recall that $I := \{1, \dots, \theta\}$. Define $G_i^{\pm 1} := K_i^{\pm 1} L_i^{\mp 1} \big(\in U_{\mathbf{q}}(\mathfrak{h} \oplus \mathfrak{h}) := U_{\mathbf{q}}^0 \big)$ for all $i \in I$, and consider inside $U_{\mathbf{q}}^0$ the $\mathcal{F}_{\mathbf{q}}$–subalgebra generated by the $K_i^{\pm 1}$'s and the $G_i^{\pm 1}$'s, namely $\boldsymbol{E}_{2\theta} := \mathcal{F}_{\mathbf{q}}\big[\big\{K_i^{\pm 1}, G_i^{\pm 1}\big\}_{i,j \in I}\big]$; note also that taking the $L_i^{\pm 1}$'s as generators instead of the $G_i^{\pm 1}$'s will give the same algebra. As a matter of fact, since the $K_i^{\pm 1}$'s and the $G_i^{\pm 1}$'s are group-like, this $\boldsymbol{E}_{2\theta}$ is indeed a Hopf $\mathcal{F}_{\mathbf{q}}$–subalgebra of $U_{\mathbf{q}}^0$.

In the dual space $\big(U_{\mathbf{q}}^0\big)^*$ we consider the $\mathcal{F}_{\mathbf{q}}$–algebra morphisms $\dot{\kappa}_i^{\pm 1}$ and $\gamma_i^{\pm 1}$ — for $i \in I$ — uniquely defined by

$$\Big\langle K_i^{z_i}, \dot{\kappa}_j^{\zeta_j} \Big\rangle := q^{\delta_{ij} z_i \zeta_j} \ , \quad \Big\langle G_i^{z_i}, \dot{\kappa}_j^{\zeta_j} \Big\rangle := 1 \ , \quad \Big\langle K_i^{z_i}, \gamma_j^{\zeta_j} \Big\rangle := 1 \ , \quad \Big\langle G_i^{z_i}, \gamma_j^{\zeta_j} \Big\rangle := q_{ii}^{\delta_{ij} z_i \zeta_j}$$

(cf. §5.1.2) for all $z_i, \zeta_j \in \mathbb{Z}$ and $i, j \in J$. Setting also $\dot{\mathcal{E}}_{2\theta} := \mathcal{F}_{\mathbf{q}}\big[\big\{\dot{\kappa}_i^{\pm 1}, \gamma_i^{\pm 1}\big\}_{i,j \in I}\big]$ for the subalgebra in $\big(U_{\mathbf{q}}^0\big)^*$ generated by the $\dot{\kappa}_i^{\pm 1}$'s and the $\gamma_i^{\pm 1}$'s, these formulas yield a non-degenerate $\mathcal{F}_{\mathbf{q}}$–pairing between $\boldsymbol{E}_{2\theta}$ and $\dot{\mathcal{E}}_{2\theta}$: in fact, the latter is a Hopf algebra (the $\dot{\kappa}_i^{\pm 1}$'s and $\gamma_i^{\pm 1}$'s being group-like), so this is actually a *Hopf* pairing.

Now $\boldsymbol{E}_{2\theta}$ and $\dot{\mathcal{E}}_{2\theta}$, paired as explained, can play the role of $\boldsymbol{E}_m$ and $\mathcal{E}_m$ in §5.2.1 above, so we apply to them the construction presented there. Taking their corresponding

$\mathcal{R}_{\mathbf{q}}$–integral form of Laurent polynomials with coefficients in $\mathcal{R}_{\mathbf{q}}$, namely

$$\boldsymbol{E}_{2\theta,\mathcal{R}_{\mathbf{q}}} := \mathcal{R}_{\mathbf{q}}\big[\{K_i^{\pm 1}, G_i^{\pm 1}\}_{i\in I}\big] \qquad \text{and} \qquad \dot{\mathcal{E}}_{2\theta,\mathcal{R}_{\mathbf{q}}} := \mathcal{R}_{\mathbf{q}}\big[\{\dot{\kappa}_i^{\pm 1}, \gamma_i^{\pm 1}\}_{i\in I}\big]$$

we have Hopf subalgebras over $\mathcal{R}_{\mathbf{q}}$, and the previously given pairing restricts to a non-degenerate $\mathcal{R}_{\mathbf{q}}$–valued pairing among these two integral forms.

Now assume in addition that the multiparameter $\mathbf{q}$ *is of strongly integral type, say* $\mathbf{q} = \big(q_{ij} = q^{d_i t_{ij}^+} = q^{d_j t_{ij}^-}\big)_{i,j\in I}$. Then besides the previous construction we can perform a second, parallel one, as follows.

Inside $\dot{\mathcal{E}}_{2\theta,\mathcal{R}_{\mathbf{q}}}$ we consider now $\kappa_i^{\pm 1} := \dot{\kappa}_i^{\pm d_i}$ (for $i \in I$), for which we have

$$\Big\langle K_i^{z_i}, \kappa_j^{\zeta_j} \Big\rangle := q^{\delta_{ij} d_i z_i \zeta_j} \quad , \qquad \Big\langle G_i^{z_i}, \kappa_j^{\zeta_j} \Big\rangle := 1$$

for all $z_i, \zeta_j \in \mathbb{Z}$ and $i, j \in J$, and set $\mathcal{E}_{2\theta} := \mathcal{F}_{\mathbf{q}}\big[\{\kappa_i^{\pm 1}, \gamma_i^{\pm 1}\}_{i,j\in I}\big]$ for the subalgebra in $\dot{\mathcal{E}}_{2\theta,\mathcal{R}_{\mathbf{q}}}$ — hence in $\big(U_{\mathbf{q}}^0\big)^*$ too — generated by the $\kappa_i^{\pm 1}$'s and the $\gamma_i^{\pm 1}$'s. Then $\mathcal{E}_{2\theta}$ is in fact a Hopf $\mathcal{F}_{\mathbf{q}}$–algebra (the $\kappa_i^{\pm 1}$'s being group-like, like the $\gamma_i^{\pm 1}$'s), and the above formulas provides a non-degenerate Hopf pairing. Taking now

$$\boldsymbol{E}_{2\theta,\mathcal{R}_{\mathbf{q}}} := \mathcal{R}_{\mathbf{q}}\big[\{K_i^{\pm 1}, G_i^{\pm 1}\}_{i\in I}\big] \qquad \text{and} \qquad \mathcal{E}_{2\theta,\mathcal{R}_{\mathbf{q}}} := \mathcal{R}_{\mathbf{q}}\big[\{\kappa_i^{\pm 1}, \gamma_i^{\pm 1}\}_{i\in I}\big]$$

we have Hopf subalgebras over $\mathcal{R}_{\mathbf{q}}$, and a non-degenerate $\mathcal{R}_{\mathbf{q}}$–valued pairing between them provided by restriction of the previous one. Basing on all the above, we can now introduce the objects we are mainly interested into:

Definition 5.2.7.

(a) $\dot{\widehat{U}}_{\mathbf{q}}^{\,0} := \big(\dot{\mathcal{E}}_{2\theta,\mathcal{R}_{\mathbf{q}}}\big)^\circ = \Big\{ f \in \boldsymbol{E}_{2\theta} \,\Big|\, \big\langle f, \dot{\mathcal{E}}_{2\theta,\mathcal{R}_{\mathbf{q}}} \big\rangle \subseteq \mathcal{R}_{\mathbf{q}} \Big\}$ if $\mathbf{q}$ is (just) *integral*,

(b) $\widehat{U}_{\mathbf{q}}^{\,0} := \big(\mathcal{E}_{2\theta,\mathcal{R}_{\mathbf{q}}}\big)^\circ = \Big\{ f \in \boldsymbol{E}_{2\theta} \,\Big|\, \big\langle f, \mathcal{E}_{2\theta,\mathcal{R}_{\mathbf{q}}} \big\rangle \subseteq \mathcal{R}_{\mathbf{q}} \Big\}$ if $\mathbf{q}$ is *strongly integral*. $\diamondsuit$

By the analysis and results in §5.2.1, applied to the present situation, we have

Proposition 5.2.8.

(a) $\dot{\widehat{U}}_{\mathbf{q}}^{\,0}$ *is a Hopf* $\mathcal{R}_{\mathbf{q}}$*–subalgebra of* $\boldsymbol{E}_{2\theta}$, *generated by all the* $\binom{K_i\,;\,c}{k}_q$*'s, the* $K_i^{\pm 1}$*'s, the* $\binom{G_i^{-1}\,;\,c}{g}_{q_{ii}}$*'s and the* $G_i^{\pm 1}$*'s.*

(b) $\dot{\widehat{U}}_{\mathbf{q}}^{\,0}$ *is a free* $\mathcal{R}_{\mathbf{q}}$*–module, with basis — cf.* (4.2) *for notation —*

$$\left\{ \prod_{i=1}^{\theta} \binom{K_i}{k_i}_q K_i^{-\lfloor k_i/2 \rfloor} \prod_{i=1}^{\theta} \binom{G_i}{g_i}_{q_{ii}} G_i^{-\lfloor g_i/2 \rfloor} \;\middle|\; k_i, g_i \in \mathbb{N} \;\; \forall\, i = 1, \dots, \theta \right\}$$

(c) $\dot{\widehat{U}}_{\mathbf{q}}^{\,0}$ *is isomorphic to the Hopf* $\mathcal{R}_{\mathbf{q}}$*–algebra with generators* $\binom{K_i\,;\,c}{k_i}_q$, $K_i^{\pm 1}$, $\binom{G_i\,;\,c}{g_i}_{q_{ii}}$, $G_i^{\pm 1}$ *(for all* $i \in I$, $k_i, g_i \in \mathbb{N}$, $c \in \mathbb{Z}$*) and relations stating that all these generators commute with each other, plus all relations like in Lemma 5.2.2 but with* $\binom{K_i\,;\,c}{k_i}_q$, $K_i^{\pm 1}$, q *and* $\binom{G_i\,;\,c}{g_i}_{q_{ii}}$, $G_i^{\pm 1}$, q_{ii} *replacing* $\binom{M\,;\,c}{n}_p$, $M^{\pm 1}$ *and* p *respectively, for all* $i \in I$. *Accordingly, the Hopf structure of* $\dot{\widehat{U}}_{\mathbf{q}}^{\,0}$ *is also given in terms of generators by formulas as in Lemma 5.2.2 now applied to the given generators.*

(d)–(e)–(f) When $\mathbf{q}$ *is strongly integral, similar claims hold true for* $\widehat{U}_{\mathbf{q}}^{\,0}$, *up to replacing everywhere each generator* $\binom{K_i\,;\,c}{k_i}_q$ *and the parameter* q *with the corresponding generator* $\binom{K_i\,;\,c}{k_i}_{q_i}$ *and the parameter* $q_i = q^{d_i}$, *respectively.*

(g) $\dot{\widehat{U}}_{\mathbf{q}}^{\,0}$ *is a Hopf* $\mathcal{R}_{\mathbf{q}}$*–subalgebra of* $\widehat{U}_{\mathbf{q}}^{\,0}$.

One last important observation is in order:

Remark 5.2.9. Definitions imply that, beside all the generators $G_i^{\pm 1}$, $\binom{G_i\,;\,c}{g_i}_{q_{ii}}$, $K_i^{\pm 1}$ and $\binom{K_i\,;\,c}{k_i}_q$, the algebra $\dot{\widehat{U}}_{\mathbf{q}}^{\,0}$ contains all $L_i^{\pm 1}$'s and $\binom{L_i\,;\,c}{l_i}_q$'s — as they give values in $\mathcal{R}_{\mathbf{q}}$ when paired with $\mathcal{E}_{2\theta,\mathcal{R}_{\mathbf{q}}}$. This restores a perfect "symmetry" in the roles of the K_i's and the L_i's, which is not apparent in the very definition of $\dot{\widehat{U}}_{\mathbf{q}}^{\,0}$. Indeed, one can easily prove that $\dot{\widehat{U}}_{\mathbf{q}}^{\,0}$ can be also generated by generators built from "L" instead of "K"; and similarly for $\widehat{U}_{\mathbf{q}}^{\,0}$. So replacing "$K$" by "$L$" everywhere yields a twin statement of Proposition 5.2.8.

5.2.10. Restricted MpQG's. We are now ready to introduce our generalization to MpQG's of the notion of restricted integral form introduced by Lusztig for $U_q(\mathfrak{g})$ (and later modified in [DL]). We keep the restriction that $\mathbf{q}$ must be of *integral type*, say $\mathbf{q} = \big(q_{ij} = q^{b_{ij}}\big)_{i,j\in I}$ with $B = \big(b_{ij}\big)_{i,j\in I} \in M_\theta(\mathbb{Z})$, as in §2.3.2.

Again, hereafter $U_{\mathbf{q}}(\mathfrak{g})$ denotes the MpQG associated with $\mathbf{q}$ as in Definition 3.1.1.

We recall from Definition 4.4.1 *(b)* the notion of *q–divided powers*: given $i \in I$, $\alpha \in \Phi^+$, $X_i \in \big\{E_i\,, F_i\big\}$, $Y_\alpha \in \big\{E_\alpha\,, F_\alpha\big\}$ and $n \in \mathbb{N}$, we call *q–divided powers* the elements $X_i^{(n)} := X_i^n \Big/ (n)_{q_{ii}}!$ and $Y_\alpha^{(n)} := Y_\alpha^n \Big/ (n)_{q_{\alpha\alpha}}!$ in $U_{\mathbf{q}}(\mathfrak{g})$.

The following result, about commutation relations between quantum binomial coefficients and quantum divided powers, is proved by straightforward induction:

Lemma 5.2.11. *Let* $\mathbf{q} = \big(q_{ij} = q^{b_{ij}}\big)_{i,j\in I}$ *be of integral type.*

(a) For any $i \in I$, $m,n,h \in \mathbb{N}$, $c \in \mathbb{Z}$, $\kappa,\lambda \in Q$, $X,Y \in \big\{K_\kappa L_\lambda \,\big|\, \kappa,\lambda \in Q\big\}$ *and* $G_i^{\pm 1} := K_i^{\pm 1} L_i^{\mp 1}$, *we have*

$$\binom{K_\kappa\, L_\lambda\,;\,c}{h}_q F_j^{(n)} \;=\; F_j^{(n)} \binom{K_\kappa\, L_\lambda\,;\,c - n\,\big(B^{\,T}\cdot\kappa - B\cdot\lambda\big)_j}{h}_q$$

$$\binom{K_\kappa\, L_\lambda\,;\,c}{h}_q E_j^{(n)} \;=\; E_j^{(n)} \binom{K_\kappa\, L_\lambda\,;\,c + n\,\big(B^{\,T}\cdot\kappa - B\cdot\lambda\big)_j}{h}_q$$

$$\binom{G_i\,;\,c}{h}_{q_{ii}} F_j^{(n)} = F_j^{(n)} \binom{G_i\,;\,c - n a_{ij}}{h}_{q_{ii}}\,, \qquad \binom{G_i\,;\,c}{h}_{q_{ii}} E_j^{(n)} = E_j^{(n)} \binom{G_i\,;\,c + n a_{ij}}{h}_{q_{ii}}$$

$$E_i^{(m)}\, F_i^{(n)} \;=\; \textstyle\sum_{s=0}^{m\wedge n} F_i^{(n-s)}\, q_{ii}^{\,s} \displaystyle\binom{G_i\,;\,2\,s-m-n}{s}_{q_{ii}} L_i^{\,s}\, E_i^{(m-s)}$$

where $\big(B^{\,T}\cdot\kappa - B\cdot\lambda\big)_j = \sum_{i\in I}\big(b_{ij}\kappa_i - b_{ji}\lambda_i\big)$ *for* $\kappa = \sum_{i\in I}\kappa_i\,\alpha_i$, $\lambda = \sum_{i\in I}\lambda_i\,\alpha_i$.

Moreover, for the Hopf structure, on q–divided powers we have formulas

$$\Delta\Big(E_i^{(n)}\Big) = \sum_{s=0}^n E_i^{(n-s)} K_i^s \otimes E_i^{(s)}\,,\ \ \epsilon\Big(E_i^{(n)}\Big) = \delta_{n,0}\,,\ \ \mathcal{S}\Big(E_i^{(n)}\Big) = (-1)^n q_{ii}^{+\binom{n}{2}} K_i^{-n} E_i^{(n)}$$

$$\Delta\Big(F_i^{(n)}\Big) = \sum_{s=0}^n F_i^{(n-s)} \otimes F_i^{(s)} L_i^{n-s}\,,\ \ \epsilon\Big(F_i^{(n)}\Big) = \delta_{n,0}\,,\ \ \mathcal{S}\Big(F_i^{(n)}\Big) = (-1)^n q_{ii}^{-\binom{n}{2}} F_i^{(n)} L_i^{-n}$$

while for $K_i^{\pm 1}$, $G_i^{\pm 1}$ *and q–binomial coefficients* $\binom{K_i\,;\,c}{h}_q$, $\binom{G_i\,;\,c}{h}_{q_{ii}}$ *we have formulas like in Lemma 5.2.2, with M and p replaced by K_i and q or by G_i and q_{ii}.*

(b) In addition, if $\mathbf{q}$ is of strongly integral type, say $\mathbf{q} = \big(q_{ij} = q^{d_i t^+_{ij}} = q^{d_j t^-_{ij}}\big)_{i,j\in I}$, *then besides all the above formulas we also have*

$$\binom{K_i\,;\,c}{h}_{q_i} F_j^{(n)} \;=\; F_j^{(n)} \binom{K_i\,;\,c - n\,t^+_{ij}}{h}_{q_i}\,, \qquad \binom{K_i\,;\,c}{h}_{q_i} E_j^{(n)} \;=\; E_j^{(n)} \binom{K_i\,;\,c + n\,t^+_{ij}}{h}_{q_i}$$

$$\binom{L_i\,;\,c}{h}_{q_i} F_j^{(n)} \,=\, F_j^{(n)} \binom{L_i\,;\,c+n\,t_{ji}^-}{h}_{q_i} \;,\qquad \binom{L_i\,;\,c}{h}_{q_i} E_j^{(n)} \,=\, E_j^{(n)} \binom{L_i\,;\,c-n\,t_{ji}^-}{h}_{q_i}$$

and formulas for the Hopf structure on q–binomial coefficients $\binom{K_i\,;\,c}{n}_{q_i}$ and $\binom{L_i\,;\,c}{n}_{q_i}$ like in Lemma 5.2.2, with M and p replaced by K_i and q_i or by L_i and q_i .

We can now extend Lusztig's definition of "*restricted* quantum universal enveloping algebra". Indeed, a straightforward extension requires that $\mathbf{q}$ be strongly integral; nevertheless, we consider also a more general definition when $\mathbf{q}$ is just integral.

Definition 5.2.12. Let $U_{\mathbf{q}}(\mathfrak{g})$ be a MpQG over the field $\mathcal{F}_{\mathbf{q}}$ as in Definition 3.1.1. We define a bunch of $\mathcal{R}_{\mathbf{q}}$–subalgebras of $U_{\mathbf{q}}(\mathfrak{g})$, with a specific a set of generators, as follows:

(a) If $\mathbf{q}$ *is of integral type*, we set

$$\dot{\widehat{U}}_{\mathbf{q}}^{-} := \Big\langle F_i^{(n)} \Big\rangle_{i\in I,\, n\in\mathbb{N}} \quad,\qquad \dot{\widehat{U}}_{\mathbf{q}}^{+} := \Big\langle E_i^{(n)} \Big\rangle_{i\in I,\, n\in\mathbb{N}}$$

$$\dot{\widehat{U}}_{\mathbf{q}}^{-,0} := \Big\langle L_i^{\pm 1} , \binom{L_i}{n}_q \Big\rangle_{i\in I\,,\, n\in\mathbb{N}} \quad,\qquad \dot{\widehat{U}}_{\mathbf{q}}^{\leq} := \Big\langle F_i^{(n)} , L_i^{\pm 1} , \binom{L_i}{n}_q \Big\rangle_{i\in I\,,\, n\in\mathbb{N}}$$

$$\dot{\widehat{U}}_{\mathbf{q}}^{+,0} := \Big\langle K_i^{\pm 1} , \binom{K_i}{n}_q \Big\rangle_{i\in I\,,\, n\in\mathbb{N}} \quad,\qquad \dot{\widehat{U}}_{\mathbf{q}}^{\geq} := \Big\langle K_i^{\pm 1} , \binom{K_i}{n}_q , E_i^{(n)} \Big\rangle_{i\in I\,,\, n\in\mathbb{N}}$$

$$\dot{\widehat{U}}_{\mathbf{q}} \,=\, \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}) \,:=\, \Big\langle \dot{\widehat{U}}_{\mathbf{q}}^{0} \bigcup \Big\{ F_i^{(n)},\, E_i^{(n)} \Big\}_{i\in I\,,\, n\in\mathbb{N}} \Big\rangle$$

(b) If $\mathbf{q}$ *is of strongly integral type*, we set

$$\widehat{U}_{\mathbf{q}}^{-} := \Big\langle F_i^{(n)} \Big\rangle_{i\in I,\, n\in\mathbb{N}} \Big(=\, \dot{\widehat{U}}_{\mathbf{q}}^{-}\Big) \quad,\qquad \widehat{U}_{\mathbf{q}}^{+} := \Big\langle E_i^{(n)} \Big\rangle_{i\in I,\, n\in\mathbb{N}} \Big(=\, \dot{\widehat{U}}_{\mathbf{q}}^{+}\Big)$$

$$\widehat{U}_{\mathbf{q}}^{-,0} := \Big\langle L_i^{\pm 1} , \binom{L_i}{n}_{q_i} \Big\rangle_{i\in I\,,\, n\in\mathbb{N}} \quad,\qquad \widehat{U}_{\mathbf{q}}^{\leq} := \Big\langle F_i^{(n)} , L_i^{\pm 1} , \binom{L_i}{n}_{q_i} \Big\rangle_{i\in I\,,\, n\in\mathbb{N}}$$

$$\widehat{U}_{\mathbf{q}}^{+,0} := \Big\langle K_i^{\pm 1} , \binom{K_i}{n}_{q_i} \Big\rangle_{i\in I\,,\, n\in\mathbb{N}} \quad,\qquad \widehat{U}_{\mathbf{q}}^{\geq} := \Big\langle K_i^{\pm 1} , \binom{K_i}{n}_{q_i} , E_i^{(n)} \Big\rangle_{i\in I\,,\, n\in\mathbb{N}}$$

$$\widehat{U}_{\mathbf{q}} \,=\, \widehat{U}_{\mathbf{q}}(\mathfrak{g}) \,:=\, \Big\langle \widehat{U}_{\mathbf{q}}^{0} \bigcup \Big\{ F_i^{(n)},\, E_i^{(n)} \Big\}_{i\in I\,,\, n\in\mathbb{N}} \Big\rangle$$

In the sequel, we shall refer to all these objects as to *restricted* MpQG's. $\diamondsuit$

The "restricted" MpQG's introduced in Definition 5.2.12 admit a presentation by generators and relations, which generalizes the one in the canonical case (cf. [DL]):

Theorem 5.2.13.

(a) Let $\mathbf{q} := \big(q_{ij} = q^{b_{ij}}\big)_{i,j\in I}$ be of integral type. Then $\dot{\widehat{U}}_{\mathbf{q}} = \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ is (isomorphic to) the associative, unital $\mathcal{R}_{\mathbf{q}}$–algebra with the following presentation by generators and relations . The generators are all elements of $\dot{\widehat{U}}_{\mathbf{q}}^{0}$ as well as all elements $F_i^{(n)}$, $E_i^{(n)}$ (for all $i \in I$, $n \in \mathbb{N}$), and the relations holding true inside $\dot{\widehat{U}}_{\mathbf{q}}^{0}$ as well as the following ones:

$$K_i^{\pm 1} E_j^{(n)} \,=\, q^{\pm n b_{ij}} E_j^{(n)} K_i^{\pm 1} \quad,\qquad K_i^{\pm 1} F_j^{(n)} \,=\, q^{\mp n b_{ij}} F_j^{(n)} K_i^{\pm 1}$$

$$L_i^{\pm 1} E_j^{(n)} \,=\, q^{\mp n b_{ji}} E_j^{(n)} L_i^{\pm 1} \quad,\qquad L_i^{\pm 1} F_j^{(n)} \,=\, q^{\pm n b_{ji}} F_j^{(n)} L_i^{\pm 1}$$

$$G_i^{\pm 1} E_j^{(n)} \,=\, q_{ii}^{\pm n a_{ij}} E_j^{(n)} G_i^{\pm 1} \quad,\qquad G_i^{\pm 1} F_j^{(n)} \,=\, q_{ii}^{\mp n a_{ij}} F_j^{(n)} G_i^{\pm 1}$$

$$\binom{K_i\,;\,c}{h}_q E_j^{(n)} = E_j^{(n)} \binom{K_i\,;\,c+nb_{ij}}{h}_q \,,\quad \binom{K_i\,;\,c}{h}_q F_j^{(n)} = F_j^{(n)} \binom{K_i\,;\,c-nb_{ij}}{h}_q$$

$$\binom{L_i\,;\,c}{h}_{\!q} E_j^{(n)} = E_j^{(n)} \binom{L_i\,;\,c-nb_{ji}}{h}_{\!q} \;,\quad \binom{L_i\,;\,c}{h}_{\!q} F_j^{(n)} = F_j^{(n)} \binom{L_i\,;\,c+nb_{ji}}{h}_{\!q}$$

$$\binom{G_i\,;\,c}{h}_{\!q_{ii}} E_j^{(n)} = E_j^{(n)} \binom{G_i\,;\,c+na_{ij}}{h}_{\!q_{ii}} \;,\quad \binom{G_i\,;\,c}{h}_{\!q_{ii}} F_j^{(n)} = F_j^{(n)} \binom{G_i\,;\,c-na_{ij}}{h}_{\!q_{ii}}$$

$$X_i^{(r)} X_i^{(s)} = \binom{r+s}{r}_{\!q_{ii}} X_i^{(r+s)} \quad,\qquad X_i^{(0)} = 1 \qquad\qquad \forall\;\, X \in \{E\,,F\}$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \, q_{ii}^{\binom{k}{2}} q_{ij}^k \, X_i^{(r)} X_j^{(1)} X_i^{(s)} \;=\; 0 \qquad\qquad \forall\;\, X \in \{E\,,F\}\,, \quad \forall\;\, i \neq j$$

$$E_i^{(m)}\, F_i^{(n)} \;=\; \sum_{s=0}^{m\wedge n} F_i^{(n-s)}\, q_{ii}^s \binom{G_i\,;\,2\,s-m-n}{s}_{\!q_{ii}} L_i^s\, E_i^{(m-s)}$$

Moreover, with respect to this presentation $\dot{\widehat{U}}_{\mathbf{q}}$ *is endowed with the Hopf algebra structure (over* $\mathcal{R}_{\mathbf{q}}$ *) uniquely given by*

$$\Delta\Big(E_i^{(n)}\Big) = \sum_{s=0}^{n} E_i^{(n-s)} K_i^s \otimes E_i^{(s)} \,,\;\; \epsilon\Big(E_i^{(n)}\Big) = \delta_{n,0}\,,\;\; \mathcal{S}\Big(E_i^{(n)}\Big) = (-1)^n q_{ii}^{+\binom{n}{2}} K_i^{-n} E_i^{(n)}$$

$$\Delta\Big(F_i^{(n)}\Big) = \sum_{s=0}^{n} F_i^{(n-s)} \otimes F_i^{(s)} L_i^s \,,\;\; \epsilon\Big(F_i^{(n)}\Big) = \delta_{n,0}\,,\;\; \mathcal{S}\Big(F_i^{(n)}\Big) = (-1)^n q_{ii}^{-\binom{n}{2}} F_i^{(n)} L_i^{-n}$$

and formulas for $\Delta\,,\epsilon\,,S$ *in Lemma 5.2.2 with* $(M,p\,) \in \big\{(L_i\,,q)\,,(K_i\,,q)\,,(G_i\,,q_{ii})\big\}$.

(b) Let $\mathbf{q} := \big(q_{ij} = q^{\,d_i\,t_{ij}^+} = q^{\,d_j\,t_{ij}^-}\big)_{i,j\in I}$ *be of* strongly integral type. *Then* $\widehat{U}_{\mathbf{q}} = \widehat{U}_{\mathbf{q}}(\mathfrak{g})$ *is (isomorphic to) the Hopf algebra over* $\mathcal{R}_{\mathbf{q}}$ *with the following presentation by generators and relations. The generators are all elements of* $\widehat{U}_{\mathbf{q}}^0$ *as well as all elements* $F_i^{(n)}$ *,* $E_i^{(n)}$ *(for all* $i \in I$ *,* $n \in \mathbb{N}$ *), and the relations are*

$$K_i^{\pm 1}\, E_j^{(n)} \;=\; q_i^{\pm n t_{ij}^+} E_j^{(n)}\, K_i^{\pm 1} \quad,\qquad K_i^{\pm 1}\, F_j^{(n)} \;=\; q_i^{\mp n t_{ij}^+} F_j^{(n)}\, K_i^{\pm 1}$$

$$L_i^{\pm 1}\, E_j^{(n)} \;=\; q_i^{\mp n t_{ji}^-} E_j^{(n)}\, L_i^{\pm 1} \quad,\qquad L_i^{\pm 1}\, F_j^{(n)} \;=\; q_i^{\pm n t_{ji}^-} F_j^{(n)}\, L_i^{\pm 1}$$

$$G_i^{\pm 1}\, E_j^{(n)} \;=\; q_{ii}^{\pm n a_{ij}} E_j^{(n)}\, G_i^{\pm 1} \quad,\qquad G_i^{\pm 1}\, F_j^{(n)} \;=\; q_{ii}^{\mp n a_{ij}} F_j^{(n)}\, G_i^{\pm 1}$$

$$\binom{K_i\,;\,c}{h}_{\!q_i} E_j^{(n)} = E_j^{(n)} \binom{K_i\,;\,c+nt_{ij}^+}{h}_{\!q_i} \;,\quad \binom{K_i\,;\,c}{h}_{\!q_i} F_j^{(n)} = F_j^{(n)} \binom{K_i\,;\,c-nt_{ij}^+}{h}_{\!q_i}$$

$$\binom{L_i\,;\,c}{h}_{\!q_i} E_j^{(n)} = E_j^{(n)} \binom{L_i\,;\,c-nt_{ji}^-}{h}_{\!q_i} \;,\quad \binom{L_i\,;\,c}{h}_{\!q_i} F_j^{(n)} = F_j^{(n)} \binom{L_i\,;\,c+nt_{ji}^-}{h}_{\!q_i}$$

$$\binom{G_i\,;\,c}{h}_{\!q_{ii}} E_j^{(n)} = E_j^{(n)} \binom{G_i\,;\,c+na_{ij}}{h}_{\!q_{ii}} \;,\quad \binom{G_i\,;\,c}{h}_{\!q_{ii}} F_j^{(n)} = F_j^{(n)} \binom{G_i\,;\,c-na_{ij}}{h}_{\!q_{ii}}$$

$$X_i^{(r)} X_i^{(s)} = \binom{r+s}{r}_{\!q_{ii}} X_i^{(r+s)} \quad,\qquad X_i^{(0)} = 1 \qquad\qquad \forall\;\, X \in \{E\,,F\}$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \, q_{ii}^{\binom{k}{2}} q_{ij}^k \, X_i^{(r)} X_j^{(1)} X_i^{(s)} \;=\; 0 \qquad\qquad \forall\;\, X \in \{E\,,F\}\,, \quad \forall\;\, i \neq j$$

$$E_i^{(m)}\, F_i^{(n)} \;=\; \sum_{s=0}^{m\wedge n} F_i^{(n-s)}\, q_{ii}^s \binom{G_i\,;\,2\,s-m-n}{s}_{\!q_{ii}} L_i^s\, E_i^{(m-s)}$$

endowed with the Hopf algebra structure (over $\mathcal{R}_{\mathbf{q}}$ *) uniquely given by*

$$\Delta\Big(E_i^{(n)}\Big) = \sum_{s=0}^{n} E_i^{(n-s)} K_i^s \otimes E_i^{(s)} \,,\;\; \epsilon\Big(E_i^{(n)}\Big) = \delta_{n,0}\,,\;\; \mathcal{S}\Big(E_i^{(n)}\Big) = (-1)^n q_{ii}^{+\binom{n}{2}} K_i^{-n} E_i^{(n)}$$

$$\Delta\Big(F_i^{(n)}\Big) = \sum_{s=0}^{n} F_i^{(n-s)} \otimes F_i^{(s)} L_i^{n-s} \,,\;\; \epsilon\Big(F_i^{(n)}\Big) = \delta_{n,0}\,,\;\; \mathcal{S}\Big(F_i^{(n)}\Big) = (-1)^n q_{ii}^{-\binom{n}{2}} F_i^{(n)} L_i^{-n}$$

formulas for $\Delta\,,\epsilon\,,S$ *in Lemma 5.2.2 for* $(M,p\,) \in \big\{(L_i\,,q_i)\,,(K_i\,,q_i)\,,(G_i\,,q_{ii})\big\}$.

(c) $\dot{\widehat{U}}_{\mathbf{q}}$ *is a Hopf* $\mathcal{R}_{\mathbf{q}}$*–subalgebra of* $\widehat{U}_{\mathbf{q}}$.

(d) Similar statements occur for the various restricted multiparameter quantum subgroups considered in Definition 5.2.12.

Proof. Everything is proved like in the canonical case (cf. [DL]), taking Lemma 5.2.11 into account, but for *(c)*, which follows from definitions and Proposition 5.2.8. □

As a first, direct consequence we have the following:

Proposition 5.2.14.

(a) $\dot{\widehat{U}}_{\mathbf{q}} = \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ *, resp.* $\dot{\widehat{U}}_{\mathbf{q}}^{\leq}$ *, resp.* $\dot{\widehat{U}}_{\mathbf{q}}^{0}$ *, resp.* $\dot{\widehat{U}}_{\mathbf{q}}^{-,0}$ *, resp.* $\dot{\widehat{U}}_{\mathbf{q}}^{0}$ *, resp.* $\dot{\widehat{U}}_{\mathbf{q}}^{+,0}$ *, resp.* $\dot{\widehat{U}}_{\mathbf{q}}^{\geq}$ *, is a Hopf* $\mathcal{R}_{\mathbf{q}}$*–subalgebra (hence, it is an* $\mathcal{R}_{\mathbf{q}}$*–integral form, as a Hopf algebra) of* $U_{\mathbf{q}}(\mathfrak{g})$ *, resp. of* $U_{\mathbf{q}}^{\leq}$ *, resp. of* $U_{\mathbf{q}}^{-,0}$ *, resp. of* $U_{\mathbf{q}}^{0}$ *, resp. of* $U_{\mathbf{q}}^{+,0}$ *, resp. of* $U_{\mathbf{q}}^{\geq}$.

Similarly, $\dot{\widehat{U}}_{\mathbf{q}}^{-}$ *, resp.* $\dot{\widehat{U}}_{\mathbf{q}}^{+}$ *, is an* $\mathcal{R}_{\mathbf{q}}$*–subalgebra — hence, it is an* $\mathcal{R}_{\mathbf{q}}$*–integral form, as an algebra — of* $U_{\mathbf{q}}^{-}$ *, resp. of* $U_{\mathbf{q}}^{+}$.

(b) If $\mathbf{q}$ *is strongly integral, similar results — as in (a) and (b) — hold true as well when "*$\dot{\widehat{U}}$*" is replaced with "*$\widehat{U}$*".*

Proof. Indeed, Theorem 5.2.13 tells us that $\dot{\widehat{U}}_{\mathbf{q}}$ is a Hopf subalgebra (over $\mathcal{R}_{\mathbf{q}}$) of $U_{\mathbf{q}}$; moreover, the scalar extension of $\dot{\widehat{U}}_{\mathbf{q}}$ from $\mathcal{R}_{\mathbf{q}}$ to $\mathcal{F}_{\mathbf{q}}$ yields $U_{\mathbf{q}}$ as an $\mathcal{F}_{\mathbf{q}}$–module, just by definition: thus $\dot{\widehat{U}}_{\mathbf{q}}$ is an integral $\mathcal{R}_{\mathbf{q}}$–form of $U_{\mathbf{q}}$, as claimed. The same argument applies to $\dot{\widehat{U}}_{\mathbf{q}}^{\leq}$, $\dot{\widehat{U}}_{\mathbf{q}}^{0}$, etc., as well as to $\widehat{U}_{\mathbf{q}}$, $\widehat{U}_{\mathbf{q}}^{\leq}$, $\widehat{U}_{\mathbf{q}}^{0}$, etc. □

An easier result, direct consequence of Lemma 5.2.11 above, is the following, about the existence of "triangular decompositions" for these restricted MpQG's over $\mathcal{R}_{\mathbf{q}}$:

Proposition 5.2.15. *(triangular decompositions for restricted MpQG's)*

The multiplication in $\dot{\widehat{U}}_{\mathbf{q}}$ *provides* $\mathcal{R}_{\mathbf{q}}$*–module isomorphisms*

$$\dot{\widehat{U}}_{\mathbf{q}}^{-,0} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{0} \cong \dot{\widehat{U}}_{\mathbf{q}}^{\leq} \cong \dot{\widehat{U}}_{\mathbf{q}}^{0} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{-,0} \quad , \qquad \dot{\widehat{U}}_{\mathbf{q}}^{+,0} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{0} \cong \dot{\widehat{U}}_{\mathbf{q}}^{\geq} \cong \dot{\widehat{U}}_{\mathbf{q}}^{0} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{+,0}$$

$$\dot{\widehat{U}}_{\mathbf{q}}^{+,0} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{-,0} \cong \dot{\widehat{U}}_{\mathbf{q}}^{0} \cong \dot{\widehat{U}}_{\mathbf{q}}^{-,0} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{+,0} \quad , \qquad \dot{\widehat{U}}_{\mathbf{q}}^{\leq} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{\geq} \cong \dot{\widehat{U}}_{\mathbf{q}} \cong \dot{\widehat{U}}_{\mathbf{q}}^{\geq} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{\leq}$$

$$\dot{\widehat{U}}_{\mathbf{q}}^{+} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{0} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{-} \cong \dot{\widehat{U}}_{\mathbf{q}} \cong \dot{\widehat{U}}_{\mathbf{q}}^{-} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{0} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}^{+}$$

and similarly with "$\dot{\widehat{U}}$*" replaced by "*$\widehat{U}$*" if* $\mathbf{q}$ *is strongly integral.*

Proof. We consider the case of $\dot{\widehat{U}}_{\mathbf{q}}$ and of the left-hand side isomorphism, namely the case $\dot{\widehat{U}}_{\mathbf{q}}^{+} \otimes_{\mathcal{R}_{\mathbf{q}}} \dot{\widehat{U}}_{\mathbf{q}}^{0} \otimes_{\mathcal{R}_{\mathbf{q}}} \dot{\widehat{U}}_{\mathbf{q}}^{-} \cong \dot{\widehat{U}}_{\mathbf{q}}$, all other cases being similar.

By definition $\dot{\widehat{U}}_{\mathbf{q}}$ is spanned over $\mathcal{R}_{\mathbf{q}}$ by monomials whose factors can be freely chosen among the elements of $\dot{\widehat{U}}_{\mathbf{q}}^{0}$, the $F_i^{(m)}$'s and the $E_j^{(n)}$'s; moreover, thanks to Proposition 5.2.8*(b)* we can replace these monomials with other monomials, say $\mathcal{M}$, in the $\binom{K_i}{k}_q$'s, the $K_i^{-\kappa}$'s, the $\binom{G_i}{g_i}_{q_{ii}}$'s, the $G_j^{-\gamma}$'s, the $F_i^{(m)}$'s and the $E_j^{(n)}$'s.

Now, by repeated use of the commutation relations among factors of this type given in Lemma 5.2.11 — plus those stating that the $\binom{K_i}{k}_q$'s, the $K_i^{-\kappa}$'s, the $\binom{G_i}{g_i}_{q_{ii}}$'s and the $G_j^{-\gamma}$'s all commute with each other — (or by the corresponding relations given in Theorem 5.2.13) one can easily see that the following holds. Each one of these monomials, say $\mathcal{M}$, can be expanded into an $\mathcal{R}_{\mathbf{q}}$–linear combination of new monomials, say $\mathcal{M}_s$, of the same type but having the following additional property: each of them has the form $\mathcal{M}_s = \mathcal{M}_s^+ \cdot \mathcal{M}_s^0 \cdot \mathcal{M}_s^-$, where $\mathcal{M}_s^+$ is a monomial in the $E_j^{(n)}$'s, $\mathcal{M}_s^0$ is a monomial in the $\binom{K_i}{k}_q$'s, the $K_i^{-\kappa}$'s, the $\binom{G_i}{g_i}_{q_{ii}}$'s and the $G_j^{-\gamma}$'s, and $\mathcal{M}_s^-$ is a monomial in the $F_j^{(m)}$'s. This means that

$$\mathcal{M}_s = \mathcal{M}_s^+ \cdot \mathcal{M}_s^0 \cdot \mathcal{M}_s^- \in \dot{\widehat{U}}_{\mathbf{q}}^+ \cdot \dot{\widehat{U}}_{\mathbf{q}}^0 \cdot \dot{\widehat{U}}_{\mathbf{q}}^-$$

hence the multiplication map $\dot{\widehat{U}}_{\mathbf{q}}^+ \otimes_{\mathcal{R}_{\mathbf{q}}} \dot{\widehat{U}}_{\mathbf{q}}^0 \otimes_{\mathcal{R}_{\mathbf{q}}} \dot{\widehat{U}}_{\mathbf{q}}^- \longrightarrow \dot{\widehat{U}}_{\mathbf{q}}$ is onto. On the other hand, the PBW Theorem 4.2.1 for $U_{\mathbf{q}} = U_{\mathbf{q}}(\mathfrak{g})$ directly implies that this map is 1:1 as well. $\square$

The previous result is improved by the following "PBW Theorem" for our restricted MpQG's (and their quantum subgroups as well):

Theorem 5.2.16. *(PBW theorem for restricted quantum groups and subgroups*

Let quantum root vectors in $U_{\mathbf{q}}(\mathfrak{g})$ be fixed as in §4.1. Then the following holds:

(a) The set of ordered monomials

$$\left\{ \prod_{k=N}^{1} F_{\beta^k}^{(f_k)} \;\middle|\; f_k \in \mathbb{N} \right\} \quad , \qquad \textit{resp.} \quad \left\{ \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \;\middle|\; e_h \in \mathbb{N} \right\} \quad ,$$

is an $\mathcal{R}_{\mathbf{q}}$–basis of $\dot{\widehat{U}}_{\mathbf{q}}^-$, resp. of $\dot{\widehat{U}}_{\mathbf{q}}^+$; in particular, both $\dot{\widehat{U}}_{\mathbf{q}}^-$ and $\dot{\widehat{U}}_{\mathbf{q}}^+$ are free $\mathcal{R}_{\mathbf{q}}$–modules. The same holds true for $\widehat{U}_{\mathbf{q}}^\pm \left(= \dot{\widehat{U}}_{\mathbf{q}}^\pm\right)$ in the strongly integral case.

(b) The set of ordered monomials

$$\left\{ \prod_{j \in I} \binom{L_j}{l_j}_q L_j^{-\lfloor l_j/2 \rfloor} \;\middle|\; l_j \in \mathbb{N} \right\} \quad , \qquad \textit{resp.} \quad \left\{ \prod_{i \in I} \binom{K_i}{k_i}_q K_i^{-\lfloor k_i/2 \rfloor} \;\middle|\; k_i \in \mathbb{N} \right\}$$

is an $\mathcal{R}_{\mathbf{q}}$–basis of $\dot{\widehat{U}}_{\mathbf{q}}^{-,0}$, resp. of $\dot{\widehat{U}}_{\mathbf{q}}^{+,0}$. Similarly, the sets

$$\left\{ \prod_{j \in I} \binom{L_j}{l_j}_q L_j^{-\lfloor l_j/2 \rfloor} \prod_{i \in I} \binom{G_i}{g_i}_{q_{ii}} G_i^{-\lfloor g_i/2 \rfloor} \;\middle|\; l_j, g_i \in \mathbb{N} \right\}$$

$$\textit{and} \quad \left\{ \prod_{i \in I} \binom{G_i}{g_i}_{q_{ii}} G_i^{-\lfloor g_i/2 \rfloor} \prod_{j \in I} \binom{K_j}{k_j}_q K_j^{-\lfloor k_j/2 \rfloor} \;\middle|\; g_i, k_j \in \mathbb{N} \right\}$$

are $\mathcal{R}_{\mathbf{q}}$–bases of $\dot{\widehat{U}}_{\mathbf{q}}^0$. In particular, all $\dot{\widehat{U}}_{\mathbf{q}}^{-,0}$, $\dot{\widehat{U}}_{\mathbf{q}}^{+,0}$ and $\dot{\widehat{U}}_{\mathbf{q}}^0$ are free $\mathcal{R}_{\mathbf{q}}$–modules.

(c) The sets of ordered monomials

$$\left\{ \prod_{k=N}^{1} F_{\beta^k}^{(f_k)} \prod_{j \in I} \binom{L_j}{l_j}_q L_j^{-\lfloor l_j/2 \rfloor} \;\middle|\; f_k, l_j \in \mathbb{N} \right\}$$

$$\textit{and} \quad \left\{ \prod_{j \in I} \binom{L_j}{l_j}_q L_j^{-\lfloor l_j/2 \rfloor} \prod_{k=N}^{1} F_{\beta^k}^{(f_k)} \;\middle|\; f_k, l_j \in \mathbb{N} \right\} \quad ,$$

$$\textit{resp.} \quad \left\{ \prod_{j \in I} \binom{K_j}{k_j}_q K_j^{-\lfloor k_j/2 \rfloor} \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \;\middle|\; k_j, e_h \in \mathbb{N} \right\}$$

$$\textit{and} \quad \left\{ \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \prod_{j \in I} \binom{K_j}{k_j}_q K_j^{-\lfloor k_j/2 \rfloor} \;\middle|\; k_j, e_h \in \mathbb{N} \right\} \quad ,$$

are $\mathcal{R}_{\mathbf{q}}$–bases of $\dot{\widehat{U}}_{\mathbf{q}}^{\leq}$, resp. $\dot{\widehat{U}}_{\mathbf{q}}^{\geq}$; in particular, $\dot{\widehat{U}}_{\mathbf{q}}^{\leq}$ and $\dot{\widehat{U}}_{\mathbf{q}}^{\geq}$ are free $\mathcal{R}_{\mathbf{q}}$–modules.

(d) The sets of ordered monomials

$$\left\{ \prod_{k=N}^{1} F_{\beta^k}^{(f_k)} \prod_{j\in I} \binom{L_j}{l_j}_q L_j^{-\lfloor l_j/2 \rfloor} \prod_{i\in I} \binom{G_i}{g_i}_{q_{ii}} G_i^{-\lfloor g_i/2 \rfloor} \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \;\middle|\; f_k, l_j, g_i, e_h \in \mathbb{N} \right\}$$

$$\left\{ \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \prod_{j\in I} \binom{L_j}{l_j}_q L_j^{-\lfloor l_j/2 \rfloor} \prod_{i\in I} \binom{G_i}{g_i}_{q_{ii}} G_i^{-\lfloor g_i/2 \rfloor} \prod_{k=N}^{1} F_{\beta^k}^{(f_k)} \;\middle|\; f_k, l_j, g_i, e_h \in \mathbb{N} \right\}$$

$$\left\{ \prod_{k=N}^{1} F_{\beta^k}^{(f_k)} \prod_{i\in I} \binom{G_i}{g_i}_{q_{ii}} G_i^{-\lfloor g_i/2 \rfloor} \prod_{j\in I} \binom{K_j}{k_j}_q K_j^{-\lfloor k_j/2 \rfloor} \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \;\middle|\; f_k, g_i, k_j, e_h \in \mathbb{N} \right\}$$

$$\left\{ \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \prod_{i\in I} \binom{G_i}{g_i}_{q_{ii}} G_i^{-\lfloor g_i/2 \rfloor} \prod_{j\in I} \binom{K_j}{k_j}_q K_j^{-\lfloor k_j/2 \rfloor} \prod_{k=N}^{1} F_{\beta^k}^{(f_k)} \;\middle|\; f_k, g_i, k_j, e_h \in \mathbb{N} \right\}$$

are $\mathcal{R}_{\mathbf{q}}$*–bases of* $\dot{\widehat{U}}_{\mathbf{q}} = \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ *; in particular,* $\dot{\widehat{U}}_{\mathbf{q}} = \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ *is a free* $\mathcal{R}_{\mathbf{q}}$*–module.*

(e) In addition, when $\mathbf{q}$ *is strongly integral similar results — akin to (b), (c) and (d) — hold true if "*$\dot{\widehat{U}}$*" is replaced with "*$\widehat{U}$ *" and every* $\binom{L_j}{l_j}_q$*, resp.* $\binom{K_j}{k_j}_q$*, is replaced with* $\binom{L_j}{l_j}_{q_j}$*, resp.* $\binom{K_j}{k_j}_{q_j}$.

Proof. *(a)* It is a classical result, due to Lusztig, that the claim holds true for $\dot{\widehat{U}}_{\check{\mathbf{q}}}^{-}$, i.e. the latter is free as an $\mathcal{R}_{\mathbf{q}}$–module with PBW-type basis given by the ordered monomials in the $\check{F}_{\beta}^{(f)}$'s, taken with respect to the product "$\cdot$" in $\dot{\widehat{U}}_{\check{\mathbf{q}}}^{-}$; the same monomials then form a PBW-like $\mathcal{R}_{\mathbf{q}}^{\surd}$–basis of $\left(\dot{\widehat{U}}_{\check{\mathbf{q}}}^{-}\right)^{\surd} := \mathcal{R}_{\mathbf{q}}^{\surd} \otimes_{\mathcal{R}_{\mathbf{q}}} \dot{\widehat{U}}_{\check{\mathbf{q}}}^{-}$ as well.

Now, the formulas in §4.4.2 show that the above mentioned "restricted" PBW monomials in $\left(\left(\dot{\widehat{U}}_{\check{\mathbf{q}}}^{-}\right)^{\surd}, \cdot\right)$ are proportional, by a coefficient which is a power of $q^{\pm 1/2}$ (hence invertible in the ground ring $\mathcal{R}_{\mathbf{q}}^{\surd}$) to their "counterparts" (with the same exponents for each root vector) in $\left(\dot{\widehat{U}}_{\mathbf{q}}^{-}\right)^{\surd} \subseteq \big(U_{\mathbf{q}}(\mathfrak{g})\big)^{\surd} = \left(\big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)^{\surd}, *\right)$, i.e. with respect to the "deformed" product "$*$" in $\big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)^{\surd} := \mathcal{R}_{\mathbf{q}}^{\surd} \otimes_{\mathcal{R}_{\mathbf{q}}} U_{\check{\mathbf{q}}}(\mathfrak{g})$. In other words, using notation of §4.4.2 we can write in short

$$\prod_{k=N}^{1} \check{F}_{\beta^k}^{\dot{\cdot}(f_k)} \;=\; q^{z/2} \prod_{k=N}^{1} F_{\beta^k}^{*(f_k)}$$

for some $z \in \mathbb{Z}$. Therefore, as the PBW monomials $\prod_{k=N}^{1} \check{F}_{\beta^k}^{\dot{\cdot}(f_k)}$ form an $\mathcal{R}_{\mathbf{q}}^{\surd}$–basis of $\left(\dot{\widehat{U}}_{\check{\mathbf{q}}}^{-}\right)^{\surd}$ we can argue that the $\prod_{k=N}^{1} F_{\beta^k}^{*(f_k)}$'s form an $\mathcal{R}_{\mathbf{q}}^{\surd}$–basis of $\left(\dot{\widehat{U}}_{\mathbf{q}}^{-}\right)^{\surd}$ too.

On the other hand, as direct consequence of Theorem 4.2.1 we have that the same $\prod_{k=N}^{1} F_{\beta^k}^{*(f_k)}$'s also form a $\mathcal{F}_{\mathbf{q}}$–basis of $U_{\mathbf{q}}^{-}$. Thus any $u_{-} \in \left(\dot{\widehat{U}}_{\mathbf{q}}^{-}\right)^{\surd}$ will have a unique expansion as $\mathcal{R}_{\mathbf{q}}^{\surd}$–linear combination of the $\prod_{k=N}^{1} F_{\beta^k}^{*(f_k)}$'s, but also a unique expansion as $\mathcal{F}_{\mathbf{q}}$–linear combination of the same "restricted" PBW-like monomials. Then the coefficients in both expansions must coincide, and since $\mathcal{R}_{\mathbf{q}}^{\surd} \cap \mathcal{F}_{\mathbf{q}} = \mathcal{R}_{\mathbf{q}}$ they must belong to $\mathcal{R}_{\mathbf{q}}$; so the $\prod_{k=N}^{1} F_{\beta^k}^{*(f_k)}$'s form an $\mathcal{R}_{\mathbf{q}}$–basis of $U_{\mathbf{q}}^{-}$, as claimed.

The same argument applies for the part of the claim concerning $\dot{\widehat{U}}_{\mathbf{q}}^{+}$.

(b) This follows by construction together with Proposition 5.2.3.

(c)–(d) These follow at once from claims *(a)–(b)* along with the existence of triangular decompositions as given in Proposition 5.2.15.

(e) This is proved by the same arguments used for claims *(a)* through *(d)*. □

Remark 5.2.17. It is worth stressing that the construction of restricted MpQG's does not "match well" with the process of cocycle deformation, even if one extends scalars from $\mathcal{R}_{\mathbf{q}}$ to $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ — and from $\mathcal{F}_{\mathbf{q}}$ to $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$ accordingly. In fact, if we label every MpQG's over $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ or $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$ by a superscript "$\sqrt{\ }$", what happens is that, although $U_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) = \big(U_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})\big)_\sigma = U_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})$ as $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$–modules, for integral forms one has in general $\dot{\hat{U}}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) \neq \dot{\hat{U}}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})$ as $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$–modules; and *a fortiori* $\dot{\hat{U}}_{\mathbf{q}}(\mathfrak{g}) \neq \dot{\hat{U}}_{\check{\mathbf{q}}}(\mathfrak{g})$. Similarly holds for all "quantum subgroups".

In order to see that, let us consider an element of $\dot{\hat{U}}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) = \Big(\dot{\hat{U}}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})\,,\,*\Big)$ of the form $E_\alpha^{*(n)} * \binom{K_j}{m}_q^*$: from §4.4.2 we have the formula

$$E_\alpha^{*(n)} * \binom{K_j}{m}_q^* = q_\alpha^{+\binom{n}{2}} \big(m_\alpha^+\big)^n \sum_{c=0}^{m} q^{-c(m-c)} \prod_{s=1}^{c} \frac{q^{1-s}\big(q_{\alpha\,\alpha_j}^{+1/2}\big)^n - 1}{q^{\,s} - 1}\, \check{E}_\alpha^{\,\check{}\,(n)} \binom{K_j}{m-c}_q^{\check{}} K_j^{\,\check{}\,c}$$

here, the right-hand side term is the expansion of $E_\alpha^{*(n)} * \binom{X_j}{m}_{p_j}^*$ into an $\mathcal{F}_{\mathbf{q}}$–linear combination of the elements of

$$\left\{ \prod_{h=1}^{N} \check{E}_{\beta^h}^{\,\check{}\,(e_h)} \prod_{i\in I} \binom{G_i}{g_i}_{q_{ii}} G_i^{-\lfloor g_i/2\rfloor} \prod_{j\in I} \binom{K_j}{k_j}_q K_j^{-\lfloor k_j/2\rfloor} \prod_{k=N}^{1} \check{F}_{\beta^k}^{\,\check{}\,(f_k)} \;\middle|\; f_k\,,g_i\,,k_j\,,e_h \in \mathbb{N} \right\}$$

which, being an $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$–basis of $\dot{\hat{U}}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g})$ — by Theorem 5.2.16*(d)* above — is also an $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$–basis of $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }} \otimes_{\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}} \dot{\hat{U}}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) = U_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g})$. Now, in the above expansion the coefficients

$$q_\alpha^{+\binom{n}{2}} \big(m_\alpha^+\big)^n\, q^{-c(m-c)} \prod_{s=1}^{c} \frac{q^{1-s}\big(q_{\alpha\,\alpha_j}^{+1/2}\big)^n - 1}{q^{\,s} - 1}$$

in general *do not belong to* $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$: therefore, we have $E_\alpha^{*(n)} * \binom{K_j}{m}_q^* \notin \dot{\hat{U}}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})$ whereas $E_\alpha^{*(n)} * \binom{K_j}{m}_q^* \in \dot{\hat{U}}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g})$ by definition. This shows that $\dot{\hat{U}}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) \neq \dot{\hat{U}}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})$ inside $U_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) = U_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})$ (as $\mathcal{F}_{\mathbf{q}}$–modules); in fact, it even proves that $\big(\dot{\hat{U}}_{\mathbf{q}}^{\geq}\big)^{\sqrt{\ }} \neq \big(\dot{\hat{U}}_{\check{\mathbf{q}}}^{\geq}\big)^{\sqrt{\ }}$, and similarly one shows that $\big(\dot{\hat{U}}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }} \neq \big(\dot{\hat{U}}_{\check{\mathbf{q}}}^{\leq}\big)^{\sqrt{\ }}$ too.

5.3. **Integral forms of "unrestricted" type.**

Beside Lusztig's "restricted" integral form, a second integral form of $U_q(\mathfrak{g})$ was introduced by De Concini, Kac and Procesi: the ground ring in that case was $\Bbbk\big[q,q^{-1}\big]$, but one can easily prove — using the analogue (in that context) of Proposition 4.3.1 — that their definition does work the same over $\mathbb{Z}\big[q,q^{-1}\big]$ too, so it yields *an integral form over* $\mathbb{Z}\big[q,q^{-1}\big]$, with suitable PBW-like basis, etc. Their construction can be easily extended to $U_{\check{\mathbf{q}}}(\mathfrak{g})$; hereafter, we extend this (obvious) generalization to any MpQG such as $U_{\mathbf{q}}(\mathfrak{g})$.

Let us fix a multiparameter matrix $\mathbf{q} := \big(q_{ij}\big)_{i,j\in I}$ and the corresponding MpQG $U_{\mathbf{q}}(\mathfrak{g})$ as in §3.1; then fix the special parameter q (depending on $\mathbf{q}$) and the "canonical" multiparameter $\check{\mathbf{q}} := \big(\check{q}_{ij} := q^{\,d_i a_{ij}}\big)_{i,j\in I}$ as in §3.2. Finally, assume quantum root vectors E_α , F_α (for all $\alpha \in \Phi^+$) have been fixed, as in §4.1, and consider for them the following "renormalizations" (where $q_{\alpha\,\alpha}$ is defined as in §3.2)

$$\overline{E}_\alpha := \big(q_{\alpha\,\alpha} - 1\big)\, E_\alpha \quad , \qquad \overline{F}_\alpha := \big(q_{\alpha\,\alpha} - 1\big)\, F_\alpha \qquad \forall\ \alpha \in \Phi^+ \tag{5.4}$$

Mimicking the construction in [DP], we introduce the following definition:

Definition 5.3.1. For any multiparameter $\mathbf{q} := \big(q_{ij}\big)_{i,j\in I}$ as in §2.3.2, fix modified quantum root vectors $\overline{E}_\alpha$ and $\overline{F}_\alpha$ (for all $\alpha \in \Phi^+$) of $U_{\mathbf{q}}(\mathfrak{g})$ as above. Then define in $U_{\mathbf{q}}(\mathfrak{g})$ the following $\mathcal{R}_{\mathbf{q}}$–subalgebras:

$$\widetilde{U}_{\mathbf{q}}^- := \big\langle\, \overline{F}_\alpha \,\big\rangle_{\alpha\in\Phi^+} \quad , \qquad \widetilde{U}_{\mathbf{q}}^0 := \big\langle\, L_i^{\pm 1},\, K_i^{\pm 1} \,\big\rangle_{i\in I} \quad , \qquad \widetilde{U}_{\mathbf{q}}^+ := \big\langle\, \overline{E}_\alpha \,\big\rangle_{\alpha\in\Phi^+}$$
$$\widetilde{U}_{\mathbf{q}}^{\leq} := \big\langle\, \overline{F}_\alpha \,,\, L_i^{\pm 1} \,\big\rangle_{\alpha\in\Phi^+,\, i\in I} \quad , \qquad \widetilde{U}_{\mathbf{q}}^{\geq} := \big\langle\, K_i^{\pm 1},\, \overline{E}_\alpha \,\big\rangle_{i\in I,\, \alpha\in\Phi^+}$$
$$\widetilde{U}_{\mathbf{q}}^{-,0} := \big\langle L_i^{\pm 1} \big\rangle_{i\in I} \,,\;\; \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) = \widetilde{U}_{\mathbf{q}} := \big\langle\, \overline{F}_\alpha \,,\, L_i^{\pm 1},\, K_i^{\pm 1},\, \overline{E}_\alpha \,\big\rangle_{i\in I,\, \alpha\in\Phi^+} \,,\;\; \widetilde{U}_{\mathbf{q}}^{+,0} := \big\langle K_i^{\pm 1} \big\rangle_{i\in I}$$

In the following, we shall refer to this kind of MpQG as *unrestricted.* ♢

Contrary to the case of restricted integral forms, if we extend scalars to $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ then all unrestricted ones are indeed 2–cocycle deformations of their canonical counterparts, just like it happens with MpQG's over $\mathcal{F}_{\mathbf{q}}$. This follows from direct analysis through the formulas in §4.4.2, as the following shows:

Proposition 5.3.2. *The Hopf algebra* $\widetilde{U}_{\mathbf{q}}^{\sqrt{\ }} = \widetilde{U}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) := \mathcal{R}_{\mathbf{q}}^{\sqrt{\ }} \otimes_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ *is a 2–cocycle deformation of its canonical counterpart, namely*

$$\widetilde{U}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) \;=\; \big(\widetilde{U}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})\big)_\sigma \;=\; \big(\widetilde{U}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})\big)^{(\tilde{\varphi})}$$

(see Theorem 3.2.2 and Proposition 3.2.4 for notation). Similarly — using a superscript "$\sqrt{\ }$" to denote scalar extension to $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ — $\big(\widetilde{U}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }}$*, resp.* $\big(\widetilde{U}_{\mathbf{q}}^{0}\big)^{\sqrt{\ }}$*, resp.* $\big(\widetilde{U}_{\mathbf{q}}^{\geq}\big)^{\sqrt{\ }}$*, is a 2–cocycle deformation of* $\big(\widetilde{U}_{\check{\mathbf{q}}}^{\geq}\big)^{\sqrt{\ }}$*, resp. of* $\big(\widetilde{U}_{\check{\mathbf{q}}}^{0}\big)^{\sqrt{\ }}$*, resp. of* $\big(\widetilde{U}_{\check{\mathbf{q}}}^{\leq}\big)^{\sqrt{\ }}$*. In particular,* $\widetilde{U}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) = \widetilde{U}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})$*,* $\big(\widetilde{U}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }} = \big(\widetilde{U}_{\check{\mathbf{q}}}^{\leq}\big)^{\sqrt{\ }}$*,* $\big(\widetilde{U}_{\mathbf{q}}^{0}\big)^{\sqrt{\ }} = \big(\widetilde{U}_{\check{\mathbf{q}}}^{0}\big)^{\sqrt{\ }}$ *and* $\big(\widetilde{U}_{\mathbf{q}}^{\geq}\big)^{\sqrt{\ }} = \big(\widehat{U}_{\check{\mathbf{q}}}^{\geq}\big)^{\sqrt{\ }}$ *as* $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$*–coalgebras, and* $\big(\widetilde{U}_{\mathbf{q}}^{\pm}\big)^{\sqrt{\ }} = \big(\widetilde{U}_{\check{\mathbf{q}}}^{\pm}\big)^{\sqrt{\ }}$ *as* $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$*–modules.*

It follows that all of $\widetilde{U}_{\mathbf{q}}^{\sqrt{\ }}$*,* $\big(\widetilde{U}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }}$*,* $\big(\widetilde{U}_{\mathbf{q}}^{0}\big)^{\sqrt{\ }}$*,* $\big(\widetilde{U}_{\mathbf{q}}^{\geq}\big)^{\sqrt{\ }}$*,* $\big(\widetilde{U}_{\mathbf{q}}^{-}\big)^{\sqrt{\ }}$ *and* $\big(\widetilde{U}_{\mathbf{q}}^{+}\big)^{\sqrt{\ }}$ *are independent of the choice of quantum root vectors* E_β *and* F_β *(for* $\beta \in \Phi^+$*).*

Proof. The same analysis as in the proof of Theorem 5.2.16*(a)* shows — looking at the proper formulas from §4.4.2 — that the identities $\big(\widetilde{U}_{\mathbf{q}}^{-}\big)^{\sqrt{\ }} = \big(\widetilde{U}_{\check{\mathbf{q}}}^{-}\big)^{\sqrt{\ }}$, $\big(\widetilde{U}_{\mathbf{q}}^{+}\big)^{\sqrt{\ }} = \big(\widetilde{U}_{\check{\mathbf{q}}}^{+}\big)^{\sqrt{\ }}$ hold as $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$–modules, and $\big(\widetilde{U}_{\mathbf{q}}^{0}\big)^{\sqrt{\ }} = \big(\widetilde{U}_{\check{\mathbf{q}}}^{0}\big)^{\sqrt{\ }}$ as $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$–coalgebras; more precisely, the latter identity can be read as $\big(\widetilde{U}_{\mathbf{q}}^{0}\big)^{\sqrt{\ }} = \Big(\big(\widetilde{U}_{\check{\mathbf{q}}}^{0}\big)^{\sqrt{\ }}\Big)_\sigma = \Big(\big(\widetilde{U}_{\check{\mathbf{q}}}^{0}\big)^{\sqrt{\ }}\Big)^{(\tilde{\varphi})}$, by the very definitions and thanks to Theorem 3.2.2 and Proposition 3.2.4.

The same argument proves also $\widetilde{U}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) = \widetilde{U}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})$, $\big(\widetilde{U}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }} = \big(\widetilde{U}_{\check{\mathbf{q}}}^{\leq}\big)^{\sqrt{\ }}$ and $\big(\widetilde{U}_{\mathbf{q}}^{\geq}\big)^{\sqrt{\ }} = \big(\widehat{U}_{\check{\mathbf{q}}}^{\geq}\big)^{\sqrt{\ }}$ as $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$–coalgebras; more precisely, one has $\widetilde{U}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) = \big(\widetilde{U}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})\big)_\sigma = \big(\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})\big)^{(\tilde{\varphi})}$, $\big(\widetilde{U}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }} = \Big(\big(\widetilde{U}_{\check{\mathbf{q}}}^{\leq}\big)^{\sqrt{\ }}\Big)_\sigma = \Big(\big(\widetilde{U}_{\check{\mathbf{q}}}^{\leq}\big)^{\sqrt{\ }}\Big)^{(\tilde{\varphi})}$ and $\big(\widetilde{U}_{\mathbf{q}}^{\geq}\big)^{\sqrt{\ }} = \Big(\big(\widetilde{U}_{\check{\mathbf{q}}}^{\geq}\big)^{\sqrt{\ }}\Big)_\sigma = \Big(\big(\widetilde{U}_{\check{\mathbf{q}}}^{\geq}\big)^{\sqrt{\ }}\Big)^{(\tilde{\varphi})}$. □

As for restricted MpQG's, we have a PBW theorem for unrestricted ones too:

Theorem 5.3.3. *(PBW theorem for unrestricted MQG's — and subgroups)*

(a) The set of ordered monomials

$$\left\{ \prod_{k=N}^{1} \overline{F}_{\beta^k}^{\,f_k} \;\middle|\; f_k \in \mathbb{N} \right\} \quad , \qquad \textit{resp.} \quad \left\{ \prod_{h=1}^{N} \overline{E}_{\beta^h}^{\,e_h} \;\middle|\; e_h \in \mathbb{N} \right\} \quad ,$$

is an $\mathcal{R}_{\mathbf{q}}$*–basis of* $\widetilde{U}_{\mathbf{q}}^{-}$*, resp. of* $\widetilde{U}_{\mathbf{q}}^{+}$*; in particular, both these are free* $\mathcal{R}_{\mathbf{q}}$*–modules.*

(b) The set of ordered monomials

$$\left\{ \prod_{j\in I} L_j^{\,a_j} \;\middle|\; a_j \in \mathbb{Z} \right\}, \quad \textit{resp.} \quad \left\{ \prod_{i\in I} K_i^{\,b_i} \;\middle|\; b_i \in \mathbb{Z} \right\}, \quad \textit{resp.} \quad \left\{ \prod_{j\in I} L_j^{\,a_j} K_i^{\,b_i} \;\middle|\; a_j, b_i \in \mathbb{Z} \right\}$$

is an $\mathcal{R}_{\mathbf{q}}$*–basis of* $\widetilde{U}_{\mathbf{q}}^{-,0}$*, resp. of* $\widetilde{U}_{\mathbf{q}}^{+,0}$*, resp. of* $\widetilde{U}_{\mathbf{q}}^{0}$*, hence all these are free* $\mathcal{R}_{\mathbf{q}}$*–modules.*

(c) The sets of ordered monomials

$$\Big\{ \prod_{k=N}^{1} \overline{F}_{\beta^k}^{\,f_k} \prod_{j \in I} {L_j}^{a_j} \Big\}_{f_k \in \mathbb{N}\,,\, a_j \in \mathbb{Z}} \quad \text{and} \quad \Big\{ \prod_{j \in I} {L_j}^{a_j} \prod_{k=N}^{1} \overline{F}_{\beta^k}^{\,f_k} \Big\}_{f_k \in \mathbb{N}\,,\, a_j \in \mathbb{Z}} \;,$$

$$\text{resp.} \quad \Big\{ \prod_{i \in I} {K_i}^{b_i} \prod_{h=1}^{N} \overline{E}_{\beta^h}^{\,e_h} \Big\}_{b_i \in \mathbb{Z}\,,\, e_h \in \mathbb{N}} \quad \text{and} \quad \Big\{ \prod_{h=1}^{N} \overline{E}_{\beta^h}^{\,e_h} \prod_{i \in I} {K_i}^{b_i} \Big\}_{b_i \in \mathbb{Z}\,,\, e_h \in \mathbb{N}} \;,$$

are $\mathcal{R}_{\mathbf{q}}$–bases of $\widetilde{U}_{\mathbf{q}}^{\leq}$, resp. of $\widetilde{U}_{\mathbf{q}}^{\geq}$; in particular, $\widetilde{U}_{\mathbf{q}}^{\leq}$ and $\widetilde{U}_{\mathbf{q}}^{\geq}$ are free $\mathcal{R}_{\mathbf{q}}$–modules.

(d) The sets of ordered monomials

$$\Big\{ \prod_{k=N}^{1} \overline{F}_{\beta^k}^{\,f_k} \prod_{j \in I} {L_j}^{a_j} \prod_{i \in I} {K_i}^{b_i} \prod_{h=1}^{N} \overline{E}_{\beta^h}^{\,e_h} \;\Big|\; f_k, e_h \in \mathbb{N}\,,\; a_j, b_i \in \mathbb{Z} \Big\}$$

$$\text{and} \qquad \Big\{ \prod_{h=1}^{N} \overline{E}_{\beta^h}^{\,e_h} \prod_{j \in I} {L_j}^{a_j} \prod_{i \in I} {K_i}^{b_i} \prod_{k=N}^{1} \overline{F}_{\beta^k}^{\,f_k} \;\Big|\; f_k, e_h \in \mathbb{N}\,,\; a_j, b_i \in \mathbb{Z} \Big\}$$

are $\mathcal{R}_{\mathbf{q}}$–bases of $\widetilde{U}_{\mathbf{q}}$; in particular, $\widetilde{U}_{\mathbf{q}} = \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ itself is a free $\mathcal{R}_{\mathbf{q}}$–module.

Proof. *(a)* Entirely similar to the proof of Theorem 5.2.16.

(b) This is obvious from definitions.

(c) We can apply once more the same ideas as for Theorem 5.2.16, thus finding that $\mathbb{B} := \Big\{ \prod_{k=N}^{1} \overline{F}_{\beta^k}^{\,f_k} \prod_{j \in I} {L_j}^{a_j} \Big\}_{f_k \in \mathbb{N}\,,\, a_j \in \mathbb{Z}}$ is an $\mathcal{R}_{\mathbf{q}}$–basis of $\widetilde{U}_{\mathbf{q}}^{\leq}$, the case for $\widetilde{U}_{\mathbf{q}}^{\geq}$ being entirely similar. The claim is true when $\mathbf{q} = \check{\mathbf{q}}$, by the results in [DP]; moreover, by Proposition 5.3.2 we have $\big(\widetilde{U}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }} = \big(\widetilde{U}_{\check{\mathbf{q}}}^{\leq}\big)^{\sqrt{\ }}$ as $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$–coalgebras, so $\mathbb{B}$ is also an $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$–basis of $\big(\widetilde{U}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }}$. On the other hand, it follows from Theorem 4.2.1 that $\mathbb{B}$ is also an $\mathcal{F}_{\mathbf{q}}$–basis of $\big(\widetilde{U}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }}$. Thus any $u \in \widetilde{U}_{\mathbf{q}}^{\leq} \Big(\subseteq \big(\widetilde{U}_{\mathbf{q}}^{\leq}\big)^{\sqrt{\ }} \cap \widetilde{U}_{\mathbf{q}}^{\leq} \Big)$ uniquely expands as an $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$–linear combination of elements in $\mathbb{B}$ but also uniquely expands as an $\mathcal{F}_{\mathbf{q}}$–linear combination of such elements: we conclude that the coefficients in these expansions belong to $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }} \cap \mathcal{F}_{\mathbf{q}} = \mathcal{R}_{\mathbf{q}}$, q.e.d.

(d) This is proved by the same arguments as *(c)* above. □

A direct fallout of the previous result is the following:

Proposition 5.3.4. *(triangular decompositions for unrestricted MpQG's)*
The multiplication in $\widetilde{U}_{\mathbf{q}}$ provides $\mathcal{R}_{\mathbf{q}}$–module isomorphisms

$$\widetilde{U}_{\mathbf{q}}^{-} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{0} \,\cong\, \widetilde{U}_{\mathbf{q}}^{\leq} \,\cong\, \widetilde{U}_{\mathbf{q}}^{0} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{-} \quad , \qquad \widetilde{U}_{\mathbf{q}}^{+} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{0} \,\cong\, \widetilde{U}_{\mathbf{q}}^{\geq} \,\cong\, \widetilde{U}_{\mathbf{q}}^{0} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{+}$$

$$\widetilde{U}_{\mathbf{q}}^{+,0} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{-,0} \,\cong\, \widetilde{U}_{\mathbf{q}}^{0} \,\cong\, \widetilde{U}_{\mathbf{q}}^{-,0} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{+,0} \;, \quad \widetilde{U}_{\mathbf{q}}^{+} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{0} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{-} \,\cong\, \widetilde{U}_{\mathbf{q}} \,\cong\, \widetilde{U}_{\mathbf{q}}^{-} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{0} \mathop{\otimes}_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}^{+}$$

Proof. Direct from Theorem 5.3.3 above. □

Here is a second consequence:

Proposition 5.3.5.

(a) $\widetilde{U}_{\mathbf{q}} = \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$, resp. $\widetilde{U}_{\mathbf{q}}^{\leq}$, resp. $\widetilde{U}_{\mathbf{q}}^{0}$, resp. $\widetilde{U}_{\mathbf{q}}^{\geq}$, is a Hopf $\mathcal{R}_{\mathbf{q}}$–subalgebra (hence is an $\mathcal{R}_{\mathbf{q}}$–integral form as a Hopf algebra) of $U_{\mathbf{q}}(\mathfrak{g})$, resp. of $U_{\mathbf{q}}^{\leq}$, resp. of $U_{\mathbf{q}}^{0}$, resp. of $U_{\mathbf{q}}^{\geq}$.

(b) $\widetilde{U}_{\mathbf{q}}^{\pm}$ is an $\mathcal{R}_{\mathbf{q}}$–subalgebra (hence an $\mathcal{R}_{\mathbf{q}}$–integral form, as an algebra) of $U_{\mathbf{q}}^{\pm}$.

Proof. Claim *(b)* is obvious, by construction, and similarly also claim *(a)* for $\widetilde{U}_{\mathbf{q}}^{0}$; the other cases are similar, so we restrict ourselves to one of them, say that of $\widetilde{U}_{\mathbf{q}}$.

Once again, the canonical case (i.e. $\mathbf{q} = \check{\mathbf{q}}$) follows from the results in [DP], suitably adapted to the present context; then by Proposition 5.3.2 above the same result also holds true for $\widetilde{U}_{\mathbf{q}}^{\sqrt{}}$ with *any* possible $\mathbf{q}$ — that is, $\widetilde{U}_{\mathbf{q}}^{\sqrt{}}$ is a Hopf $\mathcal{R}_{\mathbf{q}}^{\sqrt{}}$–subalgebra of $U_{\mathbf{q}}^{\sqrt{}} := U_{\mathbf{q}}^{\sqrt{}}(\mathfrak{g})$, for any possible $\mathbf{q}$. In particular $\widetilde{U}_{\mathbf{q}}^{\sqrt{}}$ is an $\mathcal{R}_{\mathbf{q}}^{\sqrt{}}$–subcoalgebra of $U_{\mathbf{q}}^{\sqrt{}}$, hence given any $u \in \widetilde{U}_{\mathbf{q}} \; \big(\subseteq \widetilde{U}_{\mathbf{q}}^{\sqrt{}} \big)$ we have $\Delta(u) \in \widetilde{U}_{\mathbf{q}}^{\sqrt{}} \otimes_{\mathcal{R}_{\mathbf{q}}^{\sqrt{}}} \widetilde{U}_{\mathbf{q}}^{\sqrt{}}$. By Theorem 5.3.3 the $\mathcal{R}_{\mathbf{q}}^{\sqrt{}}$–module $\widetilde{U}_{\mathbf{q}}^{\sqrt{}} \otimes_{\mathcal{R}_{\mathbf{q}}^{\sqrt{}}} \widetilde{U}_{\mathbf{q}}^{\sqrt{}}$ is free with a basis made of homogeneous tensors $v' \otimes v''$ in which both v' and v'' are PBW monomials as given in Theorem 5.3.3*(d)*: thus $\Delta(u)$ has a unique expansion of the form $\Delta(u) = \sum_s c_s \, v'_s \otimes v''_s$ for some $c_s \in \mathcal{R}_{\mathbf{q}}^{\sqrt{}}$. On the other hand, the same set of "PBW homogeneous tensors" of the form $v' \otimes v''$ as above is also an $\mathcal{F}_{\mathbf{q}}$–basis of $U_{\mathbf{q}} \otimes_{\mathcal{F}_{\mathbf{q}}} U_{\mathbf{q}}$: hence, since $U_{\mathbf{q}}$ is an $\mathcal{F}_{\mathbf{q}}$–coalgebra and $u \in \widetilde{U}_{\mathbf{q}} \subseteq U_{\mathbf{q}}$, we have also a unique $\mathcal{F}_{\mathbf{q}}$–linear expansion of $\Delta(u)$ into $\Delta(u) = \sum_s a_s \, v'_s \otimes v''_s$. Comparing both expansion inside $U_{\mathbf{q}}^{\sqrt{}} \otimes_{\mathcal{F}_{\mathbf{q}}^{\sqrt{}}} U_{\mathbf{q}}^{\sqrt{}}$ — which also has the set of all "PBW homogeneous tensors" $v' \otimes v''$ as $\mathcal{F}_{\mathbf{q}}^{\sqrt{}}$–monomials — we find $c_s = a_s \in \mathcal{R}_{\mathbf{q}}^{\sqrt{}} \cap \mathcal{F}_{\mathbf{q}} = \mathcal{R}_{\mathbf{q}}$ for all s, which means that $\Delta(u) \in \widetilde{U}_{\mathbf{q}} \otimes_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}$. So $\widetilde{U}_{\mathbf{q}}$ is an $\mathcal{R}_{\mathbf{q}}$–subcoalgebra of $U_{\mathbf{q}}$, and similar arguments prove it is stable by the antipode, hence is a Hopf $\mathcal{R}_{\mathbf{q}}$–subalgebra. □

5.4. **Integral forms for MpQG's with larger torus.**

In §3.3 we introduced generalized MpQG's, denoted $U_{\mathbf{q},\varGamma_\bullet} \equiv U_{\mathbf{q},\varGamma_\bullet}(\mathfrak{g})$, whose toral part is the group algebra of any lattice $\varGamma_\bullet = \varGamma_+ \times \varGamma_-$ with $\varGamma_\pm$ being rank θ lattices such that $Q \leq \varGamma_\pm \leq \mathbb{Q}Q$; in particular, *this required additional assumptions on the ground field* $\Bbbk$, *namely that* $\Bbbk$ *contain suitable roots of the q_{ij}'s, see §3.3.3.* We shall now consider one such generalized MpQG, say $U_{\mathbf{q},\varGamma_\bullet}$, making assumptions on $\Bbbk$ as mentioned above, and introduce integral forms for it, quickly explaining the few changes one needs in the previously described treatment of integral forms for $U_{\mathbf{q}}$, that is the case $\varGamma_\bullet = Q \times Q$.

5.4.1. Restricted integral forms for MpQG's with larger torus. *Assume that* $\mathbf{q}$ *is of integral type*. Then a $\mathbb{Z}$–bilinear form $(\ ,\ )_B$ is defined on $\mathbb{Q}Q$, and we have well-defined sublattices $\dot{Q}^{(\ell)}$ and $\dot{Q}^{(r)}$ in $\mathbb{Q}Q$ (notation of §3.3.2). *We assume in addition that* $\varGamma_+ \subseteq \dot{Q}^{(\ell)}$ *and* $\varGamma_- \subseteq \dot{Q}^{(r)}$. Then we can define a "restricted integral form" $\hat{\dot{U}}_{\mathbf{q},\varGamma_\bullet}$ for $U_{\mathbf{q},\varGamma_\bullet}$, akin to $\hat{\dot{U}}_{\mathbf{q}}$ — so that $\varGamma_\pm = Q$ yields $\hat{\dot{U}}_{\mathbf{q},\varGamma_\bullet} = \hat{\dot{U}}_{\mathbf{q}}$ — as follows.

Let $\{\gamma_i^\pm\}_{i\in I}$ be a basis of $\varGamma_\pm$: then *in Definition 5.2.12(a), replace every occurrence of "$K_i^{\pm 1}$" with "$K_{\gamma_i^+}^{\pm 1}$" and every occurrence of "$L_i^{\pm 1}$" with "$L_{\gamma_i^-}^{\pm 1}$" — so each q–binomial coefficient* $\binom{K_i\,;\,c}{n}_q$ *is replaced by* $\binom{K_{\gamma_i^+}\,;\,c}{n}_q$, *etc.; this yields the very* **definition** *of* $\hat{\dot{U}}_{\mathbf{q},\varGamma_\bullet}$.

Basing on this definition, one easily finds that *all results presented in §5.2 above for* $\hat{\dot{U}}_{\mathbf{q}}$ *have their direct counterpart for* $\hat{\dot{U}}_{\mathbf{q},\varGamma_\bullet}$ *as well.* Moreover, the natural embedding $U_{\mathbf{q}} \subseteq U_{\mathbf{q},\varGamma_\bullet}$ between MpQG's — induced by the inclusion $Q \times Q \subseteq \varGamma_\bullet$ — clearly restricts to a similar embedding $\hat{\dot{U}}_{\mathbf{q}} \subseteq \hat{\dot{U}}_{\mathbf{q},\varGamma_\bullet}$ of integral forms. Similar comments apply to the various subalgebras of $\hat{\dot{U}}_{\mathbf{q}}$ for their natural counterparts in $\hat{\dot{U}}_{\mathbf{q},\varGamma_\bullet}$.

Similarly, *assume now that* $\mathbf{q}$ *is of strictly integral type*, so that the sublattices $Q^{(\ell)}$ and $Q^{(r)}$ are defined in $\mathbb{Q}Q$ (cf. §3.3.2); concerning $\varGamma_\pm$, this time *we assume in addition that* $\varGamma_+ \subseteq Q^{(\ell)}$ *and* $\varGamma_- \subseteq Q^{(r)}$. Then we can define a second "restricted integral form" $\widehat{U}_{\mathbf{q},\varGamma_\bullet}$ for $U_{\mathbf{q},\varGamma_\bullet}$, direct analogue to $\widehat{U}_{\mathbf{q}}$, as follows.

Given bases $\{\gamma_i^\pm\}_{i\in I}$ of $\varGamma_\pm$ as above, *in Definition 5.2.12(b) replace every occurrence of "$K_i^{\pm 1}$" with "$K_{\gamma_i^+}^{\pm 1}$" and every occurrence of "$L_i^{\pm 1}$" with "$L_{\gamma_i^-}^{\pm 1}$": in particular, every q_i–divided binomial coefficient* $\binom{K_i\,;\,c}{n}_{q_i}$ *is replaced by* $\binom{K_{\gamma_i^+}\,;\,c}{n}_{q_i}$, *etc.; then read the outcome, by assumption, as the very* **definition** *of* $\widehat{U}_{\mathbf{q},\varGamma_\bullet}$.

In force of this definition, one can easily find that *all results presented in §5.2 about* $\widehat{U}_{\mathbf{q}}$ *have their direct counterpart for* $\widehat{U}_{\mathbf{q},\Gamma_\bullet}$ *as well.* In addition, the embedding $U_{\mathbf{q}} \subseteq U_{\mathbf{q},\Gamma_\bullet}$ restricts to an embedding $\widehat{U}_{\mathbf{q}} \subseteq \widehat{U}_{\mathbf{q},\Gamma_\bullet}$ between integral forms. All this applies also to the natural counterparts in $\widehat{U}_{\mathbf{q},\Gamma_\bullet}$ of the different subalgebras of $\widehat{U}_{\mathbf{q}}$.

5.4.2. Unrestricted integral forms for MpQG's with larger torus. *Let now* $\mathbf{q}$ *be of general (though Cartan) type, and make no special assumptions on* $\Gamma_\pm$. Then we can define for $U_{\mathbf{q},\Gamma_\bullet}$ an "unrestricted integral form" $\widetilde{U}_{\mathbf{q},\Gamma_\bullet}$, akin to $\widetilde{U}_{\mathbf{q}}$ — in that $\widetilde{U}_{\mathbf{q},\Gamma_\bullet} = \widetilde{U}_{\mathbf{q}}$ when $\Gamma_\pm = Q$ — in the following, very simple way.

Let $\left\{\gamma_i^\pm\right\}_{i\in I}$ be bases of $\Gamma_\pm$, as before: now, *in Definition 5.3.1, replace every occurrence of "*$K_i^{\pm 1}$*" with "*$K_{\gamma_i^+}^{\pm 1}$*" and every occurrence of "*$L_i^{\pm 1}$*" with "*$L_{\gamma_i^-}^{\pm 1}$*"; then take the final outcome as the very* **definition** *of* $\widetilde{U}_{\mathbf{q},\Gamma_\bullet}$.

Starting from this definition, one easily checks that *all results presented in §5.3 for* $\widetilde{U}_{\mathbf{q}}$ *have a direct counterpart for* $\widetilde{U}_{\mathbf{q},\Gamma_\bullet}$ *too.* Also, the natural embedding $U_{\mathbf{q}} \subseteq U_{\mathbf{q},\Gamma_\bullet}$ between MpQG's implies by restriction a similar embedding $\widetilde{U}_{\mathbf{q}} \subseteq \widetilde{U}_{\mathbf{q},\Gamma_\bullet}$ between the corresponding unrestricted integral forms. Finally, similar comments apply to the natural counterparts in $\widetilde{U}_{\mathbf{q},\Gamma_\bullet}$ of the various subalgebras considered in $\widetilde{U}_{\mathbf{q}}$.

5.5. **Duality among integral forms.**

If we take two quantum Borel subgroups $U_{\mathbf{q}}^{\geq}$ and $U_{\mathbf{q}}^{\leq}$, we know that they are in duality via a non-degenerate skew-Hopf pairing as in §4.3. Now, *assuming that* $\mathbf{q}$ *is of integral type*, if we take on either side integral forms of opposite nature, say $\dot{\widehat{U}}_{\mathbf{q}}^{\geq}$, or $\widehat{U}_{\mathbf{q}}^{\geq}$, and $\widetilde{U}_{\mathbf{q}}^{\leq}$ — or $\widetilde{U}_{\mathbf{q}}^{\geq}$ and $\dot{\widehat{U}}_{\mathbf{q}}^{\leq}$, or $\widehat{U}_{\mathbf{q}}^{\leq}$ — we find that they are "dual to each other" with respect to that pairing. To state this properly, we need to work with MpQG's with (suitably paired) larger tori. The correct statement is the following:

Proposition 5.5.1. *Let* $\Gamma_\pm$ *be rank* θ *sublattices of* $\mathbb{Q}Q$ *containing* Q, *let* $U_{\mathbf{q},\Gamma_+}^{\geq}$ *and* $U_{\mathbf{q},\Gamma_-}^{\leq}$ *be the associated Borel MpQG's, and let* $\eta : U_{\mathbf{q},\Gamma_+}^{\geq} \otimes U_{\mathbf{q},\Gamma_-}^{\leq} \longrightarrow \Bbbk$ *be the skew-Hopf pairing of §3.3.4.*

(a) Assume that $\mathbf{q} = \left(q^{b_{ij}}\right)_{i,j\in I}$ *is of integral type, and (with notation of §3.3.2) that* $\Gamma_+ = \dot{\Gamma}_-^{(\ell)}$ *and* $\Gamma_- = \dot{\Gamma}_+^{(r)}$. *Then*

$$\begin{aligned}
\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+,0} &= \left\{ u \in U_{\mathbf{q},\Gamma_+}^{+,0} \;\middle|\; \eta\Big(u\,,\widetilde{U}_{\mathbf{q},\Gamma_-}^{-,0}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \right\} \\
\widetilde{U}_{\mathbf{q},\Gamma_-}^{-,0} &= \left\{ v \in U_{\mathbf{q},\Gamma_-}^{-,0} \;\middle|\; \eta\Big(\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+,0},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \right\} \\
\dot{\widehat{U}}_{\mathbf{q}}^{+} = \dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+} &= \left\{ u \in U_{\mathbf{q},\Gamma_+}^{+} \;\middle|\; \eta\Big(u\,,\widetilde{U}_{\mathbf{q},\Gamma_-}^{-}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \right\} \\
\widetilde{U}_{\mathbf{q}}^{-} = \widetilde{U}_{\mathbf{q},\Gamma_-}^{-} &= \left\{ v \in U_{\mathbf{q},\Gamma_-}^{-} \;\middle|\; \eta\Big(\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \right\} \\
\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{\geq} &= \left\{ u \in U_{\mathbf{q},\Gamma_+}^{\geq} \;\middle|\; \eta\Big(u\,,\widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \right\} \\
\widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq} &= \left\{ v \in U_{\mathbf{q},\Gamma_-}^{\leq} \;\middle|\; \eta\Big(\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{\geq},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \right\}
\end{aligned}$$

and similarly reversing the roles of "+" and "-" and of "$\geq$*" and "*$\leq$*".*

(b) Assume that $\mathbf{q} = \left(q^{d_i t_{ij}^+} = q^{d_j t_{ij}^-}\right)_{i,j\in I}$ *is of strongly integral type (cf. §3.3.2 for notation). If* $\Gamma_- = \Gamma_+^{(r)}$ *— cf.* (3.4) *— then*

$$\begin{aligned}
\widehat{U}_{\mathbf{q},\Gamma_+}^{+,0} &= \left\{ u \in U_{\mathbf{q},\Gamma_+}^{+,0} \;\middle|\; \eta\Big(u\,,\widetilde{U}_{\mathbf{q},\Gamma_-}^{-,0}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \right\} \\
\widetilde{U}_{\mathbf{q},\Gamma_-}^{-,0} &= \left\{ v \in U_{\mathbf{q},\Gamma_-}^{-,0} \;\middle|\; \eta\Big(\widehat{U}_{\mathbf{q},\Gamma_+}^{+,0},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \right\}
\end{aligned}$$

$$\begin{aligned}
\widehat{U}_{\mathbf{q}}^{+} \; = \; \widehat{U}_{\mathbf{q},\Gamma_+}^{+} &= \Big\{ u \in U_{\mathbf{q},\Gamma_+}^{+} \;\Big|\; \eta\Big(u\,,\widetilde{U}_{\mathbf{q},\Gamma_-}^{-}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\} \\
\widetilde{U}_{\mathbf{q}}^{-} \; = \; \widetilde{U}_{\mathbf{q},\Gamma_-}^{-} &= \Big\{ v \in U_{\mathbf{q},\Gamma_-}^{-} \;\Big|\; \eta\Big(\widehat{U}_{\mathbf{q},\Gamma_+}^{+},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\} \\
\widehat{U}_{\mathbf{q},\Gamma_+}^{\geq} &= \Big\{ u \in U_{\mathbf{q},\Gamma_+}^{\geq} \;\Big|\; \eta\Big(u\,,\widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\} \\
\widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq} &= \Big\{ v \in U_{\mathbf{q},\Gamma_-}^{\leq} \;\Big|\; \eta\Big(\widehat{U}_{\mathbf{q},\Gamma_+}^{\geq},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\}
\end{aligned}$$

If instead $\Gamma_+ = \Gamma_-^{(\ell)}$ *— cf.* (3.4) *again — then*

$$\begin{aligned}
\widehat{U}_{\mathbf{q},\Gamma_-}^{-,0} &= \Big\{ v \in U_{\mathbf{q},\Gamma_-}^{-,0} \;\Big|\; \eta\Big(\widetilde{U}_{\mathbf{q},\Gamma_+}^{+,0},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\} \\
\widetilde{U}_{\mathbf{q},\Gamma_+}^{+,0} &= \Big\{ u \in U_{\mathbf{q},\Gamma_+}^{+,0} \;\Big|\; \eta\Big(u\,,\widehat{U}_{\mathbf{q},\Gamma_-}^{-,0}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\} \\
\widehat{U}_{\mathbf{q}}^{-} \; = \; \widehat{U}_{\mathbf{q},\Gamma_-}^{-} &= \Big\{ v \in U_{\mathbf{q},\Gamma_-}^{-} \;\Big|\; \eta\Big(\widetilde{U}_{\mathbf{q},\Gamma_+}^{+},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\} \\
\widetilde{U}_{\mathbf{q}}^{+} \; = \; \widetilde{U}_{\mathbf{q},\Gamma_+}^{+} &= \Big\{ u \in U_{\mathbf{q},\Gamma_+}^{+} \;\Big|\; \eta\Big(u\,,\widehat{U}_{\mathbf{q},\Gamma_-}^{-}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\} \\
\widehat{U}_{\mathbf{q},\Gamma_-}^{\leq} &= \Big\{ v \in U_{\mathbf{q},\Gamma_-}^{\leq} \;\Big|\; \eta\Big(\widetilde{U}_{\mathbf{q},\Gamma_+}^{\geq},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\} \\
\widetilde{U}_{\mathbf{q},\Gamma_+}^{\geq} &= \Big\{ u \in U_{\mathbf{q},\Gamma_+}^{\geq} \;\Big|\; \eta\Big(u\,,\widehat{U}_{\mathbf{q},\Gamma_-}^{\leq}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\}
\end{aligned}$$

Proof. (a) The assumptions imply $\big(\Gamma_+\,,\Gamma_-\big)_B \subseteq \mathbb{Z}$, hence $\eta\big(K_{\gamma_i^+}, L_{\gamma_j^-}\big) = q^{(\gamma_i^+,\gamma_j^-)_B}$ — cf. §3.3.4 — that in turn implies $\eta\left(\begin{pmatrix} K_{\gamma_i^+};0 \\ n \end{pmatrix}_q, L_{\gamma_j^-}\right) = \begin{pmatrix} (\gamma_i^+,\gamma_j^-)_B \\ n \end{pmatrix}_q \in \mathcal{R}_{\mathbf{q}}$. Taking PBW bases on both sides, this is enough to prove $\eta\Big(\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+,0}, \widetilde{U}_{\mathbf{q},\Gamma_-}^{-,0}\Big) \subseteq \mathcal{R}_{\mathbf{q}}$; therefore we get $\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+,0} \subseteq \Big\{ u \in U_{\mathbf{q},\Gamma_+}^{+,0} \;\Big|\; \eta\Big(u\,,\widetilde{U}_{\mathbf{q},\Gamma_-}^{-,0}\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\}$ and on the other hand also $\widetilde{U}_{\mathbf{q},\Gamma_-}^{-,0} \subseteq \Big\{ v \in U_{\mathbf{q},\Gamma_-}^{-,0} \;\Big|\; \eta\Big(\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+,0},v\Big) \subseteq \mathcal{R}_{\mathbf{q}} \Big\}$. This proves "half" the result we claimed true, thus we still need some additional work to do.

Since $\Gamma_+ = \dot{\Gamma}_-^{(\ell)}$ and $\Gamma_- = \dot{\Gamma}_+^{(r)}$, we can fix bases $\big\{\gamma_i^\pm\big\}_{i \in I}$ of $\Gamma_\pm$ that are dual to each other, namely $\big(\gamma_h^+\,,\gamma_k^-\big)_B = \delta_{h,k}$ for all $h, k \in I$; so we get $\eta\left(K_{\gamma_h^+}^{z_h^+}, L_{\gamma_k^-}^{z_k^-}\right) = q^{\delta_{h,k}\, z_h^+ z_k^-}$. As a consequence, the arguments used for Proposition 5.2.3 and Proposition 5.2.5 apply again (with η replacing the pairing $\langle\ ,\ \rangle$ and the $K_{\gamma_h^+}$'s, resp. the $L_{\gamma_k^-}$'s, playing the role of the X_i's, resp. of the χ_j's) now proving claim *(a)*. Indeed, the analysis developed for those results now shows that $\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+,0}$ and $\widetilde{U}_{\mathbf{q},\Gamma_-}^{-,0}$ contain bases that, up to invertible coefficients (powers of q), are dual to each other, and that is enough to conclude.

The claim about $\dot{\widehat{U}}_{\mathbf{q}}^{+} = \dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+}$ and $\widetilde{U}_{\mathbf{q}}^{-} = \widetilde{U}_{\mathbf{q},\Gamma_-}^{-}$ (both independent of $\Gamma_\pm$) is a consequence of PBW theorems for both sides and of Proposition 4.3.1. Then from this result, the one for $\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{+,0}$ and $\widetilde{U}_{\mathbf{q},\Gamma_-}^{-,0}$ and the triangular decompositions in Proposition 5.2.15 and Proposition 5.3.4, we finally get the statement concerning $\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{\geq}$ and $\widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq}$ as well.

The statement with switched "$+$" and "-" or "$\geq$" and "$\leq$" goes the same way.

(b) Up to minimal changes, this is proved much like claim *(a)*. □

Remark 5.5.2. One can use the previous result to deduce properties of a (Hopf) algebra on either side — e.g. $\widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq}$, say — out of properties on the other side — $\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{\geq}$ or $\widehat{U}_{\mathbf{q},\Gamma_+}^{\geq}$ in the example. For instance, $\widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq}$ is an $\mathcal{R}_{\mathbf{q}}$–algebra (hard to prove directly!) because $\widehat{U}_{\mathbf{q},\Gamma_+}^{\geq}$ is an $\mathcal{R}_{\mathbf{q}}$–coalgebra (that follows from its definition). Similarly, we deduce that $\widetilde{U}_{\mathbf{q}}^{+}$ *is independent of any choice of quantum root vectors* (that do enter in the definition!) because it is "the dual" of $\widehat{U}_{\mathbf{q}}^{-}$ and *the latter is independent*, by definition, of any such choice.

5.6. **Integral forms of "mixed" type.**

Let us consider two quantum Borel subgroups $U^{\geq}_{\mathbf{q},\Gamma_+}$ and $U^{\leq}_{\mathbf{q},\Gamma_-}$ as in §5.5 above, with $\mathbf{q}$ *integral*, linked by the skew-Hopf pairing η of §3.3.4. Assuming in addition that the lattices $\Gamma_\pm$ fit the conditions required in Theorem 5.5.1 (according to whether $\mathbf{q}$ is strongly integral or not), that theorem tells us that the pairing η yields by restriction $\mathcal{R}_{\mathbf{q}}$–valued skew-Hopf pairings, still denoted η, for the pairs of $\mathcal{R}_{\mathbf{q}}$–Hopf algebras $\Big(\dot{\widehat{U}}{}^{\geq}_{\mathbf{q},\Gamma_+}, \widetilde{U}^{\leq}_{\mathbf{q},\Gamma_-}\Big)$ and $\Big(\widetilde{U}^{\geq}_{\mathbf{q},\Gamma_+}, \dot{\widehat{U}}{}^{\leq}_{\mathbf{q},\Gamma_-}\Big)$, or the pairs $\Big(\widehat{U}^{\geq}_{\mathbf{q},\Gamma_+}, \widetilde{U}^{\leq}_{\mathbf{q},\Gamma_-}\Big)$ and $\Big(\widetilde{U}^{\geq}_{\mathbf{q},\Gamma_+}, \widehat{U}^{\leq}_{\mathbf{q},\Gamma_-}\Big)$ when $\mathbf{q}$ is strongly integral. Moreover, as the original pairing η is non-degenerate, the same holds true for its restrictions to $\mathcal{R}_{\mathbf{q}}$–integral forms of the original quantum Borel subgroups. Therefore, much like each MpQG $U_{\mathbf{q}}(\mathfrak{g})$ can be realized as Drinfeld double via the original pairing η (cf. Remark §3.3.5), the restrictions of the latter lead us to define the following:

Definition 5.6.1. With assumption as above — thus $\mathbf{q}$ *is of integral type* — we define the following Hopf algebras over $\mathcal{R}_{\mathbf{q}}$ as *Drinfeld doubles* (cf. §2.1)

$$\begin{aligned}\dot{\overrightarrow{U}}_{\mathbf{q},\Gamma_\bullet} &:= \dot{\overrightarrow{U}}_{\mathbf{q},\Gamma_\bullet}(\mathfrak{g}) \;=\; D\Big(\dot{\widehat{U}}{}^{\geq}_{\mathbf{q},\Gamma_+}, \widetilde{U}^{\leq}_{\mathbf{q},\Gamma_-}, \eta\Big)\\ \dot{\overleftarrow{U}}_{\mathbf{q},\Gamma_\bullet} &:= \dot{\overleftarrow{U}}_{\mathbf{q},\Gamma_\bullet}(\mathfrak{g}) \;=\; D\Big(\widetilde{U}^{\geq}_{\mathbf{q},\Gamma_+}, \dot{\widehat{U}}{}^{\leq}_{\mathbf{q},\Gamma_-}, \eta\Big)\end{aligned}$$

where $\Gamma_\bullet := \Gamma_+ \times \Gamma_-$. If in addition the multiparameter $\mathbf{q}$ is also *strongly integral*, then we define similarly also the Hopf $\mathcal{R}_{\mathbf{q}}$–algebras (again as Drinfeld doubles)

$$\begin{aligned}\overrightarrow{U}_{\mathbf{q},\Gamma_\bullet} &:= \overrightarrow{U}_{\mathbf{q},\Gamma_\bullet}(\mathfrak{g}) \;=\; D\Big(\widehat{U}^{\geq}_{\mathbf{q},\Gamma_+}, \widetilde{U}^{\leq}_{\mathbf{q},\Gamma_-}, \eta\Big)\\ \overleftarrow{U}_{\mathbf{q},\Gamma_\bullet} &:= \overleftarrow{U}_{\mathbf{q},\Gamma_\bullet}(\mathfrak{g}) \;=\; D\Big(\widetilde{U}^{\geq}_{\mathbf{q},\Gamma_+}, \widehat{U}^{\leq}_{\mathbf{q},\Gamma_-}, \eta\Big)\end{aligned} \qquad \diamondsuit$$

The following claim points out the main properties of these new objects:

Theorem 5.6.2. *Keep assumptions and notations as above. Then* $\dot{\overrightarrow{U}}_{\mathbf{q},\Gamma_\bullet}$ *, resp.* $\dot{\overleftarrow{U}}_{\mathbf{q},\Gamma_\bullet}$ *, is an* $\mathcal{R}_{\mathbf{q}}$*–integral form (as Hopf algebra) of* $U_{\mathbf{q},\Gamma_\bullet}$ *, with PBW-type basis*

$$\Bigg\{\, \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \prod_{j\in I} \binom{L_{\gamma_j^+}}{l_j}_{\!q} L_{\gamma_j^+}^{-\lfloor l_j/2\rfloor} \prod_{i\in I} K_{\gamma_i^-}^{k_i} \prod_{t=N}^{1} \overline{F}_{\beta^t}^{\,f_t} \;\Bigg|\; e_h, l_j, k_i, f_t \in \mathbb{N} \,\Bigg\} \;,$$

resp.
$$\Bigg\{\, \prod_{h=1}^{N} \overline{E}_{\beta^h}^{\,e_h} \prod_{j\in I} L_{\gamma_j^+}^{l_j} \prod_{i\in I} \binom{K_{\gamma_i^-}}{k_i}_{\!q} K_{\gamma_i^-}^{-\lfloor k_i/2\rfloor} \prod_{t=N}^{1} F_{\beta^t}^{(f_t)} \;\Bigg|\; e_h, l_j, k_i, f_t \in \mathbb{N} \,\Bigg\}$$

(notation of §5.4.1) as well as variations of these, changing the order of factors in the PBW monomials. Similarly, if $\mathbf{q}$ *is strongly integral then* $\overrightarrow{U}_{\mathbf{q},\Gamma_\bullet}$ *, resp.* $\overleftarrow{U}_{\mathbf{q},\Gamma_\bullet}$ *, is an* $\mathcal{R}_{\mathbf{q}}$*–integral form (as Hopf algebra) of* $U_{\mathbf{q},\Gamma_\bullet}$ *, with PBW-type basis*

$$\Bigg\{\, \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \prod_{j\in I} \binom{L_{\gamma_j^+}}{l_j}_{\!q_i} L_{\gamma_j^+}^{-\lfloor l_j/2\rfloor} \prod_{i\in I} K_{\gamma_i^-}^{k_i} \prod_{t=N}^{1} \overline{F}_{\beta^t}^{\,f_t} \;\Bigg|\; e_h, l_j, k_i, f_t \in \mathbb{N} \,\Bigg\} \;,$$

resp.
$$\Bigg\{\, \prod_{h=1}^{N} \overline{E}_{\beta^h}^{\,e_h} \prod_{j\in I} L_{\gamma_j^+}^{l_j} \prod_{i\in I} \binom{K_{\gamma_i^-}}{k_i}_{\!q_i} K_{\gamma_i^-}^{-\lfloor k_i/2\rfloor} \prod_{t=N}^{1} F_{\beta^t}^{(f_t)} \;\Bigg|\; e_h, l_j, k_i, f_t \in \mathbb{N} \,\Bigg\}$$

(as well as variations of these, changing the order of factors in the PBW monomials).

In addition, $\dot{\overrightarrow{U}}_{\mathbf{q},\Gamma_\bullet}$ *, resp.* $\dot{\overleftarrow{U}}_{\mathbf{q},\Gamma_\bullet}$ *, resp.* $\overrightarrow{U}_{\mathbf{q},\Gamma_\bullet}$ *, resp.* $\overleftarrow{U}_{\mathbf{q},\Gamma_\bullet}$ *, coincides with the* $\mathcal{R}_{\mathbf{q}}$*–subalgebra of* $U_{\mathbf{q},\Gamma_\bullet}$ *generated by* $\dot{\widehat{U}}{}^{\geq}_{\mathbf{q},\Gamma_+}$ *and* $\widetilde{U}^{\leq}_{\mathbf{q},\Gamma_-}$ *, resp. by* $\widetilde{U}^{\geq}_{\mathbf{q},\Gamma_+}$ *and* $\dot{\widehat{U}}{}^{\leq}_{\mathbf{q},\Gamma_-}$ *, resp. by* $\widehat{U}^{\geq}_{\mathbf{q},\Gamma_+}$ *and* $\widetilde{U}^{\leq}_{\mathbf{q},\Gamma_-}$ *, resp. by* $\widetilde{U}^{\geq}_{\mathbf{q},\Gamma_+}$ *and* $\widehat{U}^{\leq}_{\mathbf{q},\Gamma_-}$ *.*

Proof. Indeed, the result follows at once by construction, together with the fact that $\dot{\widehat{U}}{}^{\geq}_{\mathbf{q},\Gamma_+}$, $\widetilde{U}^{\leq}_{\mathbf{q},\Gamma_-}$, etc., actually are integral forms of the corresponding quantum Borel subgroups defined over $\Bbbk$, and with the PBW Theorems for them. $\square$

We are also interested in yet other mixed integral forms, defined as follows. Inside $\dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{\geq}$ (or inside $\widehat{U}_{\mathbf{q},\Gamma_+}^{\geq}$, it is the same), denote by $\widehat{U}_{\mathbf{q},\Gamma_+}^{\geq,\uparrow}$ the $\mathcal{R}_{\mathbf{q}}$–subalgebra generated by $\widehat{U}_{\mathbf{q}}^{+}$ and $\widetilde{U}_{\mathbf{q},\Gamma_+}^{0}$ (recall that by Remark 5.2.9, one has $\widetilde{U}_{\mathbf{q},\Gamma_+}^{0} \subseteq \dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{0} \subseteq \widehat{U}_{\mathbf{q},\Gamma_+}^{0}$); this is indeed an $\mathcal{R}_{\mathbf{q}}$–integral form of $U_{\mathbf{q},\Gamma_+}^{\geq}$ (as a Hopf subalgebra), and the non-degenerate skew-Hopf pairing $\eta : \dot{\widehat{U}}_{\mathbf{q},\Gamma_+}^{\geq} \otimes_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq} \longrightarrow \mathcal{R}_{\mathbf{q}}$ restricts to a similar pairing $\eta : \widehat{U}_{\mathbf{q},\Gamma_+}^{\geq,\uparrow} \otimes_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq} \longrightarrow \mathcal{R}_{\mathbf{q}}$. Similarly, we consider the $\mathcal{R}_{\mathbf{q}}$–subalgebra $\widehat{U}_{\mathbf{q},\Gamma_-}^{\leq,\uparrow}$ of $\dot{\widehat{U}}_{\mathbf{q},\Gamma_-}^{\leq}$ (or of $\widehat{U}_{\mathbf{q},\Gamma_-}^{\leq}$) generated by $\widehat{U}_{\mathbf{q}}^{-}$ and $\widetilde{U}_{\mathbf{q},\Gamma_-}^{0}$, which again is an $\mathcal{R}_{\mathbf{q}}$–integral form of $U_{\mathbf{q},\Gamma_-}^{\leq}$ for which we have a non-degenerate skew-Hopf pairing $\eta : \widetilde{U}_{\mathbf{q},\Gamma_+}^{\geq} \otimes_{\mathcal{R}_{\mathbf{q}}} \widehat{U}_{\mathbf{q},\Gamma_-}^{\leq,\uparrow} \longrightarrow \mathcal{R}_{\mathbf{q}}$ induced by the original skew-Hopf pairing between our multiparameter quantum Borel subgroups over $\Bbbk$. *In addition, we do not assume that the multiparameter* $\mathbf{q}$ *be of integral type, nor we assume* Γ_+ *and* Γ_- *to be in duality (as in §3.3.3).* All this allows the following

Definition 5.6.3. For *any* multiparameter $\mathbf{q}$ (of Cartan type) and $\Gamma_\bullet := \Gamma_+ \times \Gamma_-$, we define the following Hopf algebras over $\mathcal{R}_{\mathbf{q}}$ as *Drinfeld doubles* with respect to the above mentioned skew-Hopf pairings:

$$\check{U}_{\mathbf{q},\Gamma_\bullet}^{(+)} := \check{U}_{\mathbf{q},\Gamma_\bullet}^{(+)}(\mathfrak{g}) = D\Big(\widehat{U}_{\mathbf{q},\Gamma_+}^{\geq,\uparrow}, \widetilde{U}_{\mathbf{q},\Gamma_-}^{\leq}, \eta\Big)$$
$$\check{U}_{\mathbf{q},\Gamma_\bullet}^{(-)} := \check{U}_{\mathbf{q},\Gamma_\bullet}^{(-)}(\mathfrak{g}) = D\Big(\widetilde{U}_{\mathbf{q},\Gamma_+}^{\geq}, \widehat{U}_{\mathbf{q},\Gamma_-}^{\leq,\uparrow}, \eta\Big) \qquad \diamondsuit$$

The main properties of these more Hopf algebras are summarized as follows:

Theorem 5.6.4. *Keep notation as above.*

(a) The Hopf algebras $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(+)}$ *and* $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(-)}$ *are both* $\mathcal{R}_{\mathbf{q}}$*–integral forms (as Hopf algebras) of* $U_{\mathbf{q},\Gamma_\bullet}$*, with PBW-type basis*

$$\left\{ \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \prod_{j\in I} L_{\gamma_j^+}^{l_j} \prod_{i\in I} K_{\gamma_i^-}^{k_i} \prod_{t=N}^{1} \overline{F}_{\beta^t}^{f_t} \;\middle|\; e_h, l_j, k_i, f_t \in \mathbb{N} \right\} ,$$
$$\textit{and} \qquad \left\{ \prod_{h=1}^{N} \overline{E}_{\beta^h}^{e_h} \prod_{j\in I} L_{\gamma_j^+}^{l_j} \prod_{i\in I} K_{\gamma_i^-}^{k_i} \prod_{t=N}^{1} F_{\beta^t}^{(f_t)} \;\middle|\; e_h, l_j, k_i, f_t \in \mathbb{N} \right\}$$

(notation of §5.4.1) for $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(+)}$ *and* $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(-)}$ *respectively, as well as variations of these (changing the order of factors in the PBW monomials).*

(b) $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(+)}$*, resp.* $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(-)}$*, coincides with the* $\mathcal{R}_{\mathbf{q}}$*–subalgebra of* $U_{\mathbf{q},\Gamma_\bullet}$ *generated by* $\widehat{U}_{\mathbf{q},\Gamma_\bullet}^{+}$*,* $\widetilde{U}_{\mathbf{q},\Gamma_\bullet}^{0}$ *and* $\widetilde{U}_{\mathbf{q},\Gamma_\bullet}^{-}$*, resp. by* $\widetilde{U}_{\mathbf{q},\Gamma_\bullet}^{+}$*,* $\widetilde{U}_{\mathbf{q},\Gamma_\bullet}^{0}$ *and* $\widehat{U}_{\mathbf{q},\Gamma_\bullet}^{-}$*.*

(c) Both $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(+)}$ *and* $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(-)}$ *have obvious triangular decompositions analogous to those in Propositions 5.2.15 and 5.3.4.*

Proof. Here again, everything follows easily by construction, through our previous results on integral forms of multiparameter quantum Borel subgroups. $\square$

Remark 5.6.5. Defining $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(+)}$ and $\check{U}_{\mathbf{q},\Gamma_\bullet}^{(-)}$, as well as $\dot{\overrightarrow{U}}_{\mathbf{q},\Gamma_\bullet}$, $\dot{\overleftarrow{U}}_{\mathbf{q},\Gamma_\bullet}$, $\overrightarrow{U}_{\mathbf{q},\Gamma_\bullet}$ and $\overleftarrow{U}_{\mathbf{q},\Gamma_\bullet}$, as *Drinfeld doubles* provides great advantages, namely we get for free that

(1) they are *Hopf algebras*,

(2) they have nice PBW bases (and triangular decompositions),

(3) they are $\mathcal{R}_{\mathbf{q}}$–integral forms of $U_{\mathbf{q},\Gamma_\bullet}$ — since they are tensor products (as Drinfeld doubles!) of integral forms of multiparameter quantum Borel subgroups.

In fact, we already saw that these algebras coincide with suitable $\mathcal{R}_{\mathbf{q}}$–subalgebras in $U_{\mathbf{q},\Gamma_\bullet}$; yet, proving properties *(1)–(3)* by direct approach would *not* be trivial.

5.6.6. The link with the uniparameter case. For the uniparameter quantum group $U_q(\mathfrak{g})$ of Drinfeld and Jimbo, possibly with larger torus, one can define $\mathcal{R}_{\mathbf{q}}$–integral forms $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$, $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ and $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ much like we did with our MpQG's, constructing them as generated by quantum divided powers and binomial coefficients or by renormalized quantum root vectors; note that now $\mathcal{R}_{\mathbf{q}} = \mathbb{Z}\big[q, q^{-1}\big]$. Similarly, one can define another $\mathcal{R}_{\mathbf{q}}$–subalgebra of $U_q(\mathfrak{g})$, denoted by $\check{U}_q^{(-)}(\mathfrak{g})$, generated by $\widehat{U}_q^-$, $\widetilde{U}_q^0$ and $\widetilde{U}_q^+$, first introduced in [HL]. This is again an $\mathcal{R}_{\mathbf{q}}$–integral form of $U_q(\mathfrak{g})$, for which triangular decomposition and PBW Theorems hold true, deduced from the similar results for $\widehat{U}_q(\mathfrak{g})$ and $\widetilde{U}_q(\mathfrak{g})$. One also has its "symmetric counterpart", say $\check{U}_q^{(+)}(\mathfrak{g})$, generated by $\widetilde{U}_q^-$, $\widetilde{U}_q^0$ and $\widehat{U}_q^+$.

The construction of $\mathcal{R}_{\mathbf{q}}$–integral forms (again with $\mathcal{R}_{\mathbf{q}} = \mathbb{Z}\big[q, q^{-1}\big]$) of restricted or unrestricted type also extends to the context of *twisted quantum groups* $U_{q,M}^{\varphi}(\mathfrak{g})$ *à la Costantini-Varagnolo* (see [CV1, CV2]), still denoted $\widehat{U}_{q,M}^{\varphi}(\mathfrak{g})$ and $\widetilde{U}_{q,M}^{\varphi}(\mathfrak{g})$ in the restricted and the unrestricted case respectively. Then one has also corresponding integral forms for the various relevant (Hopf) subalgebras (Borel, nilpotent, etc.), triangular decompositions, PBW bases, etc. — see [Gav] for details. Moreover, one can define also in this context *mixed integral forms* $\check{U}_{q,M}^{(-),\varphi}(\mathfrak{g})$ and $\check{U}_{q,M}^{(+),\varphi}(\mathfrak{g})$, namely

(a) $\check{U}_{q,M}^{(-),\varphi}(\mathfrak{g})$ is the $\mathcal{R}_{\mathbf{q}}$–subalgebra of $U_{q,M}^{\varphi}(\mathfrak{g})$ generated by $\widehat{U}_q^-$, $\widetilde{U}_{q,M}^0$, $\widetilde{U}_q^+$,

(b) $\check{U}_{q,M}^{(+),\varphi}(\mathfrak{g})$ is the $\mathcal{R}_{\mathbf{q}}$–subalgebra of $U_{q,M}^{\varphi}(\mathfrak{g})$ generated by $\widetilde{U}_q^-$, $\widetilde{U}_{q,M}^0$, $\widehat{U}_q^+$,

(note that the occurrence of φ is irrelevant at the algebra level, so that $\check{U}_{q,M}^{(\pm),\varphi}(\mathfrak{g}) = \check{U}_{q,M}^{(\pm)}(\mathfrak{g})$ as $\mathcal{R}_{\mathbf{q}}$–algebras; the coalgebra structure, on the contrary, is affected).

Using the properties of $\widehat{U}_q^{\mp}$, $\widetilde{U}_{q,M}^0$ and $\widetilde{U}_q^{\pm}$ presented in [Gav], one can prove that $\check{U}_{q,M}^{(-),\varphi}(\mathfrak{g})$ and $\check{U}_{q,M}^{(+),\varphi}(\mathfrak{g})$ are again $\mathcal{R}_{\mathbf{q}}$–integral forms — as Hopf algebras — of $U_{q,M}^{\varphi}(\mathfrak{g})$. In fact, the trivial case $\varphi = 0$ gives $U_{q,M}^{\varphi=0}(\mathfrak{g}) = U_{q,M}(\mathfrak{g})$, the standard uniparameter quantum group associated with M, so the case of $U_{q,M}^{\varphi}(\mathfrak{g})$ and its $\mathcal{R}_{\mathbf{q}}$–integral forms (restricted, or unrestricted, or mixed) is a direct generalization of what occurs with $U_{q,M}(\mathfrak{g})$.

On the other hand, it is proved in [GG1] that Costantini-Varagnolo's twisted quantum groups $U_{q,M}^{\varphi}(\mathfrak{g})$ are just quotients of MpQG's $U_{\mathbf{q},M\times Q}(\mathfrak{g})$ with $\mathbf{q}$ ranging in a special subset of strongly integral type multiparameters. It follows that the same link exists among their integral forms of either type, including the mixed one.

5.6.7. Applications to topological invariants. As mentioned above, the mixed form $\check{U}_q^{(-)}(\mathfrak{g})$ was introduced in [HL]. In that paper, the authors provide a construction of a "universal quantum invariant" of integral homology spheres, call it J_M : this "lifts" the well-known Witt-Reshetikhin-Turaev (=WRT) knot invariant $\tau_M^{\mathfrak{g}}(\varepsilon)$ of M associated with $\mathfrak{g}$ and any root of unity ε , in that $\tau_M^{\mathfrak{g}}(\varepsilon)$ is obtained by evaluation of J_M at ε . Unlike the definition of the WRT invariant, the construction of this "universal" invariant J_M does not involve representations, so it provides a unified, representation-free definition of quantum invariants of integral homology spheres, performed in terms of the form $\check{U}_q^{(-)}(\mathfrak{g})$.

Now, having introduced "multiparameter mixed integral forms" $\check{U}_{q,M}^{(\pm),\varphi}(\mathfrak{g})$ and even $\check{U}_{\mathbf{q},M\times Q}^{(\pm)}(\mathfrak{g})$, *we might expect that the construction of J_M could be extended, starting from $\check{U}_{q,M}^{(\pm),\varphi}(\mathfrak{g})$ or even $\check{U}_{\mathbf{q},M\times Q}^{(\pm)}(\mathfrak{g})$ instead of $\check{U}_q^{(-)}(\mathfrak{g})$, thus providing entirely new topological invariants for knots (and links) and integral homology spheres.*

6. Specialization of MpQG's at 1

In this section we study those MpQG's for which all the q_{ii}'s are 1; in fact, as every q_{ii} is a power of a single $q \in \Bbbk^\times$, requiring $q_{ii} = 1$ for all i amounts to requiring $q = 1$.

Note that if $q_{ii} = 1$ for some i, the very definition of $U_{\mathbf{q}}(\mathfrak{g})$ makes no sense, so we have to be more subtle. First we take $U_{\mathbf{q}}(\mathfrak{g})$ as defined over a "generic" multiparameter

$\mathbf{q} := \big(q_{ij}\big)_{i,j\in I}$ of Cartan type; then we consider its $\mathbb{Z}$–forms $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$, $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$ and $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$, defined over $\mathcal{R}_{\mathbf{q}}$ (under suitable "integrality" assumptions on $\mathbf{q}$ for the first two cases); finally, for either form we specialize q — hence all the q_{ii}'s — to a root of unity (or to 1, in particular), which will make sense just because our ground ring will be set to be $\mathcal{R}_{\mathbf{q}}$.

6.1. **The "generic" ground rings.**

As a first step in the process sketched above, we formalize the loose ideas of "generic parameter of Cartan type" and of "generic parameter of (a specific) integral type". Indeed, this "universal ring of multiparameters" will be the ring of functions on the $\mathbb{Z}$–scheme of all multiparameter $\mathbf{q}$ of Cartan type, or of (fixed) integral type.

Similarly, we introduce also the (universal) rings generated by "square roots of indeterminate parameters", both for the Cartan type and for the integral type case.

6.1.1. The universal ring of multiparameters (of Cartan type). Let hereafter $\mathbb{Z}\big[\mathbf{x}^{\pm 1}\big] := \mathbb{Z}\big[\big\{x_{ij}^{\pm 1}\big\}_{i,j\in I}\big]$ be the ring of Laurent polynomials with coefficients in $\mathbb{Z}$ in the indeterminates x_{ij} $(\, i,j \in I\,)$, and let $A := \big(a_{ij}\big)_{i,j\in I}$ be an indecomposable Cartan matrix of finite type. Consider the quotient ring

$$\mathbb{Z}_A\big[\mathbf{q}^{\pm 1}\big] \;:=\; \mathbb{Z}\big[\mathbf{x}^{\pm 1}\big]\Big/\Big(\big\{x_{ij}\,x_{ji} - x_{ii}^{a_{ij}}\big\}_{i,j\in I}\Big)$$

in which we denote by q_{ij} the image of every x_{ij} (for $i,j \in I$). This is the ring of global sections of an affine scheme over $\mathbb{Z}$, call it $\mathfrak{C}_A$: by definition, the set of $\Bbbk$–points of this scheme (for any field $\Bbbk$) is just the set of all matrices $\mathbf{q} = \big(q_{ij}\big)_{i,j\in I}$ of parameters of Cartan type A with entries in $\Bbbk$ as in §2.3.2.

From all the identities $q_{ij}\,q_{ji} = q_{ii}^{a_{ij}}$ in $\mathbb{Z}_A\big[\mathbf{q}^{\pm 1}\big]$, one finds — by direct inspection of different cases of possible Cartan matrices $A = \big(a_{ij}\big)_{i,j\in I}$ — that there exists $j_0 \in I$ such that $q_{ii} = q_{j_0 j_0}^{n_i}$ for some $n_i \in \mathbb{N}$, for all $i \in I$; indeed, we can take $n_i = d_i$ $(\, i \in I\,)$ as in §2.3.2. From this and the relations between the q_{ij}'s, it is easy to argue that $\mathfrak{C}_A$ is a torus, of dimension $\binom{\theta}{2} + 1$: in particular, it is irreducible. Then $\mathbb{Z}_A\big[\mathbf{q}^{\pm 1}\big]$ is a domain, so we can take its field of fractions, denoted by $\mathbb{Q}_A(\mathbf{q})$; in the following, we denote again by q_{ij} $(\, i,j \in I\,)$ the image of x_{ij} in $\mathbb{Q}_A(\mathbf{q})$ too.

By construction, the matrix $\mathbf{q} := \big(q_{ij}\big)_{i,j\in I}$ is a Cartan type matrix of parameters in $\Bbbk := \mathbb{Q}_A(\mathbf{q})$ in the sense of §2.3.2; in addition, *none of the q_{ii}'s is a root of unity.*

Now consider the ring extension $\mathcal{R}_{\mathbf{q}}$ of $\mathbb{Z}_A\big[\mathbf{q}^{\pm 1}\big]$ generated by a (formal) square root of $q_{j_0 j_0}$ — hereafter denoted by $q := q_{j_0 j_0}^{1/2}$ — namely

$$\mathcal{R}_{\mathbf{q}} \;:=\; \big(\mathbb{Z}_A\big[\mathbf{q}^{\pm 1}\big]\big)[x]\Big/\Big(x^2 - q_{j_0 j_0}\Big) \qquad \text{so that} \qquad q_{j_0 j_0}^{1/2} := \overline{x} \in \mathcal{R}_{\mathbf{q}}$$

and then let $\mathcal{F}_{\mathbf{q}}$ be its field of fractions, such that $\mathcal{F}_{\mathbf{q}} \cong \big(\mathbb{Q}_A(\mathbf{q})\big)[x]\Big/\Big(x^2 - q_{j_0 j_0}\Big)$. We still denote by q_{ij} the images in $\mathcal{R}_{\mathbf{q}}$ and in $\mathcal{F}_{\mathbf{q}}$ of the "old" elements with same name in $\mathbb{Z}_A\big[\mathbf{q}^{\pm 1}\big]$ and $\mathbb{Q}_A(\mathbf{q})$. We shall also write $q_i^{\pm 1} := q^{\pm d_i}$ for all $i \in I$, so to be consistent with §2.3.2; in particular, $q_{j_0 j_0}^{1/2} := q = q_{j_0}$. Note in addition that we also have

$$\mathcal{R}_{\mathbf{q}} \;\cong\; \big(\mathbb{Z}\big[\,\big\{x_{ij}^{\pm 1}\big\}_{i,j\in I}\,\big]\big)[x]\Big/\Big(\,\big\{x_{ij}\,x_{ji} - x_{ii}^{a_{ij}}\big\}_{i,j\in I} \cup \big\{x^2 - x_{j_0 j_0}\big\}\Big)$$

In turn, we define also

$$\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }} \;:=\; \mathbb{Z}\big[\big\{\xi_{ij}^{\pm 1/2}\big\}_{i,j\in I}\big]\big[\xi^{\pm 1/2}\big]\Big/\Big(\Big\{\xi_{ij}^{1/2}\,\xi_{ji}^{1/2} - \big(\xi_{ii}^{1/2}\big)^{a_{ij}}\Big\}_{i,j\in I} \cup \Big\{\big(\xi^{1/2}\big)^2 - \xi_{j_0 j_0}^{1/2}\Big\}\Big)$$

which is again a domain, and $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$ as being the field of fractions of the former. In both cases, we denote by $q_{ij}^{\pm 1/2}$ and by $q^{\pm 1/2}$ the image of $\xi_{ij}^{\pm 1/2}$ and $\xi^{\pm 1/2}$ respectively (in short, one reads these symbols as " $\xi := \sqrt{x}$ "). Note that $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ and $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$ are naturally isomorphic with $\mathcal{R}_{\mathbf{q}}$ and $\mathcal{F}_{\mathbf{q}}$ respectively, but we rather see the formers as ring or field extensions of the latters via the natural embeddings $\mathcal{R}_{\mathbf{q}} \lhook\joinrel\longrightarrow \mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ and $\mathcal{F}_{\mathbf{q}} \lhook\joinrel\longrightarrow \mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$ given — in both cases — by $q_{ij}^{\pm 1} \mapsto \big(q_{ij}^{\pm 1/2}\big)^2$ and $q^{\pm 1} \mapsto \big(q^{\pm 1/2}\big)^2$.

Finally, observe also that the ring $\mathbb{Z}\big[q, q^{-1}\big]$, resp. $\mathbb{Z}\big[q^{1/2}, q^{-1/2}\big]$, of Laurent polynomial in the indeterminate q, resp. $q^{1/2}$, naturally embeds in $\mathcal{R}_{\mathbf{q}}$, resp. in $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$, and the same occurs with their corresponding fields of fractions. Then each module over $\mathbb{Z}\big[q, q^{-1}\big]$, resp. $\mathbb{Z}\big[q^{1/2}, q^{-1/2}\big]$, turns into a module over $\mathcal{R}_{\mathbf{q}}$, resp. $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$, by scalar extension.

The very reason for introducing the above definitions, which explains the "universality" of both $\mathcal{R}_{\mathbf{q}}$ and $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$, is the following. If $\Bbbk$ is any field and $\bar{\mathbf{q}} := \big(\bar{q}_{ij}\big)_{i,j\in I}$ is any multiparameter of Cartan type A chosen in $\Bbbk$ — i.e., all the $\bar{q}_{ij}$'s belong to $\Bbbk$ — then there exist unique ring morphisms $\mathcal{R}_{\mathbf{q}} \longrightarrow \Bbbk$ and $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }} \longrightarrow \Bbbk$ given by $q_{ij}^{\pm 1} \mapsto \bar{q}_{ij}^{\pm 1}$ and $q_{ij}^{\pm 1/2} \mapsto \bar{q}_{ij}^{\pm 1/2}$ (for all $i, j \in I$) respectively — and similarly if $\mathcal{R}_{\mathbf{q}}$ and $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ are replaced by the fields $\mathcal{F}_{\mathbf{q}}$ and $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$. The images of these morphisms are the subrings $\mathcal{R}_{\bar{\mathbf{q}}}$ and $\mathcal{R}_{\bar{\mathbf{q}}}^{\sqrt{\ }}$ of $\Bbbk$ respectively generated by the $\bar{q}_{ij}$'s and by the $\bar{q}_{ij}^{1/2}$'s, like in §5.1.2 — or the corresponding fields, if one starts with $\mathcal{F}_{\mathbf{q}}$ and $\mathcal{F}_{\mathbf{q}}^{\sqrt{\ }}$.

6.1.2. The universal ring of multiparameters of integral type. Let $A := \big(a_{ij}\big)_{i,j\in I}$ be a fixed indecomposable Cartan matrix of finite type as in §6.1.1, and let $B := \big(b_{ij}\big)_{i,j\in I}$ be a matrix with entries in $\mathbb{Z}$ like in §2.3.2. We consider the rings

$$\mathcal{R}_{\mathbf{q}}^{B} := \mathbb{Z}\big[q, q^{-1}\big] \quad , \qquad \mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }} := \mathbb{Z}\big[q^{1/2}, q^{-1/2}\big]$$

and the corresponding fields of fractions $\mathcal{F}_{\mathbf{q}}^{B} := \mathbb{Q}(q)$ and $\mathcal{F}_{\mathbf{q}}^{B,\sqrt{\ }} := \mathbb{Q}\big(q^{1/2}\big)$, together with the natural ring embeddings $\mathcal{R}_{\mathbf{q}}^{B} \lhook\joinrel\longrightarrow \mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }}$ and $\mathcal{F}_{\mathbf{q}}^{B} \lhook\joinrel\longrightarrow \mathcal{F}_{\mathbf{q}}^{B,\sqrt{\ }}$ given (in both cases) by $q^{\pm 1} \mapsto \big(q^{\pm 1/2}\big)^2$. In all these rings, we consider the elements $q_{ij} := q^{\,b_{ij}} \in \mathcal{R}_{\mathbf{q}}^{B} \subseteq \mathcal{F}_{\mathbf{q}}^{B}$ and $q_{ij}^{\pm 1/2} := \big(q^{\pm 1/2}\big)^{b_{ij}} \in \mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }} \subseteq \mathcal{F}_{\mathbf{q}}^{B,\sqrt{\ }}$ for all $i, j \in I$.

Much like for the previous case of $\mathcal{R}_{\mathbf{q}}$ and $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$, the rings $\mathcal{R}_{\mathbf{q}}^{B}$ and $\mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }}$ are "universal" among all those generated by multiparameters of type B in any field $\Bbbk$, in the following sense. If $\Bbbk$ is any field and $\bar{\mathbf{q}} := \big(\bar{q}_{ij}\big)_{i,j\in I}$ is any multiparameter of integral type B in $\Bbbk$, so that $\bar{q}_{ij} = \bar{q}^{\,b_{ij}}$ for some $\bar{q} \in \Bbbk$ (for all $i, j \in I$), then there exist unique ring morphisms $\mathcal{R}_{\mathbf{q}}^{B} \longrightarrow \Bbbk$ and $\mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }} \longrightarrow \Bbbk$ given by $q^{\pm 1} \mapsto \bar{q}^{\,\pm 1}$ and $q^{\pm 1/2} \mapsto \bar{q}^{\,\pm 1/2}$, so that $q_{ij} := q^{\,b_{ij}} \mapsto \bar{q}^{\,b_{ij}} = \bar{q}_{ij}$ and $q_{ij}^{\pm 1/2} \mapsto \big(\bar{q}^{\,\pm 1/2}\big)^{b_{ij}} = \bar{q}_{ij}^{\pm 1/2}$ ($i, j \in I$); similarly with the fields $\mathcal{F}_{\mathbf{q}}^{B}$ and $\mathcal{F}_{\mathbf{q}}^{B,\sqrt{\ }}$ replacing $\mathcal{R}_{\mathbf{q}}^{B}$ and $\mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }}$. The images of these morphisms are the subrings $\mathcal{R}_{\bar{\mathbf{q}}}$ and $\mathcal{R}_{\bar{\mathbf{q}}}^{\sqrt{\ }}$ (independent of B) of $\Bbbk$ respectively generated by the $\bar{q}_{ij}^{\pm 1}$'s and by the $\bar{q}_{ij}^{\pm 1/2}$'s, i.e. by $\bar{q}^{\,\pm 1}$ and by $\bar{q}^{\,\pm 1/2}$ respectively, like in §5.1.2 (or the corresponding fields, if we deal with $\mathcal{F}_{\mathbf{q}}^{B}$ and $\mathcal{F}_{\mathbf{q}}^{B,\sqrt{\ }}$).

Finally, notice that we have a natural, "hierarchical" link between our universal rings (or fields) of Cartan or integral type: namely, there exist unique epimorphisms

$$\mathcal{R}_{\mathbf{q}} \twoheadrightarrow \mathcal{R}_{\mathbf{q}}^{B} \ \big(q^{\pm 1} \mapsto q^{\pm 1}\big) \qquad \text{and} \qquad \mathcal{R}_{\mathbf{q}}^{\sqrt{\ }} \twoheadrightarrow \mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }} \ \big(q^{\pm 1/2} \mapsto q^{\pm 1/2}\big)$$

so $\mathcal{R}_{\mathbf{q}}\Big/\Big(\big\{q_{ij} - q^{\,b_{ij}}\big\}_{i,j\in I}\Big) \cong \mathcal{R}_{\mathbf{q}}^{B}$ and $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}\Big/\Big(\Big\{q_{ij}^{1/2} - \big(q^{\pm 1/2}\big)^{b_{ij}}\Big\}_{i,j\in I}\Big) \cong \mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }}$.

6.2. **Specialization at 1.**

Let $\mathbf{q}$, $\mathcal{R}_{\mathbf{q}}$ and $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ be fixed as in §6.1 above together with all related notation.

We consider the quotient ring $\mathcal{R}_{\mathbf{q},1} := \mathcal{R}_{\mathbf{q}}\big/(q-1)\,\mathcal{R}_{\mathbf{q}}$; by construction, the latter is generated by invertible elements $y_{ij}^{\pm 1} := q_{ij}^{\pm 1} \mod (q-1)\,\mathcal{R}_{\mathbf{q}}$ which obey only the relations

$y_{j_0 j_0}^{\pm 1} = 1$ and $y_{ij}^{\pm 1} y_{ji}^{\pm 1} = 1$, so that $y_{ji}^{\pm 1} = y_{ij}^{\mp 1}$. It follows that $\mathcal{R}_{\mathbf{q},1}$ is just the ring of Laurent polynomials in the $\binom{\theta}{2}$ indeterminates $y_{ij}^{1/2}$, $i < j$. In the sequel, we write $y_{\alpha\gamma}$ for an element in $\mathcal{R}_{\mathbf{q},1}$ defined like in §3.2 but for using the y_{ij}'s instead of the q_{ij}'s.

It is clear that $\mathcal{R}_{\mathbf{q},1}$ is also an $\mathcal{R}_{\mathbf{q}}$–algebra by scalar restriction through the canonical ring epimorphism $\mathcal{R}_{\mathbf{q}} \twoheadrightarrow \mathcal{R}_{\mathbf{q}} \big/ (q-1)\,\mathcal{R}_{\mathbf{q}} =: \mathcal{R}_{\mathbf{q},1}$.

For every matrix $B := \big(b_{ij}\big)_{i,j \in I}$ with entries in $\mathbb{Z}$ as in §2.3.2, we define the ring $\mathcal{R}_{\mathbf{q},1}^B := \mathcal{R}_{\mathbf{q}}^B \big/ (q-1)\,\mathcal{R}_{\mathbf{q}}^B$. Note that $\mathcal{R}_{\mathbf{q},1}^B \cong \mathbb{Z}$ since $\mathcal{R}_{\mathbf{q}}^B \cong \mathbb{Z}\big[q, q^{-1}\big]$, and the epimorphism $\mathcal{R}_{\mathbf{q}} \twoheadrightarrow \mathcal{R}_{\mathbf{q}}^B$ induces a similar epimorphism $\mathcal{R}_{\mathbf{q},1} \twoheadrightarrow \mathcal{R}_{\mathbf{q},1}^B$ at $q = 1$.

Similarly, the *"specialization at $q^{1/2} = 1$"* of both $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ and $\mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }}$ will be $\mathcal{R}_{\mathbf{q},1}^{\sqrt{\ }} := \mathcal{R}_{\mathbf{q}}^{\sqrt{\ }} \Big/ \big(q^{1/2}-1\big)\,\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ and $\mathcal{R}_{\mathbf{q},1}^{B,\sqrt{\ }} := \mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }} \Big/ \big(q^{1/2}-1\big)\,\mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }}$; we write $y_{ij}^{\pm 1/2}$ for the image of $q_{ij}^{\pm 1/2}$ in $\mathcal{R}_{\mathbf{q},1}^{\sqrt{\ }}$ and $\mathcal{R}_{\mathbf{q},1}^{B,\sqrt{\ }}$, and overall $\mathbf{y} := \big(y_{ij}\big)_{i,j \in I}$, $\mathbf{y}^{1/2} := \big(y_{ij}^{1/2}\big)_{i,j \in I}$. Again, we have an epimorphism $\mathcal{R}_{\mathbf{q},1}^{\sqrt{\ }} \twoheadrightarrow \mathcal{R}_{\mathbf{q},1}^{B,\sqrt{\ }}$ induced by $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }} \twoheadrightarrow \mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }}$. Finally, the ring extensions $\mathcal{R}_{\mathbf{q}} \hookrightarrow \mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ and $\mathcal{R}_{\mathbf{q}}^B \hookrightarrow \mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }}$ yield extensions $\mathcal{R}_{\mathbf{q},1} \hookrightarrow \mathcal{R}_{\mathbf{q},1}^{\sqrt{\ }}$ and $\mathcal{R}_{\mathbf{q},1}^B \hookrightarrow \mathcal{R}_{\mathbf{q},1}^{B,\sqrt{\ }}$; in fact, *the latter is actually an isomorphism* $\mathcal{R}_{\mathbf{q},1}^B \xrightarrow{\cong} \mathcal{R}_{\mathbf{q},1}^{B,\sqrt{\ }}$ $\big(\cong \mathbb{Z}\big)$.

Before going on and studying specializations of our objects at $q = 1$, we recall some well-known facts of quantization theory:

6.2.1. (Co-)Poisson structures on semiclassical limits. Let A be any (commutative unital) ring, let $p \in A$ be *non-invertible* in A, and $A_{p=0} := A \big/ (p) = A \big/ p\,A$. Whatever follows applies to $A \in \big\{\mathcal{R}_{\mathbf{q}}, \mathcal{R}_{\mathbf{q}}^B\big\}$ and $p := q-1$ or $A \in \big\{\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}, \mathcal{R}_{\mathbf{q}}^{B,\sqrt{\ }}\big\}$ and $p := q^{1/2}-1$.

Consider an A–module H, and let $H_{p=0} := H \big/ p\,H$ be its *specialization* at $p = 0$; clearly the latter is automatically an $A_{p=0}$–module. If in addition H has a structure of A–algebra, or of a A–coalgebra, or of a bialgebra or Hopf algebra over A, then the $H_{p=0}$ also inherits the same kind of (quotient) structure over $A_{p=0}$.

Furthermore, the following holds (see, e.g., [CP, Chapter 6]):

(a) If H has a structure of (unital, associative) A–algebra such that $H_{p=0}$ is *commutative*, then $H_{p=0}$ bears a canonically structure of (unital, associative) *Poisson algebra* over $A_{p=0}$, whose Poisson bracket is uniquely given by

$$\{x, y\} := \frac{x'\,y' - y'\,x'}{p} \mod p\,H \qquad \forall \quad x\,,\; y \in H_{p=0}$$

for any $x', y' \in H$ such that $x := x' \mod pH$, $y := y' \mod p\,H$.

If in addition H is a *bialgebra* or *Hopf algebra* over A, then the above Poisson bracket together with the quotient structure of bialgebra or Hopf algebra (over $A_{p=0}$) make $H_{p=0}$ into a *Poisson bialgebra* or *Poisson Hopf algebra* over $A_{p=0}$.

(b) If H has a structure of (counital, coassociative) A–coalgebra such that $H_{p=0}$ is *cocommutative*, then $H_{p=0}$ bears a canonically structure of (counital, coassociative) *co-Poisson algebra* over $A_{p=0}$, whose Poisson cobracket is uniquely given by

$$\nabla(x) := \frac{\Delta(x') - \Delta^{\mathrm{op}}(x')}{p} \mod p\,H \qquad \forall \quad x \in H_{p=0}$$

for any $x' \in H$ such that $x := x' \mod p\,H$.

If in addition H is a *bialgebra* or *Hopf algebra* over A, then the above Poisson cobracket together with the quotient structure of bialgebra or Hopf algebra (over $A_{p=0}$) make $H_{p=0}$ into a *co-Poisson bialgebra* or *co-Poisson Hopf algebra* over $A_{p=0}$.

As a last remark, we recall that if $\mathfrak{l}$ is a Lie algebra and the Hopf algebra $U(\mathfrak{l})$ is actually a *co-Poisson* Hopf algebra, then $\mathfrak{l}$ canonically inherits a structure of Lie bialgebra, with the original Lie bracket and the Lie cobracket given by restriction of the Poisson cobracket in $U(\mathfrak{l})$. As a consequence, if $H_{p=0} \cong U(\mathfrak{l})$ as Hopf algebras (H as above) for some Lie algebra $\mathfrak{l}$, then the latter is bears a Lie *bialgebra* structure, induced by H as explained.

Now we fix $\mathcal{F}_{\mathbf{q}}$, $\mathcal{R}_{\mathbf{q}}$, and $\mathcal{F}_{\mathbf{q}}^{B}$, $\mathcal{R}_{\mathbf{q}}^{B}$, as in §6.1. Note that their generators q_{ij} (for all i and j) form inside either field $\mathcal{F}_{\mathbf{q}}$ and $\mathcal{F}_{\mathbf{q}}^{B}$ a multiparameter matrix $\mathbf{q} := \big(q_{ij}\big)_{i,j\in I}$ of Cartan type, and even of *integral* type (namely, type B) in the case of $\mathcal{F}_{\mathbf{q}}^{B}$. We consider then the associated MpQG's defined over $\mathcal{F}_{\mathbf{q}}$ and over $\mathcal{F}_{\mathbf{q}}^{B}$, both denoted by $U_{\mathbf{q}}(\mathfrak{g})$; nevertheless, we shall loosely distinguish the two cases by saying that we are "in the *general*, resp. *integral*, case" when the ground ring is $\mathcal{F}_{\mathbf{q}}$ or $\mathcal{R}_{\mathbf{q}}$, resp. $\mathcal{F}_{\mathbf{q}}^{B}$ or $\mathcal{R}_{\mathbf{q}}^{B}$.

In the general case, we consider in $U_{\mathbf{q}}(\mathfrak{g})$ the unrestricted integral form $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$, defined over the ring $\mathcal{R}_{\mathbf{q}}$ as in §5.3. In the integral case instead, we pick in $U_{\mathbf{q}}(\mathfrak{g})$ the restricted integral forms $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ and — in the strictly integral case — $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$, defined over $\mathcal{R}_{\mathbf{q}}^{B}$ as in §5.2, and the unrestricted form $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ too — over $\mathcal{R}_{\mathbf{q}}^{B}$ again.

We can now introduce the first type of specialization we are interested into:

Definition 6.2.2.

(a) Let $\mathbf{q}$ *be of integral type.* We call *specialization of* $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ *at* $q=1$ the quotient

$$\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g}) \;:=\; \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}) \Big/ (q-1)\, \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$$

endowed with its natural (quotient) structure of Hopf algebra over $\mathcal{R}_{\mathbf{q},1}^{B} \big(= \mathbb{Z}\big)$.

As a matter of notation, setting $\dot{\widehat{U}}_{\mathbf{q}} := \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ we shall denote

$$\begin{pmatrix} \dot{\mathrm{k}}_i\,;\,c \\ n \end{pmatrix} := \begin{pmatrix} K_i\,;\,c \\ n \end{pmatrix}_{\!q} \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}} \quad , \qquad \begin{pmatrix} \dot{\mathrm{k}}_i \\ n \end{pmatrix} := \begin{pmatrix} K_i \\ n \end{pmatrix}_{\!q} \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}$$

$$\begin{pmatrix} \dot{\mathrm{l}}_i\,;\,c \\ n \end{pmatrix} := \begin{pmatrix} L_i\,;\,c \\ n \end{pmatrix}_{\!q} \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}} \quad , \qquad \begin{pmatrix} \dot{\mathrm{l}}_i \\ n \end{pmatrix} := \begin{pmatrix} L_i \\ n \end{pmatrix}_{\!q} \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}$$

$$\begin{pmatrix} \mathrm{h}_i\,;\,c \\ n \end{pmatrix} := \begin{pmatrix} G_i\,;\,c \\ n \end{pmatrix}_{\!q_{ii}} \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}} \quad , \qquad \begin{pmatrix} \mathrm{h}_i \\ n \end{pmatrix} := \begin{pmatrix} G_i \\ n \end{pmatrix}_{\!q_{ii}} \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}$$

$$\dot{\mathrm{k}}_i := \begin{pmatrix} \dot{\mathrm{k}}_i \\ 1 \end{pmatrix} \quad , \qquad \dot{\mathrm{l}}_i := \begin{pmatrix} \dot{\mathrm{l}}_i \\ 1 \end{pmatrix} \quad , \qquad \mathrm{h}_i := \begin{pmatrix} \mathrm{h}_i \\ 1 \end{pmatrix}$$

$$\mathrm{e}_\alpha^{(n)} := E_\alpha^{(n)} \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}} \quad , \qquad \mathrm{f}_\alpha^{(n)} := F_\alpha^{(n)} \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}$$

for all $i \in I$, $c \in \mathbb{Z}$, $n \in \mathbb{N}$, $\alpha \in \Phi^+$.

If $\mathbf{q}$ *is of strongly integral type*, then we call *specialization of* $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$ *at* $q=1$ the quotient

$$\widehat{U}_{\mathbf{q},1}(\mathfrak{g}) \;:=\; \widehat{U}_{\mathbf{q}}(\mathfrak{g}) \Big/ (q-1)\, \widehat{U}_{\mathbf{q}}(\mathfrak{g})$$

endowed with its natural (quotient) structure of Hopf algebra over $\mathcal{R}_{\mathbf{q},1}^{B} \big(= \mathbb{Z}\big)$. Like above, setting $\widehat{U}_{\mathbf{q}} := \widehat{U}_{\mathbf{q}}(\mathfrak{g})$ we shall write (for $i \in I$, $c \in \mathbb{Z}$, $n \in \mathbb{N}$, $\alpha \in \Phi^+$)

$$\begin{pmatrix} \mathrm{k}_i\,;\,c \\ n \end{pmatrix} := \begin{pmatrix} K_i\,;\,c \\ n \end{pmatrix}_{\!q_i} \mod (q-1)\,\widehat{U}_{\mathbf{q}} \quad , \qquad \begin{pmatrix} \mathrm{k}_i \\ n \end{pmatrix} := \begin{pmatrix} K_i \\ n \end{pmatrix}_{\!q_i} \mod (q-1)\,\widehat{U}_{\mathbf{q}}$$

$$\begin{pmatrix} \mathrm{l}_i\,;\,c \\ n \end{pmatrix} := \begin{pmatrix} L_i\,;\,c \\ n \end{pmatrix}_{\!q_i} \mod (q-1)\,\widehat{U}_{\mathbf{q}} \quad , \qquad \begin{pmatrix} \mathrm{l}_i \\ n \end{pmatrix} := \begin{pmatrix} L_i \\ n \end{pmatrix}_{\!q_i} \mod (q-1)\,\widehat{U}_{\mathbf{q}}$$

$$\begin{pmatrix} \mathrm{h}_i\,;\,c \\ n \end{pmatrix} := \begin{pmatrix} G_i\,;\,c \\ n \end{pmatrix}_{\!q_{ii}} \mod (q-1)\,\widehat{U}_{\mathbf{q}} \quad , \qquad \begin{pmatrix} \mathrm{h}_i \\ n \end{pmatrix} := \begin{pmatrix} G_i \\ n \end{pmatrix}_{\!q_{ii}} \mod (q-1)\,\widehat{U}_{\mathbf{q}}$$

$$\mathrm{k}_i := \begin{pmatrix} \mathrm{k}_i \\ 1 \end{pmatrix} \quad , \qquad \mathrm{l}_i := \begin{pmatrix} \mathrm{l}_i \\ 1 \end{pmatrix} \quad , \qquad \mathrm{h}_i := \begin{pmatrix} \mathrm{h}_i \\ 1 \end{pmatrix}$$

$$\mathrm{e}_\alpha^{(n)} := E_\alpha^{(n)} \mod (q-1)\,\widehat{U}_{\mathbf{q}} \quad , \qquad \mathrm{f}_\alpha^{(n)} := F_\alpha^{(n)} \mod (q-1)\,\widehat{U}_{\mathbf{q}}$$

(b) Let $\mathbf{q}$ be arbitrary (of Cartan type). We call *specialization of* $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ *— defined over either* $\mathcal{R}_{\mathbf{q}}$ *or* $\mathcal{R}_{\mathbf{q}}^{B}$ *— at* $q=1$ the quotient

$$\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \;:=\; \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \Big/ (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$$

endowed with its natural (quotient) structure of Hopf algebra — over $\mathcal{R}_{\mathbf{q},1}$ or $\mathcal{R}_{\mathbf{q},1}^{B}$ respectively. As a matter of notation, we shall denote (for all $\alpha \in \Phi^+$, $i \in I$)

$$f_\alpha := \overline{F}_\alpha \mod (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \quad , \qquad e_\alpha := \overline{E}_\alpha \mod (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$$
$$l_i^{\pm 1} := L_i^{\pm 1} \mod (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \ , \quad k_i^{\pm 1} := K_i^{\pm 1} \mod (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \qquad \diamondsuit$$

Remark 6.2.3. Note that the specializations introduced above can be also realized, alternatively, as scalar extensions, namely $\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g}) := \mathcal{R}_{\mathbf{q},1}^{B} \underset{\mathcal{R}_{\mathbf{q}}^{B}}{\otimes} \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$, $\widehat{U}_{\mathbf{q},1}(\mathfrak{g}) := \mathcal{R}_{\mathbf{q},1}^{B} \underset{\mathcal{R}_{\mathbf{q}}^{B}}{\otimes} \widehat{U}_{\mathbf{q}}(\mathfrak{g})$ and $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) := \mathcal{R}_{\mathbf{q},1}^{B} \underset{\mathcal{R}_{\mathbf{q}}^{B}}{\otimes} \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ or — according to what is the chosen ground ring for $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ — also $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) := \mathcal{R}_{\mathbf{q},1} \underset{\mathcal{R}_{\mathbf{q}}}{\otimes} \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$.

Our first, key result about specialization at $q=1$ is the following:

Theorem 6.2.4. *Let* $\mathbf{q} := \big(q_{ij} = q^{b_{ij}}\big)_{i,j \in I}$ *be as above, with* $B := \big(b_{ij}\big)_{i,j\in I} \in M_\theta(\mathbb{Z})$ *such that* $B + B^t = DA$ *. Then the following holds:*

(a) $\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g})$ *is a (cocommutative) co-Poisson Hopf algebra, which is isomorphic to* $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_B\big)$ *— cf. Definition 2.3.6 — the latter being endowed with the Poisson co-bracket uniquely induced by the Lie cobracket of* $\dot{\mathfrak{g}}_B$ *— cf. Definition 2.3.4(a). Indeed, an explicit isomorphism* $\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g}) \xhookrightarrow{\;\cong\;} U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_B\big)$ *is given by*

$$\binom{\dot{\mathrm{k}}_i}{n} \mapsto \binom{\dot{\mathrm{k}}_i}{n} \, , \quad \binom{\dot{\mathrm{l}}_i}{n} \mapsto \binom{\dot{\mathrm{l}}_i}{n} \, , \quad \binom{\mathrm{h}_i}{n} \mapsto \binom{\mathrm{h}_i}{n} \, , \quad \mathrm{e}_\alpha^{(n)} \mapsto \mathrm{e}_\alpha^{(n)} \, , \quad \mathrm{f}_\alpha^{(n)} \mapsto \mathrm{f}_\alpha^{(n)}$$

Similar statements hold true for the specialization at $q=1$ *of* $\dot{\widehat{U}}_{\mathbf{q}}^{\geq}$, $\dot{\widehat{U}}_{\mathbf{q}}^{\leq}$, $\dot{\widehat{U}}_{\mathbf{q}}^{0}$ *, etc.*

(b) $\widehat{U}_{\mathbf{q},1}(\mathfrak{g})$ *is a (cocommutative) co-Poisson Hopf algebra, which is isomorphic to* $U_{\mathbb{Z}}\big(\hat{\mathfrak{g}}_B\big)$ *— cf. Definition 2.3.6 — the latter being endowed with the Poisson co-bracket uniquely induced by the Lie cobracket of* $\hat{\mathfrak{g}}_B$ *— cf. Definition 2.3.4(c). Indeed, an explicit isomorphism* $\widehat{U}_{\mathbf{q},1}(\mathfrak{g}) \xhookrightarrow{\;\cong\;} U_{\mathbb{Z}}\big(\hat{\mathfrak{g}}_B\big)$ *is given by*

$$\binom{\mathrm{k}_i}{n} \mapsto \binom{\mathrm{k}_i}{n} \, , \quad \binom{\mathrm{l}_i}{n} \mapsto \binom{\mathrm{l}_i}{n} \, , \quad \binom{\mathrm{h}_i}{n} \mapsto \binom{\mathrm{h}_i}{n} \, , \quad \mathrm{e}_\alpha^{(n)} \mapsto \mathrm{e}_\alpha^{(n)} \, , \quad \mathrm{f}_\alpha^{(n)} \mapsto \mathrm{f}_\alpha^{(n)}$$

Similar statements hold true for the specialization at $q=1$ *of* $\widehat{U}_{\mathbf{q}}^{\geq}$, $\widehat{U}_{\mathbf{q}}^{\leq}$, $\widehat{U}_{\mathbf{q}}^{0}$ *, etc.*

Proof. By the definitions and the structure results for $\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g})$ and $\widehat{U}_{\mathbf{q},1}(\mathfrak{g})$ in §5.2 (in particular, Theorem 5.2.13) the proof is a straightforward check. Indeed, from the presentation of $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ and $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$ in Theorem 5.2.13 we get similar presentations of $\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g})$ and $\widehat{U}_{\mathbf{q},1}(\mathfrak{g})$: comparing these presentations with those mentioned in Remark 2.3.7*(a)* for $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_B\big)$ and $U_{\mathbb{Z}}\big(\hat{\mathfrak{g}}_B\big)$, sheer calculations show that the formulas in the above statement provide well-defined isomorphisms, as claimed.

Hereafter we give a sample of these “sheer calculations”. Out of the commutation formulas among generators of $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ — cf. Theorem 5.2.13*(a)* — we get

$$\binom{K_i}{1}_{\!q} E_j^{(n)} \;=\; E_j^{(n)} \binom{K_i\,;\, n\, b_{ij}}{1}_{\!q} \;=\; E_j^{(n)} \left(\binom{K_i}{1}_{\!q} + \big(n\, b_{ij}\big)_q K_i \right) \;=$$
$$=\; E_j^{(n)} \binom{K_i}{1}_{\!q} + \big(n\, b_{ij}\big)_q E_j^{(n)} K_i$$

Then, when we specialize this formula at $q=1$ — that is, we take it modulo $(q-1)\,\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ — the left-hand side and right-hand side become, respectively,

$$\binom{K_i}{1}_{\!q} E_j^{(n)} \;\equiv\; \binom{\dot{\mathrm{k}}_i}{1} \mathrm{e}_j^{(n)} \qquad \Big(\bmod\ (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})\Big)$$

$$E_j^{(n)} \binom{K_i}{1}_{\!q} + \big(n\,b_{ij}\big)_q E_j^{(n)} K_j \;\equiv\; \mathrm{e}_j^{(n)} \binom{\dot{\mathrm{k}}_i}{1} + n\,b_{ij}\,\mathrm{e}_j^{(n)} \qquad \Big(\bmod\ (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})\Big)$$

because $K_i = 1 + (q-1)\binom{K_i}{1}_{\!q} \equiv 1 \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$. This shows that the relation

$$\binom{K_i}{1}_{\!q} E_j^{(n)} \;=\; E_j^{(n)} \binom{K_i}{1}_{\!q} + \big(n\,b_{ij}\big)_q E_j^{(n)} K_i$$

involving some generators of $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$, through the specialization process turns into

$$\binom{\dot{\mathrm{k}}_i}{1} \mathrm{e}_j^{(n)} \;=\; \mathrm{e}_j^{(n)} \binom{\dot{\mathrm{k}}_i}{1} + n\,b_{ij}\,\mathrm{e}_j^{(n)}$$

among the corresponding elements in $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_B\big)$; but this relation is indeed one of those occurring in the presentation of $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_B\big)$ itself by generators and relations.

With a similar analysis, one sees that the generators in $\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g})$ do respect all relations that hold true among the same name generators of $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_B\big)$. In addition, *there are no extra relations* because we have PBW bases for $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ which specialize to similar bases for $\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g})$, and the latter correspond to PBW bases of $U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_B\big)$.

Finally, since $K_i = 1 + (q-1)\binom{K_i}{1}_{\!q} \equiv 1 \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ and also $L_i = 1 + (q-1)\binom{L_i}{1}_{\!q} \equiv 1 \mod (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$, it follows from Theorem 5.2.13 that $\Delta\Big(E_i^{(n)}\Big) \equiv \sum_{s=0}^{n} E_i^{(n-s)} \otimes E_i^{(s)}$ and $\Delta\Big(F_i^{(n)}\Big) \equiv \sum_{s=0}^{n} F_i^{(n-s)} \otimes F_i^{(s)}$ modulo $\Big((q-1)\,\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}) \otimes \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}) + \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}) \otimes (q-1)\,\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})\Big)$. This implies that $\dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g})$ is a cocommutative Hopf algebra. $\square$

Remark 6.2.5. In sight of Theorem 6.2.4 above, the fact that the $\dot{\mathfrak{g}}_B$'s, resp. the $\hat{\mathfrak{g}}_B$'s, for different $\mathbf{q}$'s be all isomorphic *as Lie coalgebras* — cf. Remarks 2.3.5*(c)* — is a direct consequence of the fact that all the $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$'s, resp. the $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$'s, for different $\mathbf{q}$ are isomorphic *as coalgebras*, as this happens for the $U_{\mathbf{q}}$'s.

Next, we study the structure of $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) := \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \big/ (q-1)\,\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$. For the first results, the multiparameter $\mathbf{q}$ is assumed to be generic, i.e. just of Cartan type.

Let $\widetilde{U}_{\mathbf{q}}^{\sqrt{}} := \mathcal{R}_{\mathbf{q}}^{\sqrt{}} \otimes_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}$, and let $\widetilde{U}_{\mathbf{q},1}^{\sqrt{}}(\mathfrak{g}) := \widetilde{U}_{\mathbf{q}}^{\sqrt{}} \big/ \big(q^{1/2}-1\big)\,\widetilde{U}_{\mathbf{q}}^{\sqrt{}}$ be the specialization of $\widetilde{U}_{\mathbf{q}}^{\sqrt{}}$ at $q^{1/2}=1$. For any affine Poisson group-scheme $\widetilde{G}_{DA}^{*}$ over $\mathcal{R}_{\check{\mathbf{q}},1}^{\sqrt{}}$ dual to $\tilde{\mathfrak{g}}_{DA}$, i.e. $Lie\big(\widetilde{G}_{DA}^{*}\big) \cong \tilde{\mathfrak{g}}_{DA}^{*}$, we let $\mathcal{O}^{\sqrt{}}\big(\widetilde{G}_{DA}^{*}\big)$ be its representing Hopf algebra.

Proposition 6.2.6.
$\widetilde{U}_{\mathbf{q},1}^{\sqrt{}}(\mathfrak{g})$ is a 2–cocycle deformation of $\mathcal{O}^{\sqrt{}}\big(\widetilde{G}_{DA}^{*}\big)$ for some (uniquely defined) connected, simply connected affine Poisson group-scheme $\widetilde{G}_{DA}^{*}$ over $\mathcal{R}_{\check{\mathbf{q}},1}^{\sqrt{}}$ dual to $\tilde{\mathfrak{g}}_{DA}$ (as above).

Proof. Having taken the largest ground ring $\mathcal{R}_{\mathbf{q}}^{\sqrt{}}$ instead of $\mathcal{R}_{\mathbf{q}}$, Proposition 5.3.2 applies, giving us $\widetilde{U}_{\mathbf{q}}^{\sqrt{}}(\mathfrak{g}) = \Big(\widetilde{U}_{\check{\mathbf{q}}}^{\sqrt{}}(\mathfrak{g})\Big)_{\sigma}$ for a specific 2–cocycle σ as in Definition 3.2.1 — in particular, depending only on the $q_{ij}^{\pm 1/2}$'s. By its very construction this σ induces, modding out $\big(q^{1/2}-1\big)$, a similar 2–cocycle, denoted σ_1, of the specialized Hopf algebra $\widetilde{U}^{\sqrt{\ }}_{\check{\mathbf{q}},1}(\mathfrak{g}) := \widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g}) \Big/ \big(q^{1/2}-1\big)\, \widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$; therefore we get

$$\widetilde{U}^{\sqrt{\ }}_{\mathbf{q}}(\mathfrak{g}) \Big/ \big(q^{1/2}-1\big)\, \widetilde{U}^{\sqrt{\ }}_{\mathbf{q}}(\mathfrak{g}) \;\; = \;\; \Big(\widetilde{U}^{\sqrt{\ }}_{\check{\mathbf{q}}}(\mathfrak{g}) \Big/ \big(q^{1/2}-1\big)\, \widetilde{U}^{\sqrt{\ }}_{\check{\mathbf{q}}}(\mathfrak{g}) \Big)_{\sigma_1} \tag{6.1}$$

Now, for the usual one-parameter quantum group $U_q(\mathfrak{g})$ in [DP] — cf. also [Gav] — one has a similar construction for $\widetilde{U}_q(\mathfrak{g})$ — which is nothing but the quotient of $\widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g})$ modulo $\big(L_i - K_i^{-1}\big)$ for all i — for which one has

$$\widetilde{U}_1(\mathfrak{g}) \;\; := \;\; \widetilde{U}_q(\mathfrak{g}) \Big/ \big(q^{1/2}-1\big)\, \widetilde{U}_q(\mathfrak{g}) \;\; = \;\; \mathcal{O}\big(\widetilde{G}^*\big) \tag{6.2}$$

for some (uniquely defined) connected, simply connected affine Poisson group-scheme $\widetilde{G}^*$ whose cotangent Lie bialgebra is such that $\mathrm{Lie}\big(\widetilde{G}^*\big) \;\cong\; \Big(\tilde{\mathfrak{g}}_{DA} \Big/ \big(\mathrm{k}_i + \mathrm{l}_i\big)_{i \in I} \Big)^*$ as Lie bialgebras. Once more, this result can be easily “lifted” to the level of the quantum double of $U_q(\mathfrak{g})$, which is nothing but $U_{\check{\mathbf{q}}}(\mathfrak{g})$: the resulting construction is exactly that of the integral form $\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$ within $U_{\check{\mathbf{q}}}(\mathfrak{g})$, and the results in [DP] then turn into (sort of) a “quantum double version” of (6.2), namely

$$\widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g}) \;\; := \;\; \widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g}) \Big/ (q-1)\, \widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g}) \;\; = \;\; \mathcal{O}\big(\widetilde{G}^*_{DA}\big)$$

with $\widetilde{G}^*_{DA}$ a connected Poisson group-scheme whose cotangent Lie bialgebra is $\tilde{\mathfrak{g}}_{DA}$. Extending scalars from $\mathcal{R}_{\mathbf{q},1}$ to $\mathcal{R}^{\sqrt{\ }}_{\mathbf{q},1}$ this yields

$$\widetilde{U}^{\sqrt{\ }}_{\check{\mathbf{q}},1}(\mathfrak{g}) \;\; := \;\; \widetilde{U}^{\sqrt{\ }}_{\check{\mathbf{q}}}(\mathfrak{g}) \Big/ \big(q^{1/2}-1\big)\, \widetilde{U}^{\sqrt{\ }}_{\check{\mathbf{q}}}(\mathfrak{g}) \;\; = \;\; \mathcal{O}^{\sqrt{\ }}\big(\widetilde{G}^*_{DA}\big) \tag{6.3}$$

Finally, matching (6.1) and (6.3) the claim is proved. $\square$

The previous result can be reformulated as follows: *up to scalar extension — from $\mathcal{R}_{\mathbf{q},1}$ to $\mathcal{R}^{\sqrt{\ }}_{\mathbf{q},1}$ — the Hopf algebra $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ is a 2–cocycle deformation of the Hopf algebra $\mathcal{O}\big(\widetilde{G}^*_{DA}\big)$*. Actually, we can provide the following, more precise statement:

Theorem 6.2.7. *$\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ is a $\mathbf{y}$–polynomial and Laurent $\mathbf{y}$–polynomial algebra over $\mathcal{R}_{\mathbf{q},1}$ (with $\mathbf{y}$ as in §6.2), namely*

$$\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \;\; \cong \;\; \mathcal{R}_{\mathbf{q},1}\Big[\big\{ f_\alpha, l_i^{\pm 1}, k_i^{\pm 1}, e_\alpha \big\}_{\alpha \in \Phi^+}^{i \in I} \Big]$$

where the indeterminates $\mathbf{y}$–commute among them in the following sense:

$$\begin{gathered}
f_{\alpha'}\, f_{\alpha''} \, = \, y_{\alpha''\alpha'}\, f_{\alpha''}\, f_{\alpha'} \;\;\; , \qquad e_{\alpha'}\, f_{\alpha''} \, = \, f_{\alpha''}\, e_{\alpha'} \;\;\; , \qquad e_{\alpha'}\, e_{\alpha''} \, = \, y_{\alpha'\alpha''}\, e_{\alpha''}\, e_{\alpha'} \\
k_i^{\pm 1}\, e_\alpha \, = \, y_{\alpha_i\,\alpha}^{\pm 1}\, e_\alpha\, k_i^{\pm 1} \quad , \qquad l_i^{\pm 1}\, e_\alpha \, = \, y_{\alpha\,\alpha_i}^{\mp 1}\, e_\alpha\, l_i^{\pm 1} \\
k_i^{\pm 1}\, f_\alpha \, = \, y_{\alpha_i\,\alpha}^{\mp 1}\, f_\alpha\, k_i^{\pm 1} \quad , \qquad l_i^{\pm 1}\, f_\alpha \, = \, y_{\alpha\,\alpha_i}^{\pm 1}\, f_\alpha\, l_i^{\pm 1} \\
k_i^{\pm 1}\, k_j^{\pm 1} \, = \, k_j^{\pm 1}\, k_i^{\pm 1} \;\;\; , \qquad k_i^{\pm 1}\, l_j^{\pm 1} \, = \, l_j^{\pm 1}\, k_i^{\pm 1} \;\;\; , \qquad k_i^{\pm 1}\, k_j^{\pm 1} \, = \, k_j^{\pm 1}\, k_i^{\pm 1}
\end{gathered}$$

Proof. All formulas but those in the first line are direct consequence of definitions, so we can dispose of them, and we are left with proving the first three.

We begin with the mid formula $e_{\alpha'}\, f_{\alpha''} \, = \, f_{\alpha''}\, e_{\alpha'}$, for which we have to compare $\overline{E}_{\alpha'}\, \overline{F}_{\alpha''}$ with $\overline{F}_{\alpha''}\, \overline{E}_{\alpha'}$ within $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$; and in order to do that, we shall compare these products with the similar product taken inside $\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$, where the multiplication is deformed by a 2–cocycle σ as in Proposition 5.3.2.

Indeed, in the rest of the proof we extend scalars from $\mathcal{R}_{\mathbf{q}}$ to $\mathcal{R}^{\sqrt{\ }}_{\mathbf{q}}$ and thus work with $\widetilde{U}^{\sqrt{\ }}_{\mathbf{q}}(\mathfrak{g})$ and $\widetilde{U}^{\sqrt{\ }}_{\check{\mathbf{q}}}(\mathfrak{g})$; for the former we identify $\widetilde{U}^{\sqrt{\ }}_{\mathbf{q}}(\mathfrak{g}) = \big(\widetilde{U}^{\sqrt{\ }}_{\check{\mathbf{q}}}(\mathfrak{g})\big)_\sigma$ as $\mathcal{R}^{\sqrt{\ }}_{\mathbf{q}}$–modules, which is correct by Proposition 6.2.6. In particular, inside $\widetilde{U}^{\sqrt{\ }}_{\check{\mathbf{q}}}(\mathfrak{g})$ we shall consider the original product of $\widetilde{U}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})$, hereafter denoted by "$\check{\cdot}$", and the σ–deformed product — yielding the product in $\widetilde{U}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})$ — denoted by "$*$".

By the results in [DP] — cf. also [Gav] — suitably adapted to the present "quantum double setup", we know that $\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$ is commutative modulo $(q-1)$: this implies that $\check{\overline{E}}_{\alpha'} \,\check{\cdot}\, \check{\overline{F}}_{\alpha''}$ in $\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$ can be written as

$$\check{\overline{E}}_{\alpha'} \,\check{\cdot}\, \check{\overline{F}}_{\alpha''} \;=\; \check{\overline{F}}_{\alpha''} \,\check{\cdot}\, \check{\overline{E}}_{\alpha'} \,+\, (q-1) \textstyle\sum_s c_s \, \overline{\mathcal{M}}_s^{\,\check{\cdot}} \tag{6.4}$$

for some $c_s \in \mathcal{R}_{\mathbf{q}}$, where $\check{\overline{E}}_{\alpha'}$ and $\check{\overline{F}}_{\alpha''}$ are (renormalised) quantum root vectors in $\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$ and the $\overline{\mathcal{M}}_s^{\,\check{\cdot}}$'s are PBW monomials in a PBW basis of $\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$ like in Theorem 5.3.3*(d)*.

Let us look now for the counterpart of formula (6.4) in $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ — thought of as embedded into $\widetilde{U}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g}) = \big(\widetilde{U}_{\check{\mathbf{q}}}^{\sqrt{\ }}(\mathfrak{g})\big)_\sigma$. Thanks to Proposition 4.1.1 we have

$$\overline{E}_{\alpha'} \,=\, m^+_{\alpha'}\big(\mathbf{q}^{\pm 1/2}\big) \check{\overline{E}}_{\alpha'} \quad , \qquad \overline{F}_{\alpha''} \,=\, m^-_{\alpha''}\big(\mathbf{q}^{\pm 1/2}\big) \check{\overline{F}}_{\alpha''}$$

for suitable Laurent monomials $m^+_{\alpha'}\big(\mathbf{q}^{\pm 1/2}\big)$ and $m^-_{\alpha''}\big(\mathbf{q}^{\pm 1/2}\big)$ in the $q_{ij}^{1/2}$'s (each of which is trivial if the corresponding root is simple); Now, the formulas in §4.4.2 give

$$\check{\overline{E}}_{\alpha'} * \check{\overline{F}}_{\alpha''} = \check{\overline{E}}_{\alpha'} \,\check{\cdot}\, \check{\overline{F}}_{\alpha''} \quad , \qquad \check{\overline{F}}_{\alpha''} * \check{\overline{E}}_{\alpha'} = \check{\overline{F}}_{\alpha''} \,\check{\cdot}\, \check{\overline{E}}_{\alpha'}$$

(by the same analysis as that before Proposition 3.2.4); on the other hand, again by Proposition 4.1.1 and by §2.2.2 we have that every PBW monomial in $\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$, say $\overline{\mathcal{M}}^{\,\check{\cdot}}$, has the form $\overline{\mathcal{M}}^{\,\check{\cdot}} = m_{\overline{\mathcal{M}}}\big(\mathbf{q}^{\pm 1/2}\big)\, \overline{\mathcal{M}}^{\,*}$ where $m_{\overline{\mathcal{M}}}\big(\mathbf{q}^{\pm 1/2}\big)$ is a suitable Laurent monomial in the $q_{ij}^{\pm 1/2}$'s. Tidying everything up, from (6.4) and the identities here above — writing $m^\pm_\alpha := m^\pm_\alpha\big(\mathbf{q}^{\pm 1/2}\big)$ and $m_{\overline{\mathcal{M}}} := m_{\overline{\mathcal{M}}}\big(\mathbf{q}^{\pm 1/2}\big)$ — we find

$$\begin{aligned} \overline{E}_{\alpha'} * \overline{F}_{\alpha''} \;&=\; m^+_{\alpha'}\, m^-_{\alpha''}\, \check{\overline{E}}_{\alpha'} * \check{\overline{F}}_{\alpha''} \;=\; m^+_{\alpha'}\, m^-_{\alpha''}\, \check{\overline{E}}_{\alpha'} \,\check{\cdot}\, \check{\overline{F}}_{\alpha''} \;= \\ &=\; m^+_{\alpha'}\, m^-_{\alpha''} \Big(\check{\overline{F}}_{\alpha''} \,\check{\cdot}\, \check{\overline{E}}_{\alpha'} + (q-1) \textstyle\sum_s c_s \, \overline{\mathcal{M}}_s^{\,\check{\cdot}} \Big) \;= \\ &=\; m^+_{\alpha'}\, m^-_{\alpha''} \Big(\big(m^+_{\alpha'}\big)^{-1} \big(m^-_{\alpha''}\big)^{-1} \overline{F}_{\alpha''} * \overline{E}_{\alpha'} + (q-1) \textstyle\sum_s c_s \, m_{\overline{\mathcal{M}}_s} \overline{\mathcal{M}}_s^{\,*} \Big) \;= \\ &\qquad\qquad =\; \overline{F}_{\alpha''} * \overline{E}_{\alpha'} + (q-1) \textstyle\sum_s c_s \, m^+_{\alpha'}\, m^-_{\alpha''}\, m_{\overline{\mathcal{M}}_s} \overline{\mathcal{M}}_s^{\,*} \end{aligned}$$

that is, in the end,

$$\overline{E}_{\alpha'} * \overline{F}_{\alpha''} \;=\; \overline{F}_{\alpha''} * \overline{E}_{\alpha'} + (q-1) \textstyle\sum_s c_s \, m^+_{\alpha'}\, m^-_{\alpha''}\, m_{\overline{\mathcal{M}}_s} \overline{\mathcal{M}}_s^{\,*} \tag{6.5}$$

which is almost what we need, as the right-hand side belongs to $\widetilde{U}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g})$ but possibly not to $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$. To fix this detail, we take the expansion of $\overline{E}_{\alpha'} * \overline{F}_{\alpha''}$ as an $\mathcal{R}_{\mathbf{q}}$–linear combination of the PBW basis of the $\overline{\mathcal{M}}_r^{\,*}$'s (which includes $\overline{F}_{\alpha''} * \overline{E}_{\alpha'}$ too), namely $\overline{E}_{\alpha'} * \overline{F}_{\alpha''} = \sum_r \kappa_r \, \overline{\mathcal{M}}_r^{\,*}$ for some $\kappa_r \in \mathcal{R}_{\mathbf{q}}$; comparing the latter with (6.5) we get $c_s \, m^+_{\alpha'}\, m^-_{\alpha''}\, m_{\overline{\mathcal{M}}_s} \in \mathcal{R}_{\mathbf{q}}$ for every s. Then (6.5) is an identity in $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$, which implies

$$\overline{E}_{\alpha'} * \overline{F}_{\alpha''} \;=\; \overline{F}_{\alpha''} * \overline{E}_{\alpha'} \qquad \mod (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$$

whence eventually $e_{\alpha'}\, f_{\alpha''} = f_{\alpha''}\, e_{\alpha'}$, q.e.d.

We turn now to proving the identity $e_{\alpha'}\, e_{\alpha''} = y_{\alpha'\alpha''}\, e_{\alpha''}\, e_{\alpha'}$, for which we need to compare $\overline{E}_{\alpha'}\, \overline{E}_{\alpha''}$ with $\overline{E}_{\alpha''}\, \overline{E}_{\alpha'}$ within $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$. To begin with, from the results [DP, §§9, 10 and 12] — suitably adapted, as usual, to the present, "quantum double framework" — in the standard case of $\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$ we have

$$\check{\overline{E}}_{\alpha'} \,\check{\cdot}\, \check{\overline{E}}_{\alpha''} \;=\; \check{q}_{\alpha',\alpha''}\, \check{\overline{E}}_{\alpha''} \,\check{\cdot}\, \check{\overline{E}}_{\alpha'} + (q-1) \textstyle\sum_{\underline{\alpha}} \check{c}_{\underline{\alpha}}\, \check{\overline{E}}_{\underline{\alpha}}^{\,e_{\underline{\alpha}}} \tag{6.6}$$

for all $\alpha', \alpha'' \in \Phi^+$, where $\check{q}_{\alpha',\alpha''} = q^{(\alpha',\alpha'')/2}$ by definition, $\check{c}_{\underline{\alpha}} \in \mathbb{Z}\big[q, q^{-1}\big]\ \big(\subseteq \mathcal{R}_{\mathbf{q}}\big)$ for all $\underline{\alpha}$ and the $\check{\overline{E}}_{\underline{\alpha}}^{\,e_{\underline{\alpha}}}$'s are PBW monomials in the $\check{\overline{E}}_\alpha$'s alone.

Now from (6.6) we deduce a parallel identity in $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$. Namely, acting like in the first part of the proof — basing again on the formulas in §4.4.2 — we find

$$\begin{aligned}
\overline{E}_{\alpha'} * \overline{E}_{\alpha''} \; &= \; m_{\alpha'}^{+} m_{\alpha''}^{+} \, \check{\overline{E}}_{\alpha'} * \check{\overline{E}}_{\alpha''} \; = \; m_{\alpha'}^{+} m_{\alpha''}^{-} \, q_{\alpha',\alpha''}^{+1/2} \, \check{\overline{E}}_{\alpha'} \,\check{\cdot}\, \check{\overline{E}}_{\alpha''} \; = \\
&= \; m_{\alpha'}^{+} m_{\alpha''}^{-} \, q_{\alpha',\alpha''}^{+1/2} \Big(\check{q}_{\alpha',\alpha''} \, \check{\overline{E}}_{\alpha''} \,\check{\cdot}\, \check{\overline{E}}_{\alpha'} + (q-1) \textstyle\sum_{\underline{\alpha}} \check{c}_{\underline{\alpha}} \, \check{\overline{E}}{}_{\underline{\alpha}}^{\,e_{\underline{\alpha}}} \Big) \; = \\
&= \; m_{\alpha'}^{+} m_{\alpha''}^{+} \, q_{\alpha',\alpha''}^{+1/2} \, \check{q}_{\alpha',\alpha''} \, \big(m_{\alpha''}^{+}\big)^{-1} \big(m_{\alpha'}^{+}\big)^{-1} \, q_{\alpha'',\alpha'}^{-1/2} \, \overline{E}_{\alpha''} * \overline{E}_{\alpha'} \; + \\
&\qquad + \; (q-1) \textstyle\sum_{\underline{\alpha}} \check{c}_{\underline{\alpha}} \, m_{\alpha'}^{+} m_{\alpha''}^{+} \, q_{\alpha',\alpha''}^{+1/2} \, \mu_{\underline{\alpha}} \, \overline{E}{}_{\underline{\alpha}}^{\,e_{\underline{\alpha}}} \; = \\
&= \; q_{\alpha',\alpha''}^{+1/2} \, \check{q}_{\alpha',\alpha''} \, q_{\alpha'',\alpha'}^{-1/2} \, \overline{E}_{\alpha''} * \overline{E}_{\alpha'} \; + \; (q-1) \textstyle\sum_{\underline{\alpha}} \check{c}_{\underline{\alpha}} \, m_{\alpha'}^{+} m_{\alpha''}^{+} \, q_{\alpha',\alpha''}^{+1/2} \, \mu_{\underline{\alpha}} \, \overline{E}{}_{\underline{\alpha}}^{\,e_{\underline{\alpha}}}
\end{aligned}$$

where $\mu_{\underline{\alpha}}$ is yet another Laurent monomial in the $q_{ij}^{\pm 1/2}$'s and each $\overline{E}{}_{\underline{\alpha}}^{\,e_{\underline{\alpha}}}$ is the unique PBW monomial in the $\overline{E}_{\alpha}$'s that corresponds (in an obvious sense) to $\check{\overline{E}}{}_{\underline{\alpha}}^{\,e_{\underline{\alpha}}}$. Thus

$$\overline{E}_{\alpha'} * \overline{E}_{\alpha''} \; = \; q_{\alpha',\alpha''}^{+1/2} \, \check{q}_{\alpha',\alpha''} \, q_{\alpha'',\alpha'}^{-1/2} \, \overline{E}_{\alpha''} * \overline{E}_{\alpha'} \; + \; (q-1) \textstyle\sum_{\underline{\alpha}} c_{\underline{\alpha}} \, \overline{E}{}_{\underline{\alpha}}^{\,e_{\underline{\alpha}}} \tag{6.7}$$

where $c_{\underline{\alpha}} := \check{c}_{\underline{\alpha}} \, m_{\alpha'}^{+} m_{\alpha''}^{+} \, q_{\alpha',\alpha''}^{+1/2} \, \mu_{\underline{\alpha}} \in \mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$. But we also know that the $\overline{E}_{\underline{\alpha}}$'s form a PBW basis over $\mathcal{R}_{\mathbf{q}}$ for $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$, hence $\overline{E}_{\alpha'} * \overline{E}_{\alpha''}$ uniquely expands into an $\mathcal{R}_{\mathbf{q}}$–linear combination of these monomials: comparing such an expansion with (6.7) we find that all coefficients $c_{\underline{\alpha}}$ therein necessarily belong to $\mathcal{R}_{\mathbf{q}}$: then (6.7) itself is an identity in $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ — i.e., not only in $\widetilde{U}_{\mathbf{q}}^{\sqrt{\ }}(\mathfrak{g})$. Therefore, from (6.7) we deduce

$$\overline{E}_{\alpha'} * \overline{E}_{\alpha''} \; \cong \; q_{\alpha',\alpha''}^{+1/2} \, \check{q}_{\alpha',\alpha''} \, q_{\alpha'',\alpha'}^{-1/2} \, \overline{E}_{\alpha''} * \overline{E}_{\alpha'} \qquad \mod (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \tag{6.8}$$

Finally, since $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ and $\check{q}_{ij} := q^{d_i a_{ij}} = q^{d_j a_{ji}}$ for all $i, j \in I$, we just compute that $q_{\alpha',\alpha''}^{+1/2} \, \check{q}_{\alpha',\alpha''} \, q_{\alpha'',\alpha'}^{-1/2} = q_{\alpha',\alpha''}$, whose coset in $\mathcal{R}_{\mathbf{q},1} := \mathcal{R}_{\mathbf{q}} \big/ (q-1)\, \mathcal{R}_{\mathbf{q}}$ is just $y_{\alpha'\alpha''}$; therefore (6.8) yields $e_{\alpha'} \, e_{\alpha''} = y_{\alpha'\alpha''} \, e_{\alpha''} \, e_{\alpha'}$ as claimed.

A similar procedure shows that $f_{\alpha'} \, f_{\alpha''} = y_{\alpha''\alpha'} \, f_{\alpha''} \, f_{\alpha'}$, which ends the proof. □

Remarks 6.2.8. In [An4, §3], a different construction eventually leads to a result comparable with Theorem 6.2.7 above, although slightly weaker. In general, we prefer to follow a different approach, because it exploits an independent argument and is more consistent with our global approach in the present work, mostly based on the fact that $U_{\mathbf{q}}(\mathfrak{g}) = \big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)_{\sigma}$. In addition, some results of [An4] cannot be directly applied to our context of integral forms and specializations, so we must resort to an alternative strategy.

When the multiparameter $\mathbf{q}$ is of *integral* type the last two previous results get a stronger importance from a geometrical point of view. In fact, the following is a refinement of Proposition 6.2.6 but *we provide for it an independent proof.*

Theorem 6.2.9.

Let $\mathbf{q}$ be of integral type, and $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ defined over $\mathcal{R}_{\mathbf{q}}^{B} = \mathbb{Z}\big[q, q^{-1}\big]$. Then $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ is (isomorphic to) the representing Hopf algebra $\mathcal{O}\big(\widetilde{G}_{B}^{*}\big)$ of a connected affine Poisson group-scheme over $\mathbb{Z}$ whose cotangent Lie bialgebra is $\tilde{\mathfrak{g}}_{B}$ as described in Definition 2.3.4.

Similar statements hold true for the specialization at $q = 1$ of $\widetilde{U}_{\mathbf{q}}^{\geq}$, $\widetilde{U}_{\mathbf{q}}^{\leq}$, $\widetilde{U}_{\mathbf{q}}^{0}$, etc.

Proof. First of all, when $\mathbf{q}$ is of integral type, so $q_{ij} = q^{\,b_{ij}}$ (for all i, j), we have

$$y_{ij} \; := \; q_{ij} \mod (q-1) \; = \; q^{\,b_{ij}} \mod (q-1) \; = \; 1^{\,b_{ij}} \; = \; 1 \qquad \forall \; i, j \in I$$

therefore Theorem 6.2.7 tell us that $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ is a *commutative* Hopf algebra (of Laurent polynomials); it follows then that $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) = \mathcal{O}(\mathcal{G})$ for some affine group-scheme, say $\mathcal{G}$. Moreover, from Proposition 6.2.6 (with notation as in its proof) we know that $\mathcal{O}(\mathcal{G}) = \widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) = \mathcal{O}\big(\widetilde{G}_{DA}^{*}\big)_{\sigma_1}$ where the group-scheme $\widetilde{G}_{DA}^{*}$ is connected — in other words,

$\mathcal{O}\big(\widetilde{G}^*_{DA}\big)$ has no non-trivial idempotents. Now, as $q_{ij} = q^{b_{ij}}$ for $q = 1$ the "specialized" cocycle σ_1 is trivial — namely, $\sigma_1 = \epsilon \otimes \epsilon$ — which implies that $\mathcal{O}(\mathcal{G}) = \mathcal{O}\big(\widetilde{G}^*_{DA}\big)_{\sigma_1} = \mathcal{O}\big(\widetilde{G}^*_{DA}\big)$, hence $\mathcal{G} = \widetilde{G}^*_{DA}$ as group-schemes. In addition, by Remark 6.2.1*(a)* the Hopf algebra $\mathcal{O}(\mathcal{G}) = \widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$, being commutative, inherits from $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ a *Poisson* structure, hence it is a Poisson Hopf algebra: thus $\mathcal{G}$ itself is in fact a *Poisson* group-scheme.

We point out that the Poisson structure on $\mathcal{O}(\mathcal{G}) = \widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ is induced by the multiplication in $\widetilde{U}_{\mathbf{q}}(\mathfrak{g}) = \big(\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})\big)_{\sigma_{\mathbf{q}}}$, which in turn depends on $\mathbf{q}$. Thus $\mathcal{G}$ and $\widetilde{G}^*_{DA}$, although coinciding as group-schemes, do not share, in general, the same Poisson structure.

What is still missing for having $\mathcal{G} = \widetilde{G}^*_B$ is proving that the cotangent Lie bialgebra of $\mathcal{G}$ is isomorphic to $\tilde{\mathfrak{g}}_B$, defined as in Definition 2.3.4.

First we recall the definition of the cotangent Lie bialgebra of $\mathcal{G}$. If $\mathfrak{m}_e := \operatorname{Ker}\big(\epsilon_{\mathcal{O}(\mathcal{G})}\big)$ is the augmentation ideal of $\mathcal{O}(\mathcal{G})$, the quotient $\mathfrak{m}_e \big/ \mathfrak{m}_e^2$ has a canonical structure of Lie coalgebra, such that its linear dual is the tangent Lie algebra of $\mathcal{G}$. In addition, the properties of the Poisson bracket in $\mathcal{O}(\mathcal{G})$ imply that $\mathfrak{m}_e$ is a Lie subalgebra (even a Lie ideal, indeed) of the Lie algebra $\big(\mathcal{O}(\mathcal{G}), \{\ ,\ \}\big)$, and $\mathfrak{m}_e^2$ is a Lie ideal in $\big(\mathfrak{m}_e, \{\ ,\ \}\big)$, whence $\mathfrak{m}_e \big/ \mathfrak{m}_e^2$ has a quotient Lie algebra structure; together with the Lie coalgebra structure, the latter makes $\mathfrak{m}_e \big/ \mathfrak{m}_e^2$ into a Lie bialgebra. As a matter of notation, we set $\overline{x} \in \mathfrak{m}_e \big/ \mathfrak{m}_e^2$ to denote the coset in $\mathfrak{m}_e \big/ \mathfrak{m}_e^2$ of any $x \in \mathfrak{m}_e$.

As a consequence of the PBW Theorem for $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ — i.e. Theorem 5.3.3, or directly of Theorem 6.2.7 — taking into account that the e_α's, the f_α's, the (k_i-1)'s and the (l_i-1)'s, with $\alpha \in \Phi^+$, $i \in I$, all lie in $\mathfrak{m}_e$, one has that a basis of $\mathfrak{m}_e \big/ \mathfrak{m}_e^2$ is given by the $\overline{e_\alpha}$'s, the $\overline{f_\alpha}$'s, the $\overline{(k_i-1)}$'s and the $\overline{(l_i-1)}$'s altogether. Our aim now is to prove the following

<u>*Claim:*</u> there exists a Lie bialgebra isomorphism $\phi : \mathfrak{m}_e \big/ \mathfrak{m}_e^2 \overset{\cong}{\hookrightarrow\!\!\!\!\rightarrow} \tilde{\mathfrak{g}}_B$ given by $\phi : \overline{e_\alpha} \mapsto \tilde{\mathrm{e}}_\alpha$, $\overline{f_\alpha} \mapsto \tilde{\mathrm{f}}_\alpha$, $\overline{(k_i-1)} \mapsto \dot{\mathrm{k}}_i$ and $\overline{(l_i-1)} \mapsto \dot{\mathrm{l}}_i$, for all $\alpha \in \Phi^+$, $i \in I$.

To begin with, given $\alpha, \beta \in \Phi^+$, we show that $\phi\big(\big[\overline{e_\alpha}, \overline{e_\beta}\big]\big) = \big[\phi(\overline{e_\alpha}), \phi(\overline{e_\beta})\big]$. First observe that our root vectors e_γ in $\mathfrak{g}$ come from the simple ones via a construction à la Chevalley (see [Hu, Chapter II, §25.2]), so that $[\mathrm{e}_\alpha, \mathrm{e}_\beta] = c_{\alpha,\beta}\, \mathrm{e}_{\alpha+\beta}$ for suitable $c_{\alpha,\beta} \in \mathbb{Z}$. Moreover, since (under our assumption that $\mathfrak{g}$ be simple) there are only two possible root lengths, we have $d_{\alpha+\beta} \in \{d_\alpha, d_\beta\}$; so if $d_\alpha = d_\beta$ we write $d_\delta := d_\alpha\ (= d_\beta)$ and if $d_\alpha \neq d_\beta$ we call d_δ the unique element of $\{d_\alpha, d_\beta\} \setminus \{d_{\alpha+\beta}\}$. Then recall that (for all $\gamma \in \Phi^+$)

$$e_\gamma := \overline{E}_\gamma \mod (q-1)\, \widetilde{U}_{\mathbf{q}} \qquad , \qquad \mathrm{e}_\gamma := E_\gamma \mod (q-1)\, \dot{\widehat{U}}_{\mathbf{q}}$$

$$\big[\overline{e_\alpha}, \overline{e_\beta}\big] := \big\{e_\alpha, e_\beta\big\} \mod \mathfrak{m}_e^2 \ , \quad \big\{e_\alpha, e_\beta\big\} := (q-1)^{-1} \big[\overline{E}_\alpha, \overline{E}_\beta\big] \mod (q-1)\, \widetilde{U}_{\mathbf{q}}$$

Second, since $\widetilde{U}_{\mathbf{q}}$ is commutative modulo $(q-1)$, we have $\big[\overline{E}_\alpha, \overline{E}_\beta\big] = (q-1)\, \mathcal{E}$ for some $\mathcal{E} \in \widetilde{U}_{\mathbf{q}} \cap U_{\mathbf{q}}^+ = \widetilde{U}_{\mathbf{q}}^+$ — so that $\big\{e_\alpha, e_\beta\big\} := \mathcal{E} \mod (q-1)\, \widetilde{U}_{\mathbf{q}}$. On the other hand, from $[\mathrm{e}_\alpha, \mathrm{e}_\beta] = c_{\alpha,\beta}\, \mathrm{e}_{\alpha+\beta}$ (see above) and $\mathrm{e}_\gamma := E_\gamma \mod (q-1)\, \dot{\widehat{U}}_{\mathbf{q}}$ together we get $\big[E_\alpha, E_\beta\big] = c_{\alpha,\beta}\, E_{\alpha+\beta} + (q-1)\, \mathfrak{E}$ for some $\mathfrak{E} \in \widehat{U}_{\mathbf{q}} \cap U_{\mathbf{q}}^+ = \widehat{U}_{\mathbf{q}}^+$. The latter implies

$$\begin{aligned} \big[\overline{E}_\alpha, \overline{E}_\beta\big] &= \big(q_{\alpha\alpha} - 1\big)\big(q_{\beta\beta} - 1\big)\, \big[E_\alpha, E_\beta\big] = \\ &= c_{\alpha,\beta}\, \big(q_{\alpha\alpha} - 1\big)\big(q_{\beta\beta} - 1\big)\, E_{\alpha+\beta} + (q-1)\, \big(q_{\alpha\alpha} - 1\big)\big(q_{\beta\beta} - 1\big)\, \mathfrak{E} = \\ &= c_{\alpha,\beta}\, \big(q_{\delta\delta} - 1\big)\, \overline{E}_{\alpha+\beta} + (q-1)\, \big(q_{\alpha\alpha} - 1\big)\big(q_{\beta\beta} - 1\big)\, \mathfrak{E} \end{aligned}$$

and comparing the last term with the previous identity $\big[\overline{E}_\alpha, \overline{E}_\beta\big] = (q-1)\, \mathcal{E}$ yields

$$\mathcal{E} = c_{\alpha,\beta}\, \big(2\, d_\delta\big)_q\, \overline{E}_{\alpha+\beta} + \big(q_{\alpha\alpha} - 1\big)\big(q_{\beta\beta} - 1\big)\, \mathfrak{E}$$

Then expanding $\mathfrak{E}$ w.r.t. the $\mathcal{R}_{\mathbf{q}}$–PBW basis of $\widehat{U}_{\mathbf{q}}^+$ (made of ordered products of q–divided powers $E_\gamma^{(n_\gamma)}$'s) and comparing with the expansion of $\big(q_{\alpha\alpha} - 1\big)\big(q_{\beta\beta} - 1\big)\, \mathfrak{E}$ —

which must necessarily belong to $\widetilde{U}_{\mathbf{q}}^{+}$ — in terms of the $\mathcal{R}_{\mathbf{q}}$–PBW basis of $\widetilde{U}_{\mathbf{q}}^{+}$ (made of ordered monomials in the $\overline{E}_\gamma$'s) we eventually find that

$$\big(q_{\alpha\alpha}-1\big)\big(q_{\beta\beta}-1\big)\,\mathfrak{E} \;=\; \sum_{k\geq 2}\;\sum_{\gamma_1,\dots,\gamma_k\in\Phi^+} c'_{\gamma_1,\dots,\gamma_k}\,\overline{E}_{\gamma_1}\cdots\overline{E}_{\gamma_k} \;+\; (q-1)\,\mathcal{E}'$$

for some $c'_{\gamma_1,\dots,\gamma_k}\in\mathcal{R}_{\mathbf{q}}$ and some $\mathcal{E}'\in\widetilde{U}_{\mathbf{q}}^{+}$. Therefore

$$\mathcal{E} \;=\; c_{\alpha,\beta}\,\big(2\,d_\delta\big)_q\,\overline{E}_{\alpha+\beta} \;+\; \sum_{k\geq 2}\;\sum_{\gamma_1,\dots,\gamma_k\in\Phi^+} c'_{\gamma_1,\dots,\gamma_k}\,\overline{E}_{\gamma_1}\cdots\overline{E}_{\gamma_k} \;+\; (q-1)\,\mathcal{E}'$$

which in turn implies

$$\begin{aligned}\mathcal{E}\ \bmod (q-1)\,\widetilde{U}_{\mathbf{q}} \;&=\; \Big(c_{\alpha,\beta}\,\big(2\,d_\delta\big)_q\,\overline{E}_{\alpha+\beta}\ \bmod (q-1)\,\widetilde{U}_{\mathbf{q}}\Big) \,+ \\ &\quad+\Big(\sum_{k\geq 2}\;\sum_{\gamma_1,\dots,\gamma_k\in\Phi^+} c'_{\gamma_1,\dots,\gamma_k}\,\overline{E}_{\gamma_1}\cdots\overline{E}_{\gamma_k}\ \bmod (q-1)\,\widetilde{U}_{\mathbf{q}}\Big) \;= \\ &=\; c_{\alpha,\beta}\;2\;d_\delta\;e_{\alpha+\beta} \;+\; \sum_{k\geq 2}\;\sum_{\gamma_1,\dots,\gamma_k\in\Phi^+} c_{\gamma_1,\dots,\gamma_k}\,e_{\gamma_1}\cdots e_{\gamma_k}\end{aligned}$$

with $c_{\gamma_1,\dots,\gamma_k} := \big(c'_{\gamma_1,\dots,\gamma_k}\ \bmod (q-1)\,\mathcal{R}_{\mathbf{q}}\big) \in \mathcal{R}_{\mathbf{q}}\big/(q-1)\,\mathcal{R}_{\mathbf{q}} = \mathcal{R}_{\mathbf{q},1}$. This yields

$$\begin{aligned}\big[\,\overline{e_\alpha}\,,\overline{e_\beta}\,\big] \;&=\; \Big(\big\{e_\alpha\,,e_\beta\big\}\ \bmod \mathfrak{m}_e^2\Big) \;= \\ &=\; \Big(\Big((q-1)^{-1}\big[\,\overline{E}_\alpha\,,\overline{E}_\beta\,\big]\ \bmod (q-1)\,\widetilde{U}_{\mathbf{q}}\Big)\ \bmod \mathfrak{m}_e^2\Big) \;= \\ &=\; \Big(\Big(\mathcal{E}\ \bmod (q-1)\,\widetilde{U}_{\mathbf{q}}\Big)\ \bmod \mathfrak{m}_e^2\Big) \;= \\ &=\; \Big(\Big(c_{\alpha,\beta}\;2\;d_\delta\;e_{\alpha+\beta} \;+\; \sum_{k\geq 2}\;\sum_{\gamma_1,\dots,\gamma_k\in\Phi^+} c_{\gamma_1,\dots,\gamma_k}\,e_{\gamma_1}\cdots e_{\gamma_k}\Big)\ \bmod \mathfrak{m}_e^2\Big) \;=\; c_{\alpha,\beta}\;2\;d_\delta\;\overline{e_{\alpha+\beta}}\end{aligned}$$

that is in short $\big[\,\overline{e_\alpha}\,,\overline{e_\beta}\,\big] \,=\, 2\;d_\delta\;c_{\alpha,\beta}\;\overline{e_{\alpha+\beta}}$. Now, from the last identity we compute

$$\phi\big(\big[\,\overline{e_\alpha}\,,\overline{e_\beta}\,\big]\big) \;=\; \phi\big(2\;d_\delta\;\overline{c}_{\alpha,\beta}\;\overline{e_{\alpha+\beta}}\big) \;=\; 2\;d_\delta\;\overline{c}_{\alpha,\beta}\;\phi\big(\overline{e_{\alpha+\beta}}\big) \;=\; 2\;d_\delta\;\overline{c}_{\alpha,\beta}\;\check{\mathrm{e}}_{\alpha+\beta} \tag{6.9}$$

by definition of ϕ. On the other hand, we have also

$$\big[\phi(\overline{e_\alpha})\,,\phi(\overline{e_\beta})\big] \;=\; [\check{\mathrm{e}}_\alpha\,,\check{\mathrm{e}}_\beta] \;=\; 2\,d_\alpha\,2\,d_\beta\,[\,\mathrm{e}_\alpha\,,\mathrm{e}_\beta\,] \;=\; 2\,d_\alpha\,2\,d_\beta\,c_{\alpha,\beta}\,\mathrm{e}_{\alpha+\beta} \;=\; 2\,d_\delta\,c_{\alpha,\beta}\,\check{\mathrm{e}}_{\alpha+\beta}$$

comparing this with (6.9) eventually gives $\phi\big(\big[\,\overline{e_\alpha}\,,\overline{e_\beta}\,\big]\big) \,=\, \big[\phi(\overline{e_\alpha})\,,\phi(\overline{e_\beta})\big]$, q.e.d.

Acting in the same way, one finds also

$$\begin{aligned}\big[\,\overline{k_i-1}\,,\overline{e_\alpha}\,\big] \;&=\; \Big(\big\{k_i-1\,,e_\alpha\big\}\ \bmod \mathfrak{m}_e^2\Big) \;= \\ &=\; \Big(\Big((q-1)^{-1}\big((K_i-1)\overline{E}_\alpha - \overline{E}_\alpha(K_i-1)\big)\ \bmod (q-1)\,\widetilde{U}_{\mathbf{q}}(\mathfrak{g})\Big)\ \bmod \mathfrak{m}_e^2\Big) \;= \\ &=\; \Big(\Big(\big(d^+_{i,\alpha}\big)_q\,\overline{E}_\alpha K_i\ \bmod (q-1)\,\widetilde{U}_{\mathbf{q}}(\mathfrak{g})\Big)\ \bmod \mathfrak{m}_e^2\Big) \;= \\ &=\; \Big(d^+_{i,\alpha}\,e_\alpha k_i\ \bmod \mathfrak{m}_e^2\Big) \;=\; d^+_{i,\alpha}\,\overline{e_\alpha}\end{aligned}$$

where $d^+_{i,\alpha} := +\sum_{j\in I} b_{ij}c_j$ with $\alpha = \sum_{j\in I} c_j\alpha_j$, so in the end

$$\big[\,\overline{k_i-1}\,,\overline{e_\alpha}\,\big] \;=\; d^+_{i,\alpha}\,\overline{e_\alpha} \qquad\quad \forall\ i\in I\,,\ \alpha\in\Phi^+ \tag{6.10}$$

Similarly, one finds also

$$\big[\,\overline{l_i-1}\,,\overline{e_\alpha}\,\big] \;=\; d^-_{i,\alpha}\,\overline{e_\alpha} \qquad\quad \forall\ i\in I\,,\ \alpha\in\Phi^+ \tag{6.11}$$

with $d^-_{i,\alpha} := -\sum_{j\in I} b_{ji}c_j$ for $\alpha = \sum_{j\in I} c_j\alpha_j$. Likewise, parallel formulas to (6.10) and (6.11) hold true when the $\overline{e_\alpha}$'s are replaced by the $\overline{f_\alpha}$'s.

Finally, comparing the Lie brackets (inside $\mathfrak{m}_e\big/\mathfrak{m}_e^2$) given explicitly in (6.10) and (6.11), and the similar ones where the $\overline{f_\gamma}$'s are replaced by the $\overline{e_\gamma}$'s, with the analogue brackets inside $\tilde{\mathfrak{g}}_B$ of the corresponding elements through the map ϕ as given in the *Claim*, one easily sees that the latter map is indeed a Lie algebra morphism. In addition, it is invertible because it maps a basis to a basis. Moreover, this is also an isomorphism of Lie *bi*algebras because the formulas for the Lie cobracket do correspond on either side on all elements of the form $\overline{e_i}$, $\overline{f_i}$, $\overline{k_i - 1}$ and $\overline{l_i - 1}$ (with $i \in I$), which is enough to conclude — cf. Remarks 2.3.5. In fact, this is again a matter of bookkeeping: for instance, writing $\mathfrak{m}_\otimes^{[2]} := \mathfrak{m}_e \otimes \mathfrak{m}_e^2 + \mathfrak{m}_e^2 \otimes \mathfrak{m}_e$, one has

$$\begin{aligned}
\delta(\overline{e_i}) &= \Big(\big(\Delta(e_i) - \Delta^{\mathrm{op}}(e_i) \big) \mod \mathfrak{m}_\otimes^{[2]} \Big) = \\
&= \Big(\Big(\big(\Delta(\overline{E}_i) - \Delta^{\mathrm{op}}(\overline{E}_i) \big) \mod (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g})^{\otimes 2} \Big) \mod \mathfrak{m}_\otimes^{[2]} \Big) = \\
&= \Big(\Big(\big((K_i - 1) \otimes \overline{E}_i - \overline{E}_i \otimes (K_i - 1) \big) \mod (q-1)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g})^{\otimes 2} \Big) \mod \mathfrak{m}_\otimes^{[2]} \Big) = \\
&= \Big(\big((k_i - 1) \otimes e_i - e_i \otimes (k_i - 1) \big) \mod \mathfrak{m}_\otimes^{[2]} \Big) = \\
&= \overline{(k_i - 1)} \otimes \overline{e_i} - \overline{e_i} \otimes \overline{(k_i - 1)}
\end{aligned}$$

which means $\delta(\overline{e_i}) = \overline{(k_i - 1)} \otimes \overline{e_i} - \overline{e_i} \otimes \overline{(k_i - 1)}$. Through the formulas given in the *Claim*, this last identity corresponds to $\delta(\tilde{\mathrm{e}}_i) = \dot{\mathrm{k}}_i \otimes \tilde{\mathrm{e}}_i - \tilde{\mathrm{e}}_i \otimes \dot{\mathrm{k}}_i$ given in Definition 2.3.4*(b)* for $\tilde{\mathfrak{g}}_{{}_B}$. Likewise it holds for the other cases. □

7. Specialization of MpQG's at roots of unity

In this section we study MpQG's for which all parameters q_{ii} are roots of unity. Once again, this amounts to requiring q itself to be a root of unity, or just 1. As we already considered the case $q = 1$, we assume this root to be different from 1 itself.

7.1. **Specialization at roots of unity.**

Let again $\mathcal{R}_{\mathbf{q}}$ and $\mathcal{R}_{\mathbf{q}}^{{}_B}$ be fixed as in §6.1; fix also a positive, odd integer ℓ *which is coprime with all the d_i's ($i \in I$) given in §6.1*, and let $p_\ell(x)$ be the ℓ–th cyclotomic poynomial in $\mathbb{Z}[x]$. We consider the special element $q \in \mathcal{R}_{\mathbf{q}}$ and the quotient ring $\mathcal{R}_{\mathbf{q},\varepsilon} := \mathcal{R}_{\mathbf{q}} \Big/ p_\ell(q)\, \mathcal{R}_{\mathbf{q}}$, and we call ε the image of q in $\mathcal{R}_{\mathbf{q},\varepsilon}$.

By construction, the ring $\mathcal{R}_{\mathbf{q},\varepsilon}$ is generated by invertible elements $\varepsilon_{ij}^{\pm 1}$ each of whom is the image in $\mathcal{R}_{\mathbf{q},\varepsilon}$ of the corresponding generator $q_{ij}^{\pm 1}$ of $\mathcal{R}_{\mathbf{q}}$; since $\varepsilon_{ii}^{\ell} = 1$ for all i , all these generators only obey the relations $\big(\varepsilon_{ij}^{\pm 1}\, \varepsilon_{ji}^{\pm 1}\big)^{\ell} = 1$. We denote by $\varepsilon_{\alpha\gamma}$ the element in $\mathcal{R}_{\mathbf{q},\varepsilon}$ defined like in §3.2 but for using the ε_{ij}'s instead of the $q_{ij}^{\pm 1}$'s, so that $\varepsilon_{\alpha\gamma}^{\pm 1}$ is nothing but the image in $\mathcal{R}_{\mathbf{q},\varepsilon}$ of $q_{\alpha\gamma}^{\pm 1} \in \mathcal{R}_{\mathbf{q}}$. Finally, $\mathcal{R}_{\mathbf{q},\varepsilon}$ is an $\mathcal{R}_{\mathbf{q}}$–algebra by scalar restriction via the canonical epimorphism $\mathcal{R}_{\mathbf{q}} \twoheadrightarrow \mathcal{R}_{\mathbf{q},\varepsilon}$.

Replacing $\mathcal{R}_{\mathbf{q}}$ with $\mathcal{R}_{\mathbf{q}}^{{}_B}$ everywhere, we set $\mathcal{R}_{\mathbf{q},\varepsilon}^{{}_B} := \mathcal{R}_{\mathbf{q}}^{{}_B} \Big/ p_\ell(q)\, \mathcal{R}_{\mathbf{q}}^{{}_B}$, for which we use again such notation as ε , ε_{ij} , etc., noting in addition that now $\varepsilon_{ij} = \varepsilon^{b_{ij}}$. Then the natural epimorphisms $\mathcal{R}_{\mathbf{q}} \twoheadrightarrow \mathcal{R}_{\mathbf{q}}^{{}_B}$ yields a similar one $\mathcal{R}_{\mathbf{q},\varepsilon} \twoheadrightarrow \mathcal{R}_{\mathbf{q},\varepsilon}^{{}_B}$.

Furthermore, it is worth stressing that the isomorphism $\mathcal{R}_{\mathbf{q}}^{{}_B} \cong \mathbb{Z}\big[q, q^{-1}\big]$ induces in turn $\mathcal{R}_{\mathbf{q},\varepsilon}^{{}_B} \cong \mathbb{Z}\big[q, q^{-1}\big] \Big/ p_\ell(q)\, \mathbb{Z}\big[q, q^{-1}\big] =: \mathbb{Z}[\varepsilon]$, the latter being the ring extension of $\mathbb{Z}$ by any (formal) primitive ℓ–th root of unity ε .

Similarly, we define $\mathcal{R}_{\mathbf{q},\varepsilon}^{\sqrt{\ }} := \mathcal{R}_{\mathbf{q}}^{\sqrt{\ }} \Big/ p_\ell\big(q^{1/2}\big)\, \mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ and denote by $\varepsilon^{1/2}$, $\varepsilon_{ij}^{1/2}$, etc., the image of $q^{1/2}$, $q_{ij}^{1/2}$, etc., in $\mathcal{R}_{\mathbf{q},\varepsilon}^{\sqrt{\ }}$; and likewise for $\mathcal{R}_{\mathbf{q},\varepsilon}^{{}_{B,\sqrt{\ }}} := \mathcal{R}_{\mathbf{q}}^{{}_{B,\sqrt{\ }}} \Big/ p_\ell\big(q^{1/2}\big)\, \mathcal{R}_{\mathbf{q}}^{{}_{B,\sqrt{\ }}}$, for which we have in addition $\mathcal{R}_{\mathbf{q},\varepsilon}^{{}_{B,\sqrt{\ }}} \cong \mathbb{Z}\big[\varepsilon^{1/2}\big]$ where $\varepsilon^{1/2}$ is again a primitive ℓ–th root of unity. The projection $\mathcal{R}_{\mathbf{q}}^{\sqrt{\ }} \twoheadrightarrow \mathcal{R}_{\mathbf{q}}^{{}_{B,\sqrt{\ }}}$ induces an epimorphism $\mathcal{R}_{\mathbf{q},\varepsilon}^{\sqrt{\ }} \twoheadrightarrow \mathcal{R}_{\mathbf{q},\varepsilon}^{{}_{B,\sqrt{\ }}}$, while the embeddings $\mathcal{R}_{\mathbf{q}} \hookrightarrow \mathcal{R}_{\mathbf{q}}^{\sqrt{\ }}$ and $\mathcal{R}_{\mathbf{q},\varepsilon} \hookrightarrow \mathcal{R}_{\mathbf{q},\varepsilon}^{\sqrt{\ }}$ induce embeddings $\mathcal{R}_{\mathbf{q}}^{{}_B} \hookrightarrow \mathcal{R}_{\mathbf{q}}^{{}_{B,\sqrt{\ }}}$ and $\mathcal{R}_{\mathbf{q},\varepsilon}^{{}_B} \hookrightarrow \mathcal{R}_{\mathbf{q},\varepsilon}^{{}_{B,\sqrt{\ }}}$ respectively. *In addition*, for the last map the following holds:

Lemma 7.1.1. *The morphism* $\mathcal{R}^B_{\mathbf{q},\varepsilon} \lhook\joinrel\longrightarrow \mathcal{R}^{B,\sqrt{\ }}_{\mathbf{q},\varepsilon}$ *, given by* $\varepsilon^{\pm 1} \mapsto \big(\varepsilon^{\pm 1/2}\big)^2$ *, is an isomorphism, whose inverse* $\mathcal{R}^{B,\sqrt{\ }}_{\mathbf{q},\varepsilon} \lhook\joinrel\twoheadrightarrow \mathcal{R}^B_{\mathbf{q},\varepsilon}$ *is given by* $\varepsilon^{\pm 1/2} \mapsto \varepsilon^{\pm(\ell+1)/2}$ *.*

We introduce now the "specialization at $q=\varepsilon$" of the integral forms $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$, $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$ and $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ — over the ring $\mathcal{R}^B_{\mathbf{q}}$ or $\mathcal{R}_{\mathbf{q}}$ — of our MpQG's $U_{\mathbf{q}}(\mathfrak{g})$.

Definition 7.1.2.

(a) Let $\mathbf{q}$ be a multiparameter matrix of Cartan type: given $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ over the ground ring $\mathcal{R}_{\mathbf{q}}$, we call *specialization of* $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ *at* $q=\varepsilon$ the quotient

$$\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \;:=\; \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \Big/ p_\ell(q)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \;\cong\; \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$$

endowed with its natural (quotient) structure of Hopf algebra over $\mathcal{R}_{\mathbf{q},\varepsilon}$.

(b) Let $\mathbf{q}$ in addition be of *integral type* — hence $\mathcal{R}^B_{\mathbf{q}} = \mathbb{Z}\big[q,q^{-1}\big]$. Then:

— *(b.1)* given $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ over the ground ring $\mathcal{R}^B_{\mathbf{q}}$, we call *specialization of* $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ *at* $q=\varepsilon$ the quotient

$$\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \;:=\; \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \Big/ p_\ell(q)\, \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) \;\cong\; \mathcal{R}^B_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}^B_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}(\mathfrak{g})$$

endowed with its natural (quotient) structure of Hopf algebra over $\mathcal{R}^B_{\mathbf{q},\varepsilon}$;

— *(b.2)* we call *specialization of* $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ *at* $q=\varepsilon$ the quotient

$$\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \;:=\; \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}) \Big/ p_\ell(q)\, \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g}) \;\cong\; \mathcal{R}^B_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}^B_{\mathbf{q}}} \dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$$

endowed with its natural (quotient) structure of Hopf algebra over $\mathcal{R}^B_{\mathbf{q},\varepsilon}$;

— *(b.3)* if $\mathbf{q}$ is of *strongly integral type*, we call *specialization of* $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$ *at* $q=\varepsilon$ the quotient

$$\widehat{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \;:=\; \widehat{U}_{\mathbf{q}}(\mathfrak{g}) \Big/ p_\ell(q)\, \widehat{U}_{\mathbf{q}}(\mathfrak{g}) \;\cong\; \mathcal{R}^B_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}^B_{\mathbf{q}}} \widehat{U}_{\mathbf{q}}(\mathfrak{g})$$

endowed with its natural (quotient) structure of Hopf algebra over $\mathcal{R}^B_{\mathbf{q},\varepsilon}$.

Note that, using the isomorphism $\mathcal{R}^{B,\sqrt{\ }}_{\mathbf{q},\varepsilon} \cong \mathcal{R}^B_{\mathbf{q},\varepsilon}$ of Lemma 7.1.1, *all the above mentioned specializations of MpQG's at* $q=\varepsilon$ *can be also considered as Hopf algebras over the ring* $\mathcal{R}^{B,\sqrt{\ }}_{\mathbf{q},\varepsilon}$, by scalar extension: hereafter we shall freely do that. $\diamondsuit$

The above definitions and our results in §5 yield the following:

Theorem 7.1.3. *The PBW bases (over* $\mathcal{R}_{\mathbf{q}}$ *or* $\mathcal{R}^B_{\mathbf{q}}$ *) of* $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ *, resp. of* $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ *, resp. of* $\widehat{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ *— cf. Theorems 5.2.16 and 5.3.3 — yield, through the specialization process, similar PBW-bases (over* $\mathcal{R}_{\mathbf{q},\varepsilon}$ *or* $\mathcal{R}^B_{\mathbf{q},\varepsilon}$ *) of* $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ *, resp. of* $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ *, resp. of* $\widehat{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ *.*

Basing on the remark at the end of Definition 7.1.2, consider now both $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ and $\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$ as algebras over $\mathcal{R}^{B,\sqrt{\ }}_{\mathbf{q},\varepsilon}$. Let σ_ε be the unique 2–cocycle of $\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$ naturally induced by the 2–cocycle σ of $\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})$ as given in Definition 3.2.1: that is, $\sigma_\varepsilon : \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g}) \otimes \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g}) \longrightarrow \mathcal{R}_{\mathbf{q},\varepsilon}$ is the unique $\mathcal{R}_{\mathbf{q},\varepsilon}$–linear map given by

$$\sigma_\varepsilon(x,y) \,:=\, \varepsilon_{\mu\nu}^{1/2} \quad \textbf{if} \quad x=K_\mu \;\text{ or }\; x=L_\mu \;, \quad \text{and} \;\; y=K_\nu \;\text{ or }\; y=L_\nu$$

and $\sigma_\varepsilon(x,y) \,:=\, 0$ otherwise. The results in §5 then lead us to the following

Theorem 7.1.4. *Let* $\mathbf{q}$ *be a multiparameter matrix of Cartan type. Then*

(a) The Hopf $\mathcal{R}_{\mathbf{q},\varepsilon}$ *–algebra* $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ *is a 2–cocycle deformation of* $\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$ *, namely* $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \cong \big(\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})\big)_{\sigma_\varepsilon}$ *.*

(b) Assume that $\mathbf{q}$ *is of integral type; then the Hopf* $\mathcal{R}^B_{\mathbf{q},\varepsilon}$*–algebra* $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ *is a* 2*–cocycle deformation of* $\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$ *, namely* $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \cong \big(\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})\big)_{\sigma_\varepsilon}$.

Proof. Directly from definitions along with Proposition 5.3.2, we get claim *(a)* from

$$\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) = \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\mathbf{q}}(\mathfrak{g}) = \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q}}} \big(\widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})\big)_\sigma = \Big(\mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q}}} \widetilde{U}_{\check{\mathbf{q}}}(\mathfrak{g})\Big)_{\sigma_\varepsilon} = \big(\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})\big)_{\sigma_\varepsilon}$$

and likewise we prove claim *(b)* as well. □

7.2. **Quantum Frobenius morphisms for MpQG's.**

When dealing with uniparameter quantum groups, the so-called "quantum Frobenius morphisms" set a strong link between specializations of these quantum groups (either restricted or unrestricted) at 1 and specializations at roots of unity.

When one chooses the *restricted* and the *unrestricted* integral forms, these quantum Frobenius morphisms (for uniparameter quantum groups) look as

$$\widehat{Fr}_\ell \,:\, \widehat{U}_\varepsilon(\mathfrak{g}) \relbar\joinrel\twoheadrightarrow \mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} \widehat{U}_1(\mathfrak{g}) \qquad (\textit{restricted}\ \text{case})$$

and

$$\widetilde{Fr}_\ell \,:\, \mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} \widetilde{U}_1(\mathfrak{g}) \lhook\joinrel\relbar\joinrel\rightarrow \widetilde{U}_\varepsilon(\mathfrak{g}) \qquad (\textit{unrestricted}\ \text{case})$$

where $\mathbb{Z}[\varepsilon] := \mathbb{Z}\big[q,q^{-1}\big]\Big/p_\ell(q)\,\mathbb{Z}\big[q,q^{-1}\big]$, $\widehat{U}_s(\mathfrak{g}) := \widehat{U}_q(\mathfrak{g})\Big/\big(p_\ell(q)\,\widehat{U}_q(\mathfrak{g})\big)$ and similarly also $\widetilde{U}_s(\mathfrak{g}) := \widetilde{U}_q(\mathfrak{g})\Big/\big(p_\ell(q)\,\widetilde{U}_q(\mathfrak{g})\big)$, for $s \in \{1,\varepsilon\}$. Roughly speaking, $\widehat{Fr}_\ell$ is given by taking "ℓ-th roots" of algebra generators of $\widehat{U}_\varepsilon(\mathfrak{g})$, namely quantum divided powers and quantum binomial coefficients, while (dually, in a sense) $\widetilde{Fr}_\ell$ is given by raising to the "ℓ-th power" the algebra generators of $\widetilde{U}_\varepsilon(\mathfrak{g})$, i.e. quantum root vectors and toral generators.

In the present subsection we shall show that similar quantum Frobenius morphisms do exist for MpQG's as well, with a similar description.

7.2.1. Quantum Frobenius morphisms in the restricted case. We start by considering quantum Frobenius morphisms in the restricted case, i.e. for the specializations at roots of unity of $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ and $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$. Like in the uniparameter case, they will map any specialization at a root of unity ε onto a specialization at 1.

The following provides our *quantum Frobenius morphisms* for *restricted* MpQG's:

Theorem 7.2.2. *Let* $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ *be a multiparameter matrix of integral type. Then there exists a Hopf algebra epimorphism (over* $\mathcal{R}^B_{\mathbf{q},\varepsilon} \cong \mathbb{Z}[\varepsilon]$ *)*

$$\dot{\widehat{Fr}}_\ell \,:\, \dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \relbar\joinrel\twoheadrightarrow \mathcal{R}^B_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}^B_{\mathbf{q},1}} \dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g}) \,\cong\, \mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_B\big) \tag{7.1}$$

(cf. Theorem 6.2.4(a)) given on generators by

$$E_i^{(n)} \mapsto \begin{cases} \mathrm{e}_i^{(n/\ell)} & \textit{if } \ell \,\big|\, n \\ 0 & \textit{if } \ell \not\big|\, n \end{cases} , \qquad F_i^{(n)} \mapsto \begin{cases} \mathrm{f}_i^{(n/\ell)} & \textit{if } \ell \,\big|\, n \\ 0 & \textit{if } \ell \not\big|\, n \end{cases} \tag{7.2}$$

$$\binom{K_i\,;c}{n}_{\!\varepsilon} \mapsto \begin{cases} \binom{\dot{\mathrm{k}}_i + c}{n/\ell} & \textit{if } \ell \,\big|\, n \\ 0 & \textit{if } \ell \not\big|\, n \end{cases} , \quad \binom{L_i\,;c}{n}_{\!\varepsilon} \mapsto \begin{cases} \binom{\dot{\mathrm{l}}_i + c}{n/\ell} & \textit{if } \ell \,\big|\, n \\ 0 & \textit{if } \ell \not\big|\, n \end{cases} \tag{7.3}$$

$$\binom{G_i\,;c}{n}_{\!\varepsilon_{ii}} \mapsto \begin{cases} \binom{\mathrm{h}_i + c}{n/\ell} & \textit{if } \ell \,\big|\, n \\ 0 & \textit{if } \ell \not\big|\, n \end{cases} , \qquad K_i^{\pm 1} \mapsto 1 \ , \qquad L_i^{\pm 1} \mapsto 1 \tag{7.4}$$

Moreover, the image $\mathrm{Im}\Big(\dot{\widehat{Fr}}_\ell\Big)$ *is co-central in* $\mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},1}} \dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g})$ *, that is*

$$\big(\Delta - \Delta^{\mathrm{op}}\big)(u) \in \mathit{Ker}\Big(\dot{\widehat{Fr}}_\ell\Big) \otimes \mathrm{Ker}\Big(\dot{\widehat{Fr}}_\ell\Big) \qquad \textit{for all} \quad u \in \dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \tag{7.5}$$

In addition, when $\mathbf{q}$ *is of strongly integral type, there exists yet another Hopf algebra epimorphism (over* $\mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},\varepsilon} \cong \mathbb{Z}[\varepsilon]$ *)*

$$\widehat{Fr}_\ell \,:\, \widehat{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \relbar\joinrel\relbar\joinrel\twoheadrightarrow \mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},1}} \widehat{U}_{\mathbf{q},1}(\mathfrak{g}) \cong \mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\hat{\mathfrak{g}}_{\scriptscriptstyle B}\big) \tag{7.6}$$

(cf. Theorem 6.2.4(b)) for which similar properties and a similar description hold true with each $\begin{pmatrix} L_j\,;c \\ l_j \end{pmatrix}_{\!\varepsilon}$*, resp.* $\begin{pmatrix} K_j\,;c \\ k_j \end{pmatrix}_{\!\varepsilon}$*, replaced by* $\begin{pmatrix} L_j\,;c \\ l_j \end{pmatrix}_{\!\varepsilon_j}$*, resp.* $\begin{pmatrix} K_j\,;c \\ k_j \end{pmatrix}_{\!\varepsilon_j}$.

Proof. We present the proof for $\dot{\widehat{Fr}}_\ell$ and $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$, the rest being similar.

By Theorem 5.2.13*(a)*, we have a presentation of $\dot{\widehat{U}}_{\mathbf{q},\varepsilon} := \dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ by generators and relations. Then this also yields a similar presentation for $\mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},1}} \dot{\widehat{U}}_{\mathbf{q},1}(\mathfrak{g})$, which is isomorphic to $\mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{\scriptscriptstyle B}\big)$ as a Hopf algebra, by Theorem 6.2.4*(a)*. Now, a moment's check shows that under the prescriptions given in the claim each relation in the presentation of $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}$ is mapped by $\dot{\widehat{Fr}}_\ell$ onto either a similar relation in $\dot{\widehat{U}}_{\mathbf{q},1}$ or zero, hence they do provide a well-defined algebra morphism as required.

To show what happens in a specific example, let us consider the relations

$$E_i^{(m)}\, F_i^{(n)} \;=\; \sum_{s=0}^{m \wedge n} F_i^{(n-s)}\, q_{ii}^s \begin{pmatrix} G_i\,;\, 2\,s-m-n \\ s \end{pmatrix}_{\!q_{ii}} L_i^s\, E_i^{(m-s)} \qquad \forall\;\, m,n \in \mathbb{N}$$

for every index i, holding true in $\dot{\widehat{U}}_{\mathbf{q}}$ (cf. Theorem 5.2.13). By specialization, these yield in $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}$ the relations

$$E_i^{(m)}\, F_i^{(n)} = \sum_{s=0}^{m \wedge n} F_i^{(n-s)}\, \varepsilon_{ii}^s \begin{pmatrix} G_i\,;\, 2\,s-m-n \\ s \end{pmatrix}_{\!\varepsilon_{ii}} L_i^s\, E_i^{(m-s)} \qquad \big(m,n \in \mathbb{N}\big) \tag{7.7}$$

and likewise in $\dot{\widehat{U}}_{\mathbf{q},1} \cong U_{\mathbb{Z}}\big(\dot{\mathfrak{g}}_{\scriptscriptstyle B}\big)$ the relations (cf. Definition 6.2.2 and Theorem 6.2.4)

$$\mathrm{e}_i^{(m)}\, \mathrm{f}_i^{(n)} \;=\; \sum_{s=0}^{m \wedge n} \mathrm{f}_i^{(n-s)} \begin{pmatrix} \mathrm{h}_i + (2\,s-m-n) \\ s \end{pmatrix} \mathrm{e}_i^{(m-s)} \qquad \big(m,n \in \mathbb{N}\big) \tag{7.8}$$

where one uses a bit of arithmetic of p–binomial coefficients (namely, the sixth line identity in the list of Lemma 5.2.2) and of (classical) binomial coefficients to realize that specializing $\binom{G_i\,;\,2\,s-m-n}{s}_{q_{ii}}$ at $q=1$ eventually yields $\binom{\mathrm{h}_i+(2\,s-m-n)}{s}$.

Now, a moment's thought shows that if in left-hand side of (7.7) either m or n is *not* divisible by ℓ, then for each summand in right-hand side all of $(n-s)$, s and $(m-s)$ are not divisible either; hence our prescriptions for $\dot{\widehat{Fr}}_\ell$ actually do map both sides of (7.7) to zero. If instead both m and n *are* divisible by ℓ, then there are also summands in right-hand side for which all of $(n–s)$, s and $(m–s)$ are divisible as well; more explicitly, if $m = h\,\ell$ and $n = k\,\ell$, say, then the "relevant" summands on right-hand side are exactly those with index $s = r\,\ell$ for all $r \in \big\{0\,,1\,,\dots,h \wedge k\big\}$. In this case, our prescriptions for $\dot{\widehat{Fr}}_\ell$ map the left-hand side of (7.7) to $\mathrm{e}_i^{(m/\ell)}\mathrm{f}_i^{(n/\ell)} = \mathrm{e}_i^{(h)}\mathrm{f}_i^{(k)}$ and the right-hand side to

$$\sum_{r=0}^{h \wedge k} \mathrm{f}_i^{((k\,\ell-r\,\ell)/\ell)} \binom{\mathrm{h}_i+(2\,r\,\ell-h\,\ell-k\,\ell)/\ell}{r\,\ell/\ell} \mathrm{e}_i^{((h\,\ell-r\,\ell)/\ell)} \;=\; \sum_{r=0}^{h \wedge k} \mathrm{f}_i^{(k-r)} \binom{\mathrm{h}_i+(2\,r-h-k)}{r} \mathrm{e}_i^{(h-r)}$$

where the right-hand side is equal to $\mathrm{e}_i^{(h)}\mathrm{f}_i^{(k)}$, by (7.8) for $m := h$ and $n := k$.

Therefore the given formulas do provide a well-defined morphism of algebras $\widehat{\dot{Fr}}_\ell$ as required. By construction $\widehat{\dot{Fr}}_\ell$ is clearly onto, as the generators of $\mathcal{R}^B_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}^B_{\mathbf{q},1}} \hat{\dot{U}}_{\mathbf{q},1}(\mathfrak{g}) \cong \mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\dot{\mathfrak{g}}_B)$ are the images via $\widehat{\dot{Fr}}_\ell$ of the corresponding generators of $\hat{\dot{U}}_{\mathbf{q}}$.

Finally, we must prove that $\widehat{\dot{Fr}}_\ell$ is also a *Hopf algebra* morphism and that its image is co-central. This follows from the uniparameter case, as the coalgebra structure of the integral form of these MpQG's (cf. Theorem 5.2.13*(a)* and [DL, Proposition 6.4]) is the same as in the canonical case (the cocycle deformation process does not change the coalgebra structure), and our quantum Frobenius morphism is described by the same formulas. □

7.2.3. The unrestricted case: quantum Frobenius morphisms for $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$. In the unrestricted case, i.e. that of $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$, quantum Frobenius morphisms, in comparison with the restricted case, "go the other way round". Indeed, like in the uniparameter case, we shall find them mapping the specialization at 1 (of the given unrestricted integral form of a MpQG) into any specialization at a root of unity ε.

The very construction of such quantum Frobenius morphisms requires some preparation. Mimicking what was found in [DP] for the canonical case, the first ingredient is the subalgebra of $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ generated by the ℓ–th powers of its generators.

Definition 7.2.4. We define Z_0 to be the $\mathcal{R}_{\mathbf{q},\varepsilon}$–subalgebra

$$Z_0 \;:=\; \Big\langle\, \overline{f}_\alpha^{\,\ell} \,,\, l_i^{\pm\ell} \,,\, k_i^{\pm\ell} \,,\, \overline{e}_\alpha^{\,\ell} \,\Big\rangle_{\alpha\in Q\,,\, i\in I}$$

of $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ generated by the ℓ–th powers of the generators of $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$. ♢

N.B.: the original definition of Z_0 given in [DP, Chapter 5, §19.1] reads different, but it is also proved — still in [*loc. cit.*] — to be equivalent to the one given above.

The main properties of Z_0 were investigated in [An4, §4], with a slightly more general approach. The main outcome reads as follows:

Proposition 7.2.5. *(cf.* [An4, §4]*)*

(a) Z_0 is $\boldsymbol{\varepsilon}$–central in $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$, i.e., for each monomial b in a PBW basis of $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ as in Theorem 7.1.3 and each generator $z \in \big\{\, \overline{f}_\alpha^{\,\ell} \,,\, l_i^{\pm\ell} \,,\, k_i^{\pm\ell} \,,\, \overline{e}_\alpha^{\,\ell} \,\big\}_{\alpha\in Q\,,\, i\in I}$ of Z_0 there exists a (Laurent) monomial $m_{z,b}\big(\boldsymbol{\varepsilon}^{\pm\ell}\big)$ in the $\varepsilon_{ij}^{\pm\ell}$'s such that

$$z\, b \;=\; m_{z,b}\big(\boldsymbol{\varepsilon}^{\pm\ell}\big)\, b\, z$$

In particular, when $\mathbf{q}$ is of integral type Z_0 is central, hence normal, in $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$.

(b) Z_0 is a Hopf subalgebra of $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$, which is isomorphic as an algebra over $\mathcal{R}_{\mathbf{q},\varepsilon}$ to a partially Laurent $\boldsymbol{\varepsilon}$–polynomial algebra, namely

$$Z_0 \;\cong\; \mathcal{R}_{\mathbf{q},\varepsilon}\Big[\big\{ f_\alpha^{\,\ell}, l_i^{\pm\ell}, k_i^{\pm\ell}, e_\alpha^{\,\ell} \big\}_{\alpha\in\Phi^+}^{i\in I}\Big]$$

where the indeterminates $\boldsymbol{\varepsilon}$–commute (notation as in §6.2) among them i.e.

$$f_{\alpha'}^{\,\ell}\, f_{\alpha''}^{\,\ell} \;=\; \varepsilon_{\alpha''\alpha'}^{\ell^2}\, f_{\alpha''}^{\,\ell}\, f_{\alpha'}^{\,\ell} \;\;,\qquad e_{\alpha'}^{\,\ell}\, f_{\alpha''}^{\,\ell} \;=\; f_{\alpha''}^{\,\ell}\, e_{\alpha'}^{\,\ell} \;\;,\qquad e_{\alpha'}^{\,\ell}\, e_{\alpha''}^{\,\ell} \;=\; \varepsilon_{\alpha'\alpha''}^{\ell^2}\, e_{\alpha''}^{\,\ell}\, e_{\alpha'}^{\,\ell}$$
$$k_i^{\pm\ell}\, e_\alpha^{\,\ell} \;=\; \varepsilon_{\alpha_i\,\alpha}^{\pm\ell^2}\, e_\alpha^{\,\ell}\, k_i^{\pm\ell} \;\;,\qquad l_i^{\pm\ell}\, e_\alpha^{\,\ell} \;=\; \varepsilon_{\alpha\,\alpha_i}^{\mp\ell^2}\, e_\alpha^{\,\ell}\, l_i^{\pm\ell}$$
$$k_i^{\pm\ell}\, f_\alpha^{\,\ell} \;=\; \varepsilon_{\alpha_i\,\alpha}^{\mp\ell^2}\, f_\alpha^{\,\ell}\, k_i^{\pm\ell} \;\;,\qquad l_i^{\pm\ell}\, f_\alpha^{\,\ell} \;=\; \varepsilon_{\alpha\,\alpha_i}^{\pm\ell^2}\, f_\alpha^{\,\ell}\, l_i^{\pm\ell}$$
$$k_i^{\pm\ell}\, k_j^{\pm\ell} \;=\; k_j^{\pm\ell}\, k_i^{\pm\ell} \;\;,\qquad k_i^{\pm\ell}\, l_j^{\pm\ell} \;=\; l_j^{\pm\ell}\, k_i^{\pm\ell} \;\;,\qquad k_i^{\pm\ell}\, k_j^{\pm\ell} \;=\; k_j^{\pm\ell}\, k_i^{\pm\ell}$$

In particular, if $\mathbf{q}$ is of integral type — hence $\mathcal{R}_{\mathbf{q},\varepsilon} = \mathbb{Z}[\varepsilon]]$ — then $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is a commutative Hopf algebra of partially Laurent polynomials.

(c) $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is a free (left or right) Z_0–module of rank $\ell^{\dim(\mathfrak{g})}$.

Proof. Almost everything is proved in [An4, §4], so we just stress a single detail concerning claim *(c)*. Indeed, in [An4, Proposition 4.1] yields claim *(b)* as well as *(c)*, but for the latter the involved coefficients read differently, for instance one has $f_{\alpha'}^{\,\ell} f_{\alpha''}^{\,\ell} = \varepsilon_{\ell\,\alpha'',\ell\,\alpha'} f_{\alpha''}^{\,\ell} f_{\alpha'}^{\,\ell}$. But the symbol $\varepsilon_{\alpha,\beta}$ is bimultiplicative in α and β — i.e., it is a bicharacter on $Q \times Q$ — hence $\varepsilon_{\ell\,\alpha'',\ell\,\alpha'} = \varepsilon_{\alpha''\alpha'}^{\ell^2}$, and we are done. □

We shall now compare the subalgebra Z_0, a sub-object inside $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$, which is the specialisation of $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$ at $q = \varepsilon$, with the specialization at $q = 1$, that is $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$. This leads to find a special morphism, which we call *quantum Frobenius morphism for* $\widetilde{U}_{\mathbf{q}}(\mathfrak{g})$, which links $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ with $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ — once again generalising what occurs in the uniparameter case. In order to formalise this, we need to make $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ into a Hopf algebra *over* $\mathcal{R}_{\mathbf{q},\varepsilon}$, so that we can compare it with $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$.

Let us consider the unique ring embedding

$$\mathcal{R}_{\mathbf{q},1} \longhookrightarrow \mathcal{R}^{\sqrt{\ }}_{\mathbf{q},\varepsilon}\left(\cong \mathcal{R}_{\mathbf{q},\varepsilon}\right) \quad , \qquad y_{ij}^{\pm 1} \mapsto \varepsilon_{ij}^{\ell^2} \left(\cong \left(\varepsilon_{ij}^{\pm 1/2}\right)^{2\,\ell^2}\right) \tag{7.9}$$

where in right-hand side we take into account the isomorphism $\mathcal{R}^{\sqrt{\ }}_{\mathbf{q},\varepsilon} \cong \mathcal{R}_{\mathbf{q},\varepsilon}$ given by Lemma 7.1.1; we use this embedding to perform scalar extension from $\mathcal{R}_{\mathbf{q},1}$ to $\mathcal{R}_{\mathbf{q},\varepsilon}$ for $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$, so to make $\mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q},1}} \widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ into a (Hopf) algebra over $\mathcal{R}_{\mathbf{q},\varepsilon}$.

Besides, recall — from Proposition 4.1.1 — that for any $\alpha \in \Phi^+$ there exists suitable (Laurent) monomials $m_\alpha^+\big(\mathbf{q}^{\pm 1/2}\big)$ and $m_\alpha^-\big(\mathbf{q}^{\pm 1/2}\big)$ in the $q_{ij}^{\pm 1/2}$'s such that

$$E_\alpha \,=\, m_\alpha^+\big(\mathbf{q}^{\pm 1/2}\big)\, \check{E}_\alpha \quad , \qquad F_\alpha = m_\alpha^-\big(\mathbf{q}^{\pm 1/2}\big)\, \check{F}_\alpha$$

where $\check{E}_\alpha$, resp. $\check{F}_\alpha$, is the quantum root vector associated with $\alpha \in \Phi^+$, resp. $-\alpha \in \Phi^-$, in $U_{\check{\mathbf{q}}}(\mathfrak{g})$ and E_α, resp. F_α, is the similar vector in $U_{\mathbf{q}}(\mathfrak{g}) = \big(U_{\check{\mathbf{q}}}(\mathfrak{g})\big)_\sigma$.

As a direct consequence, we have similar relations among quantum root vectors in $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) = \big(\mathcal{R}_{\mathbf{q},1} \otimes_{\mathbb{Z}[q,q^{-1}]} \widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g})\big)_\sigma$ and $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) = \big(\mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathbb{Z}[\varepsilon,\varepsilon^{-1}]} \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})\big)_\sigma$, namely

$$\overline{e}_\alpha \,=\, m_\alpha^+\big(\mathbf{y}^{\pm 1/2}\big)\, \check{\overline{e}}_\alpha \quad , \qquad \overline{f}_\alpha = m_\alpha^-\big(\mathbf{y}^{\pm 1/2}\big)\, \check{\overline{f}}_\alpha \qquad \text{in} \quad \widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \tag{7.10}$$

and

$$\overline{e}_\alpha \,=\, m_\alpha^+\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)\, \check{\overline{e}}_\alpha \quad , \qquad \overline{f}_\alpha = m_\alpha^-\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)\, \check{\overline{f}}_\alpha \qquad \text{in} \quad \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \tag{7.11}$$

Our main result in this subsection is the existence of *quantum Frobenius morphisms* for *unrestricted MpQG's*, that are the monomorphisms mentioned below:

Theorem 7.2.6. *There exists a Hopf algebra monomorphism*

$$\widetilde{Fr}_\ell \,:\, \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q},1}} \widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \longhookrightarrow \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$$

uniquely determined (still identifying $\mathcal{R}^{\sqrt{\ }}_{\mathbf{q},\varepsilon} \cong \mathcal{R}_{\mathbf{q},\varepsilon}$ *) for all* $\alpha \in \Phi^+$, $i \in I$, *by*

$$\overline{f}_\alpha \mapsto m_\alpha^-\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{\ell^2-\ell}\, \overline{f}_\alpha^{\,\ell} \,,\;\; l_i^{\pm 1} \mapsto l_i^{\pm \ell} \,,\;\; k_i^{\pm 1} \mapsto k_i^{\pm \ell} \,,\;\; \overline{e}_\alpha \mapsto m_\alpha^+\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{\ell^2-\ell}\, \overline{e}_\alpha^{\,\ell} \tag{$\star$}$$

whose image is the $\boldsymbol{\varepsilon}$*–central Hopf subalgebra* Z_0 *of* $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$*; as a consequence, the Hopf algebra* Z_0 *itself is isomorphic to* $\mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q},1}} \widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$.

In particular, when $\mathbf{q}$ *is integral the morphism* $\widetilde{Fr}_\ell$ *is described by the simpler formulas (for all* $\alpha \in \Phi^+$, $i \in I$ *)*

$$\overline{f}_\alpha \mapsto \overline{f}_\alpha^{\,\ell} \quad , \qquad l_i^{\pm 1} \mapsto l_i^{\pm \ell} \quad , \qquad k_i^{\pm 1} \mapsto k_i^{\pm \ell} \quad , \qquad \overline{e}_\alpha \mapsto \overline{e}_\alpha^{\,\ell} \quad .$$

Proof. To begin with, the morphism in (7.9) maps every $y_{\alpha,\beta}^{\pm 1/2} \in \mathcal{R}^{\sqrt{\ }}_{\mathbf{q},1} \cong \mathcal{R}_{\mathbf{q},1}$ into the corresponding $\varepsilon_{\alpha,\beta}^{\pm \ell^2/2} \in \mathcal{R}^{\sqrt{\ }}_{\mathbf{q},\varepsilon} \cong \mathcal{R}_{\mathbf{q},\varepsilon}$. Moreover, for $\mathbf{q} = \check{\mathbf{q}}$ the analysis in [DP] yields a ("quantum Frobenius") morphism $\widetilde{Fr}_\ell^{\vee} : \widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g}) \longhookrightarrow \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$ *of Hopf algebras* which is determined by the formulas given for the integral case in the above statement when $\overline{f}_\alpha = \check{\overline{f}}_\alpha$, etc., that is $\widetilde{Fr}_\ell^\vee\big(\check{\overline{f}}_\alpha\big) = \check{\overline{f}}_\alpha^{\,\ell}$ and so on; in particular, this $\widetilde{Fr}_\ell^\vee$ preserves the coproduct. Now, extending scalars, we obtain yet another Hopf algebra monomorphism $\widetilde{Fr}_\ell : \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathbb{Z}[q,q^{-1}]} \widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g}) \lhook\joinrel\longrightarrow \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathbb{Z}[\varepsilon,\varepsilon^{-1}]} \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$ that fits into the commutative diagram

$$\begin{tikzcd}
\widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g}) \arrow[rr, hook, "\widetilde{Fr}_\ell^\vee"] \arrow[d, hook] & & \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g}) \arrow[dd, hook] \\
\mathcal{R}_{\mathbf{q},1} \otimes_{\mathbb{Z}[q,q^{-1}]} \widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g}) \arrow[d, hook] & & \\
\mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q},1}} \Big(\mathcal{R}_{\mathbf{q},1} \otimes_{\mathbb{Z}[q,q^{-1}]} \widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g}) \Big) \arrow[rr, hook, "\widetilde{Fr}_\ell"'] & & \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathbb{Z}[\varepsilon,\varepsilon^{-1}]} \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})
\end{tikzcd}$$

Let us check that $\widetilde{Fr}_\ell$ satisfies the equalities in $(\star)$. First, as by (7.10) we have $\overline{e}_\alpha = m_\alpha^+\big(\mathbf{y}^{\pm 1/2}\big)\,\check{\overline{e}}_\alpha$ we find that

$$\begin{aligned}
\widetilde{Fr}_\ell\big(\overline{e}_\alpha\big) \, &= \, \widetilde{Fr}_\ell\big(m_\alpha^+\big(\mathbf{y}^{\pm 1/2}\big)\,\check{\overline{e}}_\alpha\big) \, = \, m_\alpha^+\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{\ell^2}\, \widetilde{Fr}_\ell\big(\check{\overline{e}}_\alpha\big) \, = \\
&= \, m_\alpha^+\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{\ell^2}\, \widetilde{Fr}_\ell^\vee\big(\check{\overline{e}}_\alpha\big) \, = \, m_\alpha^+\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{\ell^2}\, \check{\overline{e}}_\alpha^{\,\ell} \, = \\
&= \, m_\alpha^+\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{\ell^2}\, m_\alpha^+\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{-\ell}\, \overline{e}_\alpha^{\,\ell} \, = \, m_\alpha^+\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{\ell^2-\ell}\, \overline{e}_\alpha^{\,\ell}
\end{aligned}$$

— thanks to (7.11) — i.e. $\widetilde{Fr}_\ell\big(\overline{e}_\alpha\big) = m_\alpha^+\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{\ell^2-\ell}\,\overline{e}_\alpha^{\,\ell}$ for every root vector $\overline{e}_\alpha$ in $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$; similarly one finds $\widetilde{Fr}_\ell\big(\overline{f}_\alpha\big) = m_\alpha^-\big(\boldsymbol{\varepsilon}^{\pm 1/2}\big)^{\ell^2-\ell}\,\overline{f}_\alpha^{\,\ell}$ when dealing with the $\overline{f}_\alpha$'s, and $\widetilde{Fr}_\ell\big(l_i^{\pm 1}\big) = l_i^{\pm \ell}$, $\widetilde{Fr}_\ell\big(k_i^{\pm 1}\big) = k_i^{\pm \ell}$ for all $i \in I$.

Now recall that we have identifications of coalgebras

$$\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \, = \, \Big(\mathcal{R}_{\mathbf{q},1} \otimes_{\mathbb{Z}[q,q^{-1}]} \widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g}) \Big)_{\sigma_1} \; , \quad \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \, = \, \Big(\mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathbb{Z}[\varepsilon,\varepsilon^{-1}]} \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g}) \Big)_{\sigma_\varepsilon}$$

hence the monomorphism $\widetilde{Fr}_\ell$ defines also a monomorphism of *coalgebras* — over $\mathcal{R}_{\mathbf{q},\varepsilon}$ — from $\mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q},1}} \widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ to $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$. To prove that $\widetilde{Fr}_\ell$ is also a *Hopf algebra* morphism, it is enough to prove that $\widetilde{Fr}_\ell(x \cdot_{\sigma_1} y) = \widetilde{Fr}_\ell(x) \cdot_{\sigma_\varepsilon} \widetilde{Fr}_\ell(y)$ for all x, y in $\mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathbb{Z}[q,q^{-1}]} \widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g})$. This in turn follows from the fact that

$$\sigma_\varepsilon\Big(\widetilde{Fr}_\ell(x)\,,\widetilde{Fr}_\ell(y)\Big) \, = \, \sigma_1(x,y) \qquad \text{for all} \quad x,y \in \mathcal{R}_{\mathbf{q},\varepsilon} \otimes_{\mathbb{Z}[q,q^{-1}]} \widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g}) \tag{7.12}$$

which can be checked by direct computation with x and y being generators of $\widetilde{U}_{\check{\mathbf{q}},1}(\mathfrak{g})$ and using the ring embedding (7.9); for example, one has

$$\sigma_\varepsilon\Big(\widetilde{Fr}_\ell(k_i)\,,\widetilde{Fr}_\ell(k_j)\Big) \, = \, \sigma_\varepsilon\Big(k_i^\ell\,,k_j^\ell\Big) \, = \, \varepsilon_{ij}^{\ell^2/2} \, = \, y_{ij}^{1/2} \, = \, \sigma_1(k_i,k_j) \quad \text{for all } \; i,j \in I \; .$$

In fact, from (7.12) we get

$$\begin{aligned}
\widetilde{Fr}_\ell(x \cdot_{\sigma_1} y) \, &= \, \widetilde{Fr}_\ell\big(\sigma_1\big(x_{(1)},y_{(1)}\big)\,x_{(2)}\,y_{(2)}\,\sigma_1^{-1}\big(x_{(3)},y_{(3)}\big)\big) \, = \\
&= \, \sigma_1\big(x_{(1)},y_{(1)}\big)\,\widetilde{Fr}_\ell\big(x_{(2)}\big)\,\widetilde{Fr}_\ell\big(y_{(2)}\big)\,\sigma_1^{-1}\big(x_{(3)},y_{(3)}\big) \, = \\
&= \, \sigma_\varepsilon\Big(\widetilde{Fr}_\ell(x)_{(1)}\,,\widetilde{Fr}_\ell(y)_{(1)}\Big)\,\widetilde{Fr}_\ell(x)_{(2)}\,\widetilde{Fr}_\ell(y)_{(2)}\,\sigma_\varepsilon^{-1}\Big(\widetilde{Fr}_\ell(x)_{(3)}\,,\widetilde{Fr}_\ell(y)_{(3)}\Big) \, = \\
&= \, \widetilde{Fr}_\ell(x) \cdot_{\sigma_\varepsilon} \widetilde{Fr}_\ell(y)
\end{aligned}$$

thus the proof is completed. $\square$

7.3. **Small multiparameter quantum groups.**

In the study of uniparameter quantum groups, a relevant role is played by the so called "small quantum groups". These are usually introduced as *Hopf subalgebras of* the *restricted* quantum groups at roots of unity; nonetheless, they can also be realized as *Hopf algebra quotients of* the *unrestricted* quantum groups at roots of unity. In this subsection we extend their construction to the multiparameter context.

7.3.1. Small MpQG's: the "restricted realization". Let $\mathbf{q}$ be a multiparameter of integral type, hence possibly of strongly integral type. Correspondingly, we consider the restricted MpQG's $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}$ and $\widehat{U}_{\mathbf{q},\varepsilon}$ at a root of unity ε, like in §7.1. Inside them, we consider the following subalgebras, defined by generating sets:

$$\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} = \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) := \left\langle F_i^{(n)}, L_i^{\pm 1}, \binom{L_i}{n}_{\varepsilon}, K_i^{\pm 1}, \binom{K_i}{n}_{\varepsilon}, E_i^{(n)} \right\rangle_{i \in I}^{0 \leq n \lneqq \ell} \tag{7.13}$$

as a $\mathcal{R}_{\mathbf{q},\varepsilon}$–subalgebra of $\dot{\widehat{U}}_{\mathbf{q},\varepsilon} = \dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$, when $\mathbf{q}$ is integral, and

$$\widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon} = \widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) := \left\langle F_i^{(n)}, L_i^{\pm 1}, \binom{L_i}{n}_{\varepsilon_i}, K_i^{\pm 1}, \binom{K_i}{n}_{\varepsilon_i}, E_i^{(n)} \right\rangle_{i \in I}^{0 \leq n \lneqq \ell} \tag{7.14}$$

as a $\mathcal{R}_{\mathbf{q},\varepsilon}$–subalgebra of $\widehat{U}_{\mathbf{q},\varepsilon} = \widehat{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$, when $\mathbf{q}$ is *strongly* integral. Similarly, one defines $\dot{\widehat{\mathfrak{u}}}^{\pm}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^{\pm,0}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^{0}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^{\leq}_{\mathbf{q},\varepsilon}$ and $\dot{\widehat{\mathfrak{u}}}^{\geq}_{\mathbf{q},\varepsilon}$ inside $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$, and similarly $\widehat{\mathfrak{u}}^{\pm}_{\mathbf{q},\varepsilon}$, $\widehat{\mathfrak{u}}^{\pm,0}_{\mathbf{q},\varepsilon}$, etc., inside $\widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}$, just mimicking Definition 5.2.12 *but* working inside $\dot{\widehat{U}}_{\mathbf{q}}$ or $\widehat{U}_{\mathbf{q}}$, respectively, *and* imposing the restriction "$n \lneqq \ell$" everywhere. All these objects will be called *"restricted small multiparameter quantum (sub)groups"*.

Note that for $\mathbf{q} = \check{\mathbf{q}}$ the canonical multiparameter, the small quantum group $\widehat{\mathfrak{u}}_{\check{\mathbf{q}},\varepsilon}$ is a quantum double version of the one-parameter small quantum group by Lusztig.

Our first result is a structural one:

Theorem 7.3.2. *For any $\mathbf{q}$ of integral type, $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ is a Hopf $\mathcal{R}^{B}_{\mathbf{q},\varepsilon}$–subalgebra of $\dot{\widehat{U}}_{\mathbf{q},\varepsilon} = \dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$. In addition, if $\mathbf{q}$ of strongly integral type, then $\widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ is a Hopf $\mathcal{R}^{B}_{\mathbf{q},\varepsilon}$–subalgebra of $\widehat{U}_{\mathbf{q},\varepsilon} = \widehat{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$. Moreover, $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ admits a presentation by generators and relations that is the same as in Theorem 5.2.13 (with q specialized to ε) but for the bound on generators — i.e., they must have $0 \leq n \lneqq \ell$ as in* (7.13) *— and for the additional relations*

$$X_i^{(n)} X_i^{(m)} = 0 \ , \quad \binom{M\,;\,c}{n}_{\varepsilon} \binom{M\,;\,c-n}{m}_{\varepsilon} = 0 \qquad \forall \ n, m \lneqq \ell \, : \, n+m \geq \ell \tag{7.15}$$

for all $X \in \{F, E\}$, $M \in \{K, L\}$, $c \in \mathbb{Z}$. Similar statements hold true for all the other restricted small MpQG's, namely $\dot{\widehat{\mathfrak{u}}}^{\pm}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^{\pm,0}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^{0}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^{\leq}_{\mathbf{q},\varepsilon}$ and $\dot{\widehat{\mathfrak{u}}}^{\geq}_{\mathbf{q},\varepsilon}$, and — in the strongly integral case — $\widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}$, $\widehat{\mathfrak{u}}^{\pm}_{\mathbf{q},\varepsilon}$, $\widehat{\mathfrak{u}}^{\pm,0}_{\mathbf{q},\varepsilon}$, $\widehat{\mathfrak{u}}^{0}_{\mathbf{q},\varepsilon}$, $\widehat{\mathfrak{u}}^{\leq}_{\mathbf{q},\varepsilon}$ and $\widehat{\mathfrak{u}}^{\geq}_{\mathbf{q},\varepsilon}$.

Proof. The claim follows from the very definitions together with Theorem 5.2.13 — noting in particular that all relations between generators given there do "fit properly" with the bound $n \lneqq \ell$ on generators of the small MpQG. In particular, the additional relations in (7.15) are direct consequence of the relations

$$X_i^{(n)} X_i^{(m)} = \binom{n+m}{n}_{q_{ii}} X_i^{(n+m)} \ , \qquad \binom{M\,;\,c}{n}_{q} \binom{M\,;\,c-n}{m}_{q} = \binom{n+m}{n}_{q} \binom{M\,;\,c}{n+m}_{q}$$

(for all $X \in \{F, E\}$, $M \in \{K, L\}$ and $c \in \mathbb{Z}$) holding true in our restricted MpQG's for every $n, m \in \mathbb{N}$, that for $n, m \lneqq \ell$ such that $n+m \geq \ell$ yield (7.15) because then $\binom{n+m}{n}_{q_{ii}} = 0$ and $\binom{n+m}{n}_{q} = 0$ for $q = \varepsilon$.

Similarly, the result about the Hopf structure follows from the explicit formulas for the coalgebra structure of $\dot{\widehat{U}}_{\mathbf{q}}(\mathfrak{g})$ or $\widehat{U}_{\mathbf{q}}(\mathfrak{g})$ coming from Lemma 5.2.11. $\square$

Our second result yields triangular decompositions for restricted small MpQG's:

Proposition 7.3.3. *(triangular decompositions for restricted small MpQG's)*

The multiplication in $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ provides $\mathcal{R}^B_{\mathbf{q},\varepsilon}$–module isomorphisms

$$\dot{\widehat{\mathfrak{u}}}^-_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^0_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}^\leq_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}^0_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^-_{\mathbf{q},\varepsilon} \;, \quad \dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^0_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}^\geq_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}^0_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$$

$$\dot{\widehat{\mathfrak{u}}}^{+,0}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^{-,0}_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}^0_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}^{-,0}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^{+,0}_{\mathbf{q},\varepsilon} \;, \quad \dot{\widehat{\mathfrak{u}}}^\leq_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^\geq_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}^\geq_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^\leq_{\mathbf{q},\varepsilon}$$

$$\dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^0_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^-_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}^-_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^0_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$$

and similarly with "$\dot{\widehat{\mathfrak{u}}}$" replaced by "$\widehat{\mathfrak{u}}$" if $\mathbf{q}$ is strongly integral.

Proof. This is proved like for restricted MpQG's: one observes that the presentation of (restricted) small MpQG's given in Theorem 7.3.2 above presents the same special features that were exploited for the proof of Proposition 5.2.15, so the same arguments apply again. A quicker argument is the following: the isomorphisms of Proposition 5.2.15 restricts to maps for small quantum groups which are linear isomorphisms by Theorem 7.3.2. $\square$

The third result is a PBW-like theorem for these restricted small MpQG's:

Theorem 7.3.4. *(PBW theorem for restricted small MpQG's)*

Every restricted small MpQG is a free $\mathcal{R}^B_{\mathbf{q},\varepsilon}$–module with $\mathcal{R}^B_{\mathbf{q},\varepsilon}$–basis the subset of a PBW basis — as given in Theorem 5.2.16 — of the corresponding specialized restricted MpQG made by those PBW-like monomials in which the degree of each factor is less than ℓ. For instance, $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ has $\mathcal{R}^B_{\mathbf{q},\varepsilon}$–basis

$$\left\{ \prod_{k=N}^{1} F_{\beta^k}^{(f_k)} \prod_{j\in I} \binom{L_j}{l_j}_q L_j^{-\lfloor l_j/2\rfloor} \prod_{i\in I} \binom{G_i}{g_i}_{q_{ii}} G_i^{-\lfloor g_i/2\rfloor} \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \;\middle|\; 0 \leq f_k, l_j, g_i, e_h < \ell \right\}$$

and similarly holds for $\dot{\widehat{\mathfrak{u}}}^\pm_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^{\pm,0}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^0_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^\leq_{\mathbf{q},\varepsilon}$ and $\dot{\widehat{\mathfrak{u}}}^\geq_{\mathbf{q},\varepsilon}$, as well as — in the strongly integral case — for $\widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}$, $\widehat{\mathfrak{u}}^\pm_{\mathbf{q},\varepsilon}$, $\widehat{\mathfrak{u}}^{\pm,0}_{\mathbf{q},\varepsilon}$, $\widehat{\mathfrak{u}}^0_{\mathbf{q},\varepsilon}$, $\widehat{\mathfrak{u}}^\leq_{\mathbf{q},\varepsilon}$ and $\widehat{\mathfrak{u}}^\geq_{\mathbf{q},\varepsilon}$.

Proof. First we discuss the case of $\dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$, whose "candidate" $\mathcal{R}^B_{\mathbf{q},\varepsilon}$–basis is the set of "truncated" (ordered) PBW monomials $B^+_{\mathbf{q}} := \left\{ \prod_{h=1}^{N} E_{\beta^h}^{(e_h)} \;\middle|\; 0 \leq e_1, \dots, e_N < \ell \right\}$.

In the canonical case $\mathbf{q} = \check{\mathbf{q}}$ the required property (i.e., $B^+_{\mathbf{q}}$ is an $\mathcal{R}^B_{\mathbf{q},\varepsilon}$–basis of $\dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$) is proved by Lusztig (cf. [Lu] and references therein). For general $\mathbf{q}$, we deduce the claim from the canonical case, arguing like in the proof of Theorem 5.2.16*(a)*.

Let us consider a quantum root vector E_β for $\beta \in \Phi^+$ in $\dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$, coming (through specialization) from the same name quantum root vector in $\dot{\widehat{U}}^+_{\mathbf{q}}$. We want to prove that $E_\beta^{(n)} \in \dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$ for all $0 \leq n < \ell$: indeed, once we have $E_\beta^{(n)} \in \dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$ for all $\beta \in \Phi^+$ and $0 \leq n < \ell$ we argue that all of $\mathcal{S} := \mathit{Span}_{\mathcal{R}_{\mathbf{q},\varepsilon}}\big(B^+_{\mathbf{q}}\big)$ is included in $\dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$.

Let us resume notation as in §4.4.2. As we said, the claim being true in the canonical case implies $E_\beta^{\check{\ }(n)} \in \dot{\widehat{\mathfrak{u}}}^+_{\check{\mathbf{q}},\varepsilon}$, so that $E_\beta^{\check{\ }(n)}$ can be written as a non-commutative polynomial — with coefficients in $\mathcal{R}^B_{\mathbf{q},\varepsilon}$ — in the $E_i^{\check{\ }(c)}$'s with $i \in I$ and $c < \ell$, say

$$E_\beta^{\check{\ }(n)} = P\big(\{E_i^{\check{\ }(c)}\}_{i\in I, c<\ell}\big) \tag{7.16}$$

Now, the formulas in §4.4.2 tell us that $E_\beta^{*(n)} = \big(\varepsilon^{1/2}\big)^{z_\beta} E_\beta^{\check{\ }(n)}$ and

$$E_{i_1}^{*(s_1)} * E_{i_2}^{*(s_2)} * \cdots * E_{i_k}^{*(s_k)} = \big(\varepsilon^{1/2}\big)^{z_{\underline{i},\underline{s}}} E_{i_1}^{\check{\ }(s_1)} \,\check{\cdot}\, E_{i_2}^{\check{\ }(s_2)} \,\check{\cdot}\, \cdots \,\check{\cdot}\, E_{i_k}^{\check{\ }(s_k)} \tag{7.17}$$

for some $z_\beta, z_{\underline{i},\underline{s}} \in \mathbb{Z}$ — with $\beta \in \Phi^+$, $k \in \mathbb{N}$, $\underline{i} := (i_1, i_2, \dots, i_k) \in I^k$ and $\underline{s} := (s_1, s_2, \dots, s_k) \in \mathbb{N}^k$ — where $\varepsilon^{1/2}$ arises as specialization of $q^{1/2}$ but identifies with $\varepsilon^{(\ell+1)/2} \in \mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},\varepsilon}$. These identities together with (7.16) lead us in turn to write

$$E_\beta^{*(n)} \;=\; P_\bullet\big(\{E_i^{*(c)}\}_{i \in I, c<\ell}\big)$$

where $P_\bullet$ is again a non-commutative polynomial in the $E_i^{*(c)}$'s with coefficients in $\mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},\varepsilon}$, so that $E_\beta^{*(n)} \in \dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$, q.e.d.

We have seen above that $\mathcal{S}_{\mathbf{q}} := \mathit{Span}_{\mathcal{R}_{\mathbf{q},\varepsilon}}\big(B^+_{\mathbf{q}}\big) \subseteq \dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$, now we prove the converse. First of all, by construction $\dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$ is spanned over $\mathcal{R}_{\mathbf{q},\varepsilon}$ by monomials in the $E_i^{(n)}$'s of the form $E_{\underline{i}}^{*(\underline{s})} := E_{i_1}^{*(s_1)} * E_{i_2}^{*(s_2)} * \cdots * E_{i_k}^{*(s_k)}$ that can also be re-written as $E_{\underline{i}}^{*(\underline{s})} = \big(\varepsilon^{1/2}\big)^{z_{\underline{i},\underline{s}}} E_{i_1}^{\check{:}(s_1)} \,\check{:}\, E_{i_2}^{\check{:}(s_2)} \,\check{:}\, \cdots \,\check{:}\, E_{i_k}^{\check{:}(s_k)} =: \big(\varepsilon^{1/2}\big)^{z_{\underline{i},\underline{s}}} E_{\underline{i}}^{\check{:}(\underline{s})}$ — with notation as above; we aim to prove that each such $E_{\underline{i}}^{*(\underline{s})}$ belongs to $\mathcal{S}_{\mathbf{q}}$, as this will then entail at once that $\dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon} \subseteq \mathcal{S}_{\mathbf{q}}$. The claim is true in the canonical case, so $E_{\underline{i}}^{\check{:}(\underline{s})} \in \dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon} = \mathcal{S}_{\check{\mathbf{q}}}$ hence $E_{\underline{i}}^{\check{:}(\underline{s})}$ expands as $E_{\underline{i}}^{\check{:}(\underline{s})} = \sum_{E_{\underline{\alpha}}^{\check{:}(\underline{c})} \in B^+_{\check{\mathbf{q}}}} \kappa_{\underline{c}}\, E_{\underline{\alpha}}^{\check{:}(\underline{c})}$ for suitable $\kappa_{\underline{c}} \in \mathcal{R}_{\check{\mathbf{q}},\varepsilon} = \mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},\varepsilon}$. Using (7.17) again we get $E_{\underline{i}}^{*(\underline{s})} = \sum_{E_{\underline{\alpha}}^{*(\underline{c})} \in B^+_{\mathbf{q}}} \big(\varepsilon^{1/2}\big)^{z_{\underline{c}}} \kappa_{\underline{c}}\, E_{\underline{\alpha}}^{*(\underline{c})}$ for suitable $z_{\underline{c}} \in \mathbb{Z}$, so that $E_{\underline{i}}^{*(\underline{s})} \in \mathcal{S}_{\check{\mathbf{q}}}$, q.e.d.

Just like for $\dot{\widehat{\mathfrak{u}}}^+_{\mathbf{q},\varepsilon}$, the same arguments prove the claim is true for $\dot{\widehat{\mathfrak{u}}}^-_{\mathbf{q},\varepsilon}$ as well.

As to $\dot{\widehat{\mathfrak{u}}}^{-,0}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{\mathfrak{u}}}^{+,0}_{\mathbf{q},\varepsilon}$ and $\dot{\widehat{\mathfrak{u}}}^{0}_{\mathbf{q},\varepsilon}$, the claim follows at once from the analogous PBW theorem for $\dot{\widehat{U}}^0_{\mathbf{q},\varepsilon}$, together with the relations $\binom{M\,;\,c}{n}_{\!\varepsilon}\binom{M\,;\,c-n}{m}_{\!\varepsilon} = 0$ for all $n, m \lneqq \ell$ such that $n+m \geq \ell$ when $M \in \{K_i\,, L_i\}_{i \in I}$ (cf. Theorem 7.3.2).

Finally, the claim for $\dot{\widehat{\mathfrak{u}}}^{\geq}_{\mathbf{q},\varepsilon}$, for $\dot{\widehat{\mathfrak{u}}}^{\leq}_{\mathbf{q},\varepsilon}$ and for $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ follows from the previous results together with triangular decompositions (cf. Proposition 7.3.3).

The cases where "$\dot{\widehat{\mathfrak{u}}}$" is replaced by "$\widehat{\mathfrak{u}}$" are treated similarly. □

<u>Note</u>: from now on, for the rest of the present discussion of restricted small MpQG's, we extend our ground ring from $\mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},\varepsilon} = \mathbb{Z}[\varepsilon]$ to $\mathbb{Q}_\varepsilon$, the latter being the ℓ–th cyclotomic field over $\mathbb{Q}$ — i.e., the field extension of $\mathbb{Q}$ generated by a primitive ℓ–th root of unity ε. Thus, *all our MpQG's at a root of unity will be considered* — via scalar extension from $\mathcal{R}^{\scriptscriptstyle B}_{\mathbf{q},\varepsilon}$ to $\mathbb{Q}_\varepsilon$ — as Hopf algebras *defined over* $\mathbb{Q}_\varepsilon$.

A first, elementary result follows easily from definitions:

Proposition 7.3.5. *Assume that* $\mathbf{q}$ *is of strongly integral type. Then we have*

(a) $\dot{\widehat{U}}_{\mathbf{q},\varepsilon} = \widehat{U}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{U}}^{\pm}_{\mathbf{q},\varepsilon} = \widehat{U}^{\pm}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{U}}^{\geq}_{\mathbf{q},\varepsilon} = \widehat{U}^{\geq}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{U}}^{\leq}_{\mathbf{q},\varepsilon} = \widehat{U}^{\leq}_{\mathbf{q},\varepsilon}$, $\dot{\widehat{U}}^{0}_{\mathbf{q},\varepsilon} = \widehat{U}^{0}_{\mathbf{q},\varepsilon}$ *and* $\dot{\widehat{U}}^{\pm,0}_{\mathbf{q},\varepsilon} = \widehat{U}^{\pm,0}_{\mathbf{q},\varepsilon}$ *via natural identifications;*

(b) $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} = \widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}$, *and both algebras are generated by* $\{E_i\,, L_i^{\pm 1}, K_i^{\pm 1}, F_i\}_{i \in I}$;

(c) $\dot{\widehat{\mathfrak{u}}}^{\pm}_{\mathbf{q},\varepsilon} = \widehat{\mathfrak{u}}^{\pm}_{\mathbf{q},\varepsilon}$, *and both algebras are generated respectively by* $\{E_i\}_{i \in I}$ *— for the "+" case — and by* $\{F_i\}_{i \in I}$ *— for the "−" case;*

(d) $\dot{\widehat{\mathfrak{u}}}^{\pm,0}_{\mathbf{q},\varepsilon} = \widehat{\mathfrak{u}}^{\pm,0}_{\mathbf{q},\varepsilon}$, *and both algebras are generated respectively by* $\{K_i^{\pm 1}\}_{i \in I}$ *— for the "+" case — and by* $\{L_i^{\pm 1}\}_{i \in I}$ *— for the "−" case;*

(e) $\dot{\widehat{\mathfrak{u}}}^{0}_{\mathbf{q},\varepsilon} = \widehat{\mathfrak{u}}^{0}_{\mathbf{q},\varepsilon}$, *resp.* $\dot{\widehat{\mathfrak{u}}}^{\leq}_{\mathbf{q},\varepsilon} = \widehat{\mathfrak{u}}^{\leq}_{\mathbf{q},\varepsilon}$, *resp.* $\dot{\widehat{\mathfrak{u}}}^{\geq}_{\mathbf{q},\varepsilon} = \widehat{\mathfrak{u}}^{\geq}_{\mathbf{q},\varepsilon}$, *and both algebras are generated by* $\{K_i^{\pm 1}, L_i^{\pm 1}\}_{i \in I}$, *resp. by* $\{L_i^{\pm 1}, F_i\}_{i \in I}$, *resp. by* $\{E_i\,, K_i^{\pm 1}\}_{i \in I}$.

Proof. As to claim *(a)*, by construction it is enough to show that $\dot{\widehat{U}}^{\,0}_{\mathbf{q},\varepsilon} = \widehat{U}^{\,0}_{\mathbf{q},\varepsilon}$ or more precisely $\dot{\widehat{U}}^{\,\pm,0}_{\mathbf{q},\varepsilon} = \widehat{U}^{\,\pm,0}_{\mathbf{q},\varepsilon}$. In turn, the latter identity follows from definitions together with the following formal identity among quantum binomial coefficients

$$\binom{X}{n}_{\varepsilon} \;=\; \prod_{s=1}^{n} (d_i)_{\varepsilon^s} \binom{X}{n}_{\varepsilon_i}$$

which proves that the ε–binomial coefficients and the ε_i–binomial coefficients generate over $\mathbb{Q}_\varepsilon$ the same algebra, since $\prod_{s=1}^{n} (d_i)_{\varepsilon^s}$ is invertible in the field $\mathbb{Q}_\varepsilon$.

As to the remaining claims, everything follows again from a simple remark. Namely, definitions give

$$\prod_{r=1}^{n} \big(X_i \, \varepsilon_i^{1-r} - 1 \big) \;=\; \prod_{r=1}^{n} \big(\varepsilon_i^r - 1 \big) \binom{X_i}{n}_{\varepsilon_i} \quad , \quad \prod_{r=1}^{n} \big(X_i \, \varepsilon^{1-r} - 1 \big) \;=\; \prod_{r=1}^{n} \big(\varepsilon^r - 1 \big) \binom{X_i}{n}_{\varepsilon}$$

for all $X \in \{K, L\}$, $i \in I$ and $0 \leq n \leq \ell - 1$, and similarly $Z_i^n \;=\; [n]_{\varepsilon_i}! \, Z_i^{(n)}$ for all $Z \in \{F, E\}$, $i \in I$ and $0 \leq n \leq \ell - 1$. Now, the condition $n \leq \ell - 1$ implies that all the coefficients $\prod_{r=1}^{n} \big(\varepsilon_i^r - 1 \big)$, $\prod_{r=1}^{n} \big(\varepsilon^r - 1 \big)$ and $[n]_{\varepsilon_i}!$ that occur above are non-zero elements in $\mathbb{Q}_\varepsilon$, whence we deduce at once our claim. □

Remark 7.3.6. From the PBW Theorem 7.3.4 and Proposition 7.3.5 it follows that $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} = \widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ is a finite-dimensional $\mathbb{Q}_\varepsilon$-Hopf algebra of dimension $\ell^{\dim(\mathfrak{g}_{(D)})}$.

Next result yields a strict link (a multiparameter version of a well-known result) between small MpQG's and quantum Frobenius morphisms for restricted MpQG's; indeed, one could take it as an alternative way to introduce small MpQG's.

Theorem 7.3.7. *Let* $\mathbf{q} := \big(q_{ij} \big)_{i,j \in I}$ *be of integral type, let* $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \xrightarrow{\dot{\widehat{Fr}}_\ell} U_{\mathbb{Q}_\varepsilon}\big(\dot{\mathfrak{g}}_B\big)$ *be the scalar extension of the quantum Frobenius morphism of Theorem 7.2.2 and finally let* $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} \xrightarrow{\;\iota\;} \dot{\widehat{U}}_{\mathbf{q},\varepsilon}$ *the natural embedding of* $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ *into* $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}$ *. Then*

$$1 \longrightarrow \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} \xrightarrow{\;\iota\;} \dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \xrightarrow{\dot{\widehat{Fr}}_\ell} U_{\mathbb{Q}_\epsilon}\big(\dot{\mathfrak{g}}_B\big) \longrightarrow 1$$

is an exact sequence of Hopf $\mathbb{Q}_\epsilon$*–algebras which is cleft.*

A similar statement holds true for $\widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ *and the scalar extension of the quantum Frobenius morphism* $\widehat{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \xrightarrow{\widehat{Fr}_\ell} U_{\mathbb{Q}_\epsilon}\big(\hat{\mathfrak{g}}_B\big)$ *when* $\mathbf{q}$ *is strongly integral.*

Proof. By Theorem 7.3.4, $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is free over $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$. So, to show that the sequence is exact it is enough to prove that $\operatorname{Ker}\Big(\dot{\widehat{Fr}}_\ell\Big) = \dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \, \dot{\widehat{\mathfrak{u}}}^{\,+}_{\mathbf{q},\varepsilon}$. This follows along the same lines as for the canonical case (proved in [A, Lemma 3.4.2]), so we skip it.

To prove that the extension is cleft, we use the well-known fact that an extension of algebras is cleft if and only if it is Galois and has a normal basis (see, e.g., [DT]). Since the extension is a Hopf algebra extension, it follows that it is Galois, see [Sch2, Remark 1.6]. The normal basis property follows from [Sch1, 4.3] since $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is pointed. Indeed, by the PBW Theorem 7.1.3 one may define an algebra filtration U_n of $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ such that U_0 is the subalgebra generated by $K_i^{\pm 1}$, $L_i^{\pm 1}$ $(i \in I)$, and $E_i^{(n)}$, $F_i^{(n)}$, $\binom{M\,;\,c}{n}_{\varepsilon} \in U_n$ $(i \in I, \; n \in \mathbb{N})$. By Theorem 5.2.13 and Lemma 5.2.2, this is a coalgebra filtration, so the coradical of $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is contained in U_0 . As the latter is the linear span of group-like elements, it follows that $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is pointed. □

Remarks 7.3.8.

(a) The proof that the Hopf algebra extension above is cleft also follows by the proof of the canonical case given in [A, Lemma 3.4.3]. On the other hand, let us point out that the normal basis property means that $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is isomorphic to $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} \otimes U_{\mathbb{Q}_\epsilon}\big(\dot{\mathfrak{g}}_B\big)$ as left $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$–module and right $U_{\mathbb{Q}_\epsilon}\big(\dot{\mathfrak{g}}_B\big)$–comodule. Hence, the MpQG at a root of unity $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) = \widehat{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ can be seen as a "blend" of a restricted small MpQG, namely $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) = \widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$, and a "classical" geometrical object, namely $U_{\mathbb{Q}_\varepsilon}\big(\dot{\mathfrak{g}}_B\big) = U_{\mathbb{Q}_\varepsilon}\big(\hat{\mathfrak{g}}_B\big) = U_{\mathbb{Q}_\varepsilon}\big(\mathfrak{g}_B\big)$.

(b) Beside the canonical case, some variations of the quantum Frobenius homorphism are treated in the literature. For example, Lentner [Le] studies the quantum Frobenius map for the positive Borel algebras at small roots of unity, which are in fact Nichols algebras. Another is in [Mc], which provides the construction of the quantum Frobenius homomorphism for the positive part using Hall algebras. As in the Hopf algebra case, the quantum Frobenius map is used to study exact sequences of Nichols algebras. In [AAR2] it is shown how Nichols algebras give rise to positive parts of semisimple Lie algebras as images of the quantum Frobenius morphism.

7.3.9. Small MpQG's: the "unrestricted realization". We introduce now a second type of small MpQG's, defined in terms of unrestricted MpQG's. As in the restricted case, these are defined for a multiparameter of integral type $\mathbf{q}$. We shall eventually see that these "unrestricted" small MpQG's actually do coincide with the "restricted" ones.

Let $\mathbf{q}$ be a multiparameter of integral type, hence possibly of strongly integral type. Let $\widetilde{Fr}_\ell : \mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} \widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \cong \mathcal{R}^B_{\mathbf{q},\varepsilon} \otimes_{\mathcal{R}_{\mathbf{q},1}} \widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \longleftrightarrow \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ be the *unrestricted quantum Frobenius morphism* introduced in Theorem 7.2.6, a Hopf algebra monomorphism whose image is the central Hopf subalgebra Z_0 of $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ given in Definition 7.2.4. We consider the Hopf cokernel of $\widetilde{Fr}_\ell$, i.e. the quotient Hopf algebra

$$\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon} := \widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) := \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \Big/ \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})\, {Z_0}^+$$

— where ${Z_0}^+$ denotes the augmentation ideal of Z_0 — and similarly the cokernels of the restrictions of $\widetilde{Fr}_\ell$ to all various relevant multiparameter quantum subgroups of $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$: for instance, $\widetilde{\mathfrak{u}}^{\geq}_{\mathbf{q},\varepsilon} := \widetilde{U}^{\geq}_{\mathbf{q},\varepsilon} \Big/ \widetilde{U}^{\geq}_{\mathbf{q},\varepsilon}\big(Z_0^{\geq}\big)^+$, and so on and so forth. We call all these objects *"unrestricted small multiparameter quantum (sub)groups"*. When $\mathbf{q} = \check{\mathbf{q}}$ is the canonical multiparameter, this definition coincides with the one for the one-parameter small quantum group associated with $\mathfrak{g}$ given in [BG, III.6.4].

Since, by Proposition 7.2.5, $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is a free $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$–module of rank $\ell^{\dim(\mathfrak{g}_{(D)})}$, it follows that $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ is a finite-dimensional Hopf algebra of dimension $\ell^{\dim(\mathfrak{g}_{(D)})}$; indeed, we shall show that it actually coincides with $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} = \widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}$.

As direct consequence of definitions and previous results, we find structure results for unrestricted small MpQG's. The first one is about triangular decompositions:

Proposition 7.3.10. *(triangular decompositions for unrestricted small MpQG's)*

The multiplication in $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ provides $\mathcal{R}_{\mathbf{q},\varepsilon}$–module isomorphisms

$$\widetilde{\mathfrak{u}}^{-}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{0}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}^{\leq}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}^{0}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{-}_{\mathbf{q},\varepsilon} \;, \quad \widetilde{\mathfrak{u}}^{+}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{0}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}^{\geq}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}^{0}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{+}_{\mathbf{q},\varepsilon}$$

$$\widetilde{\mathfrak{u}}^{+,0}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{-,0}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}^{0}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}^{-,0}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{+,0}_{\mathbf{q},\varepsilon} \;, \quad \widetilde{\mathfrak{u}}^{\leq}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{\geq}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}^{\geq}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{\leq}_{\mathbf{q},\varepsilon}$$

$$\widetilde{\mathfrak{u}}^{+}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{0}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{-}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon} \cong \widetilde{\mathfrak{u}}^{-}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{0}_{\mathbf{q},\varepsilon} \underset{\mathcal{R}_{\mathbf{q},\varepsilon}}{\otimes} \widetilde{\mathfrak{u}}^{+}_{\mathbf{q},\varepsilon}$$

Proof. This can be proved like the similar result for unrestricted MpQG's, or can be deduced from the latter: details are left to the reader. ☐

The second result is a PBW-like theorem for unrestricted small MpQG's:

Theorem 7.3.11. *(PBW theorem for unrestricted small MpQG's)*

Every unrestricted small MpQG is a free $\mathcal{R}_{\mathbf{q},\varepsilon}$–module with $\mathcal{R}_{\mathbf{q},\varepsilon}$–basis made by the cosets of all PBW monomials — in the subset of a PBW basis (as given in Theorem 5.3.3) of the corresponding specialized unrestricted MpQG — in which the degree of each factor is less than ℓ . For instance, $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ has $\mathcal{R}_{\mathbf{q},\varepsilon}$–basis

$$\left\{ \prod_{k=N}^{1} \overline{F}_{\beta^k}^{f_k} \prod_{j \in I} L_j^{l_j} \prod_{i \in I} K_i^{c_i} \prod_{h=1}^{N} \overline{E}_{\beta^h}^{e_h} \;\middle|\; 0 \leq f_k, l_j, c_i, e_h < \ell \right\}$$

and similarly holds for $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{\pm}$, $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{\pm,0}$, $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{0}$, $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{\leq}$ and $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{\geq}$.

Proof. This follows at once from definitions and from Proposition 7.2.5. □

The results in §5 and Theorem 7.1.4 lead us to the following theorem.

Theorem 7.3.12. *The Hopf $\mathcal{R}_{\mathbf{q},\varepsilon}$–algebra $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ is a 2–cocycle deformation of $\widetilde{\mathfrak{u}}_{\check{\mathbf{q}},\varepsilon}$.*

Proof. Denote by $\check{Z}_0$ the subalgebra of $\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$ that defines $\widetilde{\mathfrak{u}}_{\check{\mathbf{q}},\varepsilon}$. Since $\mathbf{q}$ is of integral type, Z_0 and $\check{Z}_0$ are both central Hopf subalgebras of $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ and $\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$, respectively. By Theorem 7.1.4*(a)*, we know that the Hopf $\mathcal{R}_{\mathbf{q},\varepsilon}$–algebra $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is a 2–cocycle deformation of $\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$. As the 2–cocycle giving the deformation is

$$\begin{gathered}\sigma_\varepsilon(x,y) \;:=\; \varepsilon_{\mu\nu}^{1/2} \qquad \text{if} \quad x = K_\mu \ \text{ or } \ x = L_\mu \ , \quad y = K_\nu \ \text{ or } \ y = L_\nu \\ \sigma_\varepsilon\big(U_{\check{\mathbf{q}}}(\mathfrak{g})\,, U_{\check{\mathbf{q}}}(\mathfrak{g})^{\oplus}\big) \;:=\; 0 \;=:\; \sigma_\varepsilon\big(U_{\check{\mathbf{q}}}(\mathfrak{g})^{\oplus}\,, U_{\check{\mathbf{q}}}(\mathfrak{g})\big)\end{gathered}$$

it follows that $\sigma_\varepsilon\big|_{\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g}) \otimes \check{Z}_0 + \check{Z}_0 \otimes \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})} = \epsilon \otimes \epsilon$, the trivial 2–cocycle, with ϵ the counit of $\widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$; in particular, $Z_0 = \big(\check{Z}_0\big)_{\sigma_\varepsilon} = \check{Z}_0$ as Hopf algebras. Finally, if we define $\bar{\sigma}_\varepsilon : \widetilde{\mathfrak{u}}_{\check{\mathbf{q}},\varepsilon} \otimes \widetilde{\mathfrak{u}}_{\check{\mathbf{q}},\varepsilon} \longrightarrow \mathcal{R}_{\mathbf{q},\varepsilon}^{B}$ by $\bar{\sigma}_\varepsilon\big(\bar{x},\bar{y}\big) := \sigma_\varepsilon(x,y)$ for $x,\, y \in \widetilde{U}_{\check{\mathbf{q}},\varepsilon}(\mathfrak{g})$, a straightforward calculation shows that $\bar{\sigma}_\varepsilon$ is a 2–cocycle for $\widetilde{\mathfrak{u}}_{\check{\mathbf{q}},\varepsilon}$ and $\big(\widetilde{\mathfrak{u}}_{\check{\mathbf{q}},\varepsilon}\big)_{\sigma_\varepsilon} \cong \widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$. □

Now we extend the ground ring from $\mathcal{R}_{\mathbf{q},\varepsilon}^{B} = \mathbb{Z}[\varepsilon]$ to the cyclotomic field $\mathbb{Q}_\varepsilon$ generated over $\mathbb{Q}$ by an ℓ–th root of unity: all algebras then will be taken as defined over $\mathbb{Q}_\varepsilon$ (via scalar extension), even though we keep the same notation. In this case, we have the following structural result:

Proposition 7.3.13. *Let us consider $\widetilde{U}_{\mathbf{q},\varepsilon}$ and $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$, as well as their quantum subgroups, as defined over $\mathbb{Q}_\varepsilon$ (via scalar extension). Then we have:*

(a) $\widetilde{U}_{\mathbf{q},\varepsilon}$ is generated by $\big\{\,\overline{E}_i\,, L_i^{\pm 1}, K_i^{\pm 1}, \overline{F}_i\big\}_{i \in I}$, and $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ is generated by the corresponding set of cosets;

(b) $\widetilde{U}_{\mathbf{q},\varepsilon}^{+}$ and $\widetilde{U}_{\mathbf{q},\varepsilon}^{-}$ are generated respectively by $\big\{\,\overline{E}_i\big\}_{i \in I}$ and by $\big\{\,\overline{F}_i\big\}_{i \in I}$, and similarly $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{+}$ and $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{-}$ are generated by the corresponding sets of cosets;

(c) $\widetilde{U}_{\mathbf{q},\varepsilon}^{+,0}$, $\widetilde{U}_{\mathbf{q},\varepsilon}^{-,0}$ and $\widetilde{U}_{\mathbf{q},\varepsilon}^{0}$ are generated respectively by $\big\{K_i^{\pm 1}\big\}_{i \in I}$, by $\big\{L_i^{\pm 1}\big\}_{i \in I}$ and by $\big\{K_i^{\pm 1}, L_i^{\pm 1}\big\}_{i \in I}$, and similarly $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{+,0}$, $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{-,0}$ and $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{0}$ are generated by the corresponding sets of cosets;

(d) $\widetilde{U}_{\mathbf{q},\varepsilon}^{\leq}$, resp. $\widetilde{U}_{\mathbf{q},\varepsilon}^{\geq}$, is generated by $\big\{L_i^{\pm 1}, \overline{F}_i\big\}_{i \in I}$, resp. by $\big\{\,\overline{E}_i\,, K_i^{\pm 1}\big\}_{i \in I}$; similarly $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{\leq}$, resp. $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}^{\geq}$, is generated by the corresponding set of cosets;

(e) in all claims (a) through (d) above, one can freely replace any $\overline{E}_i$ or $\overline{F}_j$ with E_i or F_j respectively, and still have a set of generators.

Proof. It is enough to prove claim *(a)*, as the other are similar. By construction, $\widetilde{U}_{\mathbf{q},\varepsilon}$ is generated by (the specialization of) all the $K_i^{\pm 1}$'s, all the $L_i^{\pm 1}$'s and all the quantum root vectors $\overline{E}_\alpha$ and $\overline{F}_\alpha$. Now, $\overline{E}_\alpha = \big(\varepsilon_{\alpha,\alpha} - 1\big)\, E_\alpha$ so the $\overline{E}_\alpha$'s can be replaced with the E_α's, because $\big(\varepsilon_{\alpha,\alpha} - 1\big)$ is invertible in $\mathbb{Q}_\varepsilon$. Moreover, each quantum root vector E_α can be expressed, by construction (cf. §4.1), as a suitable q–iterated quantum bracket of some of the E_i's; as $E_i = \big(\varepsilon_i^2 - 1\big)^{-1}\, \overline{E}_i$, the $\overline{E}_i$'s alone are enough to generate all the $\overline{E}_\alpha$'s over $\mathbb{Q}_\varepsilon$. A similar argument works for the $\overline{F}_\alpha$'s, hence the claim for $\widetilde{U}_{\mathbf{q},\varepsilon}$ follows, and that for $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ is an obvious consequence. Claim *(e)* is clear as well from the above analysis. □

By construction, the projection π from $\widetilde{U}_{\mathbf{q},\varepsilon}$ to $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ and the scalar extension of the quantum Frobenius morphism $\widetilde{Fr}_\ell$ match together to yield a short exact sequence of Hopf $\mathbb{Q}_\varepsilon$–algebras. As before, this sequence allows to reconstruct the unrestricted MpQG $\widetilde{U}_{\mathbf{q},\varepsilon}$ as a *cleft extension*, as the following shows:

Theorem 7.3.14. *Let* $\mathbf{q} := \big(q_{ij}\big)_{i,j \in I}$ *be a multiparameter of integral type.*

Let $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \xhookrightarrow{\widetilde{Fr}_\ell} \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ *be the scalar extension to* $\mathbb{Q}_\varepsilon$ *of the unrestricted quantum Frobenius morphism of Theorem 7.2.6 and let*

$$\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \;:=\; \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \Big/ \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})\, \widetilde{U}_{\mathbf{q},1}(\mathfrak{g})^+$$

be the quotient Hopf algebra. Then

$$1 \longrightarrow \widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \xrightarrow{\;\widetilde{Fr}_\ell\;} \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \xrightarrow{\;\pi\;} \widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \longrightarrow 1 \tag{7.18}$$

is a central exact sequence of Hopf $\mathbb{Q}_\epsilon$*–algebras which is cleft.*

Proof. By Proposition 7.2.5 we know that $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is a free $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$–module of rank $\ell^{\dim(\mathfrak{g}_{(D)})}$. Since $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$ is central and $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) := \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \Big/ \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})\, \widetilde{U}_{\mathbf{q},1}(\mathfrak{g})^+$, by [Mo, Proposition 3.4.3] we have that $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) = \widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})^{\mathrm{co}\,\pi}$ and the sequence is exact. As we did before for the restricted case, to prove that the extension is cleft we show that it is Galois and has a normal basis. Since the extension is a Hopf algebra extension, it follows that it is Galois, see [Sch2, Remark 1.6]. The normal basis property follows from [Sch1, 4.3] as $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is a pointed Hopf algebra, since it is generated by group-like and skew-primitive elements. □

Remarks 7.3.15. *(a)* By the normal basis property, $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is isomorphic to $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g}) \otimes \widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ as left $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$–module and right $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$–comodule. Hence, the MpQG at a root of unity $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ can be understood as a "blend" of a *classical* geometrical object — namely $\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})$, which is $\mathcal{O}\big(\widetilde{G}_B^*\big)$ since $\mathbf{q}$ is of integral type, see Theorem 6.2.9 — and a *quantum* one — the unrestricted small MpQG $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$.

(b) Borrowing language from geometry — *without claiming to be precise, by no means* — the exact sequence (7.18) can be interpreted as follows: $\widetilde{U}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ defines a principal bundle of Hopf $\mathbb{Q}_\varepsilon$–algebras over the Poisson group $\mathit{Spec}(Z_0) = \mathit{Spec}\big(\widetilde{U}_{\mathbf{q},1}(\mathfrak{g})\big) = \mathit{Spec}\big(\mathcal{O}\big(\widetilde{G}_B^*\big)\big) = \widetilde{G}_B^*$, and, as the extension is *cleft*, that bundle is *globally trivializable*.

7.3.16. Small MpQG's: identifying the two realizations. So far we considered small MpQG's of two kinds, namely restricted and unrestricted ones. We will show now that these two types *over* $\mathbb{Q}_\varepsilon$ actually coincide, up to isomorphism:

Theorem 7.3.17. *Consider the associated small MpQG's of either type over the ground ring* $\mathbb{Q}_\varepsilon$ *(via scalar extension from* $\mathcal{R}^B_{\mathbf{q},\varepsilon} = \mathbb{Z}[\varepsilon]$ *).*

Then $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \cong \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}(\mathfrak{g})\ \big(= \widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g})\big)$ *as Hopf algebras over* $\mathbb{Q}_\varepsilon$.

A similar statement holds true for the various (small) quantum subgroups, namely $\widetilde{\mathfrak{u}}^{\geq}_{\mathbf{q},\varepsilon} \cong \dot{\widehat{\mathfrak{u}}}^{\geq}_{\mathbf{q},\varepsilon}\ \big(= \widehat{\mathfrak{u}}^{\geq}_{\mathbf{q},\varepsilon}\big)$ *, etc.*

Proof. We prove that $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \cong \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \big(= \widehat{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g}) \big)$, the rest being similar.

To begin with, from Proposition 7.3.5 we know that $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon} := \dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ — when defined over the extended ground ring $\mathbb{Q}_\varepsilon$ — is generated by $\big\{ E_i\,, L_i^{\pm 1}, K_i^{\pm 1}, F_i \big\}_{i \in I}$. Moreover, from Theorem 7.3.2 we can deduce a complete set of relations for this generating set: indeed, these relations can be also described as being of two types:

(a) the relations arising (through specialization) from those respected by the same-name elements — i.e., $E_i\,, L_i^{\pm 1}, K_i^{\pm 1}, F_i$ $(i \in I)$ *— inside the restricted MpQG* $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}$ *(before specialization) just by formally writing "ε" instead of "q",*

(b) the "singular" relations $E_i^\ell = 0\,,\ L_i^\ell - 1 = 0\,,\ K_i^\ell - 1 = 0\,,\ F_i^\ell = 0\ \ (i \in I)$ *that are induced from the relations in* $\dot{\widehat{U}}_{\mathbf{q},\varepsilon}$

$$X_i^\ell \;=\; [\ell]_{q_i}!\, X_i^{(\ell)} \quad , \qquad \prod_{s=0}^{\ell-1} \binom{Y_i\,;-s}{1}_q \;=\; \prod_{c=1}^{\ell-1} \binom{c+1}{c}_q \cdot \binom{Y_i\,;1-\ell}{\ell}_q$$

— for all $i \in I$ *,* $X \in \{E\,,F\}$ *and* $Y \in \{L\,,K\}$ *— when specializing q to ε .*

Overall, this provides another concrete, explicit presentation of $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ *over* $\mathbb{Q}_\varepsilon$ by generators and relations (with less generators than that arising from Theorem 7.3.2). In addition, as a byproduct we find — comparing with Theorem 7.3.4 — another PBW theorem for $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ (over $\mathbb{Q}_\varepsilon$), stating that $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ admits the following $\mathbb{Q}_\varepsilon$–basis

$$\Big\{ \prod_{k=N}^1 F_{\beta^k}^{f_k} \prod_{j \in I} L_j^{l_j} \prod_{i \in I} K_i^{c_i} \prod_{h=1}^N E_{\beta^h}^{e_h} \Big\}_{0 \leq f_k,\, l_j,\, c_i,\, e_h < \ell} \tag{7.19}$$

On the other hand, we know by Proposition 7.3.13 that $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon} := \widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}(\mathfrak{g})$ is generated over $\mathbb{Q}_\varepsilon$ by $\big\{ E_i\,, L_i\,, K_i\,, F_i \big\}_{i \in I}$, because $L_i^\ell = 1 = K_i^\ell$ in $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$, by definition (so that we can get rid of L_i^{-1} and K_i^{-1}); in particular, from Theorem 7.3.11 we can deduce that another possible PBW $\mathbb{Q}_\varepsilon$–basis for $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ is

$$\Big\{ \prod_{k=N}^1 F_{\beta^k}^{f_k} \prod_{j \in I} L_j^{l_j} \prod_{i \in I} K_i^{c_i} \prod_{h=1}^N E_{\beta^h}^{e_h} \Big\}_{0 \leq f_k,\, l_j,\, c_i,\, e_h < \ell} \tag{7.20}$$

Now, the generators $E_i\,, L_i, K_i, F_i$ $(i \in I)$ of $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ do respect all relations that come by straightforward rescaling from the relations respected by the generators $\overline{E}_i\,, L_i\,, K_i\,, \overline{F}_i$ $(i \in I)$. In turn, the latter are of two types:

(a) the relations arising (through specialization) from those respected by the same-name elements — i.e., $\overline{E}_i\,, L_i^{\pm 1}, K_i^{\pm 1}, \overline{F}_i$ $(i \in I)$ *— inside the unrestricted MpQG* $\widetilde{U}_{\mathbf{q},\varepsilon}$ *(before specialization) by formally writing "ε" instead of "q",*

(b) the "singular" relations $\overline{E}_i^\ell = 0\,,\ L_i^\ell - 1 = 0\,,\ K_i^\ell - 1 = 0\,,\ \overline{F}_i^\ell = 0\ \ (i \in I)$ *induced from the "relations" in* $\widetilde{U}_{\mathbf{q},\varepsilon}$

$$\overline{X}_i^{\,\ell} \;\equiv\; 0 \mod (Z_0)^+ \quad , \qquad \overline{Y}_i^{\,\ell} \;\equiv\; 1 \mod (Z_0)^+$$

— for all $X \in \{E\,,F\}$ *,* $Y \in \{L\,,K\}$ *,* $i \in I$ *— when one specializes q to ε .*

The outcome is that all this yields an explicit presentation of $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ over $\mathbb{Q}_\varepsilon$ by generators — namely $E_i\,, L_i\,, K_i\,, F_i$ $(i \in I)$ — and relations.

Comparing the previous analyses, we find that $\dot{\widehat{\mathfrak{u}}}_{\mathbf{q},\varepsilon}$ and $\widetilde{\mathfrak{u}}_{\mathbf{q},\varepsilon}$ share identical presentation: more precisely, *mapping* $E_i \mapsto E_i\,,\ L_i \mapsto L_i\,,\ K_i \mapsto K_i\,,\ F_i \mapsto F_i\ \big(i \in I\big)$ *yields a well-defined isomorphism of* $\mathbb{Q}_\varepsilon$*–algebras*; in addition, tracking the whole construction one sees at once that *this is also a morphism of Hopf algebras*. Finally, comparing (7.19) and (7.20) shows that *this is indeed an isomorphism*, q.e.d. □

Remark 7.3.18. As an application of the previous result, even for the classical (uniparameter) small quantum groups one can always make use of either realization of them: the (most widely used) restricted one, or the unrestricted one.

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